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On a global analytic Positivstellensatz

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Abstract. We consider several modified versions of the Positivstellensatz for global analytic functions that involve infinite sums of squares and/or positive semidefinite analytic functions. We obtain a general local-global criterion which localizes the obstruction to have such a global result. This criterion allows us to get completely satisfactory results for analytic curves, normal analytic surfaces and real coherent analytic sets whose connected components are all compact.

1. Introduction and statements of the results

The Positivstellensatz appears as an *algebraic certificate* in the framework of the semialgebraic geometry to determine when certain types of real sets are empty. This kind of certificate can be formulated in general for any commutative ring A via the real spectrum $\operatorname{Spec}_r(A)$ as follows:

Theorem 1.1. (Abstract Positivstellensatz) *Let $f_1, ..., f_s, g_1, ..., g_r, m_1, ..., m_k \in A$. The following assertions are equivalent*:

(a) *The set $\{f_1=0, ..., f_s=0, g_1\geq 0, ..., g_r\geq 0, m_1\neq 0, ..., m_k\neq 0\}\subset \operatorname{Spec}_r(A)$ is empty, where we abuse notation in an obvious way*;

(b) *There exists a relation in A of the form*

$$\sum_{\nu} a_\nu \prod_{k=1}^{r} g_k^{\nu_k} + \Big(\prod_{j=1}^{k} m_j^{n_j}\Big)^2 + \sum_{i=1}^{s} b_i f_i = 0,$$

where each n_i is a non-negative integer, $\nu=(\nu_1, ..., \nu_r)\in\{0,1\}^r$ is a multiindex, a_ν is a sum of squares in A for each ν, and $b_i\in A$ for $i=1, ..., s$.

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For a proof of the previous result see for instance [9, Proposition 4.4.1]. In what follows, to simplify notation, given $g_1,\dots,g_r\in A$ and a multiindex $\nu=(\nu_1,\dots,\nu_r)\in\{0,1\}^r$, we denote by g^ν the product $g_1^{\nu_1}\dots g_r^{\nu_r}$. For our purposes, the following equivalent statement will be better.

Theorem 1.2. *Let $g_1,\dots,g_r,m\in A$. The following assertions are equivalent*:
(a) $\{g_1\geq 0,\dots,g_r\geq 0\}\setminus\{m=0\}=\varnothing$ *in* $\operatorname{Spec}_r(A)$;
(b) *There exists an equation of the form*

$$m^{2n}+\sum_\nu a_\nu g^\nu=0,$$

where n is a non-negative integer and a_ν is a sum of squares in A for each ν.

Indeed, it is obvious that Theorem 1.2 is a particular case of Theorem 1.1. Let us see now that Theorem 1.1 follows from Theorem 1.2. Take $g_{r+1}=f_1,\dots$, $g_{r+s}=f_s$, $g_{r+s+1}=-f_1,\dots$, $g_{r+2s}=-f_s$ and $m=m_1\dots m_r$. A straightforward computation shows that the subsets $S_1=\{f_1=0,\dots,f_s=0,g_1\geq 0,\dots,g_r\geq 0,m_1\neq 0,\dots,m_k\neq 0\}$ and $S_2=\{g_1\geq 0,\dots,g_{r+2s}\geq 0\}\setminus\{m=0\}$ of $\operatorname{Spec}_r(A)$ are equal. If $S_2=S_1=\varnothing$, by Theorem 1.2, there exists an equation of the form

$$m^{2n}+\sum_{\nu'} a_{\nu'}(g')^{\nu'}=0, \tag{1}$$

where $n\geq 0$, $\nu'=(\nu_1,\dots,\nu_r,\nu_{r+1},\dots,\nu_{r+s},\nu_{r+s+1},\dots,\nu_{r+2s})$, each $a_{\nu'}$ is a sum of squares in A and

$$(g')^{\nu'}=g_1^{\nu_1}\dots g_r^{\nu_r}f_1^{\nu_{r+1}}\dots f_s^{\nu_{r+s}}(-f_1)^{\nu_{r+s+1}}\dots(-f_s)^{\nu_{r+2s}}.$$

Computing a little with the formula (1), one concludes that there exists $b_1,\dots,b_s\in A$ such that

$$\sum_\nu a_\nu g^\nu+\Big(\prod_{j=1}^k m_j^{n_j}\Big)^2+\sum_{i=1}^s b_if_i=0.$$

This proves that (a) implies (b) in Theorem 1.1. The other implication is easier and we here omit the details.

In what follows, whenever we refer to a Positivstellensatz we will mean an algebraic certificate of the kind described in Theorem 1.2. Nevertheless, a nice application of the Positivstellensatz is related to the representation of positive semidefinite elements over *basic constructible closed subsets* of the real spectrum:

Theorem 1.3. *Let $g_1, ..., g_r, f \in A$. The following assertions are equivalent*:
(a) $f \geq 0$ *on the basic constructible closed set* $\{g_1 \geq 0, ..., g_r \geq 0\} \subset \operatorname{Spec}_r(A)$;
(b) *There exists an equation of the form*

$$\Big(\sum_\nu a_\nu g^\nu\Big) f = f^{2m} + \sum_\nu a'_\nu g^\nu,$$

where n is a non-negative integer and a_ν and a'_ν are sums of squares in A for each ν.

Again, the previous result gives a reformulation of the Positivstellensatz, equivalent to the previous ones. Indeed, to see that Theorem 1.3 follows from Theorem 1.2 it is enough to take $g_1, ..., g_r, g_{r+1} = -f$ and $m = f$ and to compute a little. On the other hand, to check that Theorem 1.2 follows from Theorem 1.3 it is enough to take $g_1, ..., g_r$, $f = -m^2$.

It is clear that the Positivstellensatz, in any of its multiple forms, admits an obvious formulation, from the geometric viewpoint, for the ring of real functions of a certain class over a real set of a certain type, without refering to the real spectrum. Nevertheless, the most satisfactory results only appear for those situations on which the real spectrum behaves neatly, that is, when the abstract formulation and the geometric one coincides and we can apply Theorems 1.1, 1.2 and 1.3. The real spectrum tool has been proved fruitful to understand and solve Positivstellensatzes for polynomial functions, Nash functions, analytic function germs at points and compact sets...(see [7] and [9] for more details) but it has fallen short in dealing with global analytic functions without compactness assumptions. Maybe this lack of a suitable machinery is the main reason why the problem for general global analytic functions has been missing any substantial progress, even for more general formulations than we present in this article. Recall that the most relevant result for the analytic setting, which refers strongly to the compact case to use Theorem 1.2 in a determining way, goes back to the 1980s ([14], [7, Chapter VIII, Theorem 5.6]). Namely, the following result.

Theorem 1.4. *Let $g_1, ..., g_r, m\colon \mathbb{R}^n \to \mathbb{R}$ be real-analytic functions and consider the sets $S = \{g_1 \geq 0, ..., g_r \geq 0\}$ and $Y = \{m = 0\}$. Assume that S is a compact set. Then,*

$$T = S \setminus Y = \varnothing \iff \sum_\nu s_\nu g^\nu + m^{2\beta} = 0 \quad \textit{for some integer } \beta \geq 0,$$

where each s_ν is a sum of squares of analytic functions on $\mathbb{R}^n$.

Note that Theorem 1.4 also holds if we *only* assume that $S \cap Y$ is a compact set. This is a straightforward consequence of the fact that $T = S \setminus Y$ is the empty set if and only if $S = S \cap Y \subset Y$.

However, in case $S\cap Y$ is not compact the situation is more delicate. In fact, even in the simplest case (dimension $n=1$), it is not possible to have a similar result to the classical Positivstellensatz stated above. To check this, we introduce the following example strongly inspired in [6, Example 6.2]:

Example 1.5. Consider two analytic functions $f,g\colon \mathbb{R}\to\mathbb{R}$ such that their common zero set is $\{f=0\}=\{g=0\}=\{n\in\mathbb{Z}: n>0\}$. Assume moreover that the germ f_n of f at n has initial form $(-1)^n(t-n)$ and that the germ g_n of g at n has initial form $(-1)^n(t-n)^{2n-1}$. Since $\{f\geq 0\}=\{g\geq 0\}$, we have that $\{-f\geq 0, g\geq 0\}\setminus\{f=0\}=\varnothing$.

If the classical Positivstellensatz held for the line $\mathbb{R}$, there would exist sums of squares $s_0,s_1,s_2,s_3\in\mathcal{O}(\mathbb{R})$ and an integer $\alpha\geq 0$ such that

$$s_0-s_1f+s_2g-s_3fg+f^{2\alpha}=0,$$

or equivalently

$$(s_1+gs_3)f=s_0+s_2g+f^{2\alpha}.$$

If we compare orders at the point $n=\alpha+1$ in the previous equation, we achieve a contradiction. Indeed, the left-hand side has order equal to

$$\omega_1=\min\{\omega(s_1)+1,\omega(s_3)+2\alpha+2\}$$

which is either odd, or even greater than or equal to $2\alpha+2$. On the other hand, the right-hand side has order

$$\omega_2=\min\{2\alpha,\omega(s_0),\omega(s_2)+2\alpha+1\}=\min\{2\alpha,\omega(s_0)\}$$

which is an even number less than or equal to 2α, a contradiction.

Roughly speaking, the *difficulty* which appears in the previous example, and in general in the noncompact analytic case, is that the *vanishing multiplicity* of the function g can grow arbitrarily while the non-negative integer α is fixed. Note that this arbitrary growth of the *vanishing multiplicity* cannot happen in the algebraic case nor in the compact analytic case, where results relative to the Positivstellensatz are already well known as we have pointed out above. Thus, the kind of statement we can expect for the general analytic case could be the following:

PSS. *Let $g_1,\dots,g_r,m\colon \mathbb{R}^n\to\mathbb{R}$ be real-analytic functions and consider the sets $S=\{g_1\geq 0,\dots,g_r\geq 0\}$ and $Y=\{m=0\}$. Then,*

$$T=S\setminus Y=\varnothing \iff \sum_\nu s_\nu g^\nu=0,$$

where the functions s_ν are (finite) sums of squares of analytic functions whose zero set is contained in Y.

Note that since the zero set of each analytic function s_ν must be contained in Y, all this functions s_ν are nonzero whenever $m\neq 0$. Moreover, here arises the

17th Hilbert problem for the ring of global analytic functions $\mathcal{O}(\mathbb{R}^n)$ which, at the moment, is still open for $n\geq 3$ even in a more general formulation involving convergent sums of squares of meromorphic functions. For the sake of the reader, we recall here the formulation of both problems and some related terminology:

H17. *Every positive semidefinite analytic function* $f\colon \mathbb{R}^n\to\mathbb{R}$ *is a finite sum of squares of meromorphic functions on* $\mathbb{R}^n$*.*

H17$_\infty$. *Every positive semidefinite analytic function* $f\colon \mathbb{R}^n\to\mathbb{R}$ *is an infinite sum of squares of meromorphic functions on* $\mathbb{R}^n$*.*

We proceed to recall what we understand as an infinite sum of squares of meromorphic functions. First, an *infinite sum of squares of analytic functions on an open set* $\Omega\subset\mathbb{R}^n$ is a series $\sum_{k\geq 1} f_k^2$ where all $f_k\in\mathcal{O}(\Omega)$, such that

(i) the f_k's have holomorphic extensions F_k's, all defined in the same neighbourhood V of Ω in $\mathbb{C}^n$,

(ii) for every compact set $L\subset V$, $\sum_{k\geq 1}\sup_L |F_k|^2<+\infty$.

Under these assumptions, the infinite sum $\sum_{k\geq 1} f_k^2$ defines well an analytic function f on Ω and we write $f=\sum_{k\geq 1} f_k^2\in\mathcal{O}(\Omega)$; of course, this trivially includes finite sums.

Now, we say that an analytic function $f\colon \Omega\to\mathbb{R}$ is a (possibly infinite) *sum of squares* (of meromorphic functions on Ω) if there is $g\in\mathcal{O}(\Omega)$ such that $g^2 f$ is a (possibly infinite) sum of squares of analytic functions on Ω. The zero set $\{g=0\}$ is called the *bad set* of that representation as a sum of squares. As we will see later, we will often need to have a *controlled bad set*, that is, a bad set contained in the zero set $\{f=0\}$. Concerning the difference between arbitrary and controlled bad sets, we recall the following result.

Proposition 1.6. ([4, Lemma 4.1]) *Let* $\Omega\subset\mathbb{R}^n$ *be open, and let* $f\colon\Omega\to\mathbb{R}$ *be an analytic function which is a finite* (*resp. possibly infinite*) *sum of squares of meromorphic functions. Then* f *is a finite* (*resp. possibly infinite*) *sum of squares with controlled bad set.*

For more details about the 17th Hilbert problem for global analytic functions, see [4] and [12].

In what follows, we will denote by $\mathbf{PSS}_\infty$ the property PSS when we admit in its formulation that the coefficients s_ν are possible infinite sums of squares instead of only finite sums of squares. We have the following implications:

$$\mathrm{PSS} \implies \mathrm{H17},$$

$$\mathrm{PSS}_\infty \implies \mathrm{H17}_\infty.$$

Indeed, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a positive semidefinite analytic function not identically zero. Assume that PSS (resp. PSS_∞) holds for $\mathbb{R}^n$. Take $S=\{g=-f\geq 0\}=\{f=0\}$ and $m=f$, hence, $Y=\{f=0\}$. Since $T=S\setminus Y=\varnothing$, there exist finite (resp. possibly infinite) sums of squares $a,b\in\mathcal{O}(\mathbb{R}^n)$ whose zero set is contained in Y, such that $a-fb=0$. Hence, $b^2f=ab$ and f is a finite (resp. possibly infinite) sum of squares of meromorphic functions and H17 (resp. $\mathrm{H17}_\infty$) holds for $\mathcal{O}(\mathbb{R}^n)$.

Thus, since at the moment none of these facts is known we introduce also a more general statement (a weak Positivstellensatz) that involves only positive semidefinite analytic functions, which in principle might not be sums of squares:

wPSS. *Let $g_1,\ldots,g_r,m\colon \mathbb{R}^n\to\mathbb{R}$ be real-analytic functions and consider the sets $S=\{g_1\geq 0,\ldots,g_r\geq 0\}$ and $Y=\{m=0\}$. Then,*

$$T=S\setminus Y=\varnothing \quad\Longleftrightarrow\quad \sum_\nu a_\nu g^\nu=0,$$

where the functions a_ν are positive semidefinite analytic functions whose zero set is contained in Y.

One can check the following equivalences:

$$\text{wPSS and H17} \quad\Longleftrightarrow\quad \text{PSS},$$

$$\text{wPSS and H17}_\infty \quad\Longleftrightarrow\quad \text{PSS}_\infty.$$

For that, it is crucial to have representations as sums of squares with controlled bad sets of the involved positive semidefinite analytic functions (see Proposition 1.6).

Our main purpose in this work is to introduce a local-global criterion related to wPSS of the same nature as the ones introduced in [4] to approach both formulations of the 17th Hilbert problem. Before that we need to introduce some notation and terminology.

Let $Z\subset\mathbb{R}^n$ be a closed set and let $g_1,\ldots,g_r,m\colon\Omega\to\mathbb{R}$ be analytic functions defined on an open neighbourhood Ω of Z. We set $S=\{g_1\geq 0,\ldots,g_r\geq 0\}$ and $Y=\{m=0\}$. We say that:

(a) The *property* PSS (*resp.* PSS_∞) *holds for* $\{g_1,\ldots,g_r;m\}$ *at* Z if there exists a perhaps smaller open neighbourhood $\Omega'\subset\Omega$ of Z such that

$$T=(S\setminus Y)\cap\Omega'=\varnothing \quad\Longleftrightarrow\quad \sum_\nu s_\nu g^\nu=0,$$

where the functions $s_\nu\in\mathcal{O}(\Omega')$ are finite (resp. possibly infinite) sums of squares of analytic functions whose zero set is contained in Y.

(b) The *property* wPSS *holds for* $\{g_1,\dots,g_r;m\}$ *at* Z if there exists a perhaps smaller open neighbourhood $\Omega'\subset\Omega$ of Z such that

$$T=(S\setminus Y)\cap\Omega'=\varnothing \quad\Longleftrightarrow\quad \sum_{\nu} a_\nu g^\nu=0,$$

where the functions $a_\nu\in\mathcal{O}(\Omega')$ are positive semidefinite analytic functions whose zero set is contained in Y.

More generally, we say that PSS_∞ or wPSS hold at a closed set Z if they hold at Z for any family of analytic functions $\{g_1,\dots,g_r;m\}$ defined on an open neighbourhood Ω of Z.

The main result of this work is the following local-global criterion which has relevant applications as we will see in Corollary 1.10 and Theorem 1.11.

Theorem 1.7. *Let $g_1,\dots,g_r,m\colon\mathbb{R}^n\to\mathbb{R}$ be real-analytic functions and consider the sets $S=\{g_1\ge0,\dots,g_r\ge0\}$ and $Y=\bigcup_\alpha Y_\alpha=\{m=0\}$, where the Y_α's are the connected components of Y. Assume that* wPSS *holds for $\{g_1,\dots,g_r;m\}$ at Y_α for all α such that $Y_\alpha\cap S\neq\varnothing$, then* wPSS *holds for $\{g_1,\dots,g_r;m\}$ (at $\mathbb{R}^n$).*

We can also get the following result refering the property PSS_∞:

Theorem 1.8. *Let $g_1,\dots,g_r,m\colon\mathbb{R}^n\to\mathbb{R}$ be real-analytic functions and consider the sets $S=\{g_1\ge0,\dots,g_r\ge0\}$ and $Y=\bigcup_\alpha Y_\alpha=\{m=0\}$, where the Y_α's are the connected components of Y. Assume that PSS_∞ holds for $\{g_1,\dots,g_r;m\}$ at Y_α for all α such that $Y_\alpha\cap S\neq\varnothing$, then*

$$S\setminus Y=\varnothing \quad\Longleftrightarrow\quad \sum_{\nu} s_\nu g^\nu+a=0,$$

where each $s_\nu\in\mathcal{O}(\mathbb{R}^n)$ is a (possibly infinite) sum of squares of analytic functions on $\mathbb{R}^n$ with zero set contained in Y, and $a\in\mathcal{O}(\mathbb{R}^n)$ is positive semidefinite and its zero set is contained in Y.

In fact, as we have pointed out above, an affirmative solution to PSS_∞ implies an affirmative solution to $\mathrm{H17}_\infty$. Using this, we also get Theorem 1.9, which is a local-global criterion for the property PSS_∞ and can be understood as the counterpart of Theorem 1.7 for such a property. Recall that the property wPSS involves *positive semidefinite analytic functions* and that PSS_∞ involves (*possibly infinite*) *sums of squares of analytic functions.*

Before stating Theorem 1.9, we recall here that a global analytic set X in an open set $\Omega\subset\mathbb{R}^n$ is the common zero set of a finite family of global analytic functions on Ω. We define the ring of global analytic functions on X as the quotient ring $\mathcal{O}(X)=\mathcal{O}(\Omega)/\mathcal{J}(X)$, where $\mathcal{J}(X)$ is the ideal of global analytic functions of Ω vanishing on X.

Theorem 1.9. *Let $Y\subset\mathbb{R}^n$ be a global analytic set and let $\{Y_\alpha\}_\alpha$ be its connected components. Assume that* PSS_∞ *holds at Y_α for all α. Let $g_1,\dots,g_r$, $m\colon\mathbb{R}^n\to\mathbb{R}$ be real-analytic functions such that $\{m=0\}=Y$. Then* PSS_∞ *holds for $\{g_1,\dots,g_r;m\}$ (at $\mathbb{R}^n$).*

Proof. Indeed, since PSS_∞ holds at Y_α for all α, we deduce, by Theorem 1.7, that wPSS holds for $\{g_1,\dots,g_r;m\}$. Next, by [4, Theorem 1.5], wehave that any positive semidefinite analytic function $f\colon\mathbb{R}^n\to\mathbb{R}$ whose zero set is Y is a (possibly infinite) sum of squares of meromorphic functions on $\mathbb{R}^n$ with controlled bad set. Finally, putting all together and clearing denominators we conclude that PSS_∞ holds for $\{g_1,\dots,g_r;m\}$, as wanted. $\square$

Some relevant consequences of Theorems 1.7 and 1.9 are summarized in the following result:

Corollary 1.10. *Let $g_1,\dots,g_r,m\colon\mathbb{R}^n\to\mathbb{R}$ be real-analytic functions and consider the sets $S=\{g_1\geq0,\dots,g_r\geq0\}$ and $Y=\bigcup_\alpha Y_\alpha=\{m=0\}$, where the Y_α's are the connected components of Y.*

(a) *If $S\cap Y_\alpha$ is a compact set for all α, then* wPSS *holds for $\{g_1,\dots,g_r;m\}$.*

(b) *If Y_α is a compact set for each α such that $Y_\alpha\cap S\neq\varnothing$, then* PSS_∞ *holds for $\{g_1,\dots,g_r;m\}$.*

Notice that since there is no known bound for the least number of squares needed to represent a sum of squares of meromorphic functions (see [4]), we cannot state similar results to Theorems 1.7, 1.8, 1.9 and Corollary 1.10 for the property PSS. Such a least number of squares refers to the study of the *finiteness* property (that is, every sum of squares is a finite sum of squares) and the computation of Pythagoras numbers of rings of meromorphic functions (for more details see [4]).

On the other hand, Theorems 1.7 and 1.8 and Corollary 1.10 provide, in the same way as Theorem 1.3, but under the conditions of their statements, a nice representation for the positive semidefinite analytic functions over a *closed basic global semianalytic set* of $\mathbb{R}^n$. Related to this see also [1].

Furthermore, more precise results can be given in the following cases:

(a) analytic curves;

(b) normal analytic surfaces;

(c) coherent analytic subsets of $\mathbb{R}^n$ whose connected components are all compact.

In these cases Theorem 1.7 can be understood as a global Positivstellensatz in the sense of PSS for (a) and (b) and in the sense of PSS_∞ for (c).

Theorem 1.11. *Let Ω be an open set in $\mathbb{R}^n$ and $X\subset\Omega$ be an analytic set in Ω which is either a curve, a normal surface or a real coherent analytic set whose* *connected components* X_i, $i\in I$, *are all compact. Let* $g_1,\dots,g_r,m\colon X\to\mathbb{R}$ *be real-analytic functions and consider* $S=\{g_1\geq 0,\dots,g_r\geq 0\}$ *and* $Y=\{m=0\}$. *Then*

$$S\setminus Y=\varnothing \iff \sum_{\nu} s_\nu g^\nu=0,$$

where:

(A) *Each* s_ν *is a sum of two squares in* $\mathcal{O}(X)$ *whose zero set is contained in* Y, *if* X *is an analytic curve*;

(B) *Each* s_ν *is a sum of five squares in* $\mathcal{O}(X)$ *whose zero set is contained in* Y, *if* X *is a normal analytic surface*;

(C) *Each* s_ν *is a* (*possibly infinite*) *sum of squares in* $\mathcal{O}(X)$ *such that* $s_\nu|_{X_i}$ *is a finite sum of squares in* $\mathcal{O}(X_i)$ *for all* $i\in I$ *and its zero set is contained in* Y, *if* X *is a real coherent analytic set whose connected components are all compact.*

In fact, part (B) also holds true for any analytic coherent surface with isolated singularities. Part (C), is almost straightforward:

Proof of part (C). Indeed, this part of Theorem 1.11 follows from these two facts:

(1) $\mathcal{O}(X)=\prod_{i\in I}\mathcal{O}(X_i)$;

(2) If $Z\subset\mathbb{R}^n$ is a compact analytic set and $f\colon X\to\mathbb{R}$ is a positive semidefinite analytic function on X, then PSS holds for Z ([7, Chapter VIII, Theorem 5.6]). $\square$

The article is organized as follows. In Section 2 we introduce several tools that will be very relevant for the proofs of Theorems 1.7, 1.8 and Corollary 1.10. These results will be proved in Section 3. Finally, Section 4 is devoted to proving parts (A) and (B) of Theorem 1.11.

2. Preliminary results

The purpose of this section is to introduce some preliminary results that will be crucial when proving Theorems 1.7 and 1.8 and Corollary 1.10. Troughout the rest of the article, Int and Cl stand for the topological interiors and closures, respectively. If necessary, we will use a subscript to indicate where the Int and/or Cl are considered.

Lemma 2.1. *Let* $Y\subset\mathbb{R}^n$ *be a global analytic subset and let* W *be an open neighbourhood of* Y *in* $\mathbb{R}^n$. *Then, there exists an analytic function* $g\colon\mathbb{R}^n\to\mathbb{R}$ *such that* $Y\subset\{g>0\}\subset\{g\geq 0\}\subset W$.

Proof. First, since Y is a global analytic set, we can find a sum of squares $f\in\mathcal{O}(\mathbb{R}^n)$ whose zero set is Y. Let W_1,W_2 be open neighbourhoods of Y in $\mathbb{R}^n$ such that $\mathrm{Cl}_{\mathbb{R}^n}(W_1)\subset W_2\subset\mathrm{Cl}_{\mathbb{R}^n}(W_2)\subset W$. Let $\{\sigma_1,\sigma_2\colon\mathbb{R}^n\to\mathbb{R}\}$ be a smooth partition of unity subordinated to the covering $\{\Omega_1 = W_2, \Omega_2 = \mathbb{R}^n \setminus \mathrm{Cl}_{\mathbb{R}^n}(W_1)\}$ of $\mathbb{R}^n$. Recall that $0 \leq \sigma_2 \leq 1$ and that $\sigma_2|_{\mathrm{Cl}_{\mathbb{R}^n}(W_1)} \equiv 0$ and $\sigma_2|_{\mathbb{R}^n \setminus W_2} \equiv 1$.

Now, consider the continuous function on $\mathbb{R}^n \setminus Y$ defined by $\varphi = \max\{1, 1/f\}$. The product $\sigma_2\varphi$ extends to a positive semidefinite continuous function on $\mathbb{R}^n$ which is identically 0 on $\mathrm{Cl}_{\mathbb{R}^n}(W_1)$ and strictly greater than 0 on $\mathbb{R}^n \setminus W_2$. Moreover, on $\mathbb{R}^n \setminus W_2$ we have $\sigma_2\varphi f \geq 1$.

Indeed, if $x \in \mathbb{R}^n \setminus W_2$ we have $\sigma_2\varphi f(x) = \varphi f(x) = \max\{1, 1/f\} f(x)$. Thus, if $f(x) \geq 1$ it is clear that $\max\{1, 1/f\} = 1$ and $\sigma_2\varphi f(x) \geq 1$. On the other hand, if $f(x) < 1$ we have that $\max\{1, 1/f\} f(x) = (1/f) f(x) = 1$.

By Whitney's approximation theorem ([13, Section 1.6]), there exists an analytic function $\eta\colon \mathbb{R}^n \to \mathbb{R}$ such that $|\sigma_2\varphi + 1 - \eta| < \frac{1}{2}$ on $\mathbb{R}^n$. Let us see that $(\eta f)|_{\mathbb{R}^n \setminus W_2} \geq 1$. Indeed, if $x \in \mathbb{R}^n \setminus W_2$ we have

$$\begin{aligned}\eta f(x) &= (\sigma_2\varphi + 1) f(x) + (\eta - \sigma_2\varphi - 1) f(x) \geq (\sigma_2\varphi + 1) f(x) - |\sigma_2\varphi + 1 - \eta| f(x) \\ &> (\sigma_2\varphi + 1) f(x) - \tfrac{1}{2} f(x) = \big(\sigma_2\varphi + \tfrac{1}{2}\big) f(x) > \sigma_2\varphi f(x) \geq 1.\end{aligned}$$

Finally, take the function $g = \frac{1}{2} - \eta f$. It is clear that

$$Y \subset \{g > 0\} \subset \{g \geq 0\} \subset \{\eta f < 1\} \subset W_2 \subset W,$$

and therefore g fits our situation. $\square$

Lemma 2.2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an analytic function and let $\Omega \subset \mathbb{R}^n$ be an open neighbourhood of $Y = \{f = 0\}$ and $t \in \mathcal{O}(\Omega)$. Then there exists an analytic function $a\colon \mathbb{R}^n \to \mathbb{R}$ such that f divides $a|_\Omega - t$ in $\mathcal{O}(\Omega)$.*

Proof. The function t defines a global cross section of the sheaf $\mathcal{O}_{\mathbb{R}^n}/(f)$ as

$$\begin{cases} t \ \operatorname{mod}(f)\mathcal{O}_{\mathbb{R}^n,x}, & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

By Cartan's theorem B, [10], this section is just the class of an analytic function $a\colon \mathbb{R}^n \to \mathbb{R}$, such that f divides $a|_\Omega - t$ in $\mathcal{O}(\Omega)$. $\square$

The purpose of the following result is to show how we can control the zero sets of some of the involved functions of certain types of equations.

Lemma 2.3. *Let X be a global analytic set in an open set $\Omega \subset \mathbb{R}^n$ and let $g_1, \ldots, g_r, f\colon X \to \mathbb{R}$ be global analytic functions on X. Assume that we have a relation of the type*

$$\sum_\nu a_\nu g^\nu + f^2 = 0,$$

where each $a_\nu \in \mathcal{O}(X)$ is a positive semidefinite global analytic function on X. Then, there exist positive units $u, u_0 \in \mathcal{O}(X)$ such that

$$\sum_\nu (a_\nu + (fu_0)^4) g^\nu + f^2 u = 0.$$

Proof. Consider the global analytic function

$$\Delta = \frac{f^4 \sum_\nu g^\nu}{1+(f^2 \sum_\nu g^\nu)^2}.$$

Adding and substracting Δ to the given relation

$$\sum_\nu a_\nu g^\nu + f^2 = 0$$

we get that

$$\sum_\nu \left(a_\nu + \frac{f^4}{1+(f^2 \sum_\nu g^\nu)^2} \right) g^\nu + f^2 \left(1 - \frac{f^2 \sum_\nu g^\nu}{1+(f^2 \sum_\nu g^\nu)^2} \right) = 0.$$

Obviously, the function $u_0 = 1/\sqrt[4]{1+(f^2 \sum_\nu g^\nu)^2}$ is a positive unit of the ring $\mathcal{O}(X)$. Finally, the function

$$u = 1 - \frac{\Delta}{f^2} = \frac{1+(f^2 \sum_\nu g^\nu)^2 - (f^2 \sum_\nu g^\nu)}{1+(f^2 \sum_\nu g^\nu)^2} = \frac{\frac{3}{4} + \left(f^2 \sum_\nu g^\nu - \frac{1}{2}\right)^2}{1+(f^2 \sum_\nu g^\nu)^2}$$

is clearly a positive unit in $\mathcal{O}(X)$. □

Remark 2.4. Note that for each ν, the function

$$A_\nu = a_\nu + \frac{f^4}{1+(f^2 \sum_\nu g^\nu)^2}$$

is a positive semidefinite analytic function whose zero set is $\{a_\nu = 0\} \cap \{f = 0\} \subset Y$. Moreover, if a_ν is a sum of squares in $\mathcal{O}(X)$, then $A_\nu = a_\nu + (fu_0)^4$ is also a sum of squares in $\mathcal{O}(X)$.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $g_1, \dots, g_r \in \mathcal{O}(\Omega)$ be analytic functions. Suppose that the set $S = \{g_1 \geq 0, \dots, g_r \geq 0\}$ is empty. Then, there exist strictly positive analytic functions $b_\nu \in \mathcal{O}(\Omega)$ such that*

$$1 + \sum_\nu b_\nu^2 g^\nu = 0.$$

Proof. Indeed, by the strict Positivstellensatz (see [1, Corollary 2.6]), we find strictly positive analytic functions $c_1, \dots, c_r \in \mathcal{O}(\Omega)$ such that

$$1 + c_1^2 g_1 + \dots + c_r^2 g_r = 0.$$

We define the following strictly positive analytic functions on Ω:

$$c_\nu=\begin{cases} c_i, & \text{if } \nu=e_i=(0,\dots,0,\overset{(i)}{1},0,\dots,0),\\ 1/\sqrt{2^r(1+(g^\nu)^2)}, & \text{otherwise.}\end{cases}$$

We obtain the following identity

$$v+\sum_\nu c_\nu^2 g^\nu=0,$$

where

$$v=1-\sum_{\substack{\nu\neq e_i\\ i=1,\dots,r}} c_\nu^2 g^\nu=1-\sum_{\substack{\nu\neq e_i\\ i=1,\dots,r}}\frac{g^\nu}{2^r(1+(g^\nu)^2)}>1-\frac{2^r-r}{2^r}=\frac{r}{2^r}>0$$

is a strictly positive analytic function on Ω. Finally, taking $b_\nu=c_\nu/\sqrt{v}$ we are done. $\square$

The following result will allow us to control the behaviour of a finite family of analytic functions outside of an open neighbourhood of a global analytic set.

Lemma 2.6. *Let $Y\subset\mathbb{R}^n$ be a global analytic set in $\mathbb{R}^n$ and let $f_1,\dots,f_s\colon\mathbb{R}^n\to\mathbb{R}$ be analytic functions whose zero set is Y. Let $b_1,\dots,b_s\colon\mathbb{R}^n\setminus Y\to\mathbb{R}$ be strictly positive analytic functions. Then, for each open neighbourhood V of Y in $\mathbb{R}^n$ there exists a strictly positive analytic function $\rho\colon\mathbb{R}^n\to\mathbb{R}$ such that $0<\rho|f_j|<b_j/2$ on $\mathbb{R}^n\setminus V$ for $j=1,\dots,s$.*

Proof. Let U be an open neighbourhood of Y in $\mathbb{R}^n$ such that $U\subset\mathrm{Cl}_{\mathbb{R}^n}(U)\subset V$. Let $\sigma_1,\sigma_2\colon\mathbb{R}^n\to\mathbb{R}$ be a smooth partition of unit subordinated to the open covering $\{\mathbb{R}^n\setminus\mathrm{Cl}_{\mathbb{R}^n}(U),V\}$. Recall that

(a) $0\le\sigma_1,\sigma_2\le1$ and $\sigma_2=1-\sigma_1$,

(b) $\sigma_1|_{\mathrm{Cl}_{\mathbb{R}^n}(U)}\equiv0$ and $\sigma_1|_{\mathbb{R}^n\setminus V}\equiv1$.

Then the smooth functions $b'_j=b_j\sigma_1+\sigma_2$ on $\mathbb{R}^n\setminus Y$ can be extended smoothly by 1 to the whole $\mathbb{R}^n$. Note that $b'_j=b_j$ on $\mathbb{R}^n\setminus V$ and that $b'_j=1$ on U for $j=1,\dots,s$. Hence, each b'_j is strictly positive on $\mathbb{R}^n$.

Consider now the continuous function on $\mathbb{R}^n$ given by the formula

$$\varepsilon=\frac{1}{3}\min_j\left\{\frac{b'_j}{|f_j|+1}\right\},$$

which is also strictly positive on $\mathbb{R}^n$. By Whitney's approximation theorem [13, Section 1.6] there exists an analytic function $\rho\colon\mathbb{R}^n\to\mathbb{R}$ such that $|\varepsilon-\rho|<\frac{1}{2}\varepsilon$ on $\mathbb{R}^n$, or equivalently, $\frac{1}{2}\varepsilon<\rho<\frac{3}{2}\varepsilon$. Hence, ρ is strictly positive on $\mathbb{R}^n$ and

$$\rho|f_j| < \frac{3}{2}\varepsilon|f_j| \leq \frac{b'_j}{2(|f_j|+1)}|f_j| \leq \frac{b'_j}{2}.$$

Since $b'_j=b_j$ on $\mathbb{R}^n\setminus V$, we deduce that $\rho|f_j|<b_j/2$ on $\mathbb{R}^n\setminus V$, as wanted. $\square$

Lemma 2.7. *Let $r,\eta,\delta\in\mathbb{R}$ be real numbers such that $0<\delta<1$ and $|r-\eta|<\delta/(1+2|\eta|)$. Then $|r^2-\eta^2|<\delta$.*

Proof. First, note that

$$|r|-|\eta| \leq \big||r|-|\eta|\big| \leq |r-\eta| < \frac{\delta}{1+2|\eta|} < 1,$$

hence $|r|<1+|\eta|$. Thus, we get that

$$|r^2-\eta^2| = |r-\eta||r+\eta| \leq |r-\eta|\,\big||r|+|\eta|\big| < \frac{\delta}{1+2|\eta|}(1+2|\eta|) = \delta,$$

as wanted. $\square$

The following result is a slight generalization of [5, theorem on p. 454] that will be useful for the proof of Theorem 1.7.

Lemma 2.8. *Let $Z\subset\mathbb{R}^n$ be a global analytic set and A be a global semianalytic set in Z. Let $a_1,...,a_s\in\mathcal{O}(Z)$ be global analytic functions and $Y\subset Z$ be a global analytic set such that*

$$\{a_i=0\}\cap\mathrm{Cl}_Z(A) \subset Y \subset \{a_i=0\}$$

for each $i=1,...,s$. Then, there exists a positive semidefinite analytic equation λ of Y in Z such that $\lambda<|a_i|$ on $\mathrm{Cl}_Z(A)\setminus Y$ for all $i=1,...,s$ and $\lambda<1$ on Z.

Proof. First, by [5, theorem on p. 454], for each $i=1,...,s$ there exists a positive semidefinite analytic equation λ_i of Y in Z such that $\lambda_i<|a_i|$ on $\mathrm{Cl}_Z(A)\setminus Y$. Substituting λ_i by $\lambda_i/(1+\lambda_i^2)$ we may assume that $\lambda_i<1$ on Z.

Now, we take $\lambda=\prod_{i=1}^r \lambda_i$ and we have

$$\lambda < \min_i\{\lambda_i\} < \min_i\{|a_i|\}$$

on $\mathrm{Cl}_Z(A)\setminus Y$, that is, $\lambda<|a_i|$ on $\mathrm{Cl}_Z(A)\setminus Y$ for all $i=1,...,s$. Finally, since each $\lambda_i<1$ on Z, we conclude that $\lambda<1$ on Z. $\square$

3. Proofs of the main results

The purpose of this section is to prove Theorems 1.7 and 1.8 and Corollary 1.10.

Proof of Theorem 1.7. It is clear that it is enough to check that if $S\setminus Y=\varnothing$, there exist positive semidefinite analytic functions $s_\nu\in\mathcal{O}(\mathbb{R}^n)$ whose zero sets are contained in Y such that $\sum_\nu s_\nu g^\nu=0$. The proof of this fact runs in several steps:

Step 1. *Initial preparation.* We denote by Y_0 the union of the connected components Y_α of Y such that $Y_\alpha\cap S\neq\varnothing$. Since $S\setminus Y_0=S\setminus Y=\varnothing$ and wPSS holds for $\{g_1,...,g_r;m\}$ at Y_α for each α such that $Y_\alpha\cap S\neq\varnothing$, there exists an open neighbourhood Ω of Y_0 in $\mathbb{R}^n$, whose closure $\operatorname{Cl}_{\mathbb{R}^n}(\Omega)$ is contained in the open set $\mathbb{R}^n\setminus(Y\setminus Y_0)$, such that $\sum_\nu t_\nu g^\nu=0$, where each $t_\nu\in\mathcal{O}(\Omega)$ is a positive semidefinite analytic function whose zero set is contained in Y_0. Multiplying the previous equation by m^2 we may assume that $\{t_\nu=0\}=Y_0$ for each ν. This fact will be useful later to apply Lemma 2.8.

Step 2. *Local equation around Y_0 involving global analytic functions.* We will find an open global semianalytic neighbourhood Ω of Y_0 in $\mathbb{R}^n$ and a local equation of the type

$$\sum_\nu a_\nu g^\nu+h^2+hu_2=0, \tag{2}$$

where $a_\nu, h\in\mathcal{O}(\mathbb{R}^n)$, $u_2\in\mathcal{O}(\Omega)$ is a positive analytic unit, h is a positive semidefinite analytic equation of Y_0 in $\mathbb{R}^n$ and each a_ν is positive semidefinite on Ω.

Indeed, we write $\sum_{\nu,\nu\neq 0} t_\nu g^\nu+\frac{1}{2}t_0+\frac{1}{2}t_0=0$. Multiplying the previous equation by $\frac{1}{2}t_0$ we get

$$\sum_{\nu,\nu\neq 0}\left(t_\nu\frac{t_0}{2}\right)g^\nu+\left(\frac{t_0}{2}\right)^2+\left(\frac{t_0}{2}\right)^2=0.$$

Thus, we may assume from the beginning that we have an equality of the type

$$\sum_\nu t_\nu g^\nu+f^2=0,$$

where $t_\nu, f\in\mathcal{O}(\Omega)$ are positive semidefinite analytic functions whose zero set is Y_0.

Now, by Lemma 2.3, there exist units $u, u_0\in\mathcal{O}(\Omega)$ such that

$$\sum_\nu(t_\nu+(fu_0)^4)g^\nu+f^2u=0.$$

Next, consider the coherent sheaf of ideals $\mathcal{J}$ given by

$$\mathcal{J}_x=\begin{cases} f\mathcal{O}_{\mathbb{R}^n,x}, & \text{if } x\in Y_0,\\ \mathcal{O}_{\mathbb{R}^n,x}, & \text{if } x\in\mathbb{R}^n\setminus Y_0.\end{cases}$$

Since locally principal ideal subsheafs of $\mathcal{O}_{\mathbb{R}^n}$ are principal, $\mathcal{J}$ is generated by some $f_0\in\mathcal{O}(\mathbb{R}^n)$. Take $h=f_0^2$; we have that the zero set of h is Y_0 and that $u_1=h/f^2$ defines a positive unit of the ring $\mathcal{O}(\Omega)$.

Next, by Lemma 2.2, for each $t_\nu+(fu_0)^4$ there exists an analytic extension $a_\nu\colon\mathbb{R}^n\to\mathbb{R}$ such that the analytic function $t_\nu+(fu_0)^4-a_\nu$ is divisible by h^3 in $\mathcal{O}(\Omega)$. In fact, we may assume, after shrinking Ω if necessary, that each a_ν is positive semidefinite on Ω.

Indeed, note that

$$a_\nu|_\Omega=t_\nu+(fu_0)^4+h^3\zeta_\nu=t_\nu+(fu_0)^4(1+(fu_0)^2\theta_\nu)$$

for some $\zeta_\nu,\theta_\nu\in\mathcal{O}(\Omega)$. Since the function fu_0 vanishes at Y_0, we have that $1+(fu_0)^2\theta_\nu$ is strictly positive in a suitable neighbourhood of Y_0. After shrinking Ω, if necessary, we conclude that $a_\nu|_\Omega$ is positive semidefinite and that its zero set is Y_0.

Next, consider the analytic function $u_2=u_1^{-1}(u-f^2u_1^2-f^4u_1^3\sum_\nu\zeta_\nu g^\nu)\in\mathcal{O}(\Omega)$. In Ω we have

$$\begin{aligned}\sum_\nu a_\nu g^\nu+h^2+hu_2&=\sum_\nu(t_\nu+(fu_0)^4+(f^2u_1)^3\zeta_\nu)g^\nu\\&\quad+f^4u_1^2+f^2\Big(u-f^2u_1^2-f^4u_1^3\sum_\nu\zeta_\nu g^\nu\Big)\\&=\sum_\nu(t_\nu+(fu_0)^4)g^\nu+f^2u\\&=0.\end{aligned}$$

Now, shrinking again Ω and using the fact that f vanishes on Y_0 we may assume that u_2 is a positive unit in $\mathcal{O}(\Omega)$. Moreover, by Lemma 2.1, we may also assume that Ω is a global semianalytic set.

Step 3. *Global equation outside* Y_0 *and control of the behaviour of the functions* a_ν *outside of an open neighbourhood* W *of* Y_0. First, since $S\cap(\mathbb{R}^n\setminus Y_0)=S\setminus Y_0=\varnothing$, by Lemma 2.5, there exists strictly positive analytic functions $b_\nu\in\mathcal{O}(\mathbb{R}^n\setminus Y_0)$ such that

$$1+\sum_\nu b_\nu^2g^\nu=0.$$

Let W be an open neighbourhood of Y_0 on $\mathbb{R}^n$ such that $\mathrm{Cl}_{\mathbb{R}^n}(W)\subset\Omega$. By Lemma 2.6, there exists a positive unit $\rho\in\mathcal{O}(\mathbb{R}^n)$ such that $\rho h^2<\frac{1}{2}$ and $\rho|a_\nu|<b_\nu^2$ on $\mathbb{R}^n\setminus W$. Multiplying the equation (2) above by ρ we get the following identity on Ω:

$$\sum_\nu(\rho a_\nu)g^\nu+(\sqrt{\rho}h)^2+(\sqrt{\rho}h)(\sqrt{\rho}u_2)=0.$$

Note that except for $\sqrt{\rho}u_2$ (which is only defined in Ω) all the involved functions are globally analytic on $\mathbb{R}^n$. To simplify notation we denote ρa_ν, $\sqrt{\rho}h$ and $\sqrt{\rho}u_2$ again by a_ν, h and u_2, respectively. The new functions h and a_ν satisfy also the inequalities $h^2<\frac{1}{2}$ and $|a_\nu|<b_\nu^2$ on $\mathbb{R}^n\setminus W$.

Furthermore, we have the following identity, where all the involved functions are analytic on the open set $\mathbb{R}^n\setminus Y_0$:

$$\sum_\nu b_\nu^2g^\nu+h^2+w^2=0, \tag{3}$$

and $w=\sqrt{1-h^2}$ is a positive unit in $\mathcal{O}(\mathbb{R}^n)$.

Step 4. *Glueing the local and the global equations.* Consider the analytic function

$$q=-h^2-\sum_\nu a_\nu g^\nu\in\mathcal{O}(\mathbb{R}^n)$$

which is equal to hu_2 in Ω (see the identity (2)). Hence, q is positive semidefinite in Ω and $\{q=0\}\cap\Omega=Y_0$.

Since Ω is a global semianalytic subset of $\mathbb{R}^n$, by Lemma 2.8, there exists a positive semidefinite analytic equation $\lambda\colon\mathbb{R}^n\to\mathbb{R}$ of Y_0 such that $\lambda<|q|,|a_\nu|$ on $\Omega\setminus Y_0$ for all ν and $\lambda<1$ on $\mathbb{R}^n$.

Consider a smooth function $\sigma_1\colon\mathbb{R}^n\to\mathbb{R}$ with $\sigma_1^{-1}(1)=\mathbb{R}^n\setminus\Omega$ and $\sigma_1^{-1}(0)=\mathrm{Cl}_{\mathbb{R}^n}(W)$. Taking $2\sigma_1^2/(1+\sigma_1^4)$ instead of σ_1 we may assume that σ_1 is the square of a smooth function and less than or equal to 1. We take also $\sigma_2=1-\sigma_1$, which is also the square of a smooth function.

Now, consider the smooth functions

$$\psi_\nu=\sigma_1b_\nu^2+\sigma_2a_\nu\quad\text{and}\quad p=-h^2-\sum_\nu\psi_\nu g^\nu,$$

which satisfy the equation

$$h^2+\sum_\nu\psi_\nu g^\nu+p=0. \tag{4}$$

Let us see that the functions ψ_ν and p are positive semidefinite. Indeed, the positiveness of the functions ψ_ν follows straightforwardly from their definitions. Next, we proceed with p. Using (2) and (3), we rewrite p as

$$p=\sigma_1\Big(-h^2-\sum_\nu {b_\nu}^2 g^\nu\Big)+\sigma_2\Big(-h^2-\sum_\nu a_\nu g^\nu\Big)=\sigma_1 w^2+\sigma_2 q=\sigma_1 w^2+\sigma_2 h u_2.$$

From this expression it is clear that p is positive semidefinite on $\mathbb{R}^n$. Notice also that the zero sets of ψ_ν and p are equal to Y_0. Hence, (4) is the kind of equation we are looking for except for the important fact that the involved functions ψ_ν and p are not analytic.

Thus, our purpose now should be to modify the functions ψ_ν and p to obtain an analogous equation to (4), but involving in this case analytic functions whose zero sets are again Y_0. For that aim, we also introduce the continuous functions η_ν on $\mathbb{R}^n$ given by

$$\eta_\nu(x)=\begin{cases}\sqrt{(\psi_\nu(x)-a_\nu(x))/\lambda^2(x)}, & \text{if } x\in\mathbb{R}^n\setminus W,\\[2ex] 0, & \text{if } x\in W.\end{cases}$$

Since $\psi_\nu=a_\nu$ on W, the differences $\psi_\nu-a_\nu$ are flat on W. Thus, to see that the previous functions η_ν are well defined and continuous, it is enough to check that the functions $\psi_\nu-a_\nu$ are positive semidefinite on $\mathbb{R}^n\setminus W$. We have

$$\psi_\nu-a_\nu=\sigma_1 b_\nu^2+\sigma_2 a_\nu-a_\nu=\sigma_1 b_\nu^2+(\sigma_2-1)a_\nu=\sigma_1(b_\nu^2-a_\nu),$$

which is positive semidefinite on $\mathbb{R}^n\setminus W$ because $b_\nu^2-|a_\nu|>0$ on $\mathbb{R}^n\setminus W$ (see Step 3).

Next, we are going to approximate the functions η_ν, using again Whitney's approximation theorem, by suitable analytic functions which will fit our situation. For that, we need to construct a strictly positive continuous function on $\mathbb{R}^n$ which controls the approximation properly.

We extend the functions $\psi_\nu|_{\mathbb{R}^n\setminus W}$ and $p|_{\mathbb{R}^n\setminus W}$ to strictly positive smooth functions ϕ_ν and ϕ on $\mathbb{R}^n$. Such functions can be constructed (using a suitable partition of unit) because the zero sets of ψ_ν and p are equal to Y_0 which lies inside W. Next, consider the strictly positive continuous functions

$$\delta=\frac{1}{2}\min_\nu\left\{1,\phi_\nu,\frac{\min\{\phi,1\}}{2^r(1+(g^\nu)^2)}\right\}\quad\text{and}\quad\varepsilon=\frac{\delta}{1+2\max_\nu|\eta_\nu|}.$$

Let $r_\nu\colon\mathbb{R}^n\to\mathbb{R}$ be analytic approximations of η_ν such that $|\eta_\nu-r_\nu|<\varepsilon$ on $\mathbb{R}^n$. By Lemma 2.7, we have that $|\eta_\nu^2-r_\nu^2|<\delta$ for each ν.

We claim the following facts:

(a) *The global analytic functions* $a_\nu+\lambda^2r_\nu^2$ *are positive semidefinite on* $\mathbb{R}^n$ *and their zero sets are equal to* Y_0;

(b) *The global analytic function* $Q=-h^2-\sum_\nu(a_\nu+\lambda^2r_\nu^2)g^\nu$ *is positive semidefinite on* $\mathbb{R}^n$ *and its zero set is* Y_0.

We begin by proving (a). First, we have that $a_\nu+\lambda^2r_\nu$ is strictly positive outside W because if $x\in\mathbb{R}^n\setminus W$ we get

$$\begin{aligned}|\psi_\nu(x)-(a_\nu+\lambda^2r_\nu^2)(x)|&=\lambda^2(x)|(\eta_\nu^2-r_\nu^2)(x)|\\&<|(\eta_\nu^2-r_\nu^2)(x)|<\delta(x)<\frac{\phi_\nu}{2}(x)=\frac{\psi_\nu}{2}(x);\end{aligned}$$

hence, $0<(\psi_\nu/2)(x)<(a_\nu+\lambda^2r_\nu^2)(x)$.

Next, we check that $a_\nu+\lambda^2r_\nu^2$ is strictly positive on $W\setminus Y_0$. By construction, $\eta_\nu\equiv0$ on W; hence, for each $x\in W$ we have that $|r_\nu(x)^2|<\delta<1$. Thus, if $x\in W\setminus Y_0$ we get

$$(a_\nu+\lambda^2r_\nu^2)(x)\geq a_\nu(x)-\lambda^2(x)|r_\nu(x)^2|\geq\lambda(x)-\lambda^2(x)>0,$$

because $\lambda<|a_\nu|=a_\nu$ on $\Omega\setminus Y_0$ and $\lambda<1$ on $\mathbb{R}^n$. Finally, since a_ν and λ vanish on Y_0 we conclude that $a_\nu+\lambda^2r_\nu^2$ is positive semidefinite on $\mathbb{R}^n$ and that its zero set is Y_0.

Next, we check (b). We begin by proving that Q is strictly positive on $\mathbb{R}^n\setminus W$. Indeed,

$$\begin{aligned}|p(x)-Q(x)|&\leq\lambda^2(x)\Big|\sum_\nu(\eta_\nu^2-r_\nu^2)(x)g^\nu(x)\Big|<\sum_\nu|(\eta_\nu^2-r_\nu^2)(x)|\,|g^\nu(x)|\\&<\sum_\nu\delta(x)|g^\nu(x)|<\sum_\nu\frac{\phi(x)}{2^{r+1}(1+(g^\nu)^2(x))}|g^\nu(x)|<\frac{\phi(x)}{2}=\frac{p(x)}{2};\end{aligned}$$

hence, $0<p(x)/2<Q(x)$ for all $x\in\mathbb{R}^n\setminus W$. Now we check that $Q(x)>0$ on $W\setminus Y_0$. Since $\eta_\nu=0$ on W, for each $x\in W$ we have

$$|r_\nu^2(x)|<\delta(x)\leq\frac{1}{2^{r+1}(1+(g^\nu)^2(x))},$$

(recall the formula for δ above). Thus, if $x\in W\setminus Y_0$ we deduce

$$\begin{aligned}Q(x)&=q(x)-\lambda^2(x)\Big(\sum_\nu r_\nu^2(x)g^\nu(x)\Big)\\&\geq q(x)-\lambda^2(x)\Big|\sum_\nu r_\nu^2(x)g^\nu(x)\Big|\\&\geq q(x)-\lambda^2(x)\sum_\nu|r_\nu^2(x)|\,|g^\nu(x)|\end{aligned}$$

$$
\begin{aligned}
&> q(x) - \frac{\lambda^2(x)}{2^{r+1}} \sum_\nu \frac{1}{1+(g^\nu)^2(x)} |g^\nu(x)| \\
&> \lambda(x) - \lambda^2(x) \\
&> 0,
\end{aligned}
$$

because $\lambda < |q| = q$ on $\Omega \setminus Y_0$ and $\lambda < 1$ on $\mathbb{R}^n$. Finally, since Q vanishes on Y_0, we conclude that Q is positive semidefinite on $\mathbb{R}^n$ and that its zero set is Y_0.

Thus, if we take

$$
\begin{aligned}
&s_\nu = a_\nu + \lambda^2 r_\nu^2, && \text{if } \nu \neq 0, \\
&s_0 = Q + (a_0 + \lambda^2 r_0^2) + h^2,
\end{aligned}
$$

we achieve the equation

$$
\sum_\nu s_\nu g^\nu = 0,
$$

where each s_ν is a positive semidefinite analytic function on $\mathbb{R}^n$ whose zero set is Y_0, as wanted. $\square$

Remark 3.1. Note that the zero set of each s_ν is Y_0, which is the union of the connected components Y_α of Y such that $Y_\alpha \cap S \neq \varnothing$.

Next, we show how we should modify the proof of Theorem 1.7 to get Theorem 1.8.

Proof of Theorem 1.8. The only difference to the proof of Theorem 1.7 appears in the Step 2 when we extend, modulo $h^3\mathcal{O}(\Omega)$, the positive semidefinite functions $t_\nu \in \mathcal{O}(\Omega)$ to analytic functions $a_\nu \in \mathcal{O}(\mathbb{R}^n)$. At that point, for this proof we use the following fact: *If t_ν is a* (*possibly infinite*) *sum of squares in $\mathcal{O}(\Omega)$, we may assume that a_ν is a* (*possibly infinite*) *sum of squares in* $\mathcal{O}(\mathbb{R}^n)$ (see [4, Proposition 2.3]).

At the end, we take

$$
s_\nu = a_\nu + \lambda^2 r_\nu^2, \quad \text{and} \quad a = Q + h^2,
$$

and we are done. $\square$

Remarks 3.2. (a) It seems difficult to get, following the arguments above, a similar statement to Theorem 1.8 but which does not involve the positive semidefinite analytic function a. As can be checked following the proofs of Theorems 1.7 and 1.8, their developments need certain mobility to *glue analytically* the local equation around Y_0 with the global one outside Y_0 to get the global equation on $\mathbb{R}^n$.

(b) Moreover, if a in the statement of Theorem 1.8 was a sum of squares, we would obtain a local-global criterion for the PSS_∞ property.

Now, we are ready to prove Corollary 1.10.

Proof of Corollary 1.10. First, we prove (a). Since $Y_\alpha \cap S$ is a compact set, we have, by [7, Chapter VIII, Theorem 5.6], Lemma 2.3 and Remark 2.4, that PSS holds for $\{g_1, ..., g_r; m\}$ at Y_α for all α. Then, by Theorem 1.7, wPSS holds for $\{g_1, ..., g_r; m\}$.

Next, we prove (b). Since Y_α is a compact set for each α such that $S \cap Y_\alpha \neq \varnothing$, we have, by [7, Chapter VIII, Theorem 5.6], Lemma 2.3 and Remark 2.4, that PSS holds for $\{g_1, ..., g_r; m\}$ at Y_α for such α's. Then, by Theorem 1.7, wPSS holds for $\{g_1, ..., g_r; m\}$. In fact, by Remark 3.1, if $S \setminus Y = \varnothing$, there exist positive semidefinite analytic functions $s_\nu \in \mathcal{O}(\mathbb{R}^n)$ such that $\sum_\nu s_\nu g^\nu = 0$ and $\{s_\nu = 0\} = Y_0$, where Y_0 is the union of the connected components of Y whose intersections with S are nonempty. By [4, 1.7], the functions s_ν are (possibly infinite) sums of squares of meromorphic functions with controlled bad set. Denote by h_ν the denominators of such expressions as sums of squares. Multiplying the equation $\sum_\nu s_\nu g^\nu = 0$ by $h^2 = \prod_\nu h_\nu^2$ we conclude that PSS_∞ holds for $\{g_1, ..., g_r; m\}$. $\square$

Notice that the properties PSS, PSS_∞ and wPSS can be generalized in the natural way either from the global or the local viewpoint to any real global analytic set X in an open set Ω of $\mathbb{R}^n$. More precisely, we define $\mathcal{J}(X)$ as the set of the analytic functions on Ω which vanish on X. Then, we can state the following result for the ring $\mathcal{O}(\Omega)/\mathcal{J}(X)$ which is analogous to Theorem 1.7 and Corollary 1.10 for $\mathbb{R}^n$.

Corollary 3.3. *Let X be a real global analytic set in an open set Ω of $\mathbb{R}^n$ and let $g_1, ..., g_r$, $m\colon \Omega \to \mathbb{R}$ be real-analytic functions. Consider the sets*

$$S = \{g_1 \geq 0, ..., g_r \geq 0\} \cap X \quad \textit{and} \quad Y = \bigcup_\alpha Y_\alpha = \{m = 0\} \cap X,$$

where the Y_α's are the connected components of Y.

(a) *If* wPSS *holds for $\{g_1, ..., g_r; m\}$ at Y_α for all α such that $Y_\alpha \cap S \neq \varnothing$, then* wPSS *holds for $\{g_1, ..., g_r; m\}$ (at X).*

(b) *If $S \cap Y_\alpha$ is a compact set for all α, then* wPSS *holds for $\{g_1, ..., g_r; m\}$.*

(c) *If Y_α is a compact set for each α such that $Y_\alpha \cap S \neq \varnothing$, then* PSS_∞ *holds for $\{g_1, ..., g_r; m\}$.*

Proof. First, by Grauert's embedding theorem, we may assume that $\Omega = \mathbb{R}^n$. Let $g_{r+1}\colon \mathbb{R}^n \to \mathbb{R}$ be a global equation of X in $\mathbb{R}^n$ which is strictly negative on $\mathbb{R}^n \setminus X$. The result follows straightforwardly from Theorem 1.7 and Corollary 1.10 applied to the analytic functions $g_1, ..., g_r, g_{r+1}; m$ and setting, afterwards, $g_{r+1} = 0$. $\square$

Remarks 3.4. (a) In the same way, a similar result to Theorem 1.8 can be stated for a real global analytic set X.

(b) Note that for a *coherent* X the ring $\mathcal{O}(\Omega)/\mathcal{J}(X)$ is precisely the ring $\mathcal{O}(X)$ of analytic functions on X. Recall also that analytic curves and normal analytic surfaces are coherent analytic spaces.

4. Consequences for low dimension

The purpose of this section is to prove parts (A) and (B) of Theorem 1.11.

Proof of part (A) *of Theorem* 1.11. Let $\{X_i\}_{i\in I}$ be the irreducible components of X and let

$$I_1=\{i\in I : X_i\cap\{m=0\}\cap S \text{ is either empty or a discrete set}\}$$

and $I_2=I\setminus I_1$. Note that for each $i\in I_2$ the Zariski closure of $X_i\cap\{m=0\}\cap S$ is equal to X_i and that m is identically zero over the analytic curve $X_2=\bigcup_{i\in I_2}X_i$.

Let $X_1=\bigcup_{i\in I_1}X_i$. Note that the set $D=\{m|_{X_1}=0\}\cap\{g_1|_{X_1}\geq 0,\dots,g_r|_{X_1}\geq 0\}$ is a discrete set. By Corollary 3.3 (b), there exist positive semidefinite analytic functions $\sigma_\nu : X_1\to\mathbb{R}$ such that $\{\sigma_\nu=0\}=D$ and

$$\sum_\nu \sigma_\nu g^\nu|_{X_1}=0. \tag{5}$$

Next, recall that for each singular point p of an analytic curve X there exists an integer m_p only depending on p such that for each positive semidefinite analytic germ $\xi_p\in\mathcal{O}_{X_1,p}$ we have that $(\|x-p\|^2)^{2m_p}\xi_p$ is a square in $\mathcal{O}(X_{1,p})$ (see [15, Section III.3]). By Cartan's theorem B, there exists an analytic function λ on X such that$\{\lambda=0\}=D\cap\operatorname{Sing}(X_1)$ and

$$\lambda_p\equiv(\|x-p\|^2)^{2m_p}\mod \mathfrak{m}_p^{4m_p+1}$$

for all $p\in D\cap\operatorname{Sing}(X_1)$. Thus, $\lambda_p^2\sigma_{\nu,p}$ is a square in $\mathcal{O}(X_{1,p})$ for all $p\in D\cap\operatorname{Sing}(X_1)$. In view of the proof of [3, Theorem 1.2], each $\lambda^2\sigma_\nu\in\mathcal{O}(X_1)$ is a sum of two squares in $\mathcal{O}(X_1)$ whose zero set is contained in D.

Now, let $h_2 : X\to\mathbb{R}$ be a positive semidefinite equation of X_2 in X. We extend by 0 all the functions $s_\nu=h_2^2\lambda^2\sigma_\nu$ outside X_1. Thus, we get on X the equation

$$\sum_\nu s_\nu g^\nu=0,$$

where each $s_\nu\in\mathcal{O}(X)$ is a sum of two squares of analytic functions in $\mathcal{O}(X)$ whose zero set is contained in $D\cup X_2$. $\square$

Remark 4.1. In the statement of part (A) of Theorem 1.11 we may ask that the zero set of each s_ν is contained in $D\cup X_2$, which is the Zariski closure of $S\cap Y$.

Before proving part (B) of Theorem 1.11 we need a preliminary result that will give us a *square-free representation* of the set $S=\{g_1\geq 0,\dots,g_r\geq 0\}$ for a normal analytic surface X. The following proofs are strongly inspired by [8, Theorem 2, Section 2].

Lemma 4.2. *Let X be a normal analytic surface and let $Y=\bigcup_{i\in I} Y_i\subset X$ be an analytic curve in X, where the Y_i's are the irreducible components of Y. Let $a\colon X\to\mathbb{R}$ be a nonzero analytic function. Then, there exist analytic functions $\tilde{a},\Delta,b\colon X\to\mathbb{R}$ such that*:

(i) *b is a sum of squares in $\mathcal{O}(X)$ with zero set $\{b=0\}\subset Y$;*

(ii) *If $\tilde{a}$ vanishes along some Y_i, it does it with multiplicity* 1;

(iii) *Δ is a sum of squares in $\mathcal{O}(X)$ with discrete zero set contained in $\{b=0\}$;*

(iv) *$\Delta^2 a=\tilde{a}b$.*

Proof. For each $i\in I$ the ideal $\mathfrak{p}_i\subset\mathcal{O}(X)$ of all analytic functions vanishing on Y_i is a prime ideal of height 1, and, $\mathcal{O}(X)$ being normal, the localization $V_i=\mathcal{O}(X)_{\mathfrak{p}_i}$ is a discrete valuation ring. We will use freely the so-called multiplicity along Y_i, which is the real valuation m_{Y_i} associated with the discrete valuation ring V_i (see [8, Sections 1 and 2] for full details). Pick any uniformizer $g_i\in\mathfrak{p}_i$ of V_i, so that $m_{Y_i}(g_i)=1$. Write $m_{Y_i}(a)=2m_i+\varepsilon_i$, where $m_i\geq 0$ is an integer and $\varepsilon_i=0$ or $\varepsilon_i=1$. Since the valuation is real, we have that $a/g_i^{2m_i}$ is a unit in V_i or it is a meromorphic uniformizer of V_i, so that $m_{Y_i}(a/g_i^{2m_i})=1$. From this it follows that *at all points of Y_i off a discrete set* the following properties hold true:

(1) $a/g_i^{2m_i}$ is analytic;

(2) $m_{Y_i}(a/g_i^{2m_i})$ is 0 or 1;

(3) g_i generates the ideal of Y_i.

Choose for each $i\in I$ a sum of squares θ_i which is an equation for Y_i. Consider the sheaf of ideals given by

$$\mathcal{I}_x=\begin{cases}\Big(\prod_{i|x\in Y_i} g_i^{m_i},\prod_{i|x\in Y_i}\theta_i^{m_i}\Big)\mathcal{O}_{X,x}, & \text{for } x\in Y,\\ \mathcal{O}_{X,x}, & \text{otherwise.}\end{cases}$$

To see that this sheaf is well defined and coherent, one takes a neighbourhood U of $x\in X$ such that all the Y_i's that intersect U pass through x and checks that $\mathcal{I}$ is generated in U by the functions $\prod_{i|x\in Y_i} g_i^{m_i}$ and $\prod_{i|x\in Y_i}\theta_i^{m_i}$. By [11], since $\mathcal{I}$ is locally generated by at most two analytic germs, $\mathcal{I}$ is globally generated by finitely many sections $b_1,\dots,b_l\in\mathcal{O}(X)$. Consider $b=b_1^2+\dots+b_l^2$ so that $\{b=0\}=\bigcup_{m_i>0} Y_i$.

In this situation, on Y_i off a discrete set, $\mathcal{I}=(b_1,...,b_l)\mathcal{O}_X$ is generated by $g_i^{m_i}$, which readily implies that all the quotients $b_j/g_i^{m_i}$ are analytic there and at least one is a unit. Then

$$\frac{a}{b_1^2+...+b_l^2}=\frac{a}{g^{2m_i}}\Bigg/\frac{b_1^2+...+b_l^2}{g^{2m_i}}.$$

Since $a/g_i^{2m_i}$ is analytic and has $m_{Y_i}(a/g_i^{2m_i})$ is 0 or 1 off another discrete set, we deduce that

$$\frac{a}{b_1^2+...+b_l^2}=\frac{a}{g^{2m_i}}\Bigg/\frac{b_1^2+...+b_l^2}{g^{2m_i}}$$

is analytic on Y_i off a discrete set and one has that $m_{Y_i}(a/b)$ is 0 or 1 off a (bigger) discrete set $D_i\subset\bigcup_{m_i>0}Y_i\subset Y$. As the Y_i's form a locally finite family, we conclude that a/b is a meromorphic function whose zero set is contained in Y and whose poles form a discrete subset D of Y.

We are left to construct an analytic function Δ with discrete zero set D in order to *cancel* the poles of a/b. To do this, consider the coherent sheaf $(b{:}a)\mathcal{O}_X$. Recall that for each point $x\in X$ we have

$$(b:a)\mathcal{O}_{X,x}=\{f_x\in\mathcal{O}_{X,x}:f_xa_x\in b\mathcal{O}_{X,x}\}.$$

Thus, the support of the coherent sheaf $(b{:}a)\mathcal{O}_X$ is $\{x\in X{:}b_x\nmid a_x\}$, that is, the set D of poles of a/b.

The sheaf is generated in a neighbourhood of each pole x of a/b by finitely many sections $\delta_1,...,\delta_r$. We write $\delta=\sum_{k=1}^r\delta_k^2$ and we deduce that

$$a_x\delta_x=a_x\sum_{k=1}^r\delta_{k,x}^2=b_x\gamma_x$$

for some $\gamma_x\in\mathcal{O}_{X,x}$, that is, $a_x/b_x=\gamma_x/\delta_x$.

Furthermore, x is an isolated zero of δ. For that, suppose that there is $y\neq x$ arbitrarily close to x with $\delta(y)=0$. Then, all δ_k's vanish at y, and since the ideal $(b{:}a)\mathcal{O}_{X,y}$ is generated by them, we deduce that y is in the support of $(b{:}a)\mathcal{O}_{X,x}$. This means that a/b is not analytic at y, a contradiction.

Adding the square of an equation of X in $\mathbb{R}^n$, we extend δ to a sum of squares $\tilde{\delta}$ of analytic functions in a neigborhood of x in $\mathbb{R}^n$ that vanishes only at x; set $\mathcal{I}_x=\tilde{\delta}\mathcal{O}_{X,x}$. These ideals $\mathcal{I}_x$ glue to define a locally principal sheaf of ideals $\mathcal{I}$ on $\mathbb{R}^n$, whose zero set consists of the poles of a/b. Since $H^1(\mathbb{R}^n,\mathbb{Z}_2)=0$, all locally principal sheaves are globally principal, and $\mathcal{I}$ has a global generator Δ. This Δ is a non-negative analytic function on $\mathbb{R}^n$ whose zeros are the poles x of a/b, hence a discrete subset of $\{b=0\}$.

By construction $\Delta a/b$ is an analytic function and $\tilde{a}=\Delta^2 a/b$ vanishes with multiplicity 1 along all Y_i's such that $m_{Y_i}(a)$ is odd and does not vanish along any of the Y_i's such that $m_{Y_i}(a)$ is even. $\square$

Now, we are ready to prove part (B) of Theorem 1.11.

Proof of part (B) *of Theorem* 1.11. First, note that we may assume that X is irreducible. Recall that X being a normal surface, its irreducible components and its connected components coincide. We have to prove that: *If $S\setminus Y=\varnothing$, then there exist sums of five squares of analytic functions $s_\nu\in\mathcal{O}(X)$ such that $\{s_\nu=0\}\subset Y$ and* $\sum_\nu s_\nu g^\nu=0$.

We assume that $m\neq 0$ because otherwise there is nothing to prove. We write the Zariski closure of $\{g_1\geq 0,...,g_r\geq 0\}\cap\{m=0\}$ as the union of a discrete set D and a disjoint curve $Y_0=\bigcup_{i\in I}Y_i\subset Y$, where the Y_i's denote the irreducible components of Y_0.

Let $g_0=m^2$. Note that the set $\{g_0>0,g_1>0,...,g_r>0\}=\varnothing$ because $S\setminus Y=\varnothing$ implies that $\{g_1\geq 0,...,g_r\geq 0\}\subset\{g_0=0\}$.

For each $0\leq i<j\leq r$ write $g_{ij}=g_ig_j$ and apply Lemma 4.2 to $g_0,...,g_r,g_{ij}$ for $0\leq i<j\leq r$. We find analytic functions $\Delta_i,\Delta_{ij},b_i,b_{ij},\widetilde{g}_i,\tilde{g}_{ij}\in\mathcal{O}(X)$ such that

(i) b_i and b_{ij} are sums of squares in $\mathcal{O}(X)$ with zero set contained in Y_0;

(ii) if $\tilde{g}_i$ or $\tilde{g}_{ij}$ vanish along some Y_i, they do it with multiplicity 1;

(iii) Δ_i and Δ_{ij} are sums of squares in $\mathcal{O}(X)$ with discrete zero sets contained in $\{b_i=0\}$ and $\{b_{ij}=0\}$, respectively;

(iv) $\Delta_i^2 g_i=\tilde{g}_i b_i$ and $\Delta_{ij}^2 g_{ij}=\tilde{g}_{ij}b_{ij}$.

We claim that:

($*$) *The closed semianalytic set $Z=\{\tilde{g}_k\geq 0,\tilde{g}_{ij}\geq 0{:}0\leq i,j,k\leq r \text{ and } i<j\}$ is a discrete set which is contained in $D\cup Y_0$.*

Indeed, we first note that the set $\widetilde{S}=\{\tilde{g}_0>0,\tilde{g}_1>0,...,\tilde{g}_r>0\}$ must be empty. If $x\in\widetilde{S}$ then there exists $0\leq i\leq r$ such that $g_i(x)\leq 0$, otherwise, $x\in\{g_0>0,g_1>0,...,g_r>0\}$ which is empty. For such i, we have

$$0\leq\tilde{g}_i(x)b_i(x)=\Delta_i^2(x)g_i(x)\leq 0,$$

hence $b_i(x)=0$. This means that $x\in Y_0$, and therefore $\widetilde{S}\subset Y_0$. But this is impossible because the set $\widetilde{S}$ is open. Thus, $\widetilde{S}=\varnothing$.

On the other hand, Z is a set contained in

$$\begin{aligned}\{\Delta_0g_0\geq 0,\Delta_1g_1\geq 0,...,&\Delta_rg_r\geq 0,\Delta_{ij}g_ig_j\geq 0\}\\ &\subset\{\Delta_0g_0\geq 0,\Delta_1g_1\geq 0,...,\Delta_rg_r\geq 0\}\\ &=\{\Delta_0g_0=0,\Delta_1g_1\geq 0,...,\Delta_rg_r\geq 0\}\\ &\subset D\cup\bigcup_{i\in I}Y_i.\end{aligned}$$

Here we point out the reason to consider the $\tilde{g}_{ij}$'s: if both g_i and g_j change sign along an irreducible curve Y_i, then so do $\tilde{g}_i$ and $\tilde{g}_j$, but not their product, so that $\tilde{g}_{ij}=\tilde{g}_i\tilde{g}_j$ does not vanish on Y_0.

Now suppose that Z has dimension 1. Then, moving along a 1-dimensional branch of Z, we can find an open set W such that

(a) W is contained in the regular locus of X;

(b) $W\cap Z$ is a smooth connected curve that decomposes W into two connected components U and V;

(c) for each $a=\tilde{g}_i$ or $\tilde{g}_{ij}$, the open set $W\cap\{a=0\}$ is either the smooth curve $W\cap Z$ or empty.

After this preparation, we proceed as follows. Let $x\in W\cap Z$. Since the set $\widetilde{S}=\{\tilde{g}_0>0, ..., \tilde{g}_r>0\}$ is empty, there must be some $\tilde{g}_i$ such that $\tilde{g}_i(x)=0$. Hence, by (c), $\tilde{g}_i$ vanishes exactly on $W\cap Z$. Since $Z\subset D\cup Y_0$ and $\tilde{g}_i$ vanishes along each Y_i with multiplicity 0 or 1, we deduce that $\tilde{g}_i$ changes sign along $W\cap Z$, say it is >0 on U and <0 on V. But then since $Z\subset D\cup Y_0$, some other $\tilde{g}_j$ must vanish on $W\cap Z$ and be <0 on U and >0 on V. Hence $\tilde{g}_{ij}$ does not vanish on $W\cap Z$, and is <0 on W minus possibly a discrete set where it is 0. But $\tilde{g}_{ij}$ is ≥ 0 on Z, a contradiction. Thus, Z must be a discrete set. This completes the proof of the claim.

Next, the set Z being discrete we can apply Corollary 3.3 (b). Thus, if we write

$$\tilde{g}=(\tilde{g}_0, \tilde{g}_1, ..., \tilde{g}_r, \tilde{g}_{0,1}, ..., \tilde{g}_{r-1,r})$$

there exist positive semidefinite analytic functions $\sigma_\mu\colon X\to\mathbb{R}$ whose zero set is contained in Z, where $\mu\in\{0,1\}^m$ and $m=1+r+\binom{r+1}{2}$, such that

$$\sum_\mu \sigma_\mu \tilde{g}^\mu = 0. \tag{6}$$

Next, consider $b=\prod_{k=0}^r b_i\cdot\prod_{i<j} b_{ij}$ (whose zero set is contained in Y_0) and multiply the equation (6) by b. Since $\Delta_i^2 g_i=\tilde{g}_i b_i$ and $\Delta_{ij}^2 g_i g_j=\tilde{g}_{ij} b_{ij}$, each factor $b\tilde{g}^\mu$ is a positive semidefinite analytic function times the corresponding product g^ν for a certain $\nu\in\{0,1\}^r$.

Thus, the equation (6) becomes

$$\sum_\nu \sigma'_\nu g^\nu = 0, \tag{7}$$

where $\nu\in\{0,1\}^r$, each σ'_ν is a positive semidefinite analytic function on X and $\{\sigma'_0=0\}\subset D\cup Y_0$. Let

$$a_\nu = \begin{cases} \left(\dfrac{\sigma'_0}{2}\right)^2, & \text{if } \nu=0,\\[2ex] \sigma'_\nu\dfrac{\sigma'_0}{2}, & \text{otherwise,}\end{cases}$$

and $f=a_0$. Multiplying (7) by $\sigma_0'/2$ and computing a little we get that

$$\sum_\nu a_\nu g^\nu + f^2 = 0.$$

By Lemma 2.3 and Remark 2.4, we may assume that the zero set of each a_ν is contained in $D\cup Y_0$. Replacing a_0 by a_0+f^2 we achieve a formula of the type

$$\sum_\nu a_\nu g^\nu = 0,$$

where each a_ν is a positive semidefinite analytic function on X whose zero set is contained in $D\cup Y_0$.

Now, by [2, Theorem 1.4], there exists a sum of squares of analytic functions $\Lambda\colon X\to\mathbb{R}$ with $\{\Lambda=0\}\subset Z\subset D\cup Y_0$, such that after multiplying the equation (7) by Λ^2, we have

$$\sum_\nu s_\nu g^\nu = 0,$$

where each $s_\nu=\Lambda^2 a_\nu$ is a sum of five squares of analytic functions on X with zero set contained in $D\cup Y_0$, as wanted. $\square$

Remark 4.3. In the statement of part (B) of Theorem 1.11 we may ask that the zero set of each s_ν is contained in $D\cup Y_0$, which is the Zariski closure of $S\cap Y$.

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