On Nash images of Euclidean spaces

José F. Fernando [1]

Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

A R T I C L E   I N F O

A B S T R A C T

In this work we characterize the subsets of $\mathbb{R}^n$ that are images of Nash maps $f : \mathbb{R}^m \to \mathbb{R}^n$. We prove Shiota's conjecture and show that *a subset* $\mathcal{S} \subset \mathbb{R}^n$ *is the image of a Nash map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *if and only if* $\mathcal{S}$ *is semialgebraic, pure dimensional of dimension* $d \leq m$ *and there exists an analytic path* $\alpha : [0,1] \to \mathcal{S}$ *whose image meets all the connected components of the set of regular points of* $\mathcal{S}$. Two remarkable consequences are the following: (1) pure dimensional irreducible semialgebraic sets of dimension d with arc-symmetric closure are Nash images of $\mathbb{R}^d$; and (2) semialgebraic sets are projections of irreducible algebraic sets whose connected components are Nash diffeomorphic to Euclidean spaces.

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Contents

1. Introduction

Although it is usually said that the first work in Real Geometry is due to Harnack [23], who obtained an upper bound for the number of connected components of a non-singular real algebraic curve in terms of its genus, modern Real Algebraic Geometry was born with Tarski's article [36], where it is proved that the image of a semialgebraic set under a polynomial map is a semialgebraic set. A map $f := (f_1, \ldots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$ is *polynomial* if its components $f_k \in \mathbb{R}[\mathrm{x}] := \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_m]$ are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials $g_k, h_k \in \mathbb{R}[\mathrm{x}]$ such that h_k never vanishes on $\mathbb{R}^m$. A subset $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* when it has a description by a finite boolean combination of polynomial equalities and inequalities, which we will call a *semialgebraic* description. Unless stated otherwise, the topology employed in the article is the Euclidean one.

We are interested in studying what might be called the 'inverse problem' to Tarski's result. In the 1990 *Oberwolfach reelle algebraische Geometrie* week [22] Gamboa proposed:

Problem 1.1. *To characterize the (semialgebraic) subsets of $\mathbb{R}^n$ that are either polynomial or regular images of $\mathbb{R}^m$.*

During the last decade we have attempted to understand better polynomial and regular images of $\mathbb{R}^m$. Our main objectives have been the following:

- To find obstructions to be either polynomial or regular images.
- To prove (constructively) that large families of semialgebraic sets with piecewise linear boundary (convex polyhedra, their interiors, complements and the interiors of their complements) are either polynomial or regular images of Euclidean spaces.

In [9,10] we presented the first step to approach Problem 1.1. In [7] appears a complete solution to Problem 1.1 for the 1-dimensional case, whereas in [18,21,12,13,16,15,37,38] we approached constructive results concerning the representation as either polynomial or regular images of the semialgebraic sets with piecewise linear boundary commented above. A survey concerning this topic, which provides the reader a global idea of the state of the art, can be found in [20]. Articles [19,14] are of different nature. In them we find new obstructions for a semialgebraic subset of $\mathbb{R}^n$ to be either a polynomial or a regular image of $\mathbb{R}^m$. In the first one we found some properties concerning the difference $\mathrm{Cl}(\mathcal{S}) \setminus \mathcal{S}$ for a polynomial image $\mathcal{S}$ of $\mathbb{R}^m$ whereas in the second it is shown that *the set of points at infinite of $\mathcal{S}$ is a connected set.*

The rigidity of polynomial and regular maps makes really difficult to approach Problem 1.1 in its full generality. Taking into account the flexibility of Nash maps, Gamboa and Shiota discussed in 1990 the possibility of approaching the following variant of Problem 1.1.

Problem 1.2. *To characterize the (semialgebraic) subsets of $\mathbb{R}^n$ that are Nash images of $\mathbb{R}^m$.*

A *Nash function on an open semialgebraic set $U \subset \mathbb{R}^n$* is a semialgebraic smooth function on U. Recall that a (non-necessarily continuous) map $f : \mathcal{S} \to \mathcal{T}$ is *semialgebraic* if its graph is a semialgebraic set (in particular we assume that both $\mathcal{S}$ and $\mathcal{T}$ are semialgebraic sets). Given a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$, a *Nash function on $\mathcal{S}$* is the restriction to $\mathcal{S}$ of a Nash function on an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of $\mathcal{S}$.

In 1990 Shiota outlined to Gamboa and the rest of the Real Geometry team at Madrid a vague schedule that sustains the following conjecture (wrongly announced in [22,9] as proved by Shiota) in order to provide a satisfactory answer to Problem 1.2.

Conjecture 1.3 *(Shiota). Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set of dimension d. Then $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$ if and only if $\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all connected components of the set of regular points of $\mathcal{S}$.*

The set of regular points of a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is defined as follows. Let X be the Zariski closure of $\mathcal{S}$ in $\mathbb{R}^n$ and let $\widetilde{X}$ be the complexification of X, that is, the smallest complex algebraic subset of $\mathbb{C}^n$ that contains X. Define $\mathrm{Reg}(X) := X \setminus \mathrm{Sing}(\widetilde{X})$ and let $\mathrm{Reg}(\mathcal{S})$ be the interior of $\mathcal{S} \setminus \mathrm{Sing}(\widetilde{X})$ in $\mathrm{Reg}(X)$. We will explain this in more detail in 2.A.

In 2004 we met again with Shiota and discussed about possible ways to attack his conjecture. It was not clear how to follow certain parts of his 1990 schedule and we have performed strong variations and substantially simplified the architecture of the approach. However, that fruitful meeting was the starting point for the present work and some related ones [3,17]. The latter include useful tools for this article concerning:

(1) Extension of Nash functions on a Nash manifold H with boundary to a Nash manifold M of its same dimension that contains H as a closed subset [17].
(2) Approximation results on a Nash manifold relative to a Nash subset with monomial singularities [3].
(3) Equivalence of Nash classification and $\mathcal{C}^2$ semialgebraic classification for Nash manifolds with boundary [3].

Recall that an *(affine) Nash manifold with or without boundary* is a pure dimensional semialgebraic subset M of some affine space $\mathbb{R}^m$ that is a smooth submanifold with or without boundary of an open subset of $\mathbb{R}^m$. As all the Nash manifolds with or without boundary appearing in this work are affine, we will assume this property when referring to Nash manifolds with or without boundary. In addition when we refer to a Nash manifold with boundary, we assume that this boundary is smooth and in fact a Nash submanifold. The zero set of a Nash function on a Nash manifold M is called a *Nash subset of M*.

1.A. Main results

The main result of this work is Theorem 1.4 that includes a positive solution to Shiota's Conjecture. Its statement requires some preliminary definitions. Let $\alpha : [0,1] \to \mathbb{R}^n$ be a continuous semialgebraic path. Let $A \subset (0,1)$ be the smallest (finite) subset of $(0,1)$ such that the restriction $\alpha|_{(0,1)\setminus A}$ is a Nash map. Denote $\eta(\alpha) := \alpha(A)$.

A semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *well-welded* if it is pure dimensional and for each pair of points $x, y \in \mathcal{S}$ there exists a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $\eta(\alpha) \subset \operatorname{Reg}(\mathcal{S})$.

Main Theorem 1.4 *(Characterization of Nash images). Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set of dimension d. The following assertions are equivalent:*

  (i) $\mathcal{S}$ *is a Nash image of* $\mathbb{R}^d$.
  (ii) $\mathcal{S}$ *is a Nash image of* $\mathbb{R}^m$ *for some* $m \geq d$.
 (iii) $\mathcal{S}$ *is connected by Nash paths.*
 (iv) $\mathcal{S}$ *is connected by analytic paths.*
  (v) $\mathcal{S}$ *is pure dimensional and there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*
 (vi) $\mathcal{S}$ *is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*
(vii) $\mathcal{S}$ *is well-welded.*

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) are straightforward. Only the proof of the non-completely trivial implication (ii) $\Longrightarrow$ '$\mathcal{S}$ is pure dimensional' requires a comment and it is shown in Corollary 6.3. We will show in

Section 7 that a semialgebraic set $\mathcal{S}$ satisfying either condition (iii), (iv), (v) or (vi) is well-welded. Finally, we will prove in Section 8 that (vii) $\Longrightarrow$ (i). An important milestone to prove Theorem 1.4 is the following result, that will be approached in Section 6.

Theorem 1.5. *Let $H \subset \mathbb{R}^n$ be a connected d-dimensional Nash manifold with boundary. Then H is a Nash image of $\mathbb{R}^d$.*

In Section 3 we treat separately the 1-dimensional case and we characterize 1-dimensional Nash images of Euclidean spaces in terms of their irreducibility. The ring $\mathcal{N}(\mathcal{S})$ of Nash functions on a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is a noetherian ring [11, Thm.2.9] and we say that $\mathcal{S}$ is *irreducible* if and only if $\mathcal{N}(\mathcal{S})$ is an integral domain [11].

Proposition 1.6 (*The 1-dimensional case*). *Let $\mathcal{S} \subset \mathbb{R}^n$ be a 1-dimensional semialgebraic set. Then $\mathcal{S}$ is a Nash image of some $\mathbb{R}^m$ if and only if $\mathcal{S}$ is irreducible. In addition, if such is the case $\mathcal{S}$ is a Nash image of $\mathbb{R}$.*

Compare Proposition 1.6 with the more restrictive characterization results for polynomial and regular images of Euclidean spaces [7].

1.B. Two consequences

We present next two remarkable consequences of Theorem 1.4.

1.B.1. Representation of arc-symmetric semialgebraic sets

Arc-symmetric semialgebraic sets were introduced by Kurdyka in [28] and subsequently studied by many authors. Recall that a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *arc-symmetric* if for each analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\gamma((-1, 0)) \subset \mathcal{S}$ it holds that $\gamma((-1, 1)) \subset \mathcal{S}$. In particular arc-symmetric semialgebraic sets are closed subsets of $\mathbb{R}^n$. An arc-symmetric semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *irreducible* if it cannot be written as the union of two proper arc-symmetric semialgebraic subsets [28, §2]. Equivalently, $\mathcal{S}$ is irreducible if and only if the ring $\mathcal{N}(\mathcal{S})$ is an integral domain. It follows from Theorem 1.4 and [28, Cor.2.8] that a pure dimensional irreducible arc-symmetric semialgebraic set is a Nash image of $\mathbb{R}^d$ where $d := \dim(\mathcal{S})$. In addition, it holds:

Corollary 1.7. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional irreducible semialgebraic set of dimension d whose closure $\mathrm{Cl}(\mathcal{S})$ is arc-symmetric. Then $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*

1.B.2. Elimination of inequalities

Tarski–Seidenberg principle on elimination of quantifiers can be restated geometrically by saying that the projection of a semialgebraic set is again semialgebraic. A converse problem, to find an algebraic set in $\mathbb{R}^{n+k}$ whose projection is a given semialgebraic subset of $\mathbb{R}^n$, is known as the *problem of eliminating inequalities*. Motzkin proved in [30] that

this problem always has a solution for $k = 1$. However, his solution is rather complicated and is generally a reducible algebraic set. In another direction Andradas–Gamboa proved in [1,2] that if $\mathcal{S} \subset \mathbb{R}^n$ is a closed semialgebraic set whose Zariski closure is irreducible, then $\mathcal{S}$ is the projection of an irreducible algebraic set in some $\mathbb{R}^{n+k}$. In [33] Pecker gives some improvements on both results: for the first by finding a construction of an algebraic set in $\mathbb{R}^{n+1}$ that projects onto the given semialgebraic subset of $\mathbb{R}^n$, far simpler than the original construction of Motzkin; for the second by proving that if $\mathcal{S}$ is a locally closed semialgebraic subset of $\mathbb{R}^n$ with an interior point, then $\mathcal{S}$ is the projection of an irreducible algebraic subset of $\mathbb{R}^{n+1}$.

In this article we prove the following result that looks for a non-singular algebraic set with the simplest possible topology that projects onto a semialgebraic set.

Corollary 1.8. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set of dimension d. We have:*

(i) *If $\mathcal{S}$ is Nash path-connected, it is the projection of an irreducible non-singular algebraic set $X \subset \mathbb{R}^{n+k}$ (for some $k \geq 0$) whose connected components are Nash diffeomorphic to $\mathbb{R}^d$. In addition:*
  (1) *Each connected component of X projects onto $\mathcal{S}$.*
  (2) *Given any two of the connected components of X there exists an automorphism of X that swaps them.*
(ii) *In general $\mathcal{S}$ is the projection of an algebraic set $X \subset \mathbb{R}^{n+k}$ (for some $k \geq 0$) that is Nash diffeomorphic to a finite pairwise disjoint union of affine subspaces of $\mathbb{R}^{d+1}$.*

Even for dimension 1, it is not possible to impose the connectedness of X (see Example 10.1 and Remark 10.2).

1.C. Structure of the article

The article is organized as follows. In Section 2 we present some basic notions and notations used in this paper as well as some preliminary results. The reader can start directly in Section 3 and resort to the Preliminaries only when needed. In Section 3 we afford the 1-dimensional case. Its presentation is short and evidences strong differences with the polynomial and regular cases [7]. In Sections 4 and 5 we analyze with care the main properties of Nash collars, Nash doubles and the *drilling blow-up*, which is an adaptation to the Nash setting of the *oriented blow-up* [25,6] of a real analytic space with center a closed real analytic subspace. We refer the reader to [27, §5] for a presentation of the oriented blow-up of a real analytic manifold M with center a closed real analytic submanifold N whose vanishing ideal inside M is finitely generated (this happens for instance if N is compact). In our case, we take advantage of the noetherianity of the ring of Nash functions on a Nash manifold to develop the drilling blow-up and to obtain stronger global properties than in the general real analytic case. The previous tools (Nash collars, Nash doubles and drilling blow-ups) are the key to build boundaries on Nash

manifolds in Proposition 4.1 and to modify the topology of a Nash manifold in order to prove Theorem 1.5 in Section 6. We are convinced that the strategy followed to prove Theorem 1.5 will have further applications. In Section 7 we study the main properties of well-welded semialgebraic sets and we show that a semialgebraic set satisfying any of the assertions (iii), (iv), (v) or (vi) in the statement of Theorem 1.4 is well-welded. Next, in Section 8 we prove Theorem 1.4. In Section 9 we introduce the concept of *Nash path-connected components* of a semialgebraic set and we prove that each semialgebraic set can be (uniquely) written as the (finite) union of its Nash path-components. Finally, in Section 10 we prove Corollaries 1.7 and 1.8. The article ends with three Appendices. In the first one we present a miscellanea of $\mathcal{C}^2$ semialgebraic homeomorphisms between intervals that are used in the article whereas in the second we recall certain results concerning strict transforms of analytic and Nash paths under finite chains of blow-ups. The third Appendix concerns an algebrization result for a Nash normal crossing divisor of a Nash manifold.

2. Preliminaries on semialgebraic sets and Nash manifolds

In this section we introduce many concepts and notation needed in the article. We establish the following conventions: $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are Nash manifolds. Nash subsets of a Nash manifold or algebraic subsets of $\mathbb{R}^n$ are denoted with X, Y and Z. The semialgebraic sets are denoted with $\mathcal{S}, \mathcal{T}, \mathcal{R}, \dots$ On the other hand, $H \subset \mathbb{R}^m$ is a Nash manifold of dimension d with (smooth) boundary, ∂H is its boundary and $\mathrm{Int}(H) = H \setminus \partial H$ is its interior. In addition, $\mathcal{C}^r$ semialgebraic and Nash functions on a semialgebraic set are denoted with $f, g, h, \dots$.

Recall some general properties of semialgebraic sets. Semialgebraic sets are closed under Boolean combinations and by quantifier elimination they are also closed under

projections. Any set defined by a first order formula in the language of ordered fields is a semialgebraic set [4, pp. 28, 29]. Thus, the basic topological constructions as closures, interiors or boundaries of semialgebraic sets are again semialgebraic. Also images and preimages of semialgebraic sets by semialgebraic maps are again semialgebraic. The *dimension* $\dim(\mathcal{S})$ of a semialgebraic set $\mathcal{S}$ is the dimension of its Zariski closure [4, §2.8]. The *local dimension* $\dim(\mathcal{S}_x)$ *of* $\mathcal{S}$ *at a point* $x \in \mathrm{Cl}(\mathcal{S})$ is the dimension $\dim(U)$ of a small enough open semialgebraic neighborhood $U \subset \mathrm{Cl}(\mathcal{S})$ of x. The dimension of $\mathcal{S}$ coincides with the maximum of these local dimensions. For any fixed k the set of points $x \in \mathcal{S}$ such that $\dim(\mathcal{S}_x) = k$ is a semialgebraic subset of $\mathcal{S}$.

2.A. Set of regular points of a semialgebraic set

Let $Z \subset \mathbb{C}^n$ be a complex algebraic set and let $I_{\mathbb{C}}(Z)$ be the ideal of all polynomials $F \in \mathbb{C}[\mathbf{x}]$ such that $F(z) = 0$ for each $z \in Z$. A point $z \in Z$ is *regular* if the localization of the polynomial ring $\mathbb{C}[\mathbf{x}]/I_{\mathbb{C}}(Z)$ at the maximal ideal $\mathfrak{M}_z$ associated to z is a regular local ring. In this complex setting the Jacobian criterion and Hilbert's Nullstellensatz imply that $z \in Z$ is regular if and only if there exists an open neighborhood $U \subset \mathbb{C}^n$ of z such that $U \cap Z$ is an analytic manifold. We denote $\mathrm{Reg}(Z)$ the set of regular points of Z and it is an open dense subset of Z. If Z is irreducible, it is pure dimensional and $\mathrm{Reg}(Z)$ is a connected analytic manifold. In case Z is not irreducible, then the connected components of $\mathrm{Reg}(Z)$ are finitely many analytic manifolds (possibly of different dimensions). We denote $\mathrm{Sing}(Z) := Z \setminus \mathrm{Reg}(Z)$ the *set of singular points of* Z.

Let $X \subset \mathbb{R}^n$ be a (real) algebraic set and let $I_{\mathbb{R}}(X)$ be the ideal of all polynomials $f \in \mathbb{R}[\mathbf{x}]$ such that $f(x) = 0$ for each $x \in X$. A point $x \in X$ is *regular* if the localization of $\mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}(X)$ at the maximal ideal $\mathfrak{m}_x$ associated to x is a regular local ring [4, §3.3]. Let $\widetilde{X} \subset \mathbb{C}^n$ be the complex algebraic set that is the zero set of the extended ideal $I_{\mathbb{R}}(X)\mathbb{C}[\mathbf{x}]$. We call $\widetilde{X}$ the *complexification of* X. The ideal $I_{\mathbb{C}}(\widetilde{X})$ coincides with the tensorized ideal $I_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C}$, so $\widetilde{X}$ is the smallest complex algebraic subset of $\mathbb{C}^n$ that contains X and

$$\mathbb{C}[\mathbf{x}]/I_{\mathbb{C}}(\widetilde{X}) \cong (\mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}(X)) \otimes_{\mathbb{R}} \mathbb{C}.$$

The localization $(\mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}(X))_{\mathfrak{m}_x}$ is a regular local ring if and only if so is its complexification

$$(\mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}(X))_{\mathfrak{m}_x} \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C}[\mathbf{x}]/I_{\mathbb{C}}(\widetilde{X}))_{\mathfrak{M}_x}.$$

Thus, the *set of regular points* of X is $\mathrm{Reg}(X) = \mathrm{Reg}(\widetilde{X}) \cap X$ and its *set of singular points* is $\mathrm{Sing}(X) := X \setminus \mathrm{Reg}(X) = \mathrm{Sing}(\widetilde{X}) \cap X$. The connected components of the open semialgebraic subset $\mathrm{Reg}(X)$ of X is a finite union of Nash manifolds (possibly of different dimensions).

Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set of dimension d. The Zariski closure $\overline{\mathcal{S}}^{\mathrm{zar}}$ of $\mathcal{S}$ in $\mathbb{R}^n$ is the smallest algebraic subset of $\mathbb{R}^n$ that contains $\mathcal{S}$. We define

Fig. 1. $X : (x^2 + zy^2)x - y^4 = 0$.

$$\text{Reg}(\mathcal{S}) := \text{Int}_{\text{Reg}(\overline{\mathcal{S}}^{\text{zar}})}(\mathcal{S} \setminus \text{Sing}(\overline{\mathcal{S}}^{\text{zar}})) \quad \text{and} \quad \text{Sing}(\mathcal{S}) := \mathcal{S} \setminus \text{Reg}(\mathcal{S}).$$

The connected components of the open subset $\text{Reg}(\mathcal{S})$ of $\overline{\mathcal{S}}^{\text{zar}}$ is a finite union of Nash manifolds (possibly of different dimensions) and $\text{Sing}(\mathcal{S})$ is a semialgebraic set of dimension $< d$, which is closed in $\mathcal{S}$. The set $\text{Reg}_k(\mathcal{S})$ of points of dimension k of $\text{Reg}(\mathcal{S})$ is either the empty-set or a Nash manifold of dimension k for each $k = 0, 1, \ldots, d$. If $\mathcal{S}$ is pure dimensional, $\text{Reg}(\mathcal{S})$ is a dense subset of $\mathcal{S}$. A point $x \in \mathcal{S}$ is *smooth* if there exists an open neighborhood $U \subset \mathbb{R}^n$ of x such that $U \cap \mathcal{S}$ is a Nash manifold. It holds that each regular point is a smooth point, but the converse is not always true even if $\mathcal{S} = X$ is a real algebraic set, as it shows the following example.

Example 2.1. Consider the algebraic set $X := \{(x^2 + zy^2)x - y^4 = 0\} \subset \mathbb{R}^3$. The set of regular points of X is the difference $X \setminus \{x = 0, y = 0\}$, whereas the set of smooth points of X is the difference $X \setminus \{x = 0, y = 0, z \leq 0\}$ (see Fig. 1).

To prove that the points of the open half-line $\{x = 0, y = 0, z < 0\}$ are non-smooth we proceed by contradiction. Pick a point $p := (0, 0, -a^2) \in \{x = 0, y = 0, z < 0\}$ and assume that it is smooth. As the line $\{x = 0, y = 0\} \subset X$, the vector $(0, 0, 1)$ would be tangent to X at p, so the plane $z = -a^2$ would be transversal to X at p. Thus, the intersection $X \cap \{z = -a^2\}$ should be a curve that is smooth at p, but this is a contradiction because such curve $\{(x^2 - (ay)^2)x - y^4 = 0, z = -a^2\}$ has three tangent lines at p, which are those lines of equations $\{x - ay = 0\}$, $\{x + ay = 0\}$ and $\{x = 0\}$ inside the plane $\{z = -a^2\}$. The origin cannot be a smooth point of X because the set of smooth points of X is an open subset of X. Consequently, the set of non-smooth points of X contains the closed half-line $\{x = 0, y = 0, z \leq 0\}$. To finish we prove that the

points of the open half-line $\{x = 0, y = 0, z > 0\}$ are smooth. To that end, observe that the map $\varphi : \{(t,s) \in \mathbb{R}^2 : t > 0\} \to \mathbb{R}^3, (s,t) \mapsto ((s^2 + t^2)s^2, (s^2 + t^2)s, t^2)$ is a Nash embedding whose image is $X \cap \{z > 0\}$.

The set $\mathtt{Sth}(\mathcal{S})$ of smooth points of a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is by [35] a semialgebraic subset of $\mathbb{R}^n$ (and consequently a Nash submanifold of $\mathbb{R}^n$), which contains $\mathrm{Reg}(\mathcal{S})$ (maybe as a proper subset as it happens in Example 2.1), and it is open in $\mathcal{S}$. The set $\mathtt{Sth}_k(\mathcal{S})$ of points of dimension k of $\mathtt{Sth}(\mathcal{S})$ is either the empty-set or a Nash manifold of dimension k for each $k = 0, 1, \ldots, d$. Denote $\mathtt{NSth}(\mathcal{S}) := \mathcal{S} \setminus \mathtt{Sth}(\mathcal{S})$ the set of non-smooth points of $\mathcal{S}$. If X is an algebraic set, $\mathrm{Sing}(X)$ is always an algebraic subset of X whereas $\mathtt{NSth}(X)$ is in general only a semialgebraic subset of X, see Example 2.1.

Remark 2.2. Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional semialgebraic set such that $\overline{\mathcal{S}}^{\mathrm{zar}} \subset \mathbb{R}^n$ is a non-singular real algebraic set. Then $\mathrm{Reg}(\mathcal{S}) = \mathrm{Int}_{\overline{\mathcal{S}}^{\mathrm{zar}}}(\mathcal{S}) = \mathtt{Sth}(\mathcal{S})$ and $\mathrm{Sing}(\mathcal{S}) = \mathtt{NSth}(\mathcal{S})$.

As $\overline{\mathcal{S}}^{\mathrm{zar}}$ is a non-singular real algebraic set, $\mathrm{Reg}(\overline{\mathcal{S}}^{\mathrm{zar}}) = \overline{\mathcal{S}}^{\mathrm{zar}}$ and $\mathrm{Sing}(\overline{\mathcal{S}}^{\mathrm{zar}}) = \varnothing$. Thus, as $\mathcal{S}$ is pure dimensional, $\mathrm{Int}_{\overline{\mathcal{S}}^{\mathrm{zar}}}(\mathcal{S}) = \mathtt{Sth}(\mathcal{S})$ and

$$\mathrm{Reg}(\mathcal{S}) = \mathrm{Int}_{\mathrm{Reg}(\overline{\mathcal{S}}^{\mathrm{zar}})}(\mathcal{S} \setminus \mathrm{Sing}(\overline{\mathcal{S}}^{\mathrm{zar}})) = \mathrm{Int}_{\overline{\mathcal{S}}^{\mathrm{zar}}}(\mathcal{S}) = \mathtt{Sth}(\mathcal{S}),$$

$$\mathrm{Sing}(\mathcal{S}) = \mathcal{S} \setminus \mathrm{Reg}(\mathcal{S}) = \mathcal{S} \setminus \mathtt{Sth}(\mathcal{S}) = \mathtt{NSth}(\mathcal{S}).$$

Using the definition of smooth point one proves readily the following result.

Lemma 2.3. *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be Nash manifolds and let $f : M \to N$ be a Nash diffeomorphism. Let $\mathcal{S} \subset M$ be a semialgebraic set. Then $f(\mathtt{Sth}(\mathcal{S})) = \mathtt{Sth}(f(\mathcal{S}))$ and $f(\mathtt{NSth}(\mathcal{S})) = \mathtt{NSth}(f(\mathcal{S}))$.*

The previous result is no longer true if we consider the set $\mathrm{Reg}(\mathcal{S})$ of regular points of $\mathcal{S}$ instead of $\mathtt{Sth}(\mathcal{S})$.

Example 2.4. Consider the semialgebraic set $\mathcal{S} := \{x^2 + y^2 - y^3 = 0\} \subset \mathbb{R}^2$ and the Nash diffeomorphism

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (x(x^2 + 1), x^2 + 1 + y).$$

Denote $\mathcal{T} := \{(0, -1)\} \cup \{y = 0\}$ and observe that $f(\mathcal{T}) = \mathcal{S}$, $\mathrm{Sing}(\mathcal{T}) = \varnothing$ and $\mathrm{Sing}(\mathcal{S}) = \{(0, 0)\}$. Consequently, $f(\mathrm{Reg}(\mathcal{T})) = f(\mathcal{T}) = \mathcal{S} \neq \mathrm{Reg}(\mathcal{S})$.

The previous example shows that the sets $\mathrm{Reg}(\mathcal{S})$ and $\mathrm{Sing}(\mathcal{S})$ depend on how $\mathcal{S}$ is immersed in an affine space, whereas Lemma 2.3 points out that the sets $\mathtt{Sth}(\mathcal{S})$ and $\mathtt{NSth}(\mathcal{S})$ are subsets of $\mathcal{S}$ of intrinsic nature. As we will use in the sequel resolution of singularities, we will need to employ the sets $\mathrm{Reg}(\mathcal{S})$ and $\mathrm{Sing}(\mathcal{S})$ instead of the sets $\mathtt{Sth}(\mathcal{S})$

and $\mathtt{NSth}(\mathcal{S})$ in a large part of the article. However, in the proof of implication (vii) $\Longrightarrow$ (i) of Theorem 1.4 we will take advantage of the intrinsic nature of the sets $\mathtt{Sth}(\mathcal{S})$ and $\mathtt{NSth}(\mathcal{S})$. This justifies the introduction of both pairs of concepts: 'regular/singular' points and 'smooth/non-smooth' points. In Lemma 8.3 we show that under certain conditions we may embed $\mathcal{S}$ in some affine space in order to have $\mathrm{Reg}(\mathcal{S}) = \mathtt{Sth}(\mathcal{S})$ and $\mathrm{Sing}(\mathcal{S}) = \mathtt{NSth}(\mathcal{S})$.

2.B. Desingularization of algebraic sets

Let $X \subset Y \subset \mathbb{R}^n$ be algebraic sets such that Y is non-singular. Recall that X is a *normal-crossings divisor of Y* if for each point $x \in Y$ there exists a regular system of parameters $x_1, \ldots, x_d$ for Y at x such that X is given on an open Zariski neighborhood of x in Y by the equation $x_1 \cdots x_k = 0$ for some $k \leq d$. In particular, the irreducible components of X are non-singular and have codimension 1 in Y.

A rational map $f := (f_1, \ldots, f_n) : Z \to \mathbb{R}^n$ on an algebraic set $Z \subset \mathbb{R}^m$ is *regular* if its components are quotients of polynomials $f_k := \frac{g_k}{h_k}$ such that $Z \cap \{h_k = 0\} = \varnothing$.

Hironaka's desingularization results [24] are powerful tools that we will use fruitfully in Sections 7 and 8. We recall here the two results we need.

Theorem 2.5 *(Desingularization). Let $X \subset \mathbb{R}^n$ be an algebraic set. Then there exist a non-singular algebraic set $X' \subset \mathbb{R}^m$ and a proper regular map $f : X' \to X$ such that*

$$f|_{X' \setminus f^{-1}(\mathrm{Sing}(X))} : X' \setminus f^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}\, X$$

is a diffeomorphism whose inverse map is also regular.

Remark 2.6. If X is pure dimensional, $X \setminus \mathrm{Sing}\, X$ is dense in X. As f is proper, it is surjective.

Theorem 2.7. *Let $X \subset Y \subset \mathbb{R}^n$ be algebraic sets such that Y is non-singular. Then there exists a non-singular algebraic set $Y' \subset \mathbb{R}^m$ and a proper surjective regular map $g : Y' \to Y$ such that $g^{-1}(X)$ is a normal-crossings divisor of Y' and the restriction*

$$g|_{Y' \setminus g^{-1}(X)} : Y' \setminus g^{-1}(X) \to Y \setminus X$$

is a diffeomorphism whose inverse map is also regular.

2.C. Nash manifolds and Nash normal-crossings divisors

The open semialgebraic subsets of a Nash manifold $M \subset \mathbb{R}^m$ are a base of the topology and therefore Nash functions define a sheaf that we denote with $\mathcal{N}$. In particular, the sheaf $\mathcal{N}$ induces a notion of Nash function $f : U \to \mathbb{R}$ over an arbitrary open subset U of M possibly not semialgebraic. In case U is an open subset of $\mathbb{R}^m$, *Nash* means that f is

smooth and there exists a nonzero polynomial $P(\mathbf{x}, \mathbf{t}) \in \mathbb{R}[\mathbf{x}, \mathbf{t}] = \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{t}]$ such that $P(x, f(x)) = 0$ for all $x \in U$. If U is semialgebraic this definition is equivalent to the one in the Introduction [4, Prop.8.1.8]. We apparently have two definitions of Nash function $f : M \to \mathbb{R}$. Both notions are equal because *every Nash manifold has a Nash tubular neighborhood.*

2.C.1. [4, Cor.8.9.5] *Let $M \subset \mathbb{R}^m$ be a Nash manifold. Then there exists an open semialgebraic neighborhood U of M in $\mathbb{R}^m$ and a Nash retraction $\rho : U \to M$.*
   Recall the existence of finite atlas for M with domains Nash diffeomorphic to $\mathbb{R}^{\dim(M)}$:

2.C.2. [17, Lem.2.2] *A Nash manifold $M \subset \mathbb{R}^m$ of dimension d admits a finite open (semialgebraic) covering $M = \bigcup_{i=1}^r M_i$ by Nash manifolds $M_i \subset M$ each of them Nash diffeomorphic to $\mathbb{R}^d$.*
   Let $N \subset M$ be a closed Nash submanifold of dimension e. By [34, Cor.II.5.4] N is a non-singular Nash subset of M and by [3, Thm.1.4] we have:

2.C.3. *The Nash submanifold N can be covered by finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $u := (u_1, \ldots, u_d) : U \to \mathbb{R}^d$ such that $U \cap N = \{u_1 = 0, \ldots, u_{d-e} = 0\}$.*
   It is also possible to construct tubular neighborhoods of N inside M. A *Nash (vector) bundle over M* is a (vector) bundle $(\mathscr{E}, \theta, M)$ such that $\mathscr{E}$ is an (affine) Nash manifold and the projection $\theta : \mathscr{E} \to M$ is a Nash map. Examples of Nash bundles are: the trivial bundle of M, the tangent bundle of M, the normal bundle of M, etc.

2.C.4. [34, Lem.II.6.2] *There exists a Nash subbundle $(\mathscr{E}, \theta, N)$ of the trivial Nash bundle $(N \times \mathbb{R}^m, \eta, N)$, a (strictly) positive Nash function δ on N and a Nash diffeomorphism φ from a semialgebraic neighborhood V of N in M onto*

$$\mathscr{E}_\delta := \{(x, y) \in \mathscr{E} : \|y\| < \delta(x)\}$$

such that $\varphi|_N = (\mathrm{id}_N, 0)$. The tuple $(V, \varphi, \mathscr{E}, \theta, N, \delta)$ is a Nash tubular neighborhood of N in M and the composition $\theta \circ \varphi : V \to N$ is a Nash retraction.
   As an application of 2.C.3 it follows a counterpart of 2.C.2 for Nash manifolds with boundary:

2.C.5. *Let $H \subset \mathbb{R}^m$ be a d-dimensional Nash manifold with boundary. By [17, Thm.1.11] H is a closed subset of a Nash manifold $M \subset \mathbb{R}^m$ of dimension d in such a way that:*

 (i) *∂H is a closed Nash submanifold of M and so a Nash non-singular subset of M.*
 (ii) *M can be covered with finitely many open semialgebraic subsets U equipped with Nash diffeomorphisms $(u_1, \ldots, u_d) : U \to \mathbb{R}^d$ such that*

$$\begin{cases} U \subset H \text{ or } U \cap H = \varnothing & \text{if } U \text{ does not meet } \partial H, \\ U \cap H = \{u_1 \geq 0\} & \text{if } U \text{ meets } \partial H. \end{cases}$$

A *Nash normal-crossings divisor of* M is a Nash subset $X \subset M$ whose irreducible components are non-singular Nash hypersurfaces $X_1, \ldots, X_p$ of M in general position. This means that at every point $x \in X_{i_1}, \ldots, X_{i_r}$, $x \notin X_i$ for $i \neq i_1, \ldots, i_r$, the tangent hyperplanes $T_x X_{i_1}, \ldots, T_x X_{i_r}$ are linearly independent in the tangent space $T_x M$. In [17, Thm.1.6] we prove the following:

2.C.6. Let X be a Nash normal-crossings divisor of M. Then X can be covered by finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $(u_1, \ldots, u_d) : U \to \mathbb{R}^d$ such that $U \cap X = \{u_1 \cdots u_r = 0\}$, where r depends on U.

Both Nash submanifolds and Nash normal-crossings divisors are particular cases of *coherent* Nash subsets X of M (see [3, §2.B, Lem.5.1]). We can say by [3, §2.B] and [4, Prop.8.6.9] that a Nash set $X \subset M$ is *coherent* if the $\mathcal{N}$-sheaf of ideals $\mathcal{J}_x = I(X_x)$ for $x \in M$ is of *finite type*, that is, for every $x \in M$ there exists an open neighborhood U and a surjective morphism $\mathcal{N}^s|_U \to \mathcal{J}|_U$. By [3, Eq. (2.2)] it holds that if $X \subset M$ is a coherent Nash subset, then for any semialgebraic open subset U of M we have

$$I(X)\mathcal{N}(U) = I(X \cap U) \tag{2.1}$$

where $I(X \cap U) := \{f \in \mathcal{N}(U) : f|_{X \cap U} = 0\}$ and $I(X) := I(X \cap M)$. Consequently, if $x \in X$, the ideal $I(X_x) = I(X)\mathcal{N}(M_x)$. Conversely, we have [34, (I.6.5)]:

2.C.7. If $f_1, \ldots, f_r \in \mathcal{N}(M)$ generate $I(X_x)$ for all $x \in X$, then $f_1, \ldots, f_r$ generate also $I(X)$. In particular, if $f \in \mathcal{N}(M)$ satisfies $f_x \in I(X_x)$ for all $x \in X$, then $f \in I(X)$.

2.D. Approximation of differentiable semialgebraic maps by Nash maps

Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d. Denote the set of all continuous semialgebraic functions on M with $\mathcal{S}^0(M)$. For every integer $r \geq 1$ we denote the set of all semialgebraic functions $f : M \to \mathbb{R}$ that are differentiable of class r with $\mathcal{S}^r(M)$. We equip $\mathcal{S}^r(M)$ with the $\mathcal{S}^r$ *semialgebraic Whitney topology* [34, §II.1, pp. 79–80]. If $r \geq 1$, let $\xi_1, \ldots, \xi_s$ be semialgebraic $\mathcal{S}^{r-1}$ tangent fields on M that span the tangent bundle of M. For every strictly positive continuous semialgebraic function $\varepsilon : M \to \mathbb{R}$ we denote the set of all functions $g \in \mathcal{S}^r(M)$ such that

$$\begin{cases} |g| < \varepsilon & \text{if } r = 0, \\ |g| < \varepsilon \quad \text{and} \quad |\xi_{i_1} \cdots \xi_{i_\ell}(g)| < \varepsilon \text{ for } 1 \leq i_1, \ldots, i_\ell \leq s, 1 \leq \ell \leq r & \text{if } r \geq 1 \end{cases}$$

with $\mathcal{U}_\epsilon$. These sets $\mathcal{U}_\epsilon$ form a basis of neighborhoods of the zero function for a topology in $\mathcal{S}^r(M)$ that does not depend on the choice of the tangent fields if $r \geq 1$. The first important result is that the inclusion $\mathcal{N}(M) \subset \mathcal{S}^r(M)$ is dense.

2.D.1. [34, Thm.II.4.1] *Every semialgebraic $\mathcal{S}^r$ function on M can be approximated in the $\mathcal{S}^r$ topology by Nash functions.*

Let $N \subset \mathbb{R}^n$ be a Nash manifold. A semialgebraic map $f := (f_1, \ldots, f_n) : M \to N \subset \mathbb{R}^n$ is $\mathcal{S}^r$ if each component $f_k : M \to \mathbb{R}$ is $\mathcal{S}^r$. We denote the set of all $\mathcal{S}^r$ maps $M \to N$ with $\mathcal{S}^r(M, N)$. We consider in $\mathcal{S}^r(M, N)$ the subspace topology given by the canonical inclusion in the following product space endowed with the product topology [34, Rmk.II.1.3]:

$$\mathcal{S}^r(M, N) \subset \mathcal{S}^r(M, \mathbb{R}^n) = \mathcal{S}^r(M, \mathbb{R}) \times \overset{n}{\cdots} \times \mathcal{S}^r(M, \mathbb{R}) : f \mapsto (f_1, \ldots, f_n).$$

Roughly speaking, g is close to f when its components g_k are close to the components f_k of f. The previous topologies can be extended to the set $\mathcal{S}^r(\mathcal{S}, \mathcal{T})$ of $\mathcal{S}^r$ semialgebraic maps between two semialgebraic sets $\mathcal{S}$ and $\mathcal{T}$ if $\mathcal{S} \subset M$ is closed [3, §2.D-E]. This allows for instance to provide a topology on the set of $\mathcal{S}^r$ maps between two Nash manifolds with boundary.

A map $h : \mathcal{S} \to \mathcal{T}$ is an $\mathcal{S}^r$ *diffeomorphism* if it is a bijection and both h and h^{-1} are $\mathcal{S}^r$ maps. Diffeomorphisms between Nash manifolds behave well with respect to approximation if $r \geq 1$.

2.D.2. [34, Lem.II.1.7] *Let $h : M \to N$ be an $\mathcal{S}^r$ diffeomorphism of Nash manifolds. If an $\mathcal{S}^r$ map $g : M \to N$ is $\mathcal{S}^r$ close enough to h, then g is also an $\mathcal{S}^r$ diffeomorphism, and g^{-1} is $\mathcal{S}^r$ close to h^{-1}.*

From this and the existence of Nash tubular neighborhoods (2.C.1) we deduce that for all $r \geq 1$ *every $\mathcal{S}^r$ diffeomorphism $f : M \to N$ can be approximated by Nash diffeomorphisms*, hence $\mathcal{S}^1$ and Nash classifications coincide for Nash manifolds. In the case of Nash manifolds with boundary we proved in [3, Rmk.9.6] that $\mathcal{S}^2$ and Nash classifications coincide.

2.D.3. *Two Nash manifolds with boundary that are $\mathcal{S}^2$ diffeomorphic are Nash diffeomorphic.*

2.D.4. In addition, *if $f : H_1 \to H_2$ is an $\mathcal{S}^2$ diffeomorphism between two Nash manifolds with boundary and $f|_{\partial H_1} : \partial H_1 \to \partial H_2$ is a Nash diffeomorphism, then there exists a Nash diffeomorphism $g : H_1 \to H_2$ close to f such that $g|_{\partial H_1} = f|_{\partial H_1}$.*

We will deal in addition with $\mathcal{S}^0$ homeomorphisms between Nash manifolds. In this case it is not possible to approximate them by Nash diffeomorphisms, but the following result allows us to approximate them by Nash surjective maps.

Lemma 2.8. *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be Nash manifolds and let $f : M \to N$ be a semialgebraic homeomorphism. Let $g : M \to N$ be a semialgebraic map close to f in the $\mathcal{S}^0$ topology. Then g is surjective.*

Proof. We adapt the proof of [26, Thm.2.1.8]. By [34, Rmk.II.1.15] the map

$$f_*^{-1} : \mathcal{S}^0(M, N) \to \mathcal{S}^0(M, M), \ h \mapsto f^{-1} \circ h$$

is continuous. Thus, if g is close to f, then $f^{-1} \circ g$ is close to id_M. If we prove that each $g' \in \mathcal{S}^0(M, M)$ close to the identity is surjective, then each g close to f will satisfy that $f^{-1} \circ g$ is surjective, so g is surjective.

By [34, Thm.VI.2.1] there exist a compact affine non-singular algebraic set X, a non-singular algebraic subset Y of X that either has codimension 1 if M is non-compact or is empty if M is compact and a union M' of some connected components of $X \setminus Y$ such that M is Nash diffeomorphic to M' and $\mathrm{Cl}(M')$ is a compact Nash manifold with boundary Y. Let ε be a non-negative Nash equation of Y in X and let $h \in \mathcal{S}^0(M', M')$ be such that $\|h - \mathrm{id}_{M'}\| < \varepsilon|_{M'}$. We claim: *h extends continuously to $X \setminus M'$ as the identity map $\mathrm{id}_{X \setminus M'}$*. It is enough to check: *if $\{z_k\}_{k \geq 1} \subset M'$ tends to $y \in Y$, then $\{h(z_k)\}_k$ tends to y*.

Indeed, $\|h(z_k) - z_k\| < \varepsilon(z_k)$ for each $k \geq 1$. As $\{z_k\}_k$ tends to $y \in Y$, we have that $\{\varepsilon(z_k)\}_k$ tends to 0, so $\{h(z_k)\}_k$ tends to y, as claimed.

Consider next the continuous semialgebraic map

$$H : X \to X, \ x \mapsto \begin{cases} h(x) & \text{if } x \in M', \\ x & \text{if } x \in X \setminus M'. \end{cases}$$

It satisfies

$$\|H(x) - \mathrm{id}_X(x)\| \begin{cases} < \varepsilon(x) & \text{if } x \in M', \\ = 0 & \text{if } x \in X \setminus M'. \end{cases}$$

As X is compact, we know by [26, Thm.2.1.8] that there exists $\delta > 0$ such that if $\xi : X \to X$ is a continuous map and $\|\xi - \mathrm{id}_X\| < \delta$, then ξ is surjective. Consequently, if $h \in \mathcal{S}^0(M', M')$ satisfies $\|h - \mathrm{id}_{M'}\| < \min\{\varepsilon, \delta\}$, then $H : X \to X$ is surjective. Let us check that also $h \in \mathcal{S}^0(M', M')$ is surjective. Pick a point $y \in M'$. Then there exists $x \in X$, such that $H(x) = y$. If $x \in X \setminus M'$, then $H(x) = x \in X \setminus M'$, which is a contradiction. So $x \in M'$ and $h(x) = H(x) = y$, that is, h is surjective, as required. $\quad\square$

2.E. Modification of analytic arcs by Nash arcs

The handling of well-welded semialgebraic sets in Section 7 requires the modification of analytic arcs by Nash arcs that avoid certain algebraic sets. To that end we will use the following result:

Lemma 2.9. *Let $M \subset \mathbb{R}^m$ be a connected Nash manifold and let $Y \subset \mathbb{R}^m$ be an algebraic set. Let M_1, M_2 be open semialgebraic subsets of M and let $\alpha : (-1, 1) \to M_1 \cup M_2 \cup \{0\}$ be an analytic arc such that $\alpha(0) = 0$, $\alpha((0, 1)) \subset M_1$ and $\alpha((-1, 0)) \subset M_2$. Assume that*

$M \not\subset Y$. Then for every integer $\nu \geq 1$ there exist $\varepsilon > 0$ and a Nash arc $\beta : (-\varepsilon, \varepsilon) \to M_1 \cup M_2 \cup \{0\}$ such that:

(i) $\beta(0) = 0$, $\alpha(\mathbf{t}) - \beta(\mathbf{t}) \in (\mathbf{t})^\nu \mathbb{R}\{\mathbf{t}\}^m$,
(ii) $\beta((-\varepsilon, \varepsilon)) \cap Y \subset \{0\}$,
(iii) $\beta((0, \varepsilon)) \subset M_1$ and $\beta((-\varepsilon, 0)) \subset M_2$.

Proof. For simplicity we can assume $M_1 \cap M_2 = \varnothing$, $0 \notin M_1 \cup M_2$ and $0 \in Y$. Let $V \subset M$ be an open semialgebraic neighborhood of the origin equipped with a Nash diffeomorphism $\varphi : V \to \mathbb{R}^d$ such that $\varphi(0) = 0$. Shrinking the domain of α, we may assume $\mathrm{Im}(\alpha) \subset V$. Denote $\widehat{\alpha} := \varphi \circ \alpha : (-\delta, \delta) \to \mathbb{R}^d$ where $\delta > 0$ is small enough.

Shrinking M_i, V and the domain of $\widehat{\alpha}$, we may assume that $0 \notin \varphi(M_i \cap V)$ and $\varphi(M_i \cap V) = \{g_{1i} > 0, \dots, g_{\ell i} > 0\}$ for some polynomials $g_{ji} \in \mathbb{R}[\mathbf{x}]$. There exists $s \geq \nu$ large enough such that if $\gamma \in \mathbb{R}\{\mathbf{t}\}^d$ and $\gamma - \widehat{\alpha} \in (\mathbf{t})^s \mathbb{R}\{\mathbf{t}\}^d$, we have $(g_{j1} \circ \gamma)(t) > 0$ and $(g_{j2} \circ \gamma)(-t) > 0$ for $t > 0$ small enough and $j = 1, \dots, \ell$. Let $\gamma \in \mathbb{R}[\mathbf{t}]^d$ be a polynomial tuple such that $\gamma - \widehat{\alpha} \in (\mathbf{t})^s \mathbb{R}\{\mathbf{t}\}^d$.

As $Y \cap V$ has dimension $< d$, also the algebraic set $Y' := \overline{\varphi(Y \cap V)}^{\mathrm{zar}}$ has dimension $< d$. Let $h \in \mathbb{R}[\mathbf{x}]$ be a polynomial equation of Y'. Consider the surjective polynomial map

$$\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \ (t, y) \mapsto \gamma(t) + t^{s+1} y.$$

Let $y_0 \in \mathbb{R}^d$ be such that the univariate polynomial $h(\gamma(\mathbf{t}) + \mathbf{t}^{s+1} y_0) \in \mathbb{R}[\mathbf{t}]$ is not identically zero. Let $\varepsilon > 0$ be such that $\gamma_0(\mathbf{t}) := \gamma(\mathbf{t}) + \mathbf{t}^{s+1} y_0 \in \mathbb{R}[\mathbf{t}]^d$ satisfies

$$g_{j1}(\gamma_0(t)) > 0, \quad g_{j2}(\gamma_0(-t)) > 0, \quad h(\gamma_0(t)) \neq 0 \quad \text{and} \quad h(\gamma_0(-t)) \neq 0$$

for $0 < t < \varepsilon$. The Nash arc $\beta := \varphi^{-1} \circ \gamma_0 : (-\varepsilon, \varepsilon) \to M_1 \cup M_2 \cup \{0\}$ satisfies the required properties. □

3. A light start-up: the 1-dimensional case

In this short section we prove Proposition 1.6 and present some enlightening examples. Nash images of Euclidean spaces contained in the real line are its intervals and all of them are Nash images of $\mathbb{R}$. To be convinced of this fact it is enough to have a look at the following examples.

Examples 3.1. (i) The interval $(0, 1)$ is Nash diffeomorphic to $\mathbb{R}$. Consider the Nash diffeomorphism (together with its inverse):

$$f : \mathbb{R} \to (0, 1), \ t \mapsto \frac{t}{2\sqrt{1 + t^2}} + \frac{1}{2} \quad \text{and} \quad f^{-1} : (0, 1) \to \mathbb{R}, \ t \mapsto \frac{2t - 1}{2\sqrt{t(1 - t)}}.$$

In addition $f((0, +\infty)) = (\frac{1}{2}, 1)$ and $f([0, +\infty)) = [\frac{1}{2}, 1)$.

(ii) The interval $[0, 1)$ is the image of the Nash map

$$h_1 : \mathbb{R} \to \mathbb{R}, \ t \mapsto \frac{t^2}{t^2 + 1},$$

whereas $[0, 1]$ is the image of the Nash map

$$h_2 : \mathbb{R} \to \mathbb{R}, \ t \mapsto \frac{t}{t^2 + 1} + \frac{1}{2}.$$

Nash images of Euclidean spaces contained in a circumference are its connected subsets and all of them are Nash images of $\mathbb{R}$.

Examples 3.2. (i) The circumference $\mathbb{S}^1 : \ x^2 + y^2 = 1$ is a Nash image of $\mathbb{R}$. Consider the inverse of the stereographic projection from the point $(1, 0)$, which is the map

$$f : \mathbb{R} \to \mathbb{S}^1 \setminus \{(1, 0)\}, \ t \mapsto \Big(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \Big).$$

Next, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and the coordinates (x, y) with $x + \sqrt{-1}y$. Consider the map

$$g : \mathbb{C} \to \mathbb{C}, \ z := x + \sqrt{-1}y \mapsto z^2 = (x^2 - y^2) + \sqrt{-1}(2xy).$$

The image of $\mathbb{R}$ under $g \circ f$ is $\mathbb{S}^1$.

(ii) Any connected proper subset $\mathcal{S}$ of $\mathbb{S}^1$ is a Nash image of $\mathbb{R}$ because it is Nash diffeomorphic to either $(0, 1)$, $[0, 1)$ or $[0, 1]$ and these are Nash images of $\mathbb{R}$.

We are ready to prove Proposition 1.6.

Proof of Proposition 1.6. Assume $\mathcal{S}$ is irreducible. Let X be the Zariski closure of $\mathcal{S}$ in $\mathbb{R}^n$ and let $\widetilde{X}$ be its complexification. Let $(\widetilde{Y}, \pi)$ be the normalization of $\widetilde{X}$ and let $\widehat{\sigma}$ be the involution of $\widetilde{Y}$ induced by the involution σ of $\widetilde{X}$ that arises from the restriction to $\widetilde{X}$ of the complex conjugation in $\mathbb{C}^n$. We may assume that $\widetilde{Y} \subset \mathbb{C}^m$ and that $\widehat{\sigma}$ is the restriction to $\widetilde{Y}$ of the complex conjugation of $\mathbb{C}^m$. By [11, Thm.3.15] and since $\mathcal{S}$ is irreducible, $\pi^{-1}(\mathcal{S})$ has a 1-dimensional connected component $\mathcal{T}$ such that $\pi(\mathcal{T}) = \mathcal{S}$. As X has dimension 1, it is a coherent analytic set, so $\mathcal{T} \subset Y := \widetilde{Y} \cap \mathbb{R}^m$. As $\widetilde{Y}$ is a normal-curve, Y is a non-singular real algebraic curve. We claim: *the connected components of Y are Nash diffeomorphic either to $\mathbb{S}^1$ or to the real line $\mathbb{R}$.*

By [34, Thm.VI.2.1] there exist a compact affine non-singular real algebraic curve Z, a finite set F which is empty if Y is compact and a union Y' of some connected components of $Z \setminus F$ such that Y is Nash diffeomorphic to Y' and $\mathrm{Cl}(Y')$ is a compact Nash curve with boundary F. As Z is a compact affine non-singular real algebraic curve, its connected components are diffeomorphic to $\mathbb{S}^1$, so by [34, Thm.VI.2.2] the connected components of Z are in fact Nash diffeomorphic to $\mathbb{S}^1$. Now, each connected component

of Y is Nash diffeomorphic to an open connected subset of $\mathbb{S}^1$, so it is Nash diffeomorphic either to $\mathbb{S}^1$ or to the real line $\mathbb{R}$, as claimed.

Consequently, $\mathcal{T}$ is Nash diffeomorphic to a connected subset of either $\mathbb{S}^1$ or $\mathbb{R}$. By Examples 3.1 and 3.2 the semialgebraic set $\mathcal{T}$ is a Nash image of $\mathbb{R}$, so also $\mathcal{S}$ is a Nash image of $\mathbb{R}$. The converse is straightforward. $\square$

4. Building boundaries on Nash manifolds

The purpose of this section is to develop a tool to build boundaries on a non-compact Nash manifold. More precisely we will prove the following.

Proposition 4.1. *Let $H \subset \mathbb{R}^m$ be a Nash manifold with boundary. Then there exists a surjective Nash map $f : \operatorname{Int}(H) \to H$ that has local representations of the type $(x_1, \ldots, x_d) \mapsto (x_1^2, x_2, \ldots, x_d)$ at each point of $f^{-1}(\partial H)$.*

In order to ease the understanding of the strategy followed to prove Proposition 4.1 we refer the reader to Fig. 2 (b). This proof requires the use of Nash collars and Nash doubles of a Nash manifold with boundary H (see Fig. 2 (a)). These constructions for H compact are a common tool in Nash Geometry [34, §VI] but as far as we know there is no explicit reference to them in the literature when H is non-compact. In 4.A we afford the construction of Nash collars when H is non-necessarily compact. In 4.B we endow the (smooth) double of H with a Nash manifold structure. The resulting Nash manifold $\mathrm{D}(H)$ is called the *Nash double of H*. Its construction requires a Nash equation of ∂H that is strictly positive on $\operatorname{Int}(H)$ and has rank 1 at the points of ∂H.

4.A. Nash collars

Let $H \subset \mathbb{R}^m$ be a Nash manifold with boundary ∂H. A *Nash collar* of ∂H is an open semialgebraic neighborhood $W \subset H$ of ∂H equipped with a Nash diffeomorphism $\psi : W \to \partial H \times [0, 1)$ such that $\psi(x) = (x, 0)$ for all $x \in \partial H$. We recall next how Nash collars are constructed. For the smooth case see [31, Thm.I.5.9].

Lemma 4.2. *Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a Nash submanifold of dimension $d-1$. Let $U \subset M$ be an open semialgebraic neighborhood of N and $\rho : U \to N$ a Nash retraction. Let h be a Nash function on U such that $\{h = 0\} = N$ and $d_x h : T_x M \to \mathbb{R}$ is surjective for all $x \in N$. Consider the Nash map $\varphi := (\rho, h) : U \to N \times \mathbb{R}$. Then there exist an open semialgebraic neighborhood $V \subset U$ of N and a strictly positive Nash function ε on N such that $\varphi(V) = \{(x, t) \in N \times \mathbb{R} : |t| < \varepsilon(x)\}$ and $\varphi|_V : V \to \varphi(V)$ is a Nash diffeomorphism.*

Proof. We show first: *The derivative $d_x \varphi = (d_x \rho, d_x h) : T_x M \to T_x N \times \mathbb{R}$ is an isomorphism for all $x \in N$*. As $\dim(T_x M) = \dim(T_x N \times \mathbb{R})$, it is enough to show: *$d_x \varphi$ is surjective.*
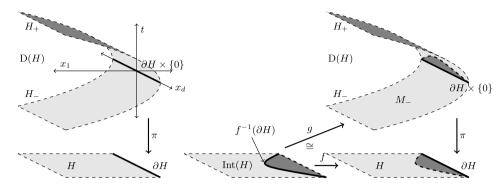
Fig. 2. (a) Nash double of H and (b) surjective Nash map $f : \mathrm{Int}(H) \to H$.

As $\varphi|_N = (\mathrm{id}_N, 0)$, we have $d_x\varphi|_{T_xN} = (\mathrm{id}_{T_xN}, 0)$, so $T_xN \times \{0\} \subset \mathrm{Im}(d_x\varphi)$. In addition $d_x h : T_x M \to \mathbb{R}$ is surjective, so there exists $v \in T_x M$ such that $d_x h(v) = 1$. Thus, $d_x\varphi(v) = (d_x\rho(v), 1)$ and $d_x\varphi$ is surjective.

Let $U' := \{x \in U : d_x\varphi \text{ is an isomorphism}\}$, which is an open semialgebraic neighborhood of N in U. Thus, $\varphi|_{U'} : U' \to N \times \mathbb{R}$ is an open map and $\varphi(U')$ is an open semialgebraic neighborhood of $N \times \{0\}$ in $N \times \mathbb{R}$. As $\varphi|_{U'} : U' \to \varphi(U')$ is a local homeomorphism and $\varphi|_N = (\mathrm{id}_N, 0)$ is a homeomorphism (onto its image), there exist by [3, Lem.9.2] open semialgebraic neighborhoods $U'' \subset U'$ of N and $W \subset N \times \mathbb{R}$ of $N \times \{0\}$ such that $\varphi|_{U''} : U'' \to W$ is a semialgebraic homeomorphism.

Consider the strictly positive semialgebraic map

$$\delta : N \to (0, +\infty), \ x \mapsto \mathrm{dist}((x, 0), (N \times \mathbb{R}) \setminus W).$$

By 2.D.1 there exists a strictly positive Nash function ε on N such that $\frac{1}{2}\delta < \varepsilon < \delta$. Consider the open semialgebraic neighborhood $W' := \{(x, t) \in N \times \mathbb{R} : |t| < \varepsilon(x)\} \subset W$ of $N \times \mathbb{R}$ and define $V := (\varphi|_{U''})^{-1}(W')$. The restriction $\varphi|_V : V \to W'$ is a Nash diffeomorphism, as required. $\square$

Lemma 4.3. *Let $H \subset \mathbb{R}^m$ be a d-dimensional Nash manifold with boundary ∂H. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d that contains H as a closed subset. Then there exist an open semialgebraic neighborhood $M' \subset M$ of H and a Nash equation h of ∂H in M' such that $\mathrm{Int}(H) = \{h > 0\}$ and $d_x h : T_x H \to \mathbb{R}$ is surjective for all $x \in \partial H$.*

Proof. We may assume $\partial H \neq \varnothing$. The proof is conducted in several steps:

4.A.1. We construct first an $\mathcal{S}^2$ semialgebraic function h_0 on M such that $\partial H \subset \{h_0 = 0\}$ and $d_x h_0(v) > 0$ for each $x \in \partial H$ and each non-zero vector $v \in T_x M$ pointing 'inside M'.

By 2.C.5 we can cover ∂H with finitely many open semialgebraic subsets U_i of M that are equipped with Nash diffeomorphisms $u_i := (u_{i1}, \ldots, u_{id}) : U_i \to \mathbb{R}^d$ such that

$U_i \cap H = \{u_{i1} \geq 0\}$ for $i = 1, \ldots, r$. Let $\{\theta_i\}_{i=1}^{r+1}$ be an $\mathcal{S}^2$ partition of unity subordinated to the finite covering $\{U_i\}_{i=1}^r \cup \{M \setminus \partial H\}$ of M and consider the $\mathcal{S}^2$ function $h_0 := \sum_{i=1}^r \theta_i u_{i1}$. It holds $\partial H \subset \{h_0 = 0\}$.

Fix $x \in \partial H$ and let $v \in T_x M$ be a non-zero vector pointing 'inside M', that is, $d_x u_{i1}(v) > 0$ if $x \in U_i$. We have

$$d_x h_0 = \sum_{x \in U_i} u_{i1}(x) d_x \theta_i + \sum_{x \in U_i} \theta_i(x) d_x u_{i1} = \sum_{x \in U_i} \theta_i(x) d_x u_{i1} \quad \rightsquigarrow$$

$$d_x h_0(v) = \sum_{x \in U_i} \theta_i(x) d_x u_{i1}(v) > 0$$

because $\sum_{x \in U_i} \theta_i(x) = 1$, $\theta_i(x) \geq 0$ and $d_x u_{i1}(v) > 0$ if $x \in U_i$.

4.A.2. By [3, Prop.8.2] there exists a Nash function h_1 on M close to h_0 in the $\mathcal{S}^2$ topology such that $\partial H \subset \{h_1 = 0\}$ and $d_x h_1(v) > 0$ for each $x \in \partial H$ and each non-zero vector $v \in T_x M$ pointing 'inside M'. We claim: *there exists an open semialgebraic neighborhood $W \subset M$ of ∂H such that $\{h_1 > 0\} \cap W = \mathrm{Int}(H) \cap W$ and $\{h_1 = 0\} \cap W = \partial H$.*

Pick a point $x \in \partial H$ and assume $x \in U_1$. As h_1 vanishes identically at ∂H, we may write $h_1|_{U_1} = u_{11} a_1$ where a_1 is a Nash function on U_1. Pick $y \in \partial H \cap U_1$ and observe that $d_y h_1 = a_1(y) d_y u_{11}$. Let $v \in T_y M$ be a non-zero vector pointing 'inside M'. As $d_y u_{11}(v) > 0$ and $d_y h_1(v) > 0$, we deduce $a_1(y) > 0$. Define $W_1 := \{a_1 > 0\} \subset U_1$ and notice that $\partial H \cap U_1 \subset W_1$, $\{h_1 > 0\} \cap W_1 = \mathrm{Int}(H) \cap W_1$ and $\{h_1 = 0\} \cap W_1 = \partial H \cap W_1$. Construct analogously $W_2, \ldots, W_r$ and observe that $W := \bigcup_{i=1}^r W_i$ satisfies the required properties.

4.A.3. Next, we construct h. If $W = M$, it is enough to set $h := h_1$. Suppose $W \neq M$. Let ε_0 be a (continuous) semialgebraic function whose value is 1 on ∂H and -1 on $M \setminus W$. Let ε be a Nash approximation of ε_0 such that $|\varepsilon - \varepsilon_0| < \frac{1}{2}$. Then

$$\varepsilon(x) \begin{cases} > \frac{1}{2} & \text{if } x \in \partial H, \\ < \frac{-1}{2} & \text{if } x \in M \setminus W. \end{cases}$$

Thus, $\{\varepsilon > 0\} \subset W$ is an open semialgebraic neighborhood of ∂H. By [34, Prop.II.5.3] ∂H is a Nash subset of M. Let f be a Nash equation of ∂H in M. Substituting f by $\frac{f^2}{\varepsilon^2 + f^2}$ we may assume that f is non-negative and $f(x) = 1$ if $\varepsilon(x) = 0$.

Consider the (continuous) semialgebraic function on M given by

$$\delta(x) := \begin{cases} 1 & \text{if } \varepsilon(x) > 0, \\ \frac{1}{f^2(x)} & \text{if } \varepsilon(x) \leq 0. \end{cases}$$

Let g be a Nash function on M such that $\delta < g^2$. Consider the Nash function

$$h := h_1 + f^2 g^2 (h_1^2 + 1)$$

and let us prove that it satisfies the required conditions.

4.A.4. We claim: *h is positive on* $\text{Int}(H)$.

Let $x \in \text{Int}(H)$. If $h_1(x) > 0$, then $h(x) > 0$. If $h_1(x) \leq 0$, then $\varepsilon(x) \leq 0$ and

$$h(x) = h_1(x) + g^2(x) f^2(x) (h_1^2(x) + 1)$$
$$> h_1(x) + \frac{1}{f^2(x)} f^2(x) (h_1^2(x) + 1) = h_1^2(x) + h_1(x) + 1 > 0.$$

4.A.5. It holds: $\{h = 0\} \cap W = \partial H$, $\{h > 0\} \cap W = \text{Int}(H) \cap W$ *and* $d_x h : T_x H \to \mathbb{R}$ *is surjective for all* $x \in \partial H$.

Recall that $W = \bigcup_{i=1}^r W_i$ and fix $i = 1$. We have seen in 4.A.2 that there exists a Nash function a_1 on U_1 such that $h_1|_{U_1} = u_{11} a_1$ and $\partial H \cap U_1 \subset W_1 := \{a_1 > 0\}$. As f vanishes identically at ∂H, we deduce that $f|_{U_1} = u_{11} b_1$ where b_1 is a Nash function on U_1. Consequently,

$$h|_{U_1} = u_{11} a_1 + g^2|_{U_1} u_{11}^2 b_1^2 (u_{11}^2 a_1^2 + 1) = u_{11}(a_1 + g^2|_{U_1} u_{11} b_1^2 (u_{11}^2 a_1^2 + 1))$$
$$\text{and} \quad d_x h = a_1(x) d_x u_{11} = d_x h_1 \quad \forall\, x \in \partial H \cap U_1.$$

Define $W_1' := \{a_1 + g^2|_{U_1} u_{11} b_1^2 (u_{11}^2 a_1^2 + 1) > 0\} \cap W_1$, which is an open semialgebraic subset of M. We have $\partial H \cap U_1 \subset W_1'$ and $\{h > 0\} \cap W_1' = \text{Int}(H) \cap W_1'$. Construct analogously $W_2', \ldots, W_r'$ and observe that $W' := \bigcup_{i=1}^r W_i' \subset M$ is an open neighborhood of ∂H that satisfies $\{h = 0\} \cap W' = \partial H$, $\{h > 0\} \cap W' = \text{Int}(H) \cap W'$ and $d_x h : T_x H \to \mathbb{R}$ is surjective for all $x \in \partial H$.

Consequently, $M' := \text{Int}(H) \cup W'$ satisfies the required conditions. $\quad\square$

Lemma 4.4. *Let* $H \subset \mathbb{R}^m$ *be a d-dimensional Nash manifold with boundary* ∂H. *Then every semialgebraic neighborhood* $U \subset H$ *of* ∂H *contains a Nash collar* (W, ψ) *of* ∂H *in* H.

Proof. By 2.C.5 there exists a Nash manifold $M \subset \mathbb{R}^m$ of dimension d that contains H as a closed subset and such that ∂H is a Nash submanifold of M of dimension $d - 1$. Let $U' \subset M$ be an open semialgebraic neighborhood of ∂H such that $U' \cap H = U$.

As ∂H is a Nash manifold without boundary, there exists a Nash tubular neighborhood A of ∂H in $\mathbb{R}^m$ together with a Nash retraction $\rho : A \to \partial H$. Substituting U' by $U' \cap A$, we may assume $U' \subset A$. By Lemma 4.3 we can shrink M to have a Nash equation h of ∂H in M such that $\{h > 0\} \cap M = \text{Int}(H)$ and $d_x h : T_x H \to \mathbb{R}$ is surjective for all $x \in \partial H$. Denote

$$\varphi := (\rho|_{U'}, h|_{U'}) : U' \to \partial H \times \mathbb{R}.$$

By Lemma 4.2 there exists an open semialgebraic neighborhood $V \subset U'$ of ∂H and a strictly positive Nash function ε on ∂H such that $\varphi(V) = \{(y, t) \in \partial H \times \mathbb{R} : \ |t| < \varepsilon(y)\}$ and $\varphi|_V : V \to \varphi(V)$ is a Nash diffeomorphism. Consider the Nash diffeomorphism

$$\psi : V \to \partial H \times (-1, 1), \ \ x \mapsto \left(\rho(x), \frac{h(x)}{\varepsilon(\rho(x))} \right)$$

and $W := \psi^{-1}(\partial H \times [0, 1)) \subset U' \cap H$. The restriction $\psi|_W : W \to \partial H \times [0, 1)$ provides a collar of ∂H such that $W \subset U$, as required. $\square$

4.B. Nash doubles

Let $H \subset \mathbb{R}^m$ be a d-dimensional Nash manifold with boundary ∂H. Let h be a Nash equation of ∂H such that $\mathrm{Int}(H) = \{h > 0\}$ and $d_x h : T_x H \to \mathbb{R}$ is surjective for all $x \in \partial H$ (see Lemma 4.3). We have:

4.B.1. $\mathrm{D}(H) := \{(x, t) \in H \times \mathbb{R} : \ t^2 - h(x) = 0\}$ *is a Nash manifold of dimension d that contains $\partial H \times \{0\}$ as the Nash subset $\{t = 0\}$.*

Proof. The semialgebraic set $\mathrm{D}(H)$ satisfies

$$D(H) \setminus (\partial H \times \{0\}) = \{(x, t) \in \mathrm{Int}(H) \times \mathbb{R} : \ t = \pm\sqrt{h(x)}\},$$

which is the union of two disjoint graphs of Nash functions on the Nash manifold $\mathrm{Int}(H)$. Consequently, it is enough to show: *for each $(x_0, 0) \in \partial H \times \{0\}$ there exists an open semialgebraic neighborhood $W \subset \mathrm{D}(H)$ of $(x_0, 0)$ such that W is a Nash manifold.*

Let $M' \subset \mathbb{R}^m$ be a Nash manifold of dimension d that contains H as a closed subset. Pick a point $x_0 \in \partial H$ and let $U \subset M'$ be an open semialgebraic neighborhood of x_0 equipped with a Nash diffeomorphism $u := (u_1, \dots, u_d) : U \to (-1, 1) \times \mathbb{R}^{d-1}$ such that $u(x_0) = 0$ and $U \cap H = \{u_1 \geq 0\}$. We may assume $u_1 = h|_U$. Observe that $V := (U \cap H) \times \mathbb{R}$ is an open semialgebraic subset of $H \times \mathbb{R}$ and

$$W := \mathrm{D}(H) \cap V = \{(x, \pm\sqrt{h(x)}) : \ x \in U \cap H\}$$

is an open semialgebraic neighborhood of $(x_0, 0)$ in $\mathrm{D}(H)$. Consider the Nash map

$$u' := (u'_1, \dots, u'_d) : W \to (-1, 1) \times \mathbb{R}^{d-1}, \ (x, t) \mapsto (t, u_2(x), \dots, u_d(x))$$

and let us check that it is a Nash diffeomorphism. As

$$(u'_2, \dots, u'_d)(W) = (u_2, \dots, u_d)(U \cap H) = \mathbb{R}^{d-1},$$

we have $u'(W) = (-1, 1) \times \mathbb{R}^{d-1}$, so u' is surjective.

Pick $(x_1, t_1), (x_2, t_2) \in W$ such that $u'(x_1, t_1) = u'(x_2, t_2)$. Then $t_1 = t_2$, so

$$u_1(x_1) = h(x_1) = t_1^2 = t_2^2 = h(x_2) = u_1(x_2)$$

and $u(x_1) = u(x_2)$. As u is injective, we have $x_1 = x_2$, so $(x_1, t_1) = (x_2, t_2)$. Thus, u' is injective. Denote $u^{-1} := \phi := (\phi_1, \ldots, \phi_m)$. The inverse of u' is the Nash map

$$\zeta : (-1, 1) \times \mathbb{R}^{d-1} \to W, \ (t, y') := (t, y_2, \ldots, y_d) \mapsto (\phi(t^2, y'), t).$$

The differential of ζ at a point $(t, y') \in (-1, 1) \times \mathbb{R}^{d-1}$ is

$$\begin{pmatrix} 2t \frac{\partial \phi_1}{\partial y_1}(t^2, y') & \frac{\partial \phi_1}{\partial y_2}(t^2, y') & \cdots & \frac{\partial \phi_1}{\partial y_d}(t^2, y') \\ \vdots & \vdots & \ddots & \vdots \\ 2t \frac{\partial \phi_m}{\partial y_1}(t^2, y') & \frac{\partial \phi_m}{\partial y_2}(t^2, y') & \cdots & \frac{\partial \phi_m}{\partial y_d}(t^2, y') \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and it has rank d. Consequently, u' is a Nash diffeomorphism as required. $\square$

4.B.2. Consider the surjective Nash map $\pi : \mathrm{D}(H) \to H$, $(x, t) \to x$ and write $\epsilon = \pm$. We have:

(i) *The restriction* $\pi|_{\mathrm{D}(H) \cap \{\epsilon t > 0\}} : \mathrm{D}(H) \cap \{\epsilon t > 0\} \to \mathrm{Int}(H)$ *is a Nash diffeomorphism.*
(ii) $\pi(x, 0) = x$ *for all* $(x, 0) \in \partial H \times \{0\} = \mathrm{D}(H) \cap \{t = 0\}$.
(iii) π *has local representations* $(y_1, \ldots, y_d) \mapsto (y_1^2, y_2, \ldots, y_d)$ *at each point of* $\mathrm{D}(H) \cap \{t = 0\}$.

Proof. (i) The intersection $\mathrm{D}(H) \cap \{\epsilon t > 0\}$ is the graph of the strictly positive Nash function $\epsilon \sqrt{h}$ on $\mathrm{Int}(H)$. Consequently $\pi|_{\mathrm{D}(H) \cap \{\epsilon t > 0\}} : \mathrm{D}(H) \cap \{\epsilon t > 0\} \to \mathrm{Int}(H)$ is Nash diffeomorphism.

Statement (ii) is evident.

(iii) Consider the Nash diffeomorphism u' constructed in the proof of 4.B.1 and its inverse ζ. We have

$$(-1, 1) \times \mathbb{R}^{d-1} \overset{\zeta}{\to} W \overset{\pi}{\to} H \cap U \overset{u}{\to} (-1, 1) \times \mathbb{R}^{d-1},$$
$$(t, y') \mapsto (\phi(t^2, y'), t) \mapsto \phi(t^2, y') \mapsto (t^2, y'),$$

as required. $\square$

4.B.3. *H is Nash diffeomorphic to* $H_\epsilon := \mathrm{D}(H) \cap \{\epsilon t \geq 0\}$ *for* $\epsilon = \pm$ *and* $\mathrm{D}(H)$ *is a Nash manifold structure for the double of H. In addition, the Nash map* $\tau : \mathrm{D}(H) \to \mathrm{D}(H)$, $(x, t) \mapsto (x, -t)$ *is an involution such that* $\tau(H_+) = H_-$ *and whose set of fixed points is* $\partial H \times \{0\}$.

Proof. The proof is conducted in several steps:

Step 1. To prove that H_ϵ *and* H *are Nash diffeomorphic* we need to construct first suitable neighborhoods $U \subset H$ of ∂H and $V \subset D(H)$ of $\partial H \times \{0\}$.

Let $U \subset H$ be an open semialgebraic neighborhood of ∂H equipped with a Nash retraction $\rho : U \to \partial H$. Define $\varphi := (\rho, h) : U \to \partial H \times \mathbb{R}$. By Lemma 4.2 we may assume there exists a strictly positive Nash function ε on ∂H such that

$$\varphi(U) = \{(y, s) \in \partial H \times \mathbb{R} : \ 0 \le s < \varepsilon(y)\}$$

and $\varphi : U \to \varphi(U)$ is a Nash diffeomorphism.

Let $V := \pi^{-1}(U)$ and $V' := \{(y, t) \in \partial H \times \mathbb{R} : \ |t| < \sqrt{\varepsilon(y)}\}$. We claim: *The Nash map*

$$\psi : V \to \partial H \times \mathbb{R}, \ (x, t) \mapsto (\rho(x), t) \tag{4.1}$$

is a Nash diffeomorphism onto its image V'.

(1) ψ *is injective.* If $(x_1, t_1), (x_2, t_2) \in V$ satisfy $\psi(x_1, t_1) = \psi(x_2, t_2)$, then $\rho(x_1) = \rho(x_2)$ and $h(x_1) = t_1^2 = t_2^2 = h(x_2)$. Consequently, $\varphi(x_1) = \varphi(x_2)$, so $x_1 = x_2$. Thus, $(x_1, t_1) = (x_2, t_2)$.

(2) $\psi(V) = V'$. Pick $(x, t) \in V$. Then $x \in U$ and $\varphi(x) = (\rho(x), h(x)) \in \varphi(U)$, so $t^2 = h(x) < \varepsilon(\rho(x))$ and $\psi(x, t) \in V'$. Conversely, let $(y, t) \in V'$. As $(y, t^2) \in \varphi(U)$, there exists $x \in U$ such that $\varphi(x) := (\rho(x), h(x)) = (y, t^2)$. Then $(x, t) \in V$ and $\psi(x, t) := (\rho(x), t) = (y, t)$.

(3) *The derivative* $d_z\psi : T_z\mathrm{D}(H) \to T_{\rho(z)}\partial H \times \mathbb{R}$ *is an isomorphism for each* $z \in V$. Write $z := (x, t)$ and notice that $T_z\mathrm{D}(H) = \{(v, r) \in T_xH \times \mathbb{R} : \ d_xh(v) - 2tr = 0\}$ and $d_z\psi(v, r) = (d_x\rho(v), r)$. If $t \ne 0$,

$$d_z\psi(v, r) = \Big(d_x\rho(v), \frac{1}{2t}d_xh(v)\Big).$$

As $d_x\varphi = (d_x\rho, d_xh)$ is an isomorphism, also $d_z\psi$ is an isomorphism. If $t = 0$, that is, $z = (x, 0) \in \partial H \times \mathbb{R}$, then $T_z\mathrm{D}(H) = \{(v, r) \in T_xH \times \mathbb{R} : \ d_xh(v) = 0\} = T_x\partial H \times \mathbb{R}$ and $d_z\psi(v, r) = (v, r)$ because $\rho|_{\partial H} = \mathrm{id}_{\partial H}$. Consequently, $d_z\psi$ is an isomorphism also in this case.

Step 2. Define

$$H^\bullet := H \setminus \varphi^{-1}\Big(\Big\{(y, s) \in \partial H \times \mathbb{R} : \ 0 \le |s| < \frac{\varepsilon(y)}{4}\Big\}\Big),$$

$$H_\epsilon^\bullet := H_\epsilon \setminus \psi^{-1}\Big(\Big\{(y, s) \in \partial H \times \mathbb{R} : \ 0 \le |s| < \frac{\sqrt{\varepsilon(y)}}{2}\Big\}\Big),$$

and let us show: *The restriction* $\varpi_\epsilon := \pi|_{H_\epsilon^\bullet} : H_\epsilon^\bullet \to H^\bullet$ *is a Nash diffeomorphism.*

Indeed, ϖ_ϵ is clearly injective. Let $x \in H^\bullet$. As $x \in \mathrm{Int}(H)$, we have $h(x) > 0$ and write $t := \epsilon\sqrt{h(x)}$. It holds that $(x, t) \in \mathrm{D}(H)$ and $\pi(x, t) = x$. We want to check that

$(x,t) \in H_\epsilon^\bullet$. If $x \notin U$, then $(x,t) \in H_\epsilon \setminus V \subset H_\epsilon^\bullet$. If $x \in U$, then $\psi(x,t) = (\rho(x),t)$. As $x \in H^\bullet$, it holds $\frac{\varepsilon(\rho(x))}{4} \leq h(x)$, so $\frac{\sqrt{\varepsilon(\rho(x))}}{2} \leq \sqrt{h(x)} = \epsilon t$. Consequently, $(x,t) \in H_\epsilon^\bullet$ and ϖ_ϵ is surjective. In addition, by 4.B.2(i) $d_z\varpi_\epsilon = d_z\pi$ is an isomorphism for each $z \in H_\epsilon^\bullet \subset D(H) \cap \{\epsilon t > 0\}$. Consequently, ϖ_ϵ is a Nash diffeomorphism.

Step 3. $H^\bullet$ *is Nash diffeomorphic to* H *and* $H_\epsilon^\bullet$ *is Nash diffeomorphic to* H_ϵ. By 2.D.3 it is enough to show that the pairs of objects above are $\mathcal{S}^2$ diffeomorphic.

As τ is an involution such that $\tau(H_+) = H_-$, we assume $\epsilon = +$. Denote

$$U' := \{(y,s) \in \partial H \times \mathbb{R} : \ 0 \leq s < \varepsilon(y)\},$$
$$W' := \{(y,t) \in \partial H \times \mathbb{R} : \ 0 \leq t < \sqrt{\varepsilon(y)}\},$$
$$W'' := \{(y,t) \in \partial H \times \mathbb{R} : \ 0 \leq -t < \sqrt{\varepsilon(y)}\}.$$

Observe that $U = \varphi^{-1}(U')$, $V = \psi^{-1}(W' \cup W'')$ and set $W := \psi^{-1}(W')$. Consider the Nash diffeomorphisms

$$\Lambda : U' \to \partial H \times [0,1), \ (y,s) \mapsto \left(y, \frac{s}{\varepsilon(y)}\right),$$
$$\Delta : W' \to \partial H \times [0,1), \ (y,t) \mapsto \left(y, \frac{t}{\sqrt{\varepsilon(y)}}\right).$$

Observe that $(\Lambda \circ \varphi)(H^\bullet \cap U) = \partial H \times [\frac{1}{4}, 1)$ and $(\Delta \circ \psi)(H_+^\bullet \cap W) = \partial H \times [\frac{1}{2}, 1)$.

Let $f_1 : [\frac{1}{4}, 1) \to [0,1)$ and $f_2 : [\frac{1}{2}, 1) \to [0,1)$ be $\mathcal{S}^2$ diffeomorphisms such that $f_1|_{[\frac{3}{4},1)} = \mathrm{id}_{[\frac{3}{4},1)}$ and $f_2|_{[\frac{3}{4},1)} = \mathrm{id}_{[\frac{3}{4},1)}$ (see Examples A.1(i) and (ii) in Appendix A). Consider the $\mathcal{S}^2$ diffeomorphisms

$$F_1 : \partial H \times [\tfrac{1}{4}, 1) \to \partial H \times [0,1), \ (y,t) \mapsto (y, f_1(t)),$$
$$F_2 : \partial H \times [\tfrac{1}{2}, 1) \to \partial H \times [0,1), \ (y,t) \mapsto (y, f_2(t))$$

It holds $F_i|_{\partial H \times [\frac{3}{4},1)} = \mathrm{id}_{\partial H \times [\frac{3}{4},1)}$ for $i = 1, 2$. Denote again φ and ψ the respective restrictions of these Nash maps to U and W. Define

$$g : H^\bullet \to H, \ x \mapsto \begin{cases} x & \text{if } x \in H^\bullet \setminus U, \\ (\Lambda \circ \varphi)^{-1}(F_1((\Lambda \circ \varphi)(x))) & \text{if } x \in H^\bullet \cap U, \end{cases}$$

$$g_+ : H_+^\bullet \to H_+, \ z \mapsto \begin{cases} z & \text{if } z \in H_+^\bullet \setminus W, \\ (\Delta \circ \psi)^{-1}(F_2((\Delta \circ \psi)(z))) & \text{if } z \in H_+^\bullet \cap W. \end{cases}$$

Both g and g_+ are $\mathcal{S}^2$ diffeomorphisms.

Step 4. Consequently, both H_+ and $H_- = \tau(H_+)$ are Nash diffeomorphic to H and have as common boundary $\partial H \times \{0\}$. In addition $\tau|_{\partial H \times \{0\}} = \mathrm{id}_{\partial H \times \{0\}}$, so $D(H)$ is by [31, §I.6] the double of H. $\quad\square$

4.B.4.   There exists an open semialgebraic neighborhood V of $\partial H \times \{0\}$ in $D(H)$ such that $M_\epsilon := H_\epsilon \cup V$ is a Nash manifold Nash diffeomorphic to $\mathrm{Int}(H)$ that contains H_ϵ as a closed subset. In particular $\pi(M_\epsilon) = H$.

Proof. Define $V' := \{(y, s) \in \partial H \times \mathbb{R} : |s| < \sqrt{\varepsilon(y)}\}$ and $V := \psi^{-1}(V')$ where ψ is the Nash map defined in (4.1). Consider the Nash diffeomorphism

$$\Theta : V' \to \partial H \times (-1, 1), \ (y, s) \to \left(y, \frac{s}{\sqrt{\varepsilon(y)}}\right).$$

By 4.B.3 it is enough to show that $M_\epsilon := H_\epsilon \cup V$ and $\mathrm{Int}(H_\epsilon)$ are Nash diffeomorphic. As τ is an involution such that $\tau(H_-) = H_+$, we assume $\epsilon = -$. By 2.D.3 it is enough to show that the objects above are $\mathcal{S}^2$ diffeomorphic.

Let $f_3 : (-1, 1) \to (-1, 0)$ be an $\mathcal{S}^2$ diffeomorphism such that $f_3|_{(-1, -\frac{1}{2}]} = \mathrm{id}_{(-1, -\frac{1}{2}]}$ (see Example A.1(iii)). Consider the $\mathcal{S}^2$ diffeomorphism

$$F_3 : \partial H \times (-1, 1) \to \partial H \times (-1, 0), \ (y, t) \mapsto (y, f_3(t))$$

and define

$$h : M_- \to \mathrm{Int}(H_-), \ z \mapsto \begin{cases} z & \text{if } z \in M_- \setminus V, \\ (\Theta \circ \psi)^{-1}(F_3((\Theta \circ \psi)(z))) & \text{if } x \in V, \end{cases}$$

which is an $\mathcal{S}^2$ diffeomorphism.

Observe $H_- \subset M_-$ and $H = \pi(H_-) \subset \pi(M_-) \subset \pi(D(H)) \subset H$, as required. $\square$

4.C. Proof of Proposition 4.1

Let $D(H)$ be the Nash double of H. By 4.B.4 there exist

- a Nash manifold $M_- \subset D(H)$ such that $H_- \subset M_-$ is a closed subset and $\pi(M_-) = H$,
- a Nash diffeomorphism $g : \mathrm{Int}(H) \to M_-$.

Let us check that $f := \pi \circ g$ satisfy the required conditions. First $f(\mathrm{Int}(H)) = \pi(M_-) = H$. Next, pick a point $y_0 \in \partial H$ and let $x_0 \in f^{-1}(y_0)$. We have to prove that f has a local representation $(x_1, \ldots, x_d) \mapsto (x_1^2, x_2, \ldots, x_d)$ at x_0. It holds $z_0 := g(x_0) = (y_0, 0)$. As g is a Nash diffeomorphism, it is enough to prove that π has a local representation $(x_1, \ldots, x_d) \mapsto (x_1^2, x_2, \ldots, x_d)$ at z_0. But this follows by 4.B.2(iii), as required. $\square$

5. A main ingredient: the drilling blow-up

In this section we construct the *drilling blow-up* of a Nash manifold M with center a closed Nash submanifold N. We refer the reader to [25,6] for the oriented blow-up of a

real analytic space with center a closed subspace, which is the counterpart of the tool we present in this section for the real analytic setting. In [27, §5] appears a presentation of the oriented blow-up in the analytic case closer to the drilling blow-up we provide here. The authors consider there the case of the oriented blow-up of a real analytic manifold M with center a closed real analytic submanifold N whose vanishing ideal inside M is finitely generated (this happens for instance if N is compact). In [8, §3] we present a similar construction in the semialgebraic setting, which is used to 'appropriately embed' semialgebraic sets in Euclidean spaces. The following result, whose proof makes use of the drilling blow-up and which will be a key tool for our purposes, will allow to erase a closed Nash submanifold from a Nash manifold (see Fig. 5).

Lemma 5.1 *(Erasing a closed Nash submanifold). Let M be a Nash manifold and let $N \subset M$ be a closed Nash submanifold. Then there exists a surjective Nash map $h : M \setminus N \to M$.*

5.A. Local structure of the drilling blow-up

Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. Assume that there exists a Nash diffeomorphism $u := (u_1, \ldots, u_d) : M \to \mathbb{R}^d$ such that $N = \{u_{e+1} = 0, \ldots, u_d = 0\}$. Denote $\psi := u^{-1} : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to M$. Let $\zeta_{e+1}, \ldots, \zeta_d : \mathbb{R}^d \to \mathbb{R}^k$ be Nash maps such that the vectors $\zeta_{e+1}(y, 0), \ldots, \zeta_d(y, 0)$ are linearly independent for each $y \in \mathbb{R}^e$. Write $z \in \mathbb{R}^{d-e}$ as $z := (z_{e+1}, \ldots, z_d)$. Consider the Nash maps

$$\varphi : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to \mathbb{R}^k, \ (y, z) \mapsto \zeta_{e+1}(y, z)z_{e+1} + \ldots + \zeta_d(y, z)z_d,$$

$$\phi : \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1} \to \mathbb{R}^k, \ (y, \rho, w) \mapsto \zeta_{e+1}(y, \rho w)w_{e+1} + \cdots + \zeta_d(y, \rho w)w_d$$

and assume that $\varphi(y, z) = 0$ if and only if $z = 0$. Consider the projections

$$\theta_1 : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to \mathbb{R}^e, \ (y, z) \mapsto y,$$

$$\theta_2 : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to \mathbb{R}^{d-e}, \ (y, z) \mapsto z.$$

Define

$$\Phi : \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1} \to M \times \mathbb{S}^{k-1}, \ (y, \rho, w) \mapsto \left(\psi(y, \rho w), \frac{\phi(y, \rho, w)}{\|\phi(y, \rho, w)\|} \right).$$

5.A.1. Φ is a well-defined Nash map.

Proof. Observe first that $\varphi(y, \rho w) = \rho\phi(y, \rho, w)$ for all $(y, \rho, w) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$. If $\rho \neq 0$, the product $\rho w \neq 0$, so $\rho\phi(y, \rho, w) = \varphi(y, \rho w) \neq 0$ and $\phi(y, \rho, w) \neq 0$. If $\rho = 0$, then

$$\phi(y, 0, w) = \zeta_{e+1}(y, 0)w_{e+1} + \cdots + \zeta_d(y, 0)w_d.$$

As $\zeta_{e+1}(y, 0), \ldots, \zeta_d(y, 0)$ are linearly independent and $\|w\| = 1$, we conclude $\phi(y, 0, w) \neq 0$. Consequently, Φ is a well-defined Nash map, as required. $\quad\square$

5.A.2. Fix $\epsilon = \pm$ and denote

$$I_\epsilon := \begin{cases} [0, +\infty) & \text{if } \epsilon = +, \\ (-\infty, 0] & \text{if } \epsilon = -. \end{cases}$$

The closure $\widetilde{M}_\epsilon$ in $M \times \mathbb{S}^{k-1}$ of the set

$$\Gamma_\epsilon := \left\{ \left(\psi(y, z), \epsilon \frac{\varphi(y, z)}{\|\varphi(y, z)\|} \right) \in M \times \mathbb{S}^{k-1} : z \neq 0 \right\}$$

is a Nash manifold with boundary such that:

(i) *$\widetilde{M}_\epsilon \subset \mathrm{Im}(\Phi)$.*
(ii) *The restriction of Φ to $\mathbb{R}^e \times I_\epsilon \times \mathbb{S}^{d-e-1}$ induces a Nash diffeomorphism between $\mathbb{R}^e \times I_\epsilon \times \mathbb{S}^{d-e-1}$ and $\widetilde{M}_\epsilon$. Consequently, $\partial\widetilde{M}_\epsilon = \Phi(\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})$ and $\Gamma_\epsilon = \mathrm{Int}(\widetilde{M}_\epsilon) = \Phi(\mathbb{R}^e \times (I_\epsilon \setminus \{0\}) \times \mathbb{S}^{d-e-1})$.*

Proof. (i) Recall $\varphi(y, \rho w) = \rho\phi(y, \rho, w)$ for all $(y, \rho, w) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$. As $\mathrm{sign}(\rho) = \frac{\rho}{|\rho|}$ for $\rho \neq 0$, we have

$$\frac{\phi(y, \rho, w)}{\|\phi(y, \rho, w)\|} = \mathrm{sign}(\rho) \frac{\varphi(y, \rho w)}{\|\varphi(y, \rho w)\|}. \tag{5.1}$$

Consequently, $\Gamma_+ \sqcup \Gamma_- \subset \mathrm{Im}(\Phi)$. Let us check: $\widetilde{M}_\epsilon \setminus \Gamma_\epsilon \subset \mathrm{Im}(\Phi)$.

Pick a point $(a, b) \in \widetilde{M}_\epsilon \setminus \Gamma_\epsilon$. By the Nash curve selection lemma [4, Prop.8.1.13] there exists a Nash arc $\gamma : (-1, 1) \to M \times \mathbb{S}^{k-1}$ such that $\gamma(0) = (a, b)$ and $\gamma((0, 1)) \subset \Gamma_\epsilon$. Let $\pi : M \times \mathbb{S}^{k-1} \to M$ be the projection onto the first factor and let $(\alpha, \beta) : (-1, 1) \to \mathbb{R}^e \times \mathbb{R}^{d-e} \equiv \mathbb{R}^d$ be a Nash arc such that $\psi(\alpha, \beta) = \pi \circ \gamma$. We write

$$\gamma|_{(0,1)} = \left(\psi, \epsilon \frac{\varphi}{\|\varphi\|} \right) \circ (\alpha, \beta)|_{(0,1)}.$$

Note that $\beta(0) = 0$ because otherwise $(a, b) \in \Gamma_\epsilon$. In addition, $a = \psi(\alpha(0), 0)$. Observe that

$$b = \lim_{t \to 0^+} \epsilon \frac{\varphi(\alpha(t), \beta(t))}{\|\varphi(\alpha(t), \beta(t))\|}. \tag{5.2}$$

As $\gamma((0, 1)) \subset \Gamma_\epsilon$, the Nash arc β is not identically 0 and we may assume $\beta(t) = 0$ if and only if $t = 0$. Let $\xi \in \mathbb{R}[[\mathtt{t}]]_{\mathrm{alg}}$ be a Nash series such that $\|\beta(t)\| = \xi(t)$ for $t > 0$ small. As

$$\text{order}(\xi) = \min_{e+1 \le i \le d} \{\text{order}(\beta_i)\},$$

the quotient $\frac{\beta}{\|\beta\|}|_{(0,1)}$ extends to a Nash arc $\eta : (-1,1) \to \mathbb{S}^{d-e-1}$. We have $\beta = (\epsilon\xi)(\epsilon\eta)$. By (5.1) and (5.2)

$$b = \lim_{t \to 0^+} \frac{\phi(\alpha(t), \epsilon\xi(t), \epsilon\eta(t))}{\|\phi(\alpha(t), \epsilon\xi(t), \epsilon\eta(t))\|}$$

Consequently, as $\xi(0) = 0$,

$$b = \frac{\phi(\alpha(0), 0, \epsilon\eta(0))}{\|\phi(\alpha(0), 0, \epsilon\eta(0))\|} \quad \text{and} \quad (a,b) = \Phi(\alpha(0), 0, \epsilon\eta(0)) \in \text{Im}(\Phi).$$

(ii) The proof of this statement is conducted in several steps:

Step 1. Denote $R := \Phi(\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})$ and consider the open semialgebraic subset of $\text{Im}(\Phi)$

$$U := \{(a,b) \in \text{Im}(\Phi) : \text{rk}((\zeta_{e+1} \circ u)(a), \dots, (\zeta_d \circ u)(a)) = d - e\}. \tag{5.3}$$

As $\zeta_{e+1}(y,0), \dots, \zeta_d(y,0)$ are linearly independent for each $y \in \mathbb{R}^e$ we have $R \subset U$. In addition,

$$\Phi^{-1}(U) = \{(y, \rho, w) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1} : \text{rk}(\zeta_{e+1}(y, \rho w), \dots, \zeta_d(y, \rho w)) = d - e\}.$$

We claim: $\Phi|_{\Phi^{-1}(U)} : \Phi^{-1}(U) \to U$ *is a Nash diffeomorphism. In particular, U is a Nash manifold that contains R.*

Let $(a,b) \in U$ and let $(y, \rho, w) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$ be such that $\Phi(y, \rho, w) = (a,b)$, that is, $a = \psi(y, \rho w)$ and $b = \frac{\phi(y,\rho,w)}{\|\phi(y,\rho,w)\|}$. Consequently, $(y, \rho w) = u(a) = (\theta_1(u(a)), \theta_2(u(a)))$. Consider the system of linear equations

$$\mathrm{v}_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + \mathrm{v}_d(\zeta_d \circ u)(a) = b. \tag{5.4}$$

As $(a,b) \in U$ and $b \in \mathbb{S}^{k-1}$, the system (5.4) has a unique non-zero solution $v \in \mathbb{R}^{d-e}$. If we solve (5.4) using Cramer's rule, one sees that the map $U \to \mathbb{S}^{d-e-1}$, $(a,b) \mapsto w := (w_{e+1}, \dots, w_d) := \frac{v}{\|v\|}$ is Nash. As $\|b\| = 1$,

$$b = v_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + v_d(\zeta_d \circ u)(a) = \frac{v_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + v_d(\zeta_d \circ u)(a)}{\|v_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + v_d(\zeta_d \circ u)(a)\|}$$

$$= \frac{\frac{v_{e+1}}{\|v\|}(\zeta_{e+1} \circ u)(a) + \cdots + \frac{v_d}{\|v\|}(\zeta_d \circ u)(a)}{\|\frac{v_{e+1}}{\|v\|}(\zeta_{e+1} \circ u)(a) + \cdots + \frac{v_d}{\|v\|}(\zeta_d \circ u)(a)\|}$$

$$= \frac{w_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + w_d(\zeta_d \circ u)(a)}{\|w_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + w_d(\zeta_d \circ u)(a)\|}.$$

As $\rho w = \theta_2(u(a))$, the vectors $\theta_2(u(a))$ and w are linearly dependent. As w is a unitary vector, $\theta_2(u(a)) = \langle \theta_2(u(a)), w \rangle w$. It holds

$$\Phi(\theta_1(u(a)), \langle \theta_2(u(a)), w \rangle, w)$$
$$= \left(\psi(\theta_1(u(a)), \theta_2(u(a))), \frac{w_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + w_d(\zeta_d \circ u)(a)}{\|w_{e+1}(\zeta_{e+1} \circ u)(a) + \cdots + w_d(\zeta_d \circ u)(a)\|} \right) = (a, b)$$

Write $w(a, b) := w$. The Nash map

$$\Psi_0 : U \to \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}, \ (a, b) \mapsto (\theta_1(u(a)), \langle \theta_2(u(a)), w(a, b) \rangle, w(a, b)) \qquad (5.5)$$

satisfies: $\text{Im}(\Psi_0) \subset \Phi^{-1}(U)$ *and* $\Phi \circ \Psi_0 = \text{id}_U$. Let us check in addition $\Psi_0 \circ \Phi|_{\Phi^{-1}(U)} = \text{id}_{\Phi^{-1}(U)}$ to conclude $\Phi|_{\Phi^{-1}(U)}$ *is a Nash diffeomorphism*.

Let $(y, \rho, w) \in \Phi^{-1}(U)$. We have

$$(\Psi_0 \circ \Phi)(y, \rho, w) = \Psi_0 \left(\psi(y, \rho w), \frac{\phi(y, \rho, w)}{\|\phi(y, \rho, w)\|} \right) = \left(y, \rho \left\langle w, \frac{v}{\|v\|} \right\rangle, \frac{v}{\|v\|} \right)$$

where $v \in \mathbb{R}^{d-e}$ is the unique solution of the system

$$\mathrm{v}_{e+1} \zeta_{e+1}(y, \rho w) + \cdots + \mathrm{v}_d \zeta_d(y, \rho w) = \frac{w_{e+1} \zeta_{e+1}(y, \rho w) + \cdots + w_d \zeta_d(y, \rho w))}{\|w_{e+1} \zeta_{e+1}(y, \rho w) + \cdots + w_d \zeta_d(y, \rho w))\|}.$$

Consequently,

$$v = \frac{w}{\|w_{e+1} \zeta_{e+1}(y, \rho w) + \cdots + w_d \zeta_d(y, \rho w))\|}$$

and $\frac{v}{\|v\|} = w$ (recall that $\|w\| = 1$). Thus, $(\Psi_0 \circ \Phi)(y, \rho, w) = (y, \rho, w)$, as claimed.

Step 2. Next, let us check: $\Phi^{-1}(\Gamma_\epsilon) = \mathbb{R}^e \times (I_\epsilon \setminus \{0\}) \times \mathbb{S}^{d-e-1}$ *and the restriction*

$$\Phi| : \mathbb{R}^e \times (I_\epsilon \setminus \{0\}) \times \mathbb{S}^{d-e-1} \to \Gamma_\epsilon$$

is a Nash diffeomorphism.

Define

$$\Psi_\epsilon : \Gamma_\epsilon \to \mathbb{R}^e \times (I_\epsilon \setminus \{0\}) \times \mathbb{S}^{d-e-1}, (a, b) \mapsto \left(\theta_1(u(a)), \epsilon \|\theta_2(u(a))\|, \frac{\theta_2(u(a))}{\epsilon \|\theta_2(u(a))\|} \right).$$

The previous map is well-defined and Nash because $\theta_2(u(a)) \neq 0$ for all $(a, b) \in \Gamma_\epsilon$. It follows that $\Phi|_{\mathbb{R}^e \times (I_\epsilon \setminus \{0\}) \times \mathbb{S}^{d-e-1}}$ is a Nash diffeomorphism whose inverse is Ψ_ϵ.

Step 3. We deduce from Steps 1 & 2 that the restriction $\Phi| : \mathbb{R}^e \times I_\epsilon \times \mathbb{S}^{d-e-1} \to \widetilde{M_\epsilon}$ is a Nash diffeomorphism. As $\mathbb{R}^e \times I_\epsilon \times \mathbb{S}^{d-e-1}$ is a Nash manifold with boundary, $\widetilde{M_\epsilon}$ is also a Nash manifold with boundary. In addition $\partial \widetilde{M_\epsilon} = R$ and $\text{Int}(\widetilde{M_\epsilon}) = \Gamma_\epsilon$, as required. $\quad\square$

5.A.3. Denote $R := \partial \widetilde{M}_+ = \partial \widetilde{M}_-$ and $\widehat{M} := \widetilde{M}_+ \cup \widetilde{M}_- = \Gamma_+ \sqcup R \sqcup \Gamma_-$. Then Φ induces a Nash diffeomorphism between $\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$ and $\widehat{M}$. In addition, the Nash map $\sigma : M \times \mathbb{S}^{k-1} \to M \times \mathbb{S}^{k-1}, (a,b) \to (a,-b)$ induces a Nash involution on $\widehat{M}$ without fixed points such that $\sigma(\widetilde{M}_+) = \widetilde{M}_-$ and $\Phi(y,-\rho,-w) = (\sigma \circ \Phi)(y,\rho,w)$ for each $(y,\rho,w) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$. We call $\widehat{M}$ the *twisted Nash double* of $\widetilde{M}_+$.

Proof. By 5.A.2 it holds

$$\Phi(\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}) = \Gamma_- \sqcup R \sqcup \Gamma_+ = \widehat{M}. \tag{5.6}$$

In addition, it follows from the proof of 5.A.2(ii) (Steps 1 & 2) that Φ is injective and a local Nash diffeomorphism (for the points of $\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1}$ use the map Ψ_0 introduced in (5.5)). Consequently, Φ is a Nash diffeomorphism onto its image $\widehat{M}$ and the latter is a Nash manifold.

The second part of the statement is straightforward. $\square$

5.A.4. Consider the projection $\pi : M \times \mathbb{S}^{k-1} \to M$ onto the first factor and denote $\pi_\epsilon := \pi|_{\widetilde{M}_\epsilon}$. Then

(i) π_ϵ is proper, $\pi_\epsilon(\widetilde{M}_\epsilon) = M$ and $R = \pi_\epsilon^{-1}(N)$.
(ii) The restriction $\pi_\epsilon|_{\Gamma_\epsilon} : \Gamma_\epsilon \to M \setminus N$ is a Nash diffeomorphism.
(iii) For each $q \in N$ it holds $\pi_\epsilon^{-1}(q) = \{q\} \times \mathbb{S}_q^{d-e-1}$ where $\mathbb{S}_q^{d-e-1}$ is the sphere of dimension $d - e - 1$ obtained when intersecting the sphere $\mathbb{S}^{k-1}$ with the linear subspace L_q generated by $(\zeta_{e+1} \circ u)(q), \dots, (\zeta_d \circ u)(q)$.

Proof. (i) Let $K \subset M$ be a compact set. Then $\pi^{-1}(K) = K \times \mathbb{S}^{k-1}$, which is a compact set. As $\widetilde{M}_\epsilon = \mathrm{Cl}(\Gamma_\epsilon)$ is a closed subset of $M \times \mathbb{S}^{k-1}$, the intersection $\pi^{-1}(K) \cap \widetilde{M}_\epsilon$ is a compact set, so π_ϵ is proper. Thus, $\pi(\widetilde{M}_\epsilon) = \mathrm{Cl}(\pi(\Gamma_\epsilon)) = \mathrm{Cl}(M \setminus N) = M$. In addition $\pi_\epsilon^{-1}(N) = \widetilde{M}_\epsilon \setminus \Gamma_\epsilon = R$.

(ii) We have the commutative diagram

$$
\begin{array}{ccc}
\mathbb{R}^d \setminus \{z = 0\} & \xrightarrow{\;\;\Theta\;\;} & \Gamma_\epsilon \\
\text{id} \downarrow & & \downarrow \pi|_{\Gamma_\epsilon} \\
\mathbb{R}^d \setminus \{z = 0\} & \xrightarrow[\cong]{\;\;\psi|\;\;} & M \setminus N
\end{array}
\qquad
\begin{array}{ccc}
(y,z) & \longmapsto & \left(\psi(y,z), \epsilon \frac{\varphi(y,z)}{\|\varphi(y,z)\|}\right) \\
\downarrow & & \downarrow \\
(y,z) & \longmapsto & \psi(y,z)
\end{array}
$$

As Θ is a Nash diffeomorphism, we conclude that $\pi|_{\Gamma_\epsilon}$ is also a Nash diffeomorphism.

(iii) Let $q \in N$. We have

$$\pi_\epsilon^{-1}(q) = \pi^{-1}(q) \cap R = \pi^{-1}(q) \cap \Phi(\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1}) = \{q\} \times \Delta_u(\mathbb{S}^{d-e-1}),$$
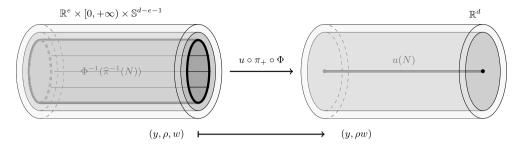
where

Fig. 3. Local structure of the drilling blow-up $\widetilde{M}_+$ of M of center N.

$$\Delta_u : \mathbb{S}^{d-e-1} \to \mathbb{S}^{k-1}, \ w := (w_{e+1}, \ldots, w_d) \mapsto \frac{(\zeta_{e+1} \circ u)(q)w_{e+1} + \cdots + (\zeta_d \circ u)(q)w_d}{\|(\zeta_{e+1} \circ u)(q)w_{e+1} + \cdots + (\zeta_d \circ u)(q)w_d\|}.$$

The map Δ_u is a Nash diffeomorphism between $\mathbb{S}^{d-e-1}$ and the sphere obtained when intersecting $\mathbb{S}^{k-1}$ with the linear subspace L_q generated by $(\zeta_{e+1} \circ u)(q), \ldots, (\zeta_d \circ u)(q)$. $\square$

5.A.5.   Denote $\widehat{\pi} := \pi|_{\widehat{M}}$ and consider the commutative diagram (see also Figs. 3 and 4).

$$(5.7)$$

As a consequence, we have: *The Nash maps π_ϵ and $\widehat{\pi}$ have local representations*

$$x := (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_e, x_{e+1}, x_{e+1}x_{e+2}, \ldots, x_{e+1}x_d) \quad (5.8)$$

in an open neighborhood of each point $p \in R$. In addition, $d\pi_p(T_p\widehat{M}) \not\subset T_{\pi(p)}N$.

Proof. After a change of coordinates in $\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$, we may assume that $p \in R$ is the image of the point $(0, 0, (1, 0, \ldots, 0))$. Consider the local parametrization around $(0, 0, (1, 0, \ldots, 0))$ of the set $\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$ given by

$$\eta : \qquad \mathbb{R}^e \times \mathbb{R} \times \mathcal{B} \qquad \to \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1},$$
$$(y, \rho, v := (v_{e+2}, \ldots, v_d)) \mapsto (y, \rho, (\sqrt{1 - \|v\|^2}, v))$$

where $\mathcal{B}$ is the open ball of center the origin and radius 1 in $\mathbb{R}^{d-e-1}$. It holds

$$u \circ \widehat{\pi} \circ \Phi \circ \eta : \mathbb{R}^e \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}^d,$$
$$(y, \rho, v) \quad \mapsto (y, \rho\sqrt{1 - \|v\|^2}, \rho v).$$

Consider the Nash diffeomorphism

$$f : \mathbb{R}^e \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}^d, (y, \rho, v) \to \left(y, \rho\sqrt{1 - \|v\|^2}, \frac{v}{\sqrt{1 - \|v\|^2}}\right),$$

whose inverse is

$$f^{-1} : \mathbb{R}^d \to \mathbb{R}^e \times \mathbb{R} \times \mathcal{B}, \ (y, \rho', v') \mapsto \left(y, \rho'\sqrt{1 + \|v'\|^2}, \frac{v'}{\sqrt{1 + \|v'\|^2}}\right).$$

The Nash map

$$\pi' := u \circ \widehat{\pi} \circ \Phi \circ \eta \circ f^{-1} : \mathbb{R}^d \to \mathbb{R}^d, (y, \rho', v') \mapsto (y, \rho', \rho'v')$$

represents $\widehat{\pi}$ locally around p and the restriction

$$\pi'_\epsilon := \pi'|_{\{\epsilon\rho' \geq 0\}} : \{\epsilon\rho \geq 0\} \to \mathbb{R}^d, (y, \rho', v') \mapsto (y, \rho', \rho'v')$$

represents π_ϵ locally around p.

To prove that $d\pi_p(T_p\widehat{M}) \not\subset T_{\pi(p)}N$, it is enough to show $d\pi'_0(\mathbb{R}^d) \not\subset \{z = 0\} = u(N)$.

If $\vec{\mathsf{e}}_{e+1} := (0, \ldots, 0, \overset{(e+1)}{1}, 0, \ldots, 0)$, we have $d\pi'_0(\mathsf{e}_{e+1}) = \mathsf{e}_{e+1} \notin \{z = 0\}$, as required. $\square$

5.B. Global structure of the drilling blow-up

We construct next the *drilling blow-up of a Nash manifold with center a closed Nash submanifold.* We refer the reader to Fig. 4 in order to get a global idea of the involved strategy. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. Let $f_1, \ldots, f_k \in \mathcal{N}(M)$ be a finite system of generators of the ideal $I(N)$ of Nash functions on M vanishing identically on N. Consider the Nash map

$$F : M \setminus N \to \mathbb{S}^{k-1}, x \mapsto \frac{(f_1(x), \ldots, f_k(x))}{\|(f_1(x), \ldots, f_k(x))\|}. \tag{5.9}$$

We have:

5.B.1. Fix $\epsilon = \pm$. The closure $\widetilde{M}_\epsilon$ in $M \times \mathbb{S}^{k-1}$ of the graph

$$\Gamma_\epsilon := \{(x, \epsilon F(x)) \in M \times \mathbb{S}^{k-1} : \ x \in M \setminus N\}$$

is a Nash manifold with boundary. Denote $R := \partial\widetilde{M}_+ = \partial\widetilde{M}_-$ and $\widehat{M} := \widetilde{M}_+ \cup \widetilde{M}_- = \Gamma_+ \sqcup R \sqcup \Gamma_-$. In addition, $\widehat{M}$ is a Nash manifold and the Nash map $\sigma : M \times \mathbb{S}^{k-1} \to M \times \mathbb{S}^{k-1}, (a, b) \to (a, -b)$ induces a Nash involution on $\widehat{M}$ without fixed points that maps $\widetilde{M}_+$ onto $\widetilde{M}_-$. We call $\widehat{M}$ the twisted Nash double of $\widetilde{M}_+$.

Proof. The Nash manifold M can be covered by 2.C.3 (applied to N) and by 2.C.2 (applied to $M \setminus N$) with finitely many open semialgebraic subsets U_i equipped with Nash diffeomorphisms $u_i := (u_{i,1}, \ldots, u_{i,d}) : U_i \to \mathbb{R}^d$ such that either $U_i \cap N = \varnothing$ or $U_i \cap N = \{u_{i,e+1} = 0, \ldots, u_{i,d} = 0\}$. Denote $\psi_i := u_i^{-1}$.

The collection $\{U_i \times \mathbb{S}^{k-1}\}_i$ is a finite open semialgebraic covering of $M \times \mathbb{S}^{k-1}$. To prove that $\widetilde{M}_\epsilon$ is a Nash manifold with boundary, we will show that $\widetilde{M}_\epsilon \cap (U_i \times \mathbb{S}^{k-1})$ is a Nash manifold with (possibly empty) boundary for $i = 1, \ldots, r$. Analogously, to check that $\widehat{M}$ is a Nash manifold, we will show that each intersection $\widehat{M} \cap (U_i \times \mathbb{S}^{k-1})$ is a Nash manifold.

Case 1. If $N \cap U_i = \varnothing$, the intersection $\widetilde{M}_\epsilon \cap (U_i \times \mathbb{S}^{k-1})$ is the graph of the Nash map $\epsilon F|_{U_i}$, so it is a Nash manifold. Observe that $(\widetilde{M}_+ \cap \widetilde{M}_-) \cap (U_i \times \mathbb{S}^{k-1}) = \varnothing$, so

$$\widehat{M} \cap (U_i \times \mathbb{S}^{k-1}) = (\widetilde{M}_+ \cap (U_i \times \mathbb{S}^{k-1})) \sqcup (\widetilde{M}_- \cap (U_i \times \mathbb{S}^{k-1}))$$

is also a Nash manifold.

Case 2. If $N \cap U_i \neq \varnothing$, the intersection $\widetilde{M}_\epsilon \cap (U_i \times \mathbb{S}^{k-1})$ is the closure in $U_i \times \mathbb{S}^{k-1}$ of the set

$$\Gamma_{i,\epsilon} := \Gamma_\epsilon \cap (U_i \times \mathbb{S}^{k-1}) = \{(\psi_i(y,z), \epsilon F(\psi_i(y,z))) \in U_i \times \mathbb{S}^{k-1} : z \neq 0\}$$

and $\widehat{M} \cap (U_i \times \mathbb{S}^{k-1}) = (\widetilde{M}_+ \cap (U_i \times \mathbb{S}^{k-1})) \cup (\widetilde{M}_- \cap (U_i \times \mathbb{S}^{k-1}))$.

By (2.1) it holds

$$I(N \cap U_i) = I(N)\mathcal{N}(U_i) = (f_1|_{U_i}, \ldots, f_k|_{U_i})\mathcal{N}(U_i).$$

As ψ_i is a Nash diffeomorphism, $f_1 \circ \psi_i, \ldots, f_k \circ \psi_i$ generate the ideal $I(\{z = 0\})$ of Nash functions on $\mathbb{R}^d$ vanishing identically on $\psi_i^{-1}(N \cap U_i) = \{z = 0\}$. The ideal $I(\{z = 0\})$ is also generated by $z_{e+1}, \ldots, z_d$. Thus, there exist Nash functions $\zeta_{j,\ell} \in \mathcal{N}(\mathbb{R}^d)$ such that

$$f_\ell \circ \psi_i = \zeta_{e+1,\ell}(y,z)z_{e+1} + \cdots + \zeta_{d,\ell}(y,z)z_d.$$

Notice that

$$\frac{\partial}{\partial z_j}(f_\ell \circ \psi_i)(y,0) = \zeta_{j,\ell}(y,0) \tag{5.10}$$

for each $y \in \mathbb{R}^e$. Write $\zeta_j := (\zeta_{j,1}, \ldots, \zeta_{j,k}) : \mathbb{R}^d \to \mathbb{R}^k$. We have

$$((f_1 \circ \psi_i)(y,z), \ldots, (f_k \circ \psi_i)(y,z)) = \zeta_{e+1}(y,z)z_{e+1} + \cdots + \zeta_d(y,z)z_d.$$

As $f_1 \circ \psi_i, \ldots, f_k \circ \psi_i$ generate the ideal $I(\{z = 0\})$, we deduce:

- $\zeta_{e+1}(y,z)z_{e+1} + \cdots + \zeta_d(y,z)z_d = 0$ if and only if $z = 0$.

- The vectors $\zeta_{e+1}(y,0),\dots,\zeta_d(y,0)$ are linearly independent for all $y \in \mathbb{R}^e$ (by (5.10)) and the Jacobian criterion).

By 5.A.2 and 5.A.3 we deduce that $\widetilde{M}_\epsilon \cap (U_i \times \mathbb{S}^{k-1})$ is a Nash manifold with boundary and $\widehat{M} \cap (U_i \times \mathbb{S}^{k-1})$ is a Nash manifold.

The rest of the statement follows from 5.A.3. $\quad\square$

5.B.2. We keep the notations already introduced in 5.B.1. *Consider the projection* $\pi : M \times \mathbb{S}^{k-1} \to M$ *onto the first factor. Denote* $\pi_\epsilon := \pi|_{\widetilde{M}_\epsilon}$ *and* $\widehat{\pi} := \pi|_{\widehat{M}}$. *We have:*

(i) π_ϵ *is proper,* $\pi_\epsilon(\widetilde{M}_\epsilon) = M$ *and* $R = \pi_\epsilon^{-1}(N)$.
(ii) *The restriction* $\pi_\epsilon|_{\Gamma_\epsilon} : \Gamma_\epsilon \to M \setminus N$ *is a Nash diffeomorphism.*
(iii) *Consider the Nash map* $f := (f_1,\dots,f_k) : M \to \mathbb{R}^k$ *(whose coordinates generate* $I(N)$*). Fix* $q \in N$ *and let* E_q *be any complementary linear subspace of* $T_q N$ *in* $T_q M$. *Then* $\pi_\epsilon^{-1}(q) = \{q\} \times \mathbb{S}_q^{d-e-1}$, *where* $\mathbb{S}_q^{d-e-1}$ *denotes the sphere of dimension* $d-e-1$ *obtained when intersecting* $\mathbb{S}^{k-1}$ *with the* $(d-e)$*-dimensional linear subspace* $d_q f(E_q)$.
(iv) *The Nash maps* π_ϵ *and* $\widehat{\pi}$ *have local representations of the type*

$$x := (x_1,\dots,x_d) \mapsto (x_1,\dots,x_e,x_{e+1},x_{e+1}x_{e+2},\dots,x_{e+1}x_d)$$

in an open neighborhood of each point $p \in R$. *In addition,* $d\pi_p(T_p\widehat{M}) \not\subset T_{\pi(p)}N$.

Proof. This statement follows straightforwardly from 5.A.4 and 5.A.5. To prove (iii) we use in addition (5.10) and the equality $\ker(d_q f) = T_q N$. $\quad\square$

5.B.3. *The pairs* $(\widetilde{M}_\epsilon, \pi_\epsilon)$ *and* $(\widehat{M}, \widehat{\pi})$ *do not depend on the generators* $f_1,\dots,f_k$ *of* $I(N)$ *up to Nash diffeomorphisms compatible with the respective projections. Moreover, such Nash diffeomorphisms are unique.*

Proof. Let $f_{k+1} := g_1 f_1 + \dots + g_k f_k \in I(N)$ for some $g_j \in \mathcal{N}(M)$. Let $(\widetilde{M}_\epsilon', \pi_\epsilon')$ and $(\widehat{M}', \widehat{\pi}')$ be the pairs associated to the system of generators $f_1,\dots,f_k,f_{k+1}$ of $I(N)$. Let us construct a Nash diffeomorphism $\Theta : \widehat{M} \to \widehat{M}'$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\widehat{M} & \xrightarrow{\;\;\Theta\;\;} & \widehat{M}' \\
& & \\
{\scriptstyle \widehat{\pi}}\searrow & & \swarrow{\scriptstyle \widehat{\pi}'} \\
& M &
\end{array}
\tag{5.11}
$$

Denote $R' := (\widehat{\pi}')^{-1}(N)$. Consider the Nash maps

$$g := (g_1, \dots, g_k) : M \to \mathbb{R}^k, \quad f := (f_1, \dots, f_k) : M \to \mathbb{R}^k, \quad \text{and}$$

$$f' := (f, f_{k+1}) : M \to \mathbb{R}^{k+1}$$

and let $F : M \setminus N \to \mathbb{S}^{k-1}$ be the Nash map introduced in (5.9).

If $(a, b) \in \widehat{M}_\epsilon \setminus R$, then $b = \epsilon F(a) = \epsilon \frac{f(a)}{\|f(a)\|}$ and

$$\left(a, \epsilon \frac{f'(a)}{\|f'(a)\|} \right) = \left(a, \frac{(\epsilon F(a), \langle g(a), \epsilon F(a) \rangle)}{\sqrt{1 + \langle g(a), \epsilon F(a) \rangle^2}} \right) = \left(a, \frac{(b, \langle g(a), b \rangle)}{\sqrt{1 + \langle g(a), b \rangle^2}} \right) \in \widetilde{M}'_\epsilon \setminus R'. \quad (5.12)$$

Consider the Nash map

$$\Theta : \widehat{M} \to \widehat{M}', \ (a, b) \mapsto \left(a, \frac{(b, \langle g(a), b \rangle)}{\sqrt{1 + \langle g(a), b \rangle^2}} \right)$$

and let us check that it is the Nash diffeomorphism we are looking for.

We claim: If $(a, c := (c_1, \dots, c_k, c_{k+1})) \in \widehat{M}'$, then $c' := (c_1, \dots, c_k) \neq 0$ and $c = (c', \langle g(a), c' \rangle)$.

We distinguish two cases:

Case 1. If $p \notin N$, there exist $\epsilon = \pm$ such that

$$c = \epsilon \frac{f'(a)}{\|f'(a)\|} = \epsilon \frac{(f(a), \langle g(a), f(a) \rangle)}{\|f'(a)\|} = (c', \langle g(a), c' \rangle).$$

Case 2. If $p \in N$, there exists by the Nash curve selection lemma a Nash arc $\gamma : (-1, 1) \to \widehat{M}'$ such that $\gamma(0) = (a, c)$ and $\gamma((0, 1)) \subset \widetilde{M}'_\epsilon \setminus R'$. It holds

$$c = \epsilon \lim_{t \to 0} \frac{(f' \circ \widehat{\pi}' \circ \gamma)(t)}{\|(f' \circ \widehat{\pi}' \circ \gamma)(t)\|}$$

$$= \epsilon \lim_{t \to 0} \frac{((f \circ \widehat{\pi}' \circ \gamma)(t), \langle (g \circ \widehat{\pi}' \circ \gamma)(t), (f \circ \widehat{\pi}' \circ \gamma)(t) \rangle)}{\|(f' \circ \widehat{\pi}' \circ \gamma)(t)\|} = (c', \langle g(a), c' \rangle).$$

As in both cases $c \in \mathbb{S}^k$, we have $c' \neq 0$, as claimed.

The Nash map

$$\Psi : \widehat{M}' \to \widehat{M}, (a, c) \mapsto \left(a, \frac{c'}{\|c'\|} \right)$$

is the inverse of Θ, so Θ is a Nash diffeomorphism. In addition, $\widehat{\pi} = \widehat{\pi}' \circ \Theta$ and $\Theta(\widetilde{M}_\epsilon) = \widetilde{M}'_\epsilon$ for $\epsilon = \pm$, as required. The unicity of Θ follows from (5.11) and (5.12). $\quad \square$

Definition 5.2. The pair $(\widetilde{M}_+, \pi_+)$ will be called the *drilling blow-up of the Nash manifold M with center the closed Nash submanifold* $N \subset M$.

Remark 5.3. We will see in 5.D that the twisted Nash double $\widehat{M}$ of $\widetilde{M}_+$ together with $\widehat{\pi} : \widehat{M} \to M$ relates the drilling blow-up of M with center N with the classical blow-up of M with center N.

We are ready to prove Lemma 5.1. We will make a strong use of the global structure of the drilling blow-up of a Nash manifold M with center a closed Nash submanifold N.

Proof of Lemma 5.1. Let $(\widetilde{M}_+, \pi)$ be the drilling blow-up of M with center N and let $R := \pi_+^{-1}(N)$, see Fig. 5. The Nash map $\pi_+ : \widetilde{M}_+ \to M$ is surjective. By Proposition 4.1 there exists a surjective Nash map

$$f : \operatorname{Int}(\widetilde{M}_+) = \widetilde{M}_+ \setminus R \to \widetilde{M}_+.$$

By 5.B.2 the restriction $\pi|_{\widetilde{M}_+ \setminus R} : \widetilde{M}_+ \setminus R \to M \setminus N$ is a Nash diffeomorphism. Consequently,

$$h := \pi_+ \circ f \circ (\pi_+|_{\widetilde{M}_+ \setminus R})^{-1} : M \setminus N \to \widetilde{M}_+ \setminus \pi^{-1}(N) \to M$$

is a surjective Nash map, as required. □

5.C. Alternative presentation of the drilling blow-up

Our purpose next is to extend the construction in diagram (5.7) to an open semialgebraic neighborhood of the center N of the drilling blow-up of M. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. Let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{M}_+, \pi_+)$ of M with center N. In the next lemma we will use notations already introduced in 2.C.4. We refer the reader to Fig. 4 to appreciate its importance.

Lemma 5.4. *Let $U_1 \subset M$ and $U_2 \subset \widehat{M}$ be respective open semialgebraic neighborhoods of N and $R := \widehat{\pi}^{-1}(N)$. Then there exist Nash tubular open neighborhoods $(V_1, \varphi_1, \mathscr{E}, \theta_1, N, \delta)$ of N in U_1 and $(V_2, \varphi_2, \mathscr{F}, \theta_2, R, \delta \circ \widehat{\pi}|_R)$ of R in U_2 and a Nash embedding $\psi := (\psi_1, \psi_2) : R \to N \times \mathbb{S}^{m-1}, \ (x, v) \mapsto (x, \psi_2(x, v))$ such that $\widehat{\pi}(V_2) = V_1$,*

$$\varphi_1 \circ \widehat{\pi} \circ \varphi_2^{-1} : \mathscr{F}_{\delta \circ \widehat{\pi}|_R} \to \mathscr{E}_\delta, \ (x, v, t) \mapsto (x, t\psi_2(x, v))$$

and V_2 is the Nash double of a collar of R in $\widetilde{M}_+$.

Proof. By 2.C.4 there exists a Nash tubular neighborhood $(V, \varphi, \mathscr{E}, \theta, N, \delta)$ of N in U_1. Recall that $(\mathscr{E}, \theta, N)$ is a Nash subbundle of the trivial bundle $(N \times \mathbb{R}^m, \theta', N)$. Write $\varphi := (\phi_1, \phi_2) : V \to N \times \mathbb{R}^m$ and observe that $\phi_1|_N = \operatorname{id}_N$ and $\phi_2|_N = 0$.
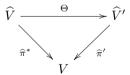
5.C.1. For each $x \in N$ we have $d_x \varphi = (d_x \phi_1, d_x \phi_2) : T_x M \to T_x N \times \mathbb{R}^m$ and $d_x \phi_1|_{T_x N} = (\mathrm{id}_{T_x N}, 0)$. Let H_x be a complementary linear subspace of $T_x N$ in $T_x M$. As $d_x \varphi$ is an isomorphism, $d_x \phi_1|_{T_x N} = \mathrm{id}_{T_x N}$ and $d_x \phi_1(H_x) \subset T_x N$, we have

$$T_{(x,0)} \mathscr{E} = d_x \varphi(T_x M) = d_x \varphi(T_x N \oplus H_x) = (T_x N \times \{0\}) \oplus (d_x \phi_1, d_x \phi_2)(H_x)$$

$$= (T_x N \times \{0\}) \oplus (\{0\} \times d_x \phi_2(H_x)) = T_x N \times d_x \phi_2(H_x), \tag{5.13}$$

so $\dim(d_x \phi_2(H_x)) = d - e$. By the Jacobian criterion the tuple $\phi_2 := (\phi_{21}, \dots, \phi_{2m})$ generates $I(N_x)$ for each $x \in N$, that is, $(\phi_2)\mathcal{N}_x := (\phi_{21}, \dots, \phi_{2m})\mathcal{N}_x = I(N_x)$ for each $x \in N$. Write $I(N) := \{f \in \mathcal{N}(M) : f|_N = 0\}$ and $I_V(N) := \{f \in \mathcal{N}(V) : f|_N = 0\}$. By (2.1) and 2.C.7

$$(\phi_2)\mathcal{N}(V) := (\phi_{21}, \dots, \phi_{2m})\mathcal{N}(V) = I_V(N) = I(N)\mathcal{N}(V).$$

Let $(\widehat{V}, \widehat{\pi}^*)$ be the twisted Nash double of the drilling blow-up of V with center N that arises from a finite system of generators of $I(N)$ restricted to V and let $(\widehat{V}', \widehat{\pi}')$ be the twisted Nash double of the drilling blow-up of V centered at N that arises from ϕ_2. It holds $\widehat{V} = \widehat{\pi}^{-1}(V)$ and $\widehat{\pi}^* = \widehat{\pi}|_{\widehat{V}}$ if $(\widehat{M}, \widehat{\pi})$ is the twisted Nash double of the drilling blow-up of M with center N. By 5.B.3 there exists a Nash diffeomorphism $\Theta : \widehat{V} \to \widehat{V}'$ that makes the following diagram commutative.

5.C.2. Denote the coordinates in $\mathbb{R}^m$ with $\mathrm{y} := (\mathrm{y}_1, \dots, \mathrm{y}_m)$. The restriction to $\mathscr{E}_\delta$ of the tuple $(\mathrm{y}_1, \dots, \mathrm{y}_m)$ generates the ideal $I(N \times \{0\})$ of $\mathcal{N}(\mathscr{E}_\delta)$. Let $(\widehat{\mathscr{E}_\delta}, \widehat{\Pi})$ be the twisted Nash double of the drilling blow-up $(\widehat{\mathscr{E}}_{\delta,+}, \Pi_+)$ of $\mathscr{E}_\delta$ with center $N \times \{0\}$ that arises from the tuple $(\mathrm{y}_1, \dots, \mathrm{y}_m)$. It holds

$$\widehat{\mathscr{E}_\delta} := \mathrm{Cl}(\{(x, y, \pm\tfrac{y}{\|y\|}) \in \mathscr{E}_\delta \times \mathbb{S}^{m-1} : y \neq 0\}).$$

By 5.B.2(iii) $\widehat{\Pi}^{-1}(x, 0) = \{(x, 0)\} \times \mathbb{S}_{(x,0)}$ where $\mathbb{S}_{(x,0)}$ is the intersection of $\mathbb{S}^{m-1}$ with the linear subspace $\theta^{-1}(x)$ of $\mathbb{R}^m$. Consequently,

$$\widehat{\mathscr{E}_\delta} \setminus \widehat{\Pi}^{-1}(N \times \{0\}) = \{(x, y, \pm\tfrac{y}{\|y\|}) \in \mathscr{E}_\delta \times \mathbb{S}^{m-1} : y \neq 0\}, \tag{5.14}$$

$$\widehat{\Pi}^{-1}(N \times \{0\}) = \{(x, 0, \pm\tfrac{y}{\|y\|}) \in \mathscr{E}_\delta \times \mathbb{S}^{m-1} : (x, y) \in \mathscr{E}_\delta, y \neq 0\}. \tag{5.15}$$

5.C.3. It holds $\widehat{V}' \subset V \times \mathbb{S}^{m-1}$ and $\widehat{\mathscr{E}}_\delta \subset \mathscr{E}_\delta \times \mathbb{S}^{m-1}$. We claim: *The image of the Nash map*

$$\Lambda : \widehat{V}' \to \mathscr{E}_\delta \times \mathbb{S}^{m-1}, \ (a,b) \mapsto (\varphi(a), b)$$

is $\widehat{\mathscr{E}}_\delta$. *The image of the Nash map*

$$\Delta : \widehat{\mathscr{E}}_\delta \to V \times \mathbb{S}^{m-1}, \ (x,y,w) \mapsto (\varphi^{-1}(x,y), w)$$

is $\widehat{V}'$. *Thus,* $\Delta : \widehat{\mathscr{E}}_\delta \to \widehat{V}'$ *and* $\Lambda : \widehat{V}' \to \widehat{\mathscr{E}}_\delta$ *are mutually inverse Nash diffeomorphisms.*
  Pick $(a,b) \in \widehat{V}' \setminus (\widehat{\pi}')^{-1}(N)$. We have

$$(a,b) = \left(a, \pm\frac{\phi_2(a)}{\|\phi_2(a)\|} \right).$$

Consequently,

$$\Lambda(a,b) = \left(\varphi(a), \pm\frac{\phi_2(a)}{\|\phi_2(a)\|} \right) = \left(\phi_1(a), \phi_2(a), \pm\frac{\phi_2(a)}{\|\phi_2(a)\|} \right) \in \widehat{\mathscr{E}}_\delta.$$

By continuity $\Lambda(\widehat{V}') \subset \widehat{\mathscr{E}}_\delta$.
  Pick now $(x,y,w) \in \widehat{\mathscr{E}}_\delta \setminus \widehat{\Pi}^{-1}(N \times \{0\})$. We have $w = \pm\frac{y}{\|y\|} = \pm\frac{\phi_2(\varphi^{-1}(x,y))}{\|\phi_2(\varphi^{-1}(x,y))\|}$ and

$$\Delta(x,y,w) = \left(\varphi^{-1}(x,y), \pm\frac{\phi_2(\varphi^{-1}(x,y))}{\|\phi_2(\varphi^{-1}(x,y))\|} \right) \in \widehat{V}'.$$

By continuity $\Delta(\widehat{\mathscr{E}}_\delta) \subset \widehat{V}'$.

5.C.4. The maps in the rows of the following commutative diagram are Nash diffeomorphisms:

$$(5.16)$$

5.C.5. It holds: *The Nash maps*

$$\rho : \widehat{\mathscr{E}}_\delta \to \widehat{\Pi}^{-1}(N \times \{0\}), \ (x,y,w) \mapsto (x, 0, w),$$
$$\varrho : \mathscr{E}_\delta \to N \times \{0\}, \ (x,y) \mapsto (x, 0)$$

are Nash retractions such that $\widehat{\Pi} \circ \rho = \varrho \circ \widehat{\Pi}$.

5.C.6. Define

$$h : \widehat{\mathscr{E}_\delta} \to \mathbb{R}, \quad (x, y, w) \mapsto \begin{cases} +\|y\| & \text{if } (x, y, w) \in \widetilde{\mathscr{E}_{\delta,+}}, \\ -\|y\| & \text{if } (x, y, w) \in \widetilde{\mathscr{E}_{\delta,-}}. \end{cases}$$

We claim: *The semialgebraic function h is Nash and $d_q h : T_q \widehat{\mathscr{E}_\delta} \to \mathbb{R}$ is surjective for all $q \in \widehat{\Pi}^{-1}(N \times \{0\})$. In addition, $|h(x, y, w)| = \|y\|$ for all $(x, y, w) \in \widehat{\mathscr{E}_\delta}$, so $\{h = 0\} = \widehat{\Pi}^{-1}(N \times \{0\})$.*

Pick a point $q \in \widehat{\Pi}^{-1}(N \times \{0\})$. By 5.B.2(iv) there exist semialgebraic neighborhoods $A_1 \subset \widehat{\mathscr{E}_\delta}$ of q and $A_2 \subset \mathscr{E}_\delta$ of $\widehat{\Pi}(q)$ and Nash diffeomorphisms

$$u := (u_1, \ldots, u_d) : A_1 \to \mathbb{R}^d,$$
$$v := (v_1, \ldots, v_d) : A_2 \to \mathbb{R}^d$$

such that $u(q) = 0$, $v(\widehat{\Pi}(q)) = 0$, $v((N \times \{0\}) \cap A_2) = \{v_{e+1} = 0, \ldots, v_d = 0\}$ and

$$\widehat{\Pi}_0 := v \circ \widehat{\Pi} \circ u^{-1} : \mathbb{R}^d \to \mathbb{R}^d,$$
$$(x_1, \ldots, x_d) \mapsto (z_1, \ldots, z_d) := (x_1, \ldots, x_e, x_{e+1}, x_{e+2} x_{e+1}, \ldots, x_d x_{e+1}).$$

As $\mathsf{y}_1, \ldots, \mathsf{y}_m$ generate the ideal $I(N \times \{0\})$ of $\mathcal{N}(\mathscr{E}_\delta)$, their restrictions to $(N \times \{0\}) \cap A_2$ generate by (2.1) the ideal $I((N \times \{0\}) \cap A_2)$ of $\mathcal{N}(A_2)$. We have $\mathsf{y}_i(v^{-1}(z_1, \ldots, z_e, 0, \ldots, 0)) = 0$ for $(z_1, \ldots, z_e) \in \mathbb{R}^e \times \{0\}$ and $i = 1, \ldots, m$ and

$$z_{e+1} = (\mathsf{y}_1 \circ v^{-1})\xi_1 + \cdots + (\mathsf{y}_m \circ v^{-1})\xi_m$$

for some $\xi_1, \ldots, \xi_m \in \mathcal{N}(\mathbb{R}^d)$. By Schwarz's inequality

$$z_{e+1}^2 \leq (\xi_1^2 + \cdots + \xi_m^2)((\mathsf{y}_1 \circ v^{-1})^2 + \cdots + (\mathsf{y}_m \circ v^{-1})^2).$$

Composing with $\widehat{\Pi}_0$, we have

$$x_{e+1}^2 \leq ((\xi_1 \circ \widehat{\Pi}_0)^2 + \cdots + (\xi_m \circ \widehat{\Pi}_0)^2)((\mathsf{y}_1 \circ v^{-1} \circ \widehat{\Pi}_0)^2 + \cdots + (\mathsf{y}_m \circ v^{-1} \circ \widehat{\Pi}_0)^2).$$

Comparing orders at the origin we deduce that the Nash series $(\mathsf{y}_1 \circ v^{-1} \circ \widehat{\Pi}_0)^2 + \cdots + (\mathsf{y}_m \circ v^{-1} \circ \widehat{\Pi}_0)^2$ has order 2 at the origin. As $\mathsf{y}_i(v^{-1}(z_1, \ldots, z_e, 0, \ldots, 0)) = 0$ for each $(z_1, \ldots, z_e) \in \mathbb{R}^e \times \{0\}$,

$$\mathsf{y}_i \circ v^{-1} \circ \widehat{\Pi}_0 = x_{e+1} \gamma_i$$

where γ_i is a Nash function on $\mathbb{R}^d$. Thus,

$$(\mathsf{y}_1 \circ v^{-1} \circ \widehat{\Pi}_0)^2 + \cdots + (\mathsf{y}_m \circ v^{-1} \circ \widehat{\Pi}_0)^2 = x_{e+1}^2(\gamma_1^2 + \cdots + \gamma_m^2).$$

As $(\mathbf{y}_1 \circ v^{-1} \circ \widehat{\Pi}_0)^2 + \cdots + (\mathbf{y}_m \circ v^{-1} \circ \widehat{\Pi}_0)^2$ is a Nash series of order 2 at the origin, $\gamma_1^2 + \cdots + \gamma_m^2$ is a unit at the origin. Consequently, $h \circ u^{-1} = x_{e+1} \sqrt{\gamma_1^2 + \cdots + \gamma_m^2}$ on an open semialgebraic neighborhood of the origin. In particular,

$$\frac{\partial(h \circ u^{-1})}{\partial x_{e+1}}(0) = \sqrt{\gamma_1^2 + \cdots + \gamma_m^2}(0) > 0.$$

Thus, h is a Nash function and $d_q h : T_q \widehat{M} \to \mathbb{R}$ is surjective for all $q \in \widehat{\Pi}^{-1}(N \times \{0\})$.

5.C.7. Consider the trivial Nash bundle $\mathscr{G} := \widehat{\Pi}^{-1}(N \times \{0\}) \times \mathbb{R}$ over $\widehat{\Pi}^{-1}(N \times \{0\})$ and let $\theta_2' : \mathscr{G} \to \widehat{\Pi}^{-1}(N \times \{0\})$ be the projection onto the first factor. Define

$$g := (\rho, h) : \widehat{\mathscr{E}}_\delta \to \mathscr{G}, \ (x, y, w) \mapsto \begin{cases} (x, 0, w, +\|y\|) & \text{if } (x, y, w) \in \widetilde{\mathscr{E}}_{\delta,+}, \\ (x, 0, w, -\|y\|) & \text{if } (x, y, w) \in \widetilde{\mathscr{E}}_{\delta,-}. \end{cases}$$

By Lemma 4.2 there exists an open semialgebraic neighborhood $W \subset \widehat{\mathscr{E}}_\delta$ of $\widehat{\Pi}^{-1}(N \times \{0\})$ such that $g(W) \subset \mathscr{G}$ is an open semialgebraic neighborhood of $\widehat{\Pi}^{-1}(N \times \{0\}) \times \{0\}$ and the restriction map $g|_W : W \to g(W)$ is a Nash diffeomorphism.

As $\widehat{\Pi}$ is proper, $\widehat{\Pi}(\widehat{\mathscr{E}}_\delta \setminus W)$ is a closed subset of $\mathscr{E}_\delta$ that does not meet $N \times \{0\}$. As the restriction $\widehat{\Pi}|_{\widehat{\mathscr{E}}_\delta \setminus \widehat{\Pi}^{-1}(N \times \{0\})}$ is a Nash diffeomorphism, $\widehat{\Pi}(\widehat{\mathscr{E}}_\delta \setminus W) = \mathscr{E}_\delta \setminus \widehat{\Pi}(W)$. Consequently, $W' := \widehat{\Pi}(W)$ is an open semialgebraic neighborhood of $N \times \{0\}$ in $\mathscr{E}_\delta$.

5.C.8. We may assume taking a smaller strictly positive $\delta \in \mathcal{N}(N)$ that $\mathscr{E}_\delta = W'$ and $W = \widehat{\Pi}^{-1}(\mathscr{E}_\delta) = \widehat{\mathscr{E}}_\delta$. Define $\delta' := \delta \circ \theta|_{N \times \{0\}} \in \mathcal{N}(N \times \{0\})$. We claim:

$$g(\widehat{\mathscr{E}}_\delta) = \mathscr{G}_{\delta' \circ \widehat{\Pi}} := \{(x, 0, w, t) \in \mathscr{G} : \ |t| < (\delta' \circ \widehat{\Pi})(x, 0, w)\}.$$

Consequently, $\widehat{\mathscr{E}}_\delta$ *is a Nash tubular neighborhood of* $\widehat{\Pi}^{-1}(N \times \{0\})$ *and* $g : \widehat{\mathscr{E}}_\delta \to \mathscr{G}_{\delta' \circ \widehat{\Pi}}$ *is a Nash diffeomorphism.*

We have

$$\begin{aligned} \widehat{\mathscr{E}}_\delta &= \widehat{\Pi}^{-1}(\mathscr{E}_\delta) = \{(x, y, w) \in \widehat{\mathscr{E}}_\delta : \ \|y\| < \delta(x)\} \\ &= \{(x, y, w) \in \widehat{\mathscr{E}}_\delta : \ |h(x, y, w)| < (\delta' \circ \widehat{\Pi})(x, 0, w)\} \\ &= g^{-1}(\{(x, 0, w, t) \in \mathscr{G} : \ |t| < (\delta' \circ \widehat{\Pi})(x, 0, w)\}) = g^{-1}(\mathscr{G}_{\delta' \circ \widehat{\Pi}}), \end{aligned}$$

as claimed.

5.C.9. The composition $\widehat{\Pi} \circ g^{-1}$ is a Nash map that satisfies

$$\widehat{\Pi} \circ g^{-1} : g(\widehat{\mathscr{E}}_\delta) \to \mathscr{E}_\delta, \ (x, 0, w, t) \mapsto (x, tw). \tag{5.17}$$

5.C.10. Consider the trivial Nash vector bundle $\mathscr{F} := R \times \mathbb{R}$ over $R := \widehat{\pi}^{-1}(N)$ and let $\theta_2 : R \times \mathbb{R} \to R$ be the projection onto the first factor. By (5.15) and (5.16) the Nash map

$$\psi_0 := (\Lambda \circ \Theta)|_R : R \subset \widehat{V} \to \widehat{\Pi}^{-1}(N \times \{0\}) \subset \widehat{\mathscr{E}}_\delta$$

is a Nash diffeomorphism and there exists a Nash map $\psi_2 : R \to \mathbb{S}^{m-1}$ such that $\psi_0(x, v) = (x, 0, \psi_2(x, v))$ for each $(x, v) \in R$. In addition, ψ_0 induces a Nash isomorphism of Nash vector bundles
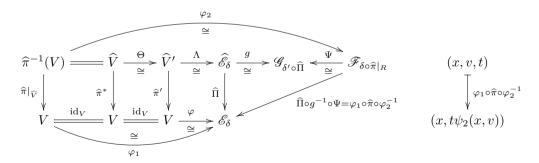
$$\Psi : \mathscr{F} \to \mathscr{G}, (x, v, t) \mapsto (x, 0, \psi_2(x, v), t)$$

such that the following diagram is commutative.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\ \Psi\ } & \mathscr{G} \\ {\scriptstyle \theta_2}\downarrow & {\scriptstyle \cong} & \downarrow{\scriptstyle \theta_2'} \\ R & \xrightarrow[\ \cong\ ]{\psi_0} & \widehat{\Pi}^{-1}(N \times \{0\}) \end{array}$$

By (5.16) $\widehat{\Pi} \circ \psi_0 = \varphi \circ \widehat{\pi}|_R$, so $\delta' \circ \widehat{\Pi} \circ \psi_0 = \delta \circ \theta \circ \varphi \circ \widehat{\pi}|_R = \delta \circ \widehat{\pi}|_R$ because $\theta \circ \varphi|_N = \mathrm{id}_N$. Consequently, $\Psi(\mathscr{F}_{\delta \circ \widehat{\pi}|_R}) = \mathscr{G}_{\delta' \circ \widehat{\Pi}}$.

5.C.11. Define $\varphi_1 := \varphi$ and $\varphi_2 := \Psi^{-1} \circ g \circ \Lambda \circ \Theta$. We have the following commutative diagram:

$$\begin{array}{c} \xrightarrow{\qquad\qquad\qquad \varphi_2 \ \cong \qquad\qquad\qquad} \\ \widehat{\pi}^{-1}(V) = \widehat{V} \xrightarrow[\cong]{\Theta} \widehat{V}' \xrightarrow[\cong]{\Lambda} \widehat{\mathscr{E}}_\delta \xrightarrow[\cong]{g} \mathscr{G}_{\delta' \circ \widehat{\Pi}} \xleftarrow[\cong]{\Psi} \mathscr{F}_{\delta \circ \widehat{\pi}|_R} \qquad (x, v, t) \\ {\scriptstyle \widehat{\pi}|_{\widehat{V}}}\downarrow \quad {\scriptstyle \widehat{\pi}^*}\downarrow \quad {\scriptstyle \widehat{\pi}'}\downarrow \quad {\scriptstyle \widehat{\Pi}}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \varphi_1 \circ \widehat{\pi} \circ \varphi_2^{-1}} \\ V = V \xrightarrow[\cong]{\mathrm{id}_V} V \xrightarrow[\cong]{\varphi} \mathscr{E}_\delta \quad {\scriptstyle \widehat{\Pi} \circ g^{-1} \circ \Psi = \varphi_1 \circ \widehat{\pi} \circ \varphi_2^{-1}} \qquad (x, t\psi_2(x, v)) \\ \xrightarrow[\qquad\qquad\qquad \varphi_1 \qquad\qquad\qquad]{} \end{array}$$

Take $V_1 := V$, $\theta_1 := \theta$, $V_2 := \widehat{\pi}^{-1}(V)$. The Nash tubular neighborhoods $(V_1, \varphi_1, \mathscr{E}, \theta_1, N, \delta)$ of N in U_1 and $(V_2, \varphi_2, \mathscr{F}, \theta_2, R, \delta \circ \widehat{\pi}|_R)$ of R in U_2 and the Nash embedding

$$\psi : R \to N \times \mathbb{S}^{m-1}, \ (x, v) \mapsto (x, \psi_2(x, v))$$

satisfy the required conditions. $\square$

The following result justifies the first part of the name of the drilling blow-up $(\widetilde{M}_+, \pi_+)$, see also Figs. 4 and 5.

Corollary 5.5 *(Alternative description of the drilling blow-up). Let $M \subset \mathbb{R}^m$ be a Nash manifold, let $N \subset M$ be a closed Nash submanifold and let U be an open semialgebraic neighborhood of N in M. Then there exist a Nash tubular neighborhood $(V, \varphi, \mathscr{E}, \theta, N, \delta)$ of N in U such that $M \setminus V$ is a Nash manifold with boundary ∂V and a Nash diffeomorphism $g : M \setminus V \to \widetilde{M}_+$.*

Proof. By Lemma 5.4 there exist Nash tubular open neighborhoods $(V_1, \varphi_1, \mathscr{E}, \theta_1, N, \delta_1)$ of N in U and $(V_2, \varphi_2, \mathscr{F}, \theta_2, R, \varepsilon_1 := \delta_1 \circ \widehat{\pi}|_R)$ of $R := \widehat{\pi}^{-1}(N)$ in $\widehat{M}$ and a Nash embedding $\psi : R \to N \times \mathbb{S}^{m-1}$, $(x, v) \mapsto (x, \psi_2(x, v))$ such that

$$\varphi_1 \circ \widehat{\pi} \circ \varphi_2^{-1} : \mathscr{F}_{\varepsilon_1} \to \mathscr{E}_{\delta_1}, \ (x, v, t) \mapsto (x, t\psi_2(x, v))$$

and $V_2 \subset \widehat{M}$ is the Nash double of a collar of R in $\widetilde{M}_+$. By 5.C.10 we may write $\mathscr{F} := R \times \mathbb{R}$ and $\theta_2 : R \times \mathbb{R} \to R$ is the projection onto the first factor. Recall that $\pi_+|_{\widetilde{M}_+ \setminus R} : \widetilde{M}_+ \setminus R \to M \setminus N$ is a Nash diffeomorphism. Define $\delta := \frac{\delta_1}{4}$, $V := \varphi_1^{-1}(\mathscr{E}_\delta)$, $\varphi := \varphi_1|_V$ and $\theta := \theta_1$. We claim: $(V, \varphi, \mathscr{E}, \theta, N, \delta)$ *is the Nash tubular neighborhood we sought.*

Notice that $\pi_+^{-1}(V) = \varphi_2^{-1}(\mathscr{F}_\varepsilon) \cap \widetilde{M}_+$ where $\varepsilon := \delta \circ \widehat{\pi}|_R$. Define $M^\bullet := M \setminus V$ and $\widetilde{M}_+^\bullet := \pi_+^{-1}(M^\bullet) = \widetilde{M}_+ \setminus \varphi_2^{-1}(\mathscr{F}_\varepsilon)$. It holds $\pi_+|_{\widetilde{M}_+^\bullet} : \widetilde{M}_+^\bullet \to M^\bullet$ is a Nash diffeomorphism.

Denote $W := \{(z, t) \in \mathscr{F} : \ 0 \le t < \varepsilon_1(z)\} = \varphi_2(\varphi_2^{-1}(\mathscr{F}_{\varepsilon_1}) \cap \widetilde{M}_+)$ and consider the Nash diffeomorphism

$$\Lambda : W \to R \times [0, 1), \ (z, t) \to \left(z, \frac{t}{\varepsilon_1(z)}\right).$$

As $\frac{\varepsilon}{\varepsilon_1} = \frac{1}{4}$, we have $(\Lambda \circ \varphi_2)(\widetilde{M}_+^\bullet \cap \varphi_2^{-1}(W)) = R \times [\frac{1}{4}, 1)$. Let $f_1 : [\frac{1}{4}, 1) \to [0, 1)$ be an $\mathcal{S}^2$ diffeomorphism such that $f_1|_{[\frac{3}{4}, 1)} = \mathrm{id}_{[\frac{3}{4}, 1)}$ (see Example A.1(i)). Consider the $\mathcal{S}^2$ diffeomorphism

$$F_1 : R \times [\tfrac{1}{4}, 1) \to R \times [0, 1), \ (z, t) \mapsto (z, f_1(t)).$$

It holds $F_1|_{R \times [\frac{3}{4}, 1)} = \mathrm{id}_{R \times [\frac{3}{4}, 1)}$. Denote again φ_2 the restriction of this Nash map to $\varphi_2^{-1}(W)$ and define

$$\Phi : \widetilde{M}_+^\bullet \to \widetilde{M}_+, \ x \mapsto \begin{cases} x & \text{if } x \in \widetilde{M}_+^\bullet \setminus \varphi_2^{-1}(W), \\ (\Lambda \circ \varphi_2)^{-1}(F_1((\Lambda \circ \varphi_2)(x))) & \text{if } x \in \widetilde{M}_+^\bullet \cap \varphi_2^{-1}(W), \end{cases}$$

which is an $\mathcal{S}^2$ diffeomorphisms. The restriction $\Phi|_{\pi_+^{-1}(\partial V)} : \pi_+^{-1}(\partial V) = \partial \widetilde{M}_+^\bullet \to \pi_+^{-1}(N) = \partial \widetilde{M}_+$ is a Nash diffeomorphism. By 2.D.4 there exists a Nash diffeomorphism $\Phi' : \widetilde{M}_+^\bullet \to \widetilde{M}_+$ such that $\Phi'|_{\partial \widetilde{M}_+^\bullet} = \Phi|_{\partial \widetilde{M}_+^\bullet}$.

The composition $g := \Phi' \circ (\pi_+|_{\widetilde{M}_+^\bullet}^{-1}) : M^\bullet \to \widetilde{M}_+$ is a Nash diffeomorphism, so $M^\bullet$ is a Nash manifold with boundary $\partial V = g^{-1}(\partial \widetilde{M}_+)$, as required. $\square$

Fig. 4. Full picture of the drilling blow-up $\widetilde{M}_+$ of M with center N.

Fig. 5. Geometry of the tool introduced in Lemma 5.1 to erase a closed Nash submanifold N from the Nash manifold M.

5.D. Relationship between drilling blow-up and classical blow-up

We justify next the second part of the name of the drilling blow-up relating it with the classical blow-up. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. Let $f_1, \ldots, f_k \in \mathcal{N}(M)$ be a system of generators of $I(N)$. Define

$$\Gamma' := \{(x, (f_1(x) : \ldots : f_k(x))) \in M \times \mathbb{RP}^{k-1} : \ x \in M \setminus N\}.$$

The closure $B(M, N)$ of Γ' in $M \times \mathbb{RP}^{k-1}$ together with the restriction π' to $B(M, N)$ of the projection $M \times \mathbb{RP}^{k-1} \to M$ is the classical *blow-up of M with center N*.

Corollary 5.6. *Let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{M}_+, \pi_+)$. Let $\sigma : \widehat{M} \to \widehat{M}$, $(a, b) \mapsto (a, -b)$ be the involution of $\widehat{M}$ without fixed points. Consider the Nash map*

$$\Theta : M \times \mathbb{S}^{k-1} \to M \times \mathbb{RP}^{k-1}, (p, q) \to (p, [q])$$

and its restriction $\theta : \widehat{M} \to B(M, N)$. We have

(i) *$\theta(\widehat{M}) = B(M, N)$, $\theta \circ \sigma = \theta$, $\pi' \circ \theta = \widehat{\pi}$ and $\theta^{-1}(a, [b]) = \{(a, b), (a, -b)\}$ for each $(a, [b]) \in B(M, N)$.*
(ii) *θ is an unramified two to one Nash covering of $B(M, N)$.*

Remark 5.7. Many well-known properties of $(B(M, N), \pi')$ concerning: the fibers of π', the local representations (5.8) of π' at the points of $\pi'^{-1}(N)$, finite coverings of $B(M, N)$ whose members are Nash diffeomorphic either to $\mathbb{R}^e \times \mathbb{RP}^{d-e}$ or to $\mathbb{R}^d$, the fact that π' is proper and the restriction $\pi'| : B(M, N) \setminus \pi'^{-1}(N) \to M \setminus N$ is a Nash diffeomorphism, the fact that $B(M, N)$ does not depend on the generators of $I(N)$, etc., follow at once from 5.A, 5.B and Corollary 5.6.

6. Connected Nash manifolds with boundary as Nash images of Euclidean spaces

In this section we prove Theorem 1.5. By Proposition 4.1 every Nash manifold H with boundary is the image under a Nash map of its interior $\text{Int}(H)$. Consequently, we are reduced to prove the following.

Theorem 6.1. *Let $M \subset \mathbb{R}^m$ be a connected d-dimensional Nash manifold. Then M is a Nash image of $\mathbb{R}^d$.*

The proof of Theorem 6.1 still requires some preliminary work that we develop next. We prove first that connected Nash manifolds with boundary are connected by Nash paths, so they are under the assumptions of Theorem 1.4.

Lemma 6.2. *Let $M \subset \mathbb{R}^m$ be a connected Nash manifold. Then M is connected by Nash paths.*

Proof. By [4, Thm.2.4.5 & Prop.2.5.13] M is semialgebraically path connected. Let $x, y \in M$ and let $\alpha : [0, 1] \to M$ be a continuous semialgebraic path that connects x and y. Let $\varepsilon > 0$ and let $\widehat{\alpha} : (-\varepsilon, 1 + \varepsilon) \to M$ be any (continuous) semialgebraic extension of α to the interval $(-\varepsilon, 1 + \varepsilon)$. By [34, Cor.II.5.7] there exists a Nash approximation

$\beta : (-\varepsilon, 1 + \varepsilon) \to M$ of α such that $\beta(0) = \alpha(0) = x$ and $\beta(1) = \alpha(1) = y$. Thus, M is connected by Nash paths.　$\square$

Corollary 6.3. *Let $M \subset \mathbb{R}^m$ be a connected Nash manifold and let $f : M \to \mathbb{R}^n$ be a Nash map. Then $\mathcal{S} := f(M)$ is pure dimensional and connected by Nash paths.*

Proof. Assume $\mathcal{S}$ is not pure dimensional. Let $B \subset \mathbb{R}^n$ be a small ball such that $\dim(B \cap \mathcal{S}) < \dim(\mathcal{S})$. Let Y be the Zariski closure of $B \cap \mathcal{S}$ and let $P \in \mathbb{R}[\mathbf{x}]$ be a polynomial equation of Y. As the Nash function $P \circ f$ vanish on the open set $f^{-1}(B)$ of the connected Nash manifold M, the composition $P \circ f$ is identically zero on M. Consequently, $\mathcal{S} \subset Y$, which is a contradiction. Thus, $\mathcal{S}$ is pure dimensional.

To prove that $\mathcal{S}$ is connected by Nash paths pick $x, y \in \mathcal{S}$ and let $a, b \in M$ be such that $f(a) = x$ and $f(b) = y$. As M is connected by Nash paths, there exists a Nash path $\beta : [0, 1] \to M$ connecting a and b. Thus, $\alpha := f \circ \beta$ is a Nash path connecting x and y, as required.　$\square$

Remark 6.4. By Proposition 4.1 and Corollary 6.3 connected Nash manifolds with boundary are connected by Nash paths.

6.A. Managing semialgebraic triangulations

The proof of Theorem 6.1 involves an inductive argument on the number of simplices of a suitable (semialgebraic) triangulation of the connected Nash manifold M. Let $\sigma \subset \mathbb{R}^n$ be a simplex of dimension d. The *facets of σ* are the faces of σ of dimension $d - 1$. As usual, we denote the relative interior of σ with σ^0 and we will say that σ is a d-simplex. The first step of the inductive argument concerns the following statement: *The interior of a simplex is a Nash image of an Euclidean space.*

Lemma 6.5. *Let $\sigma \subset \mathbb{R}^n$ be a simplex of dimension n and let σ^0 be its interior. Then σ^0 is Nash diffeomorphic to $\mathbb{R}^n$.*

Proof. It is enough to consider the simplex $\sigma := \{\mathbf{x}_1 \geq 0, \ldots, \mathbf{x}_n \geq 0, \mathbf{x}_1 + \cdots + \mathbf{x}_n \leq 1\} \subset \mathbb{R}^n$. Consider the open orthant $\mathcal{Q} := \{\mathbf{y}_1 > 0, \ldots, \mathbf{y}_n > 0\}$ and the Nash diffeomorphism

$$f : \sigma^0 \to \mathcal{Q}, \quad (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{1 - \sum_{i=1}^n x_i}, \ldots, \frac{x_n}{1 - \sum_{i=1}^n x_i} \right)$$

whose inverse is the Nash map

$$f^{-1} : \mathcal{Q} \to \sigma^0, \quad (y_1, \ldots, y_n) \mapsto \left(\frac{y_1}{1 + \sum_{i=1}^n y_i}, \ldots, \frac{y_n}{1 + \sum_{i=1}^n y_i} \right).$$

We are reduced to prove that $\mathcal{Q}$ is Nash diffeomorphic to $\mathbb{R}^n$. To that end, we show that *the open interval $(0, +\infty)$ is Nash diffeomorphic to $\mathbb{R}$.* Consider
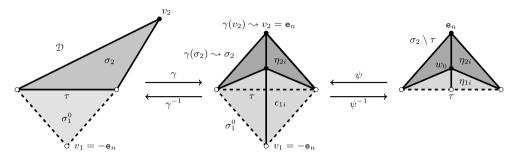
Fig. 6. Construction of the homeomorphisms between $\mathcal{D}$ and $\sigma_2 \setminus \tau$.

$$h_1 : (0, +\infty) \to \mathbb{R}, \ t \mapsto t - \frac{1}{t} \quad \text{and} \quad h_1^{-1} : \mathbb{R} \to (0, +\infty), \ t \mapsto \frac{t + \sqrt{t^2 + 4}}{2}.$$

We are done. $\quad \square$

The following result is the clue to erase a simplex from a semialgebraic triangulation of a Nash manifold.

Lemma 6.6 (*Erasing simplices*). *Let $\sigma_1, \sigma_2 \subset \mathbb{R}^n$ be two simplices of dimension n that only share a facet τ. Let $\mathcal{D} := \sigma_1^0 \cup (\sigma_2 \setminus \partial\tau) = (\tau^0 \cup \sigma_1^0) \cup (\sigma_2 \setminus \tau)$ and $\overline{\mathcal{D}} := \sigma_1 \cup \sigma_2$. Then there exists a semialgebraic homeomorphism $\psi : \sigma_2 \to \overline{\mathcal{D}}$ such that $\psi(\sigma_2 \setminus \tau) = \mathcal{D}$ and $\psi|_{\partial\sigma_2 \setminus \tau^0} = \mathrm{id}_{\partial\sigma_2 \setminus \tau^0}$.*

Proof. The proof is conducted in several steps. Fig. 6 summarizes the followed strategy.

6.A.1. Let v_i be the vertex of σ_i not contained in τ. We may assume $\tau \subset \{x_n = 0\}$, its barycenter is the origin of $\mathbb{R}^n$, $v_2 := (v_{21}, \ldots, v_{2n}) \in \{x_n > 0\}$ and v_1 is $-e_n := (0, \ldots, 0, -1)$. We claim: *after a semialgebraic homeomorphism of $\mathbb{R}^n$ that keeps σ_1 invariant, we may assume $v_2 = e_n := (0, \ldots, 0, 1)$.*

Consider the semialgebraic homeomorphism

$$\gamma : \mathbb{R}^n \to \mathbb{R}^n, \ x := (x_1, \ldots, x_n) \mapsto \begin{cases} x - \frac{x_n}{v_{2n}}(v_{2,1}, \ldots, v_{2,n-1}, v_{2,n} - 1) & \text{if } x_n > 0, \\ x & \text{if } x_n \leq 0, \end{cases}$$

whose inverse map is

$$\gamma^{-1} : \mathbb{R}^n \to \mathbb{R}^n, \ y := (y_1, \ldots, y_n) \mapsto \begin{cases} y + y_n(v_{2,1}, \ldots, v_{2,n-1}, v_{2,n} - 1) & \text{if } y_n > 0, \\ y & \text{if } y_n \leq 0. \end{cases}$$

We have $\gamma(v_2) = e_n$ and $\gamma|_{\{x_n \leq 0\}} = \mathrm{id}_{\{x_n \leq 0\}}$. As $\tau \subset \{x_n = 0\}$, it holds $\gamma(\sigma_2)$ is the simplex whose vertices are those of τ and e_n.

6.A.2. Denote the simplex whose vertices are the affinely independent points $p_1, \ldots, p_k \in \mathbb{R}^n$ with $[p_1, \ldots, p_k]$. Let $w_1, \ldots, w_n$ be the vertices of the $(n-1)$-simplex τ. The barycenter of τ is the origin, so the barycenter of σ_2 is $w_0 := \frac{1}{n+1}\mathsf{e}_n$. Consider the n-simplices

$$\eta_{1i} := [0, w_0, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n] \quad i = 1, \ldots, n,$$

$$\eta_{2i} := [\mathsf{e}_n, w_0, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n] \quad i = 1, \ldots, n.$$

The family $\{\eta_{11}, \ldots, \eta_{1n}, \eta_{21}, \ldots, \eta_{2n}\}$ provides a triangulation of σ_2. Consider also the n-simplices

$$\epsilon_{1i} := [-\mathsf{e}_n, w_0, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n] \quad i = 1, \ldots, n.$$

Our choices done in 6.A.1 assure that the collection $\{\epsilon_{11}, \ldots, \epsilon_{1n}, \eta_{21}, \ldots, \eta_{2n}\}$ provides a triangulation of $\overline{\mathcal{D}}$.

6.A.3. Fix $i = 1, \ldots, n$ and take barycentric coordinates in $\mathbb{R}^n$ with respect to the affine basis $\mathcal{B}_i := \{0, w_0, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n\}$ and consider the affine isomorphism of $\mathbb{R}^n$ given by

$$\psi_i : \mathbb{R}^n \to \mathbb{R}^n, \quad \left(1 - \sum_{k \neq i} \lambda_k\right) 0 + \sum_{k \neq i} \lambda_k w_k \mapsto \left(1 - \sum_{k \neq i} \lambda_k\right)(-\mathsf{e}_n) + \sum_{k \neq i} \lambda_k w_k.$$

Observe that $\psi_i(\eta_{1i}) = \epsilon_{1i}$, $\psi_i|_{\eta_{1i} \cap \eta_{1j}} = \psi_j|_{\eta_{1i} \cap \eta_{1j}}$ and $\psi_i|_{\eta_{1i} \cap \eta_{2j}} = \mathrm{id}_{\eta_{1i} \cap \eta_{2j}}$ for $1 \leq i, j \leq n$. Consequently, the semialgebraic map

$$\psi : \sigma_2 \to \overline{\mathcal{D}}, \quad x \mapsto \begin{cases} \psi_i(x) & \text{if } x \in \eta_{1i} \text{ for } i = 1, \ldots, n, \\ x & \text{if } x \in \eta_{2i} \text{ for } i = 1, \ldots, n \end{cases}$$

is a well-defined homeomorphism such that $\psi(\sigma_2 \setminus \tau) = \mathcal{D}$ and $\psi|_{\partial\sigma_2 \setminus \tau^0} = \mathrm{id}_{\partial\sigma_2 \setminus \tau^0}$, as required. $\quad\square$

To take advantage of the full strength of Lemma 6.6 we need the following result to subdivide simplices in the appropriate way, see Fig. 7.

Lemma 6.7. *Let $\sigma \subset \mathbb{R}^n$ be a simplex of dimension n and let $\tau_1, \ldots, \tau_k$ be facets of σ for some $k = 1, \ldots, n+1$. Let ϵ be either the intersection of the remaining facets $\tau_{k+1}, \ldots, \tau_{n+1}$ if $k < n+1$ or σ if $k = n+1$. Let b be the barycenter of ϵ and let η_i be the convex hull of $\tau_i \cup \{b\}$ for $i = 1, \ldots, k$. Then the simplices $\eta_1, \ldots, \eta_k$ provide a triangulation of σ.*

Proof. We have to prove: $\sigma = \bigcup_{i=1}^{k} \eta_i$ and if ρ_1 is a face of η_i and ρ_2 is a face of η_j, then $\rho_1 \cap \rho_2$ is either the empty-set or a common face of ρ_1 and ρ_2.
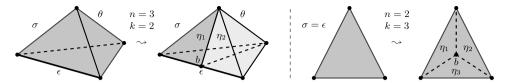
Fig. 7. Triangulation of σ induced by $\epsilon := \tau_{k+1} \cap \cdots \cap \tau_{n+1}$.

6.A.4. If $k = n+1$, then $\sigma = \epsilon$ and b is the barycenter of σ, so $\eta_1, \ldots, \eta_{n+1}$ provide a triangulation of σ. We assume in the following $k < n+1$. Let W be the affine subspace of $\mathbb{R}^n$ generated by ϵ and let L be the affine subspace generated by $\theta := \tau_1 \cap \cdots \cap \tau_k$. We have $\dim(W) = k-1$, $\dim(L) = n-k$ and $W \cap L = \varnothing$. Let E be the affine hyperplane of $\mathbb{R}^n$ that contains L and is parallel to W. Consider the projection

$$\pi : \mathbb{R}^n \setminus E \to W, \ x \mapsto (\{x\} + L) \cap W,$$

where $\{x\} + L$ denotes the affine subspace of $\mathbb{R}^n$ generated by $\{x\} \cup L$. We claim: $\pi(\sigma \setminus E) = \epsilon$. As $\pi|_W = \mathrm{id}_W$ and $\epsilon \subset W$, it is enough to check that $\pi(\sigma \setminus E) \subset \epsilon$.

It holds: *the vertices of σ are either vertices of ϵ or θ.*

Both ϵ and θ are faces of σ. The k vertices of ϵ (it is a simplex of dimension $k-1$) are vertices of σ and the $n-k+1$ vertices of θ (it is a simplex of dimension $n-k$) are vertices of σ. We have $k + (n-k+1) = n+1$ vertices. As $\epsilon \cap \theta = \varnothing$, we have all the vertices of σ.

Denote the vertices of ϵ with $v_1, \ldots, v_k$ and the vertices of θ with $v_{k+1}, \ldots, v_{n+1}$. For each $x \in \sigma \setminus E$ there exist $\lambda_i \geq 0$ such that $\sum_{i=1}^{n+1} \lambda_i = 1$ and $x = \sum_{i=1}^{n+1} \lambda_i v_i$. As $\sigma \cap E = \theta$ and $x \in \sigma \setminus E$, we have $\mu := \sum_{i=1}^{k} \lambda_i > 0$. If $\mu = 1$, then $x \in \epsilon$ and $\pi(x) = x \in \epsilon$. If $\mu < 1$, consider the points

$$p = \sum_{i=1}^{k} \frac{\lambda_i}{\mu} v_i \in \epsilon \subset W \quad \text{and} \quad q = \sum_{i=k+1}^{n+1} \frac{\lambda_i}{1-\mu} v_i \in \theta \subset L$$

that satisfy $x = \mu p + (1-\mu)q$. Consequently,

$$p = \tfrac{1}{\mu} x + \tfrac{\mu-1}{\mu} q \in (\{x\} + L) \cap W \quad \text{and} \quad \pi(x) = p \in \epsilon.$$

6.A.5. We are ready to prove $\sigma = \bigcup_{i=1}^{k} \eta_i$. Let $x \in \sigma$. If $x \in \theta$, then $x \in \eta_1 \cap \cdots \cap \eta_k$. So we assume $x \notin \theta$. Consider the simplex ρ of base θ and vertex $\pi(x) \in \epsilon$. Observe that $x \in \rho$ and we may write $x = \alpha \pi(x) + (1-\alpha)y$ for some $y \in \theta$ and $\alpha \in [0,1]$. The k facets of ϵ are the intersections of ϵ with the facets $\tau_1, \ldots, \tau_k$ of σ. In addition, $\eta_i \cap \epsilon$ is the cone of base $\tau_i \cap \epsilon$ (a facet of ϵ) and vertex b (the barycenter of ϵ). Consequently, $\epsilon = \bigcup_{i=1}^{k} (\eta_i \cap \epsilon)$ and we assume that $\pi(x) \in \eta_1 \cap \epsilon$. As $\pi(x) \in \eta_1$ and $y \in \theta \subset \tau_1 \subset \eta_1$, we conclude $x = \alpha \pi(x) + (1-\alpha)y \in \eta_1$.

6.A.6.   Let ρ_1 be a face of η_i and ρ_2 a face of η_j. We have to prove that $\rho_1 \cap \rho_2$ is either the empty-set or a common face of ρ_1 and ρ_2.

Observe that ρ_i is either a face of σ, the vertex b or the cone of basis a face of σ and vertex b.

(1) Suppose $\rho_1 = \{b\}$. Then $\rho_1 \cap \rho_2$ is either the empty-set or $\{b\}$.

(2) Suppose ρ_1 is a face of σ. Then $\rho_1 \cap \rho_2 = \rho_1 \cap \rho_2 \cap \sigma$ is the intersection of two faces of σ, so it is either the empty-set or a face of σ, which is a common face of ρ_1 and ρ_2.

(3) Suppose ρ_1 and ρ_2 are cones over faces δ_1 and δ_2 of σ with vertex $\{b\}$. Consequently, $\rho_1 \cap \rho_2$ is either $\{b\}$ or the cone of base $\delta_1 \cap \delta_2$ and vertex $\{b\}$, which is a face of ρ_1 and ρ_2, as required. $\square$

6.B. Proof of Theorem 6.1

The proof is conducted in several steps. Fig. 8 summarizes the followed strategy.

6.B.1.   We may assume M is bounded in $\mathbb{R}^m$. By [4, Thm.9.2.1 & Rmk.9.2.3] there exist a finite simplicial complex K and a semialgebraic homeomorphism $\Phi : |K| \to \mathrm{Cl}(M)$ such that

- The semialgebraic sets M and $\mathrm{Cl}(M) \setminus M$ are finite unions of images $\Phi(\sigma^0)$ where $\sigma \in K$.
- The restriction $\Phi|_{\sigma^0} : \sigma^0 \to \mathrm{Cl}(M)$ is a Nash embedding for each $\sigma \in K$.

Denote the simplices of dimension d of K with $\sigma_1, \dots, \sigma_r$.

6.B.2.   Let E be the union of the simplices of K of dimension $\leq d - 2$. Denote $\sigma_i' := (\Phi^{-1}(M) \setminus E) \cap \sigma_i$. Let us *reorder the indices* $1, \dots, r$ *in such a way that* σ_i' *shares the interior of a face of dimension* $d - 1$ *with* $\bigcup_{j=1}^{i-1} \sigma_j'$ *for* $i = 2, \dots, r$. Consequently, $M \setminus \bigcup_{j=i+1}^{r} \Phi(\sigma_i)$ *is connected for* $i = 1, \dots, r - 1$.

Indeed, as M is a connected Nash manifold, $\Phi^{-1}(M) \setminus E$ is connected. Fix $1 \leq s < r$ and assume that we have ordered the simplices $\sigma_1, \dots, \sigma_s$ in such a way that σ_i' shares the interior of a face of dimension $d-1$ with $\bigcup_{j=1}^{i-1} \sigma_j'$ for $i = 2, \dots, s$. Let $C_1 := \bigcup_{j=1}^{s} \sigma_j'$ and $C_2 := \bigcup_{j=s+1}^{r} \sigma_j'$. As $\Phi^{-1}(M) \setminus E = C_1 \cup C_2$ is connected and C_1, C_2 are closed in $\Phi^{-1}(M) \setminus E$, we may assume that $C_1 \cap \sigma_{s+1}' \neq \varnothing$, so σ_{s+1}' shares the interior of a face of dimension $d - 1$ with $\bigcup_{j=1}^{s} \sigma_j'$. This is so because K is a triangulation of $\mathrm{Cl}(M)$ and $(\Phi^{-1}(M) \setminus E) \cap \eta = \varnothing$ for each face η of dimension $\leq d - 2$. Proceeding recursively 6.B.2 follows.

6.B.3.   We proceed by induction on r to prove the statement. If $r = 1$, then M is Nash diffeomorphic to an open simplex, so by Lemma 6.5 M is Nash diffeomorphic to $\mathbb{R}^d$. Assume the result true for $r - 1$. As $M \setminus \Phi(\sigma_r)$ is connected, there exists a surjective Nash map $h_1 : \mathbb{R}^d \to M \setminus \Phi(\sigma_r)$. Let us check that the statement is also true for r.
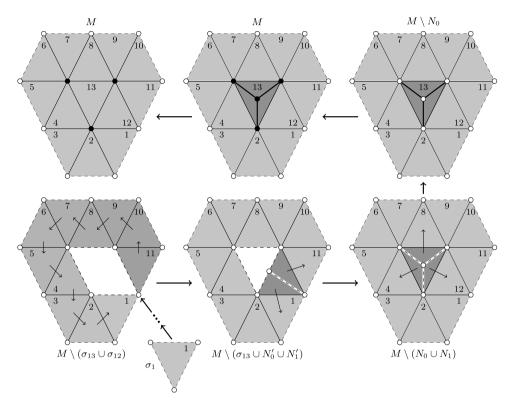
Fig. 8. Strategy to prove Theorem 6.1.

To that end, it is enough to show that *M is the image under a surjective Nash map* $h_0 : M \setminus \Phi(\sigma_r) \to M$. Once this is done, $h := h_0 \circ h_1 : \mathbb{R}^d \to M$ is the surjective Nash map we sought.

6.B.4. Let $\tau_1, \ldots, \tau_s$ be the facets of σ_r that are facets of another simplex σ_k for some $k = 1, \ldots, r-1$. Let ϵ be either the intersection of the remaining facets $\tau_{s+1}, \ldots, \tau_{d+1}$ of σ_r if $s < d+1$ or σ_r if $s = d+1$. Thus, ϵ is a face of σ_r of dimension $s-1$. The s facets of ϵ are the intersections of ϵ with the facets $\tau_1, \ldots, \tau_s$ of σ_r. Let b be the barycenter of ϵ and let η_i be the convex hull of $\tau_i \cup \{b\}$ for $i = 1, \ldots, s$. By Lemma 6.7 the simplices $\eta_1, \ldots, \eta_s$ provides a triangulation of σ.

Let $\mathfrak{H}_\ell$ be the collection of all the faces of $\eta_1, \ldots, \eta_s$ of dimension ℓ for $\ell = 0, \ldots, d-2$ and let $\mathfrak{H}_{d-1}$ be the collection of all the facets of $\eta_1, \ldots, \eta_s$ different from $\tau_1, \ldots, \tau_s$. Define

$$N_\ell := \bigcup_{\sigma \in \mathfrak{H}_\ell} \Phi(\sigma^0) \cap M$$

and observe that N_ℓ is a closed Nash submanifold of $M \setminus \bigsqcup_{j=0}^{\ell-1} N_j$, where $\bigsqcup_{j=0}^{\ell-1} N_j := \varnothing$ if $\ell = 0$. By Lemma 5.1 there exists a surjective Nash map $g_\ell : M \setminus \bigsqcup_{j=0}^{\ell} N_j \to M \setminus \bigsqcup_{j=0}^{\ell-1} N_j$.

Thus,

$$g := g_{d-1} \circ \cdots \circ g_0 : M \setminus \bigsqcup_{j=0}^{d-1} N_j \to M$$

is a surjective Nash map.

6.B.5. Observe that $M \setminus \bigsqcup_{j=0}^{d-1} N_j = (M \setminus \Phi(\sigma_r)) \cup \bigcup_{j=1}^{s} \Phi(\tau_j^0 \cup \eta_j^0)$. For each $j = 1, \ldots, s$ let σ_{i_j} be the simplex of K such that $\sigma_{i_j} \cap \sigma_r = \tau_j$. By Lemma 6.6 there exists a semialgebraic homeomorphism $f_j : \sigma_{i_j} \setminus \tau_j \to (\sigma_{i_j} \setminus \tau_j) \cup (\tau_j^0 \cup \eta_j^0)$ such that $f_j|_{\partial \sigma_{i_j} \setminus \tau_j} = \mathrm{id}_{\partial \sigma_{i_j} \setminus \tau_j}$.

Consider the semialgebraic homeomorphism

$$f : M \setminus \Phi(\sigma_r) \to (M \setminus \Phi(\sigma_r)) \cup \bigcup_{j=1}^{s} \Phi(\tau_j^0 \cup \eta_j^0), \quad x \mapsto \begin{cases} x & \text{if } x \notin \bigcup_{j=1}^{s} \Phi(\sigma_{i_j}), \\ \Phi(f_j(\Phi^{-1}(x))) & \text{if } x \in \Phi(\sigma_{i_j}). \end{cases}$$

Let $f' : M \setminus \Phi(\sigma_r) \to M \setminus \bigsqcup_{j=0}^{d-1} N_j$ be a close Nash approximation of f (use 2.D.1). As f is a semialgebraic homeomorphism, f' is by Lemma 2.8 surjective. Consequently $h_0 := f' \circ g : M \setminus \Phi(\sigma_r) \to M$ is a surjective Nash map, as required. $\quad\square$

7. Main properties of well-welded semialgebraic sets

In this section we describe the main properties of well-welded semialgebraic sets. Given a continuous semialgebraic path $\alpha : [0,1] \to \mathbb{R}^n$ we define $\eta(\alpha)$ as the image $\alpha(A)$ of the smallest (finite) subset $A \subset (0,1)$ such that $\alpha|_{(0,1)\setminus A}$ is a Nash map. Recall that a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *well-welded* if $\mathcal{S}$ is pure dimensional and for each pair of points $x, y \in \mathcal{S}$ there exists a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathcal{S})$.

The following two results provide examples of well-welded semialgebraic sets. Once we prove Theorem 1.4 we will conclude that there are no more well-welded semialgebraic sets.

Lemma 7.1. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set that is connected by Nash paths. Then $\mathcal{S}$ is well-welded.*

Proof. As $\mathcal{S}$ is connected by Nash paths, we only have to check that $\mathcal{S}$ is pure dimensional. Suppose by contradiction that $\mathcal{S}$ is not pure dimensional. Pick $x \in \mathcal{S}$ and an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of x such that $\dim(\mathcal{S} \cap U) < \dim(\mathcal{S})$. Let Y be the Zariski closure of $\mathcal{S} \cap U$ in $\mathbb{R}^n$ and pick a point $y \in \mathcal{S} \setminus Y$. Let $\alpha : [0,1] \to \mathcal{S}$ be a Nash path such that $\alpha(0) = x$ and $\alpha(1) = y$. As $\alpha^{-1}(\mathcal{S} \cap Y) \subset [0,1]$ is a neighborhood of the origin and Y is an algebraic set, we deduce by the identity principle that $y \in \mathrm{Im}(\alpha) \subset Y$, which is a contradiction. Thus, $\mathcal{S}$ is pure dimensional, as required. $\quad\square$
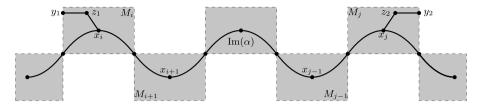
Fig. 9. Sketch of proof of Lemma 7.2.

Lemma 7.2. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional semialgebraic set. Assume that there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of $\mathrm{Reg}(\mathcal{S})$. Then $\mathcal{S}$ is well-welded.*

Proof. Let $M_1, \dots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. Let $x_i \in M_i \cap \mathrm{Im}(\alpha)$ and let $t_i \in (0,1)$ be such that $\alpha(t_i) = x_i$. We may assume $t_1 < \cdots < t_r$. As $\mathcal{S}$ is pure dimensional, $\mathcal{S} = \bigcup_{i=1}^r \mathrm{Cl}(M_i) \cap \mathcal{S}$. Let $y_1, y_2 \in \mathcal{S}$ and assume $y_1 \in \mathrm{Cl}(M_i) \cap \mathcal{S}$ and $y_2 \in \mathrm{Cl}(M_j) \cap \mathcal{S}$ with $i < j$. By the Nash curve selection lemma there exist Nash arcs $\alpha_k : (-1,1) \to \mathbb{R}^n$ such that $\alpha_1((-1,0)) \subset M_i$, $\alpha_1((-1,0)) \subset M_j$ and $\alpha_k(0) = y_k$ for $k = 1,2$. As M_i and M_j are connected Nash manifolds, there exist Nash paths $\gamma_1 : [0,1] \to M_i$ and $\gamma_2 : [0,1] \to M_j$ such that

$$\gamma_k(0) = z_k := \alpha_k(-\tfrac{1}{2}) \quad \text{and} \quad \gamma_k(1) = \begin{cases} x_i & \text{if } k = 1, \\ x_j & \text{if } k = 2. \end{cases}$$

Consider the continuous semialgebraic path

$$\beta := (\alpha_1|_{[-\frac{1}{2},0]})^{-1} * \gamma_1 * \alpha|_{[t_i,t_j]} * \gamma_2^{-1} * \alpha_2|_{[-\frac{1}{2},0]}$$

that connects the points y_1, y_2 and satisfies

$$\eta(\beta) \subset \{z_1, x_i, x_j, z_2\} \subset \mathrm{Reg}(\mathcal{S}),$$

see Fig. 9. Consequently, $\mathcal{S}$ is well-welded, as required. $\square$

7.A. Basic properties of well-welded semialgebraic sets I

We show next that well-welded semialgebraic sets have irreducible Zariski closure and well-welded closure. We will prove later in 7.C that they are in fact irreducible and that the image of a well-welded semialgebraic set under a Nash map is again well-welded.

Lemma 7.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a well-welded semialgebraic set. Then $\overline{\mathcal{S}}^{\mathrm{zar}}$ is irreducible.*

Proof. Suppose by contradiction that $\overline{\mathcal{S}}^{\mathrm{zar}}$ is reducible. Let $X_1, \dots, X_r$ be the irreducible components of $\overline{\mathcal{S}}^{\mathrm{zar}}$. We have

$$\mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}}) = \bigcup_{i=1}^{r} \mathrm{Sing}(X_i) \cup \bigcup_{i \neq j} (X_i \cap X_j).$$

We claim: $\mathbb{S} \cap X_i \setminus \mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}}) \neq \varnothing$ *for* $i = 1, \ldots, r$.

Suppose that $\mathbb{S} \cap X_1 \setminus \mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}}) = \varnothing$. Then $\mathbb{S} \subset \mathrm{Sing}(X_1) \cup \bigcup_{i=2}^{r} (\mathbb{S} \cap X_i)$, so

$$\bigcup_{i=1}^{r} X_i = \overline{\mathbb{S}}^{\mathrm{zar}} \subset \mathrm{Sing}(X_1) \cup \bigcup_{i=2}^{r} X_i,$$

which is a contradiction.

Let $x_i \in \mathbb{S} \cap X_i \setminus \mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}})$ for $i = 1, 2$ and let $f_1 \in \mathbb{R}[\mathbf{x}]$ be a polynomial equation of X_1. Let $\alpha : [0, 1] \to \mathbb{S}$ be a (continuous) semialgebraic map such that $\alpha(0) = x_1$, $\alpha(1) = x_2$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathbb{S})$. Let

$$t_0 := \inf\{t \in (0, 1) : \ \alpha(t) \notin X_1\}.$$

Note that $\alpha(t_0) \notin \eta(\alpha)$ because $\alpha(t_0) \in X_1 \cap X_i \subset \mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}})$ for some $i \neq 1$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathbb{S})$. Let $\varepsilon > 0$ be such that $\alpha|_{(t_0-\varepsilon, t_0+\varepsilon)}$ is a Nash map, $\alpha((t_0 - \varepsilon, t_0)) \subset X_1$ and $\alpha((t_0, t_0 + \varepsilon)) \cap X_1 = \varnothing$. As $f_1 \circ \alpha|_{(t_0-\varepsilon, t_0+\varepsilon)}$ is a Nash function such that $f_1 \circ \alpha|_{(t_0-\varepsilon, t_0)} \equiv 0$, we have $f_1 \circ \alpha|_{(t_0-\varepsilon, t_0+\varepsilon)} \equiv 0$, which is a contradiction. Consequently, $\overline{\mathbb{S}}^{\mathrm{zar}}$ is irreducible, as required. $\square$

Lemma 7.4. *Let* $\mathbb{S} \subset \mathbb{T} \subset \mathbb{R}^n$ *be semialgebraic sets such that* $\mathbb{S}$ *is well-welded and* $\mathbb{T} \subset \mathrm{Cl}(\mathbb{S})$. *Then* $\mathbb{T}$ *is well-welded.*

Proof. Let $x_1, x_2 \in \mathbb{T}$. As $\mathbb{S}$ is pure dimensional, $\mathrm{Cl}(\mathrm{Reg}(\mathbb{S})) = \mathrm{Cl}(\mathbb{S}) = \mathrm{Cl}(\mathbb{T})$. By the Nash curve selection lemma there exist Nash arcs $\alpha_k : (-1, 1) \to \mathbb{R}^n$ such that $\alpha_k((-1, 0)) \subset \mathrm{Reg}(\mathbb{S})$, $\alpha_k(0) = x_k$ for $k = 1, 2$. Consider the points $y_k := \alpha_k(-\frac{1}{2})$ for $k = 1, 2$. As $\mathbb{S}$ is well-welded, there exists a continuous semialgebraic path $\alpha_3 : [0, 1] \to \mathbb{S}$ such that $\alpha_3(0) = y_1$, $\alpha_3(1) = y_2$ and $\eta(\alpha_3) \subset \mathrm{Reg}(\mathbb{S})$. As $\overline{\mathbb{S}}^{\mathrm{zar}} = \overline{\mathbb{T}}^{\mathrm{zar}}$, we have

$$\mathrm{Reg}(\mathbb{S}) = \mathrm{Int}_{\mathrm{Reg}(\overline{\mathbb{S}}^{\mathrm{zar}})}(\mathbb{S} \setminus \mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}})) \subset \mathrm{Int}_{\mathrm{Reg}(\overline{\mathbb{S}}^{\mathrm{zar}})}(\mathbb{T} \setminus \mathrm{Sing}(\overline{\mathbb{S}}^{\mathrm{zar}})) = \mathrm{Reg}(\mathbb{T}).$$

The continuous semialgebraic path $\alpha := (\alpha_1|_{[-\frac{1}{2}, 0]})^{-1} * \alpha_3 * \alpha_2|_{[-\frac{1}{2}, 0]}$ connects x_1 with x_2 and $\eta(\alpha) \subset \eta(\alpha_3) \cup \{y_1, y_2\} \subset \mathrm{Reg}(\mathbb{S}) \subset \mathrm{Reg}(\mathbb{T})$. Consequently, $\mathbb{T}$ is well-welded. $\square$

7.B. Description of well-welding in terms of bridges

It is difficult to handle the definition of well-welded semialgebraic set. Our purpose next is to provide a characterization of well-welding in terms of the existence of Nash arcs between the connected components of the set of regular points of a semialgebraic set.

Definition 7.5. Let $M_1, M_2 \subset \mathbb{R}^n$ be two Nash manifolds. A *(Nash) bridge between M_1 and M_2* is the image Γ of a Nash arc $\alpha : (-1, 1) \to \mathbb{R}^n$ such that $\alpha((-1, 0)) \subset M_1$ and $\alpha((0, 1)) \subset M_2$.

To lighten the presentation we write bridge when referring to Nash bridge. As a straightforward consequence of Proposition 7.8 below we have the following.

Corollary 7.6. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional semialgebraic set and let $M_1, \ldots M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is well-welded.*
(ii) *We can reorder the indices $i = 1, \ldots, r$ in such a way that there exist bridges $\Gamma_i \subset \mathcal{S}$ between M_i and $\bigsqcup_{j=1}^{i-1} M_j$ for $i = 2, \ldots, r$.*

When dealing with well-welded semialgebraic sets, we need often that a continuous semialgebraic path connecting two points avoids certain algebraic set. In this direction we present the following three results.

Lemma 7.7. *Let $\alpha : [0, 1] \to \mathbb{R}^n$ be a continuous semialgebraic path and let $Y \subset \mathbb{R}^n$ be an algebraic set. Assume that $\mathrm{Im}(\alpha) \not\subset Y$ and $\eta(\alpha) \cap Y = \varnothing$. Then $\alpha^{-1}(Y)$ is a finite set.*

Proof. Suppose $\alpha^{-1}(Y)$ is not a finite set. Then $\alpha^{-1}(Y)$ is a closed semialgebraic subset of the interval $[0, 1]$ of dimension 1. Let $A \subset (0, 1)$ be the smallest (finite) subset of $(0, 1)$ such that $\alpha|_{(0,1)\setminus A}$ is a Nash map. Let C be a connected component of $[0, 1] \setminus (A \cup \{0, 1\})$ such that $\dim(C \cap \alpha^{-1}(Y)) = 1$. As $\alpha|_C$ is a Nash map, $C \subset \alpha^{-1}(Y)$. As $\alpha^{-1}(Y)$ is closed, $\mathrm{Cl}(C) \subset \alpha^{-1}(Y)$. Notice that $\mathrm{Cl}(C) \setminus C \subset A \cup \{0, 1\}$. If $\mathrm{Cl}(C) \setminus C = \{0, 1\}$, we have $\mathrm{Cl}(C) = [0, 1]$, so $\mathrm{Im}(\alpha) \subset Y$, which is a contradiction. Consequently, $A \neq \varnothing$ and $(\mathrm{Cl}(C) \setminus C) \cap A \neq \varnothing$, so $\eta(\alpha) \cap Y \neq \varnothing$, against the hypothesis. Thus, $\alpha^{-1}(Y)$ is a finite set, as required. $\square$

Proposition 7.8. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional semialgebraic set and let $M_1, \ldots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. Let $Z \subset \mathbb{R}^n$ be an algebraic set such that $M_i \not\subset Z$ for $i = 1, \ldots, r$. The following assertions are equivalent:*

(i) *For each pair of points $x_1, x_2 \in \mathcal{S}$ there exists a continuous semialgebraic path $\alpha : [0, 1] \to \mathcal{S}$ such that $\alpha(0) = x_1$, $\alpha(1) = x_2$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathcal{S}) \setminus Z$.*
(ii) *The indices $i = 1, \ldots, r$ can be reordered to have bridges $\Gamma_i \subset \mathcal{S}$ between $M_i \setminus Z$ and $\bigsqcup_{j=1}^{i-1}(M_j \setminus Z)$ for $i = 2, \ldots, r$.*

Proof. (ii) $\implies$ (i) Let $x_1, x_2 \in \mathcal{S}$. As $\mathcal{S}$ is pure dimensional, $\mathrm{Reg}(\mathcal{S}) \setminus Z$ is dense in $\mathcal{S}$ and we may assume $x_1 \in \mathrm{Cl}(M_1 \setminus Z)$ and $x_2 \in \mathrm{Cl}(M_\ell \setminus Z)$ for some $\ell = 1, \ldots, r$. By the Nash curve selection lemma there exist Nash paths $\alpha_k : (-1, 1) \to \mathbb{R}^n$ such that
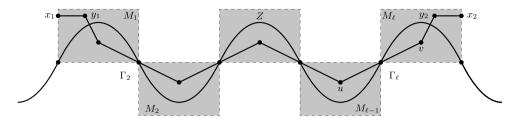
Fig. 10. Sketch of proof of implication (ii) $\Longrightarrow$ (i) in Proposition 7.8.

$\alpha_1((-1,0)) \subset M_1 \setminus Z$, $\alpha_2((-1,0)) \subset M_\ell \setminus Z$ and $\alpha_k(0) = x_k$ for $k = 1, 2$. Consider the points $y_k := \alpha_k(-\frac{1}{2})$ for $k = 1, 2$. We proceed by induction on ℓ.

If $\ell = 1$, then $y_1, y_2 \in M_1 \setminus Z$. As M_1 is a connected Nash manifold, there exists a Nash path $\beta : [0,1] \to M_1$ such that $\beta(0) = y_1$ and $\beta(1) = y_2$. Consider the continuous semialgebraic path $\alpha := (\alpha_1|_{[-\frac{1}{2},0]})^{-1} * \beta * \alpha_2|_{[-\frac{1}{2},0]}$ that connects the points x_1 and x_2 and satisfies $\eta(\alpha) \subset \{y_1, y_2\} \subset \operatorname{Reg}(\mathcal{S}) \setminus Z$ and $\operatorname{Im}(\alpha) \subset M_1 \cup \{x_1, x_2\} \subset \mathcal{S}$.

Assume that for each point $y \in M_i$ for $i = 2, \ldots, \ell - 1$ there exists a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x_1$, $\alpha(1) = y$ and $\eta(\alpha) \subset \operatorname{Reg}(\mathcal{S}) \setminus Z$. Let $\gamma : (-1,1) \to \mathcal{S}$ be a Nash arc such that $\gamma((-1,1)) = \Gamma_\ell$, $\gamma((-1,0)) \subset \bigsqcup_{j=1}^{\ell-1}(M_j \setminus Z)$ and $\gamma((0,1)) \subset M_\ell \setminus Z$. Let $u := \gamma(-\frac{1}{2}) \in \bigsqcup_{j=1}^{\ell-1}(M_j \setminus Z)$ and $v := \gamma(\frac{1}{2}) \in M_\ell \setminus Z$. By induction hypothesis there exists a continuous semialgebraic path $\rho_1 : [0,1] \to \mathcal{S}$ such that $\rho_1(0) = x_1$, $\rho_1(1) = u$ and $\eta(\rho_1) \subset \operatorname{Reg}(\mathcal{S}) \setminus Z$. As M_ℓ is a connected Nash manifold, there exists a Nash path $\rho_2 : [0,1] \to M_\ell$ such that $\rho_2(0) = v$ and $\rho_2(1) = y_2$. Consider the continuous semialgebraic path $\alpha := \rho_1 * \gamma|_{[-\frac{1}{2},\frac{1}{2}]} * \rho_2 * \alpha_2|_{[-\frac{1}{2},0]}$ that connects the points x_1 and x_2 and satisfies $\eta(\alpha) \subset \eta(\rho_1) \cup \{u, v, y_2\} \subset \operatorname{Reg}(\mathcal{S}) \setminus Z$ and $\operatorname{Im}(\alpha) \subset \mathcal{S}$, see Fig. 10.

(i) $\Longrightarrow$ (ii) If $\operatorname{Reg}(\mathcal{S})$ is connected, there is nothing to prove. Assume that the result is true if the number of connected components of $\operatorname{Reg}(\mathcal{S})$ is $< r$ and let us check that the result is also true if the number of connected components of $\operatorname{Reg}(\mathcal{S})$ is r.

Let $x_i \in M_i \setminus Z$ for $i = 1, \ldots, r$. By hypothesis one can construct a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x_1$, $x_i \in \operatorname{Im}(\alpha)$ for $i = 2, \ldots, r$ and $\eta(\alpha) \subset \operatorname{Reg}(\mathcal{S}) \setminus Z$. By Lemma 7.7 $\operatorname{Im}(\alpha) \cap Z$ is a finite set.

We may reorder the connected components M_i in such a way that if $i < j$, then

$$t_i := \inf\{t \in (0,1) : \; \alpha(t) \in M_i\} < \inf\{t \in (0,1) : \; \alpha(t) \in M_j\} =: t_j.$$

Let us check: $\mathcal{S}' := \operatorname{Cl}(\mathcal{S} \setminus \operatorname{Cl}(M_r)) \cap \mathcal{S}$ *is pure dimensional, satisfies the hypothesis of* (i) *and* $\operatorname{Reg}(\mathcal{S}') = \bigsqcup_{i=1}^{r-1} M_i$. It is enough to show:

(a) $\operatorname{Reg}(\mathcal{S}') = \bigsqcup_{i=1}^{r-1} M_i$ *is dense in* $\mathcal{S}'$. *Consequently,* $\mathcal{S}'$ *is pure dimensional.*
(b) *For each point* $x \in \mathcal{S}'$ *there exists a continuous semialgebraic path* $\beta : [0,1] \to \mathcal{S}'$ *such that* $\beta(0) = x_1$, $\beta(1) = x$ *and* $\eta(\beta) \subset \operatorname{Reg}(\mathcal{S}') \setminus Z$.

Let us prove first (a). By Lemma 7.3 $\overline{\mathcal{S}}^{\mathrm{zar}}$ is irreducible because $\mathcal{S}$ is well-welded. As $\bigsqcup_{i=1}^{r-1} M_i$ is a non-empty open subset of $\mathcal{S}$, we conclude $\overline{\mathcal{S}'}^{\mathrm{zar}} = \overline{\mathcal{S}}^{\mathrm{zar}}$. Thus,

$$\mathrm{Reg}(\mathcal{S}') = \mathrm{Int}_{\mathrm{Reg}(\overline{\mathcal{S}}^{\mathrm{zar}})}(\mathcal{S}' \setminus \mathrm{Sing}(\overline{\mathcal{S}}^{\mathrm{zar}})) \subset \mathrm{Reg}(\mathcal{S}) = \bigsqcup_{i=1}^{r} M_i.$$

We deduce

$$\bigsqcup_{i=1}^{r-1} M_i \subset \mathrm{Reg}(\mathcal{S}') \subset \mathcal{S}' \cap \mathrm{Reg}(\mathcal{S}) = \mathrm{Cl}(\mathrm{Reg}(\mathcal{S}) \setminus \mathrm{Cl}(M_r)) \cap \mathrm{Reg}(\mathcal{S})$$

$$= \mathrm{Cl}\left(\bigsqcup_{i=1}^{r} M_i \setminus \mathrm{Cl}(M_r) \right) \cap \bigsqcup_{i=1}^{r} M_i \subset \mathrm{Cl}\left(\bigsqcup_{i=1}^{r-1} M_i \right) \cap \bigsqcup_{i=1}^{r} M_i = \bigsqcup_{i=1}^{r-1} M_i,$$

so $\mathrm{Reg}(\mathcal{S}') = \bigsqcup_{i=1}^{r-1} M_i$.

As $M_1, \ldots, M_r$ are the connected components of $\mathrm{Reg}(\mathcal{S})$, we have $M_i \subset (\mathrm{Cl}(M_i) \setminus \mathrm{Cl}(M_r)) \cap \mathcal{S}$, so $\mathrm{Cl}((\mathrm{Cl}(M_i) \setminus \mathrm{Cl}(M_r)) \cap \mathcal{S}) = \mathrm{Cl}(M_i)$ for $i = 1, \ldots, r-1$. As $\mathcal{S} = \bigcup_{i=1}^{r} \mathrm{Cl}(M_i) \cap \mathcal{S}$,

$$\mathcal{S}' = \mathrm{Cl}\left(\bigcup_{i=1}^{r} \mathrm{Cl}(M_i) \cap \mathcal{S} \setminus \mathrm{Cl}(M_r) \right) \cap \mathcal{S} = \mathrm{Cl}\left(\bigcup_{i=1}^{r-1} \mathrm{Cl}(M_i) \cap \mathcal{S} \setminus \mathrm{Cl}(M_r) \right) \cap \mathcal{S}$$

$$= \bigcup_{i=1}^{r-1} \mathrm{Cl}((\mathrm{Cl}(M_i) \setminus \mathrm{Cl}(M_r)) \cap \mathcal{S}) \cap \mathcal{S} = \bigcup_{i=1}^{r-1} \mathrm{Cl}(M_i) \cap \mathcal{S} = \mathrm{Cl}(\mathrm{Reg}(\mathcal{S}')) \cap \mathcal{S}',$$

so $\mathrm{Reg}(\mathcal{S}')$ is dense in $\mathcal{S}'$.

Let us show next (b). As $x \in \mathcal{S}'$, there exists an index $1 \leq i \leq r-1$ such that $x \in \mathrm{Cl}(M_i \setminus Z)$. By the Nash curve selection lemma there exists a Nash arc $\gamma_1 : (-1,1) \to \mathbb{R}^n$ such that $\gamma_1((-1,0)) \subset M_i \setminus Z$ and $\gamma_1(0) = x$. Let $s_i \in (0, t_{i+1})$ be such that $\alpha(s_i) \in M_i \setminus Z$. As M_i is a connected Nash manifold, there exists a Nash path $\gamma_2 : [0,1] \to M_i$ such that $\gamma_2(0) = \alpha(s_i)$ and $\gamma_2(1) = \gamma_1(-\frac{1}{2})$. The continuous semialgebraic path $\beta := \alpha|_{[0,s_i]} * \gamma_2 * \gamma_1|_{[-\frac{1}{2},0]}$ connects x_1 with x and satisfies $\mathrm{Im}(\beta) \subset \mathcal{S}'$ and

$$\eta(\beta) \subset \eta(\alpha|_{[0,s_i]}) \cup \{\alpha(s_i), \gamma_1(-\tfrac{1}{2})\} \subset \mathrm{Reg}(\mathcal{S}') \setminus Z.$$

Consequently, $\mathcal{S}'$ satisfies the desired conditions.

By induction hypothesis we may assume that there exist bridges $\Gamma_i \subset \mathcal{S}$ between $M_i \setminus Z$ and $\bigsqcup_{j=1}^{i-1}(M_j \setminus Z)$ for $i = 2, \ldots, r-1$. Recall that $\mathrm{Im}(\alpha) \cap Z$ is a finite set. Let $\varepsilon > 0$ be such that the restriction $\alpha|_{(t_r - \varepsilon, t_r + \varepsilon)}$ is a Nash map, $\alpha((t_r, t_r + \varepsilon)) \subset M_r \setminus Z$ and $\alpha((t_r - \varepsilon, t_r)) \subset M_i \setminus Z$ for some $i = 1, \ldots, r-1$. It holds (after reparameterizing) that $\Gamma_r := \alpha((t_r - \varepsilon, t_r + \varepsilon))$ is a bridge between $\bigsqcup_{j=1}^{r-1} M_j \setminus Z$ and $M_r \setminus Z$, as required. $\quad\square$

Corollary 7.9. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a well-welded semialgebraic set and let $Z \subset \mathbb{R}^n$ be an algebraic set that does not contain $\mathcal{S}$. Then for each pair of points $x, y \in \mathcal{S}$ there exists a continuous semialgebraic path $\alpha : [0, 1] \to \mathcal{S}$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathcal{S}) \setminus Z$.*

Proof. Let $M_1, \dots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. As $\mathcal{S}$ is well-welded, we can reorder the indices $i = 1, \dots, r$ in such a way that there exist bridges $\Gamma_i \subset \mathcal{S}$ between M_i and $\bigsqcup_{j=1}^{i-1} M_j$ for $i = 2, \dots, r$ (see Corollary 7.6). Denote $X = \overline{\mathcal{S}}^{\mathrm{zar}}$, which is an irreducible algebraic set not contained in Z. By Theorem 2.5 there exist a non-singular algebraic set X_1 and a proper regular map $f : X_1 \to X$ such that $\mathcal{S} \subset f(X_1)$ and

$$f|_{X_1 \setminus f^{-1}(\mathrm{Sing}(X))} : X_1 \setminus f^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}(X)$$

is a diffeomorphism whose inverse map is also regular.

Denote $\Lambda_i := \mathrm{Cl}(f^{-1}(\Gamma_i \setminus \mathrm{Sing}(X))) \cap f^{-1}(\Gamma_i)$ and $N_i := f^{-1}(M_i)$ for $i = 1, \dots, r$. As $M_i \subset X \setminus \mathrm{Sing}(X)$, we have that $N_i \subset X_1 \setminus f^{-1}(\mathrm{Sing}(X))$ is a Nash manifold. After shrinking Γ_i if necessary, we may assume by Lemma B.2 that Λ_i is a bridge between N_i and $\bigsqcup_{j=1}^{i-1} N_j$ such that $\Lambda_i \setminus \bigsqcup_{j=1}^{r} N_j = \{q_i\}$ and $f(q_i) \in \mathcal{S}$ for some $q_i \in X_1$.

Consider the algebraic set $Z' := f^{-1}(Z \cap X) \subset X_1$. As X is irreducible, also X_1 is irreducible. If Z' contains N_i, then Z' contains X_1, so Z contains X, which is a contradiction. Consequently, Z' contains no N_i. As X_1 is a Nash manifold, there exists by Lemma 2.9 a bridge Λ_i' between $N_i \setminus Z'$ and $\bigsqcup_{j=1}^{i-1}(N_j \setminus Z')$ such that $\Lambda_i' \setminus \bigsqcup_{j=1}^{r} N_j = \{q_i\}$. Consequently, $\Gamma_i' := f(\Lambda_i')$ is a bridge between $M_i \setminus Z$ and $\bigsqcup_{j=1}^{i-1}(M_j \setminus Z)$ such that $\Gamma_i' \setminus \mathrm{Reg}(\mathcal{S}) = \{f(q_i)\} \subset \mathcal{S}$. By Proposition 7.8 the result follows. $\square$

7.C. Basic properties of well-welded semialgebraic sets II

We prove next the remaining announced properties of well-welded semialgebraic sets.

Corollary 7.10. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a well-welded semialgebraic set. Then $\mathcal{S}$ is irreducible.*

Proof. Denote $d := \dim(\mathcal{S})$. By [11, Lem.3.6] it is enough to prove that if f is a Nash function on $\mathcal{S}$ whose zero-set has dimension d, then f is identically zero.

Let f be a Nash function on $\mathcal{S}$ whose zero-set has dimension d. Let $M_1, \dots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. We may assume that f is identically zero only on $M_1, \dots, M_k$. Observe that $k \geq 1$ and assume by contradiction that $k < r$. Let $\mathcal{T} := \{f = 0\} \cap \bigsqcup_{j=k+1}^{r} M_j$, which is a semialgebraic set of dimension $< d$. Let Y be the Zariski closure of $\mathcal{T}$, which has dimension $< d$.

Pick points $x_1 \in \bigsqcup_{i=1}^{k}(M_i \setminus Y)$ and $x_2 \in \bigsqcup_{i=k+1}^{r}(M_i \setminus Y)$. By Corollary 7.9 there exists a continuous semialgebraic path $\alpha : [0, 1] \to \mathcal{S}$ such that $\alpha(0) = x_1$, $\alpha(1) = x_2$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathcal{S}) \setminus Y$. By Lemma 7.7 the inverse image $\alpha^{-1}(Y \cap \mathcal{S})$ is a finite set, so there exist $\varepsilon > 0$ and $t_0 \in (0, 1)$ such that $\alpha|_{(t_0 - \varepsilon, t_0 + \varepsilon)}$ is a Nash map,
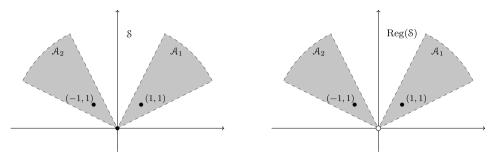
Fig. 11. Semialgebraic set $\mathcal{S} := \{(4x^2 - y^2)(4y^2 - x^2) > 0, y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$ and its set of regular points.

$$\alpha((t_0 - \varepsilon, t_0)) \subset \bigsqcup_{i=1}^{k} (M_i \setminus Y) \quad \text{and} \quad \alpha((t_0, t_0 + \varepsilon)) \subset \bigsqcup_{i=k+1}^{r} (M_i \setminus Y).$$

As $f \circ \alpha|_{(t_0 - \varepsilon, t_0)}$ is identically zero, $f \circ \alpha|_{(t_0 - \varepsilon, t_0 + \varepsilon)}$ is identically zero, so $\alpha((t_0, t_0 + \varepsilon)) \subset \mathcal{T} \subset Y$, which is a contradiction. Consequently, $k = r$ and f is identically zero, as required. $\square$

Corollary 7.11. *Let $\mathcal{S}_1 \subset \mathbb{R}^m$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be semialgebraic sets. Assume that there exists a surjective Nash map $f : \mathcal{S}_1 \to \mathcal{S}_2$ and that $\mathcal{S}_1$ is well-welded. Then $\mathcal{S}_2$ is well-welded.*

Proof. Let $y_1, y_2 \in \mathcal{S}_2$ and $x_1, x_2 \in \mathcal{S}_1$ be such that $f(x_i) = y_i$. Let Y be the Zariski closure of $f^{-1}(\mathrm{Sing}(\mathcal{S}_2))$, which has dimension $< \dim(\mathcal{S}_1)$ because $\mathcal{S}_1$ is irreducible. By Corollary 7.9 there exists a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}_1$ such that $\alpha(0) = x_1$, $\alpha(1) = x_2$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathcal{S}_1) \setminus Y$. The (continuous) semialgebraic map $\beta := f \circ \alpha$ satisfies $\beta(0) = y_1$, $\beta(1) = y_2$ and $\eta(\beta) \subset f(f^{-1}(\mathcal{S}_2 \setminus \mathrm{Sing}(\mathcal{S}_2))) = \mathrm{Reg}(\mathcal{S}_2)$. Consequently, $\mathcal{S}_2$ is well-welded. $\square$

Example 7.12. There exist pure dimensional, irreducible semialgebraic sets that are not well-welded. Let

$$\mathcal{S} := \{(4x^2 - y^2)(4y^2 - x^2) > 0, y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2,$$

which is a pure dimensional irreducible semialgebraic set, see Fig. 11. Let us check that it is not well-welded.

Observe first that $\mathrm{Sing}(\mathcal{S}) = \{(0,0)\}$. Pick the points $x := (1,1), y := (-1,1) \in \mathcal{S}$ and assume that $\mathcal{S}$ is well-welded. There exists a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $\eta(\alpha) \subset \mathcal{S} \setminus \{(0,0)\}$. Consider the open semialgebraic sets $\mathcal{A}_1 := \mathrm{Reg}(\mathcal{S}) \cap \{x > 0\}$ and $\mathcal{A}_2 := \mathrm{Reg}(\mathcal{S}) \cap \{x < 0\}$, which satisfy $\mathcal{S} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{(0,0)\}$. Let $t_0 := \inf(\alpha^{-1}(\mathcal{A}_2)) > 0$. Note that $\alpha(t_0) = (0,0)$ and there exists $\varepsilon > 0$ such that

(i) $\alpha((t_0 - \varepsilon, t_0)) \subset \mathcal{A}_1$, $\alpha((t_0, t_0 + \varepsilon)) \subset \mathcal{A}_2$,
(ii) $\alpha'(t) \neq 0$ and $\alpha(t) \neq \alpha(t_0)$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \setminus \{t_0\}$.

The tangent line to $\mathrm{Im}(\alpha|_{(t_0-\varepsilon,t_0+\varepsilon)})$ at $\alpha(t_0)$ is the line generated by the vector

$$w = \lim_{t \to t_0} \frac{\alpha(t) - \alpha(t_0)}{(t - t_0)^k}$$

where $2k$ is the order of the series $\|\alpha\|^2$ at t_0. Note that

$$w = \lim_{t \to t_0^+} \frac{\alpha(t) - \alpha(t_0)}{(t - t_0)^k} \in \mathrm{Cl}(\mathcal{A}_1) \setminus \{(0,0)\},$$

$$w = \lim_{t \to t_0^-} \frac{\alpha(t) - \alpha(t_0)}{(t - t_0)^k} \in \mathrm{Cl}(\mathcal{A}_2 \cup -\mathcal{A}_2) \setminus \{(0,0)\},$$

which is a contradiction because $\mathrm{Cl}(\mathcal{A}_1) \cap \mathrm{Cl}(\mathcal{A}_2 \cup -\mathcal{A}_2) = \{(0,0)\}$. Thus, $\mathcal{S}$ is not well-welded. $\square$

7.D. Alternative description of well-welded semialgebraic sets

We describe next well-welded semialgebraic sets using piecewise analytic paths instead of continuous semialgebraic paths. We say that a continuous path $\alpha : [0,1] \to \mathbb{R}^n$ is *piecewise analytic* if there exists a finite set $A' \subset (0,1)$ such that $\alpha|_{(0,1)\setminus A'}$ is an analytic map. Let A be the smallest set with the previous property and define $\eta(\alpha) = \alpha(A)$.

Lemma 7.13. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set. Then the following assertions are equivalent:*

(i) *$\mathcal{S}$ is well-welded.*

(ii) *For each pair of points $x, y \in \mathcal{S}$ there exists a piecewise analytic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $\eta(\alpha) \subset \mathrm{Reg}(\mathcal{S})$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is immediate. For the converse, we proceed as follows.

Let $M_1, \ldots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. An *analytic bridge* between M_i and M_j is the image Λ of an analytic arc $\alpha : (-1,1) \to \mathbb{R}^n$ such that $\alpha((-1,0)) \subset M_i$ and $\alpha((0,1)) \subset M_j$. Proceeding as in the proof of Proposition 7.8 we can reorder the indices $i = 1, \ldots, r$ in such a way that there exist analytic bridges $\Lambda_i \subset \mathcal{S}$ between M_i and $\bigsqcup_{j=1}^{i-1} M_j$ for $i = 2, \ldots, r$. By Lemma 2.9 we can substitute the analytic bridges $\Lambda_i \subset \mathcal{S}$ by (Nash) bridges $\Gamma_i \subset \mathcal{S}$ between M_i and $\bigsqcup_{j=1}^{i-1} M_j$ for $i = 2, \ldots, r$. By Corollary 7.6 $\mathcal{S}$ is well-welded. $\square$

The following two results are the counterpart of Lemmas 7.1 and 7.2 for analytic paths. As the proofs are pretty similar to those of Lemmas 7.1 and 7.2, we leave the details to the reader.

Lemma 7.14. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set that is connected by analytic paths. Then $\mathcal{S}$ is well-welded.*

Lemma 7.15. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional semialgebraic set. Assume that there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of $\mathrm{Reg}(\mathcal{S})$. Then $\mathcal{S}$ is well-welded.*

7.E. Strict transforms of well-welded semialgebraic sets

We prove next that as it happens with irreducible arc-analytic sets [28, Thm.2.6] the strict transform of a well-welded semialgebraic set under a sequence of blow-ups is a well-welded semialgebraic set of its same dimension.

Lemma 7.16. *Let $X \subset \mathbb{R}^m$ and $Z \subset Y \subset \mathbb{R}^n$ be algebraic sets. Let $f : X \to Y$ be a proper regular map such that the restriction $f|_{X \setminus f^{-1}(Z)} : X \setminus f^{-1}(Z) \to Y \setminus Z$ is bijective. Let $\mathcal{S} \subset Y$ be a well-welded semialgebraic set of dimension d such that $\mathcal{S} \not\subset Z$. Then the strict transform $\mathcal{S}_1 := \mathrm{Cl}(f^{-1}(\mathcal{S} \setminus Z)) \cap f^{-1}(\mathcal{S})$ of $\mathcal{S}$ under f is a well-welded semialgebraic set of dimension d.*

Proof. The proof is conducted in several steps:

7.E.1.   We may assume $Y = \overline{\mathcal{S}}^{\mathrm{zar}}$ and $X = \overline{X \setminus f^{-1}(Z)}^{\mathrm{zar}}$.
Let $Y' := \overline{\mathcal{S}}^{\mathrm{zar}}$, $Z' = \overline{\mathcal{S}}^{\mathrm{zar}} \cap Z$ and $X' = \overline{f^{-1}(Y' \setminus Z')}^{\mathrm{zar}}$. Consider the proper regular map $f' := f|_{X'} : X' \to Y'$. The restriction $f'|_{X' \setminus f'^{-1}(Z')} : X' \setminus f'^{-1}(Z') \to Y' \setminus Z'$ is bijective because $X' \setminus f'^{-1}(Z') = f^{-1}(Y' \setminus Z')$.

7.E.2.   We claim: $\mathcal{S}_1$ is pure dimensional. As the restriction $f|_{X \setminus Z} : X \setminus f^{-1}(Z) \to Y \setminus Z$ is proper and bijective, it is a semialgebraic homeomorphism. As $\mathcal{S} \setminus Z$ is pure dimensional of dimension d, also $f^{-1}(\mathcal{S} \setminus Z)$ is pure dimensional of dimension d. As $f^{-1}(\mathcal{S} \setminus Z) \subset \mathcal{S}_1 \subset \mathrm{Cl}(f^{-1}(\mathcal{S} \setminus Z))$, we conclude that $\mathcal{S}_1$ is pure dimensional of dimension d as well.

7.E.3.   Observe that $\dim(X) = \dim(Y) = d$ and $\dim(Z) < d$. The algebraic set

$$Z_1 := \overline{f(\mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{S}_1)) \cup \mathrm{Sing}(Y)}^{\mathrm{zar}} \cup Z$$

has by [4, Thm.2.8.8] dimension $< d$. We claim: $f^{-1}(Z_1)$ has dimension $< d$.
As the restriction $f|_{X \setminus Z} : X \setminus f^{-1}(Z) \to Y \setminus Z$ is bijective, $\dim(f^{-1}(Z_1 \setminus Z)) = \dim(Z_1 \setminus Z) < d$ by [4, Thm.2.8.8]. If $f^{-1}(Z)$ has dimension d, it contains an irreducible component of X, which is a contradiction because $X = \overline{X \setminus f^{-1}(Z)}^{\mathrm{zar}}$. Thus, $\dim(f^{-1}(Z)) < d$, so $\dim(f^{-1}(Z_1)) < d$.

7.E.4. The restriction $f|_{X \setminus f^{-1}(Z_1)} : X \setminus f^{-1}(Z_1) \to Y \setminus Z_1$ is a bijective proper regular map between the Nash manifolds $X \setminus f^{-1}(Z_1)$ and $Y \setminus Z_1$. Let C be the set of critical points of $f|_{X \setminus f^{-1}(Z_1)}$, which is by [4, Thm.9.6.2 & Lem.9.6.3] a semialgebraic set of dimension $< d$. Let $Z_2 := Z_1 \cup \overline{f(C)}^{\text{zar}}$, which is an algebraic set of dimension $< d$. Again $f^{-1}(Z_2) = f^{-1}(Z_2 \setminus Z) \cup f^{-1}(Z)$ has dimension $< d$. Consequently, the restriction map $f|_{X \setminus f^{-1}(Z_2)} : X \setminus f^{-1}(Z_2) \to Y \setminus Z_2$ is a Nash diffeomorphism between the Nash manifolds $X \setminus f^{-1}(Z_2)$ and $Y \setminus Z_2$.

7.E.5. As $\mathcal{S}_1$ has dimension d, $\mathrm{Sing}(\mathcal{S}_1)$ has dimension $< d$ and $\overline{f(\mathrm{Sing}(\mathcal{S}_1))}^{\text{zar}}$ has dimension $< d$. Define $Z_3 := \overline{f(\mathrm{Sing}(\mathcal{S}_1))}^{\text{zar}} \cup Z_2$ and observe that $f^{-1}(Z_3) = f^{-1}(Z_3 \setminus Z) \cup f^{-1}(Z)$ is an algebraic set of dimension $< d$.

7.E.6. Let $M_1, \ldots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S})$. By Proposition 7.8 and Corollary 7.9, we may reorder the indices $1, \ldots, r$ in such a way that for each $j = 2, \ldots, r$ there exists a bridge $\Gamma_j \subset \mathcal{S}$ between $M_j \setminus Z_3$ and $\bigsqcup_{k=1}^{j-1}(M_k \setminus Z_3)$. We have $\Gamma_j \setminus \mathrm{Reg}(\mathcal{S}) = \{p_j\}$ for $j = 2, \ldots, r$. Shrinking each bridge Γ_j, we may assume that $\Gamma_j \cap Z_3 \subset \{p_j\}$ for $j = 2, \ldots, r$.

7.E.7. Let $C_{i1}, \ldots, C_{is_i}$ be the connected components of $M_i \setminus Z_3$. As $\Gamma_i \cap Z_3 \subset \{p_i\}$, we may assume that Γ_i is a bridge between C_{i1} and $\bigsqcup_{k=1}^{i-1} \bigsqcup_{j=1}^{s_k} C_{kj}$. As M_i is a connected Nash manifold, we may construct using Lemmas 2.9 and 6.2 bridges $\Gamma_{ij} \subset M_i$ between C_{ij} and $\bigsqcup_{k=1}^{j-1} C_{ik}$.

7.E.8. Denote the connected components of $\mathrm{Reg}(\mathcal{S}) \setminus Z_3$ with $N_1, \ldots, N_s$. By 7.E.6 and 7.E.7 we may assume that there exist bridges $\Lambda_j \subset \mathcal{S}$ between N_j and $\bigsqcup_{k=1}^{j-1} N_k$ such that the intersection $\Lambda_j \cap Z_3$ is either the empty-set or a singleton for $j = 2, \ldots, s$.

Denote $\Lambda_j' := \mathrm{Cl}(f^{-1}(\Lambda_j \setminus Z)) \cap f^{-1}(\Lambda_j) \subset \mathcal{S}_1$. By Lemma B.2 $\Lambda_j' \cap f^{-1}(Z)$ is either the empty-set or a singleton $\{z_j\}$ and the curve germ Λ_{j,z_j}' is irreducible. Denote $N_j' := f^{-1}(N_j)$. As $\mathcal{S} \setminus Z_3$ is Nash diffeomorphic to $f^{-1}(\mathcal{S} \setminus Z_3)$, it holds that $N_1', \ldots, N_s'$ are the connected components of the Nash manifold $f^{-1}(\mathcal{S} \setminus Z_3)$. Shrinking Λ_j', we may assume that it is a bridge between N_j' and $\bigsqcup_{i=1}^{j-1} N_i'$ for $j = 2, \ldots, s$.

7.E.9. As $\mathrm{Sing}(\mathcal{S}_1) \subset f^{-1}(Z_3)$, we have $\mathcal{S}_1 \setminus f^{-1}(Z_3) \subset \mathrm{Reg}(\mathcal{S}_1)$. Note that

$$\mathcal{S}_1 \setminus f^{-1}(Z_3) = f^{-1}(\mathcal{S} \setminus Z_3) = \bigsqcup_{j=1}^{s} N_j'$$

because $Z \subset Z_3$, $f^{-1}(\mathcal{S} \setminus Z) \subset \mathcal{S}_1 \subset f^{-1}(\mathcal{S})$ and

$$f^{-1}(\mathcal{S} \setminus Z_3) = f^{-1}(\mathcal{S} \setminus Z) \setminus f^{-1}(Z_3) \subset \mathcal{S}_1 \setminus f^{-1}(Z_3) \subset f^{-1}(\mathcal{S} \setminus Z_3).$$

Let $M'_1, \ldots, M'_\ell$ be the connected components of $\mathrm{Reg}(\mathbb{S}_1)$. As $\mathrm{Sing}(\mathbb{S}_1) \subset f^{-1}(Z_3)$,

$$\bigsqcup_{j=1}^{s} N'_j = \mathbb{S}_1 \setminus f^{-1}(Z_3) \subset \mathrm{Reg}(\mathbb{S}_1) = \bigsqcup_{k=1}^{\ell} M'_k,$$

so each N'_j is contained in some M'_k. As $\dim(f^{-1}(Z_3)) < d$, we have $\mathrm{Reg}(\mathbb{S}_1) \subset \mathrm{Cl}(\bigsqcup_{j=1}^{s} N'_j)$. Thus, for each $k = 1, \ldots, \ell$ there exists $1 \leq j \leq s$ such that $N'_j \subset M'_k$. Define

$$j(k) := \min\{j = 1, \ldots, s : \ N'_j \subset M'_k\}$$

and note that $j(k_1) \neq j(k_2)$ if $k_1 \neq k_2$. We may assume $j(k_1) < j(k_2)$ if $k_1 < k_2$. Observe that $\Lambda'_{j(k)} \subset \mathbb{S}_1$ is a bridge between M'_k and $\bigsqcup_{i=1}^{k-1} M'_i$. We conclude by Corollary 7.6 that $\mathbb{S}_1$ is well-welded, as required. $\quad\square$

Corollary 7.17. *Let $Y \subset X \subset \mathbb{R}^n$ be algebraic sets and let $\mathbb{S} \subset X$ be a well-welded semialgebraic set of dimension d such that $\mathbb{S} \not\subset Y$. Let $(B(X,Y), \pi)$ be the blow-up of X with center Y and let $\mathbb{S}_1 := \mathrm{Cl}(\pi^{-1}(\mathbb{S} \setminus Y)) \cap \pi^{-1}(\mathbb{S})$ be the strict transform of $\mathbb{S}$ under π. Then $\mathbb{S}_1$ is a well-welded semialgebraic set of dimension d.*

8. Well-welded semialgebraic sets as Nash images of Euclidean spaces

In this section we prove Theorem 1.4. The most difficult part, implication (vii) $\Longrightarrow$ (i), is approached in two steps. We prove first that each well-welded semialgebraic set is the image under a Nash map of a 'checkerboard set' of its same dimension. Afterwards we show that a checkerboard set of dimension d is the image under a Nash map of its set of regular points, which is a connected Nash manifold of dimension d and consequently a Nash image of $\mathbb{R}^d$ by Theorem 6.1. We define the boundary of a semialgebraic set $\mathbb{S} \subset \mathbb{R}^n$ as $\partial \mathbb{S} := \mathrm{Cl}(\mathbb{S}) \setminus \mathrm{Reg}(\mathbb{S})$. It holds $\mathrm{Sing}(\mathbb{S}) \subset \partial \mathbb{S}$.

8.A. Checkerboard sets

A pure dimensional semialgebraic set $\mathbb{S} \subset \mathbb{R}^n$ is a *checkerboard set* (see Fig. 12) if it satisfies the following properties:

- $\overline{\mathbb{S}}^{\mathrm{zar}}$ is a non-singular real algebraic set.
- $\overline{\partial \mathbb{S}}^{\mathrm{zar}}$ is a normal-crossings divisor of $\overline{\mathbb{S}}^{\mathrm{zar}}$.
- $\mathrm{Reg}(\mathbb{S})$ is connected.

Remark 8.1. The difference $\mathbb{S} \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}}$ is a union of connected components of $\overline{\mathbb{S}}^{\mathrm{zar}} \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}}$.

Observe that $\mathbb{S} \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}} = \mathrm{Cl}(\mathbb{S}) \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}} = \mathrm{Reg}(\mathbb{S}) \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}}$. Consequently, $\mathbb{S} \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}}$ is an open and closed subset of $\overline{\mathbb{S}}^{\mathrm{zar}} \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}}$, so it is a union of connected components of $\overline{\mathbb{S}}^{\mathrm{zar}} \setminus \overline{\partial \mathbb{S}}^{\mathrm{zar}}$.
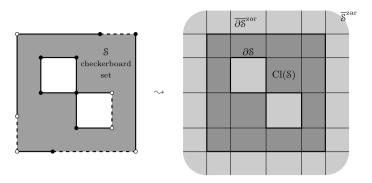
Fig. 12. Checkerboard set $\mathcal{S}$, its closure $\mathrm{Cl}(\mathcal{S})$, its Zariski closure $\overline{\mathcal{S}}^{\mathrm{zar}}$, $\partial\mathcal{S}$ and the Zariski closure of $\partial\mathcal{S}$.

Each checkerboard set is a well-welded semialgebraic set by the following result.

Lemma 8.2. *Let $X \subset \mathbb{R}^n$ be a non-singular algebraic set and let $Y \subset X$ be a normal-crossings divisor. Let $\mathcal{C} \subset X \setminus Y$ be a union of connected components of $X \setminus Y$ and let $\mathcal{S} \subset X$ be a semialgebraic set such that $\mathcal{C} \subset \mathcal{S} \subset \mathrm{Cl}(\mathcal{C})$. Then $\mathcal{S}$ is well-welded if and only if $\mathcal{S}$ is connected.*

Proof. The 'only if' implication is straightforward. We prove next the 'if' implication.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the connected components of $\mathcal{C}$. We claim: *there exist points $p_2, \ldots, p_r \in Y \cap \mathcal{S}$ such that, after reordering indices, $p_i \in \mathcal{S} \cap \mathrm{Cl}(\mathcal{C}_i) \cap \bigcup_{j=1}^{i-1} \mathrm{Cl}(\mathcal{C}_j)$ for $i = 2, \ldots, r$.*

We proceed by induction on r. If $r = 1$, the claim is clear. Assume the result true if the number of connected components of $\mathcal{C}$ is $< r$ and let us see that it is also true when it is equal to r. As $\mathcal{S}$ is connected, there exists a continuous semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components $\mathcal{C}_i$. We may reorder the indices $1, \ldots, r$ in such a way that if $i < j$, then

$$t_i := \inf\{t \in (0,1): \ \alpha(t) \in \mathrm{Cl}(\mathcal{C}_i) \cap \mathcal{S}\} \leq \inf\{t \in (0,1): \ \alpha(t) \in \mathrm{Cl}(\mathcal{C}_j) \cap \mathcal{S}\} =: t_j.$$

Consider the semialgebraic set $\mathcal{S}' := \mathrm{Cl}(\mathcal{S} \setminus \mathrm{Cl}(\mathcal{C}_r)) \cap \mathcal{S}$. It holds: $\mathcal{S}'$ *is connected.*

As $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are the connected components of $\mathcal{C}$, we have $\mathrm{Cl}(\mathcal{C}_i) = \mathrm{Cl}(\mathrm{Cl}(\mathcal{C}_i) \setminus \mathrm{Cl}(\mathcal{C}_r))$ for $i \neq r$. Consequently,

$$\mathcal{S}' = \mathrm{Cl}(\mathcal{S} \setminus \mathrm{Cl}(\mathcal{C}_r)) \cap \mathcal{S} = \bigcup_{i=1}^{r-1} \mathrm{Cl}(\mathrm{Cl}(\mathcal{C}_i) \setminus \mathrm{Cl}(\mathcal{C}_r)) \cap \mathcal{S} = \bigcup_{i=1}^{r-1} \mathrm{Cl}(\mathcal{C}_i) \cap \mathcal{S} \qquad (8.1)$$

Consider the connected semialgebraic set $\mathcal{T} := \alpha([0, t_r])$. Observe that $\mathcal{T} \subset \mathcal{S}'$ and $\mathcal{T} \cap \mathrm{Cl}(\mathcal{C}_i) \neq \varnothing$ for $i = 1, \ldots, r-1$. Consequently, $\mathcal{S}'$ is connected.

As $\mathcal{S}' \setminus Y = \bigsqcup_{i=1}^{r-1} \mathcal{C}_i$, there exist by induction hypothesis points $p_2, \ldots, p_{r-1} \in Y \cap \mathcal{S}'$ such that, after reordering indices, $p_i \in \mathcal{S} \cap \mathrm{Cl}(\mathcal{C}_i) \cap \bigcup_{j=1}^{i-1} \mathrm{Cl}(\mathcal{C}_j)$ for $i = 2, \ldots, r-1$. As

$\mathcal{S} = \mathcal{S}' \cup (\mathrm{Cl}(\mathcal{C}_r) \cap \mathcal{S})$ is connected and $\mathcal{S}'$ and $\mathrm{Cl}(\mathcal{C}_r) \cap \mathcal{S}$ are closed in $\mathcal{S}$, there exists $p_r \in \mathcal{S}' \cap \mathrm{Cl}(\mathcal{C}_r)$. By (8.1) there exists $1 \leq i \leq r-1$ such that $p_r \in \mathrm{Cl}(\mathcal{C}_i) \cap \mathrm{Cl}(\mathcal{C}_r) \cap \mathcal{S}$. Notice that $\mathrm{Cl}(\mathcal{C}_i) \cap \mathrm{Cl}(\mathcal{C}_r) \subset Y$ because $\mathcal{C}$ is a union of connected components of $X \setminus Y$. Thus, the claim follows.

Next we prove: $\mathcal{S}_0 := \mathcal{C} \cup \{p_2, \ldots, p_r\}$ is a well-welded semialgebraic set.

Fix $i = 2, \ldots, r$ and let $1 \leq j \leq i-1$ be such that $p_i \in \mathrm{Cl}(\mathcal{C}_j) \cap \mathrm{Cl}(\mathcal{C}_i) \subset Y$. As Y is a normal-crossings divisor of X, there exists an open semialgebraic neighborhood $U \subset X$ of p_i and a Nash diffeomorphism $\psi : U \to \mathbb{R}^d$ such that $\psi(0) = p_i$ and $\psi(U \cap Y) = \{x_1 \cdots x_s = 0\}$. We may assume

$$\{x_1 > 0, \ldots, x_s > 0\} \subset \psi(\mathcal{C}_i \cap U) \quad \text{and} \quad \{\varepsilon_1 x_1 > 0, \ldots, \varepsilon_s x_s > 0\} \subset \psi(\mathcal{C}_j \cap U)$$

where $\varepsilon_k = \pm 1$ for $k = 1, \ldots, s$. Consider the Nash curve $\beta := (\beta_1, \ldots, \beta_d) : (-1, 1) \to \mathbb{R}^d$ where

$$\beta_k(t) = \begin{cases} t & \text{if } \varepsilon_k = -1 \\ t^2 & \text{if } \varepsilon_k = 1 \end{cases}$$

for $k = 1, \ldots, s$ and $\beta_k(t) = 0$ for $k = s+1, \ldots, d$. Observe that

$$\beta((-1, 0)) \subset \{\varepsilon_1 x_1 > 0, \ldots, \varepsilon_s x_s > 0\} \quad \text{and} \quad \beta((0, 1)) \subset \{x_1 > 0, \ldots, x_s > 0\}.$$

Consider the Nash curve $\gamma := \psi^{-1} \circ \beta : (-1, 1) \to \mathcal{C}_i \cup \mathcal{C}_j \cup \{p_i\}$ that satisfies $\gamma((-1, 0)) \subset \mathcal{C}_j$ and $\gamma((0, 1)) \subset \mathcal{C}_i$. Thus, $\Gamma_i := \gamma((-1, 1)) \subset \mathcal{S}$ is a bridge between $\mathcal{C}_j$ and $\mathcal{C}_i$. By Corollary 7.6 $\mathcal{S}_0$ is well-welded.

As $\mathcal{S}_0 \subset \mathcal{S} \subset \mathrm{Cl}(\mathcal{S}_0)$, we conclude by Lemma 7.4 that $\mathcal{S}$ is well-welded, as required. $\quad\square$

The following result will allow us to lighten the presentation of the proof of implication (vii) $\Longrightarrow$ (i) of Theorem 1.4.

Lemma 8.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional semialgebraic set of dimension d. Suppose that $\mathrm{Sth}(\mathcal{S})$ is connected, there exists a Nash manifold M of dimension d that contains $\mathcal{S}$ and the smallest Nash subset Y of M that contains $(\mathrm{Cl}(\mathcal{S}) \cap M) \setminus \mathrm{Sth}(\mathcal{S})$ is a Nash normal crossings divisor of M. Then there exists a Nash embedding $\varphi : M \hookrightarrow \mathbb{R}^m$ for some $m \geq 1$ such that $\varphi(\mathcal{S})$ is a checkerboard subset of $\mathbb{R}^m$, $\mathrm{Reg}(\varphi(\mathcal{S})) = \varphi(\mathrm{Sth}(\mathcal{S}))$ and $\mathrm{Sing}(\varphi(\mathcal{S})) = \varphi(\mathrm{NSth}(\mathcal{S}))$.*

Proof. As $\mathcal{S}$ is connected, we may assume that M is connected. By Lemma C.1 there exists a Nash embedding $\varphi : M \hookrightarrow \mathbb{R}^m$ such that

(i) $\varphi(M)$ is a connected component of its Zariski closure V in $\mathbb{R}^m$ that is a non-singular real algebraic set of dimension d.

(ii) The Zariski closure X of $\varphi(Y)$ in $\mathbb{R}^m$ is a normal crossing divisor of V such that $\varphi(M) \cap X = \varphi(Y)$.

Observe that $\overline{\varphi(S)}^{\mathrm{zar}} = \overline{\varphi(M)}^{\mathrm{zar}} = V$. By Remark 2.2 and Lemma 2.3 we deduce $\varphi(\mathtt{Sth}(S)) = \mathtt{Sth}(\varphi(S)) = \mathrm{Reg}(\varphi(S))$ and $\partial\varphi(S) = \mathrm{Cl}(\varphi(S)) \setminus \mathrm{Reg}(\varphi(S)) = \varphi((\mathrm{Cl}(S) \cap M) \setminus \mathtt{Sth}(S))$. Thus, $\varphi(Y)$ is the smallest Nash subset of $\varphi(M)$ that contains $\partial\varphi(S)$, so $\overline{\partial\varphi(S)}^{\mathrm{zar}}$ is X, which is a normal crossing divisor of $V = \overline{\varphi(S)}^{\mathrm{zar}}$. Consequently, $\varphi(S)$ is a checkerboard subset of $\mathbb{R}^m$, $\mathrm{Reg}(\varphi(S)) = \varphi(\mathtt{Sth}(S))$ and $\mathrm{Sing}(\varphi(S)) = \varphi(\mathtt{NSth}(S))$, as required. $\square$

8.B. Well-welded semialgebraic sets as Nash images of checkerboard sets

Our purpose next is to prove that a well-welded semialgebraic set is the image under a proper surjective regular map of a checkerboard set of its same dimension.

Theorem 8.4. *Let $S \subset \mathbb{R}^n$ be a well-welded semialgebraic set of dimension d and denote $X := \overline{S}^{\mathrm{zar}}$. Then there exist a checkerboard set $T \subset \mathbb{R}^m$ of dimension d and a proper regular map $f : Y := \overline{T}^{\mathrm{zar}} \to X$ such that $f(T) = S$.*

Remark 8.5. *As f is proper, if S is in addition bounded, then T is also bounded.*

The proof of Theorem 8.4 is quite involved and requires some preliminary work.

Proposition 8.6. *Let $X \subset \mathbb{R}^n$ be a non-singular algebraic set of dimension d and let $Z \subset X$ be a normal-crossings divisor. Let $S \subset X$ be a connected semialgebraic set such that $\mathcal{C} := S \setminus Z$ is a union of connected components of $X \setminus Z$ and $\mathcal{C} \subset S \subset \mathrm{Cl}(\mathcal{C})$. Then there exist a checkerboard set $T \subset \mathbb{R}^m$ of dimension d and a proper surjective regular map $f : Y := \overline{T}^{\mathrm{zar}} \to X$ such that the restriction $f|_{Y \setminus f^{-1}(Z)} : Y \setminus f^{-1}(Z) \to X \setminus Z$ is a regular diffeomorphism and $f(T) = S$.*

Proof of Proposition 8.6 when S is closed. Let $M_1, \ldots, M_s$ be the connected components of the Nash manifold $\mathrm{Reg}(S)$ and define $\mathcal{R} := \bigcup_{i \neq j} \mathrm{Cl}(M_i) \cap \mathrm{Cl}(M_j)$. Notice that

$$\mathcal{R} = \{x \in S : \ \mathrm{Reg}(S)_x \not\subset M_{i,x} \ \forall i = 1, \ldots, s\} \subset \{x \in S : \ \mathrm{Reg}(S)_x \text{ is not connected}\}.$$

The irreducible components of Z are non-singular. Denote them with $Z_1, \ldots, Z_r$.

8.B.1. We claim: *each irreducible component of $A := \overline{\mathcal{R}}^{\mathrm{zar}}$ is an irreducible component of some intersection $Z_{i_1} \cap \cdots \cap Z_{i_\ell}$.*

As $\mathcal{R} \subset Z$, also $A \subset Z$. Let A_1 be an irreducible component of A and let $\mathcal{R}_1 := \mathcal{R} \cap A_1$. Let $A_2, \ldots, A_r$ be the remaining irreducible components of A. Observe that

$$\mathcal{R}_1 \setminus \bigcup_{j=2}^{r} A_j = \mathcal{R} \setminus \bigcup_{j=2}^{r} A_j \neq \varnothing \quad \text{and} \quad \dim(A_1) = \dim\left(\mathcal{R}_1 \setminus \bigcup_{j=2}^{r} A_j\right) = \dim(\mathcal{R}_1)$$

because A is the Zariski closure of $\mathcal{R}$. In addition, we have $A_1 \cap \operatorname{Reg}(A) = \operatorname{Reg}(A_1) \cap \operatorname{Reg}(A)$ and $\mathcal{R}_1 \cap \operatorname{Reg}(\mathcal{R}) = A_1 \cap \operatorname{Reg}(\mathcal{R}) = \operatorname{Reg}(\mathcal{R}_1) \cap \operatorname{Reg}(\mathcal{R})$.

Assume that $A_1 \subset Z_k$ exactly for $k = 1, \ldots, \ell$. Note that $\ell \leq d$. Pick a point

$$x \in \mathcal{R}_1 \cap \operatorname{Reg}(\mathcal{R}) \setminus \bigcup_{j=\ell+1}^{r} Z_j \subset \operatorname{Reg}(A).$$

As Z is a normal-crossings divisor of X, the intersection $Z_1 \cap \cdots \cap Z_\ell$ is a non-singular algebraic set and there exists an open semialgebraic neighborhood $U \subset X \setminus \bigcup_{j=\ell+1}^{r} Z_j$ of x equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(x) = 0$ and $u(Z \cap U) = \{x_1 \cdots x_\ell = 0\}$. We may assume in addition:

- $\mathcal{R} \cap U = A \cap U = A_1 \cap U = \mathcal{R}_1 \cap U$ is a connected closed submanifold of U.
- $u(Z_i \cap U) = \{x_i = 0\}$ for $i = 1, \ldots, \ell$.
- $\mathcal{S}_x \cap M_{1,x} \neq \varnothing$ and $\mathcal{S}_x \cap M_{2,x} \neq \varnothing$.

As $\mathcal{C}$ is a union of connected components of $X \setminus Z$, we have that $u(\mathcal{C} \cap U)$ is a union of sets of the type $\{\varepsilon_1 x_1 > 0, \ldots, \varepsilon_\ell x_\ell > 0\}$ where $\varepsilon_i = \pm 1$. Consider the projection $\pi : \mathbb{R}^\ell \times \mathbb{R}^{d-\ell} \to \mathbb{R}^\ell$, $(x, y) \mapsto x$ and observe that $u(\mathcal{C} \cap U) = \pi(u(\mathcal{C} \cap U)) \times \mathbb{R}^{d-\ell}$. Consequently, $\mathcal{S} = \operatorname{Cl}(\mathcal{C})$ satisfies $u(\mathcal{S} \cap U) = \pi(u(\mathcal{S} \cap U)) \times \mathbb{R}^{d-\ell}$. As $\operatorname{Reg}(\mathcal{S})_x$ is not connected, $\operatorname{Reg}(\pi(u(\mathcal{S} \cap U)))_0$ is not connected. Consequently, for each $y \in (Z_1 \cap \cdots \cap Z_\ell) \cap U$ the germ $\operatorname{Reg}(\mathcal{S})_y$ is not connected. As $\mathcal{S}_x \cap M_{1,x} \neq \varnothing$ and $\mathcal{S}_x \cap M_{2,x} \neq \varnothing$, we deduce $\mathcal{S}_y \cap M_{1,y} \neq \varnothing$ and $\mathcal{S}_y \cap M_{2,y} \neq \varnothing$ for each $y \in (Z_1 \cap \cdots \cap Z_\ell) \cap U$. Thus,

$$A_1 \cap U \subset (Z_1 \cap \cdots \cap Z_\ell) \cap U \subset \mathcal{R} \cap U = A_1 \cap U.$$

As $Z_1 \cap \cdots \cap Z_\ell$ is pure dimensional, we conclude that $\dim(A_1) = \dim(Z_1 \cap \cdots \cap Z_\ell)$. As A_1 is irreducible, it is an irreducible component of $Z_1 \cap \cdots \cap Z_\ell$.

8.B.2. Next, we prove: $\dim(\mathcal{R}) \leq \dim(X) - 2$. Assume by contradiction that $\dim(\mathcal{R}) = \dim(X) - 1$. Let $x \in \operatorname{Reg}(Z) \cap \operatorname{Reg}(\mathcal{R})$. There exists an open semialgebraic neighborhood $U \subset X$ of x such that $U \cap Z$ is Nash diffeomorphic to $\{x_1 = 0\}$, so $\mathcal{C}_x$ is either Nash equivalent to $\{x_1 > 0\}$ or to $\{x_1 > 0\} \cup \{x_1 < 0\}$. Consequently, $\mathcal{S}_x$ is either Nash equivalent to $\{x_1 \geq 0\}$ or to $\mathbb{R}^d$. But in both cases $\operatorname{Reg}(\mathcal{S})_x$ is connected, a contradiction.

8.B.3. Let A_1 be an irreducible component of A and let (Y_1, f_1) be the blow-up of X with center A_1. Denote $\mathcal{T}_1 := \operatorname{Cl}(f_1^{-1}(\mathcal{S} \setminus A_1))$ and observe that $f_1(\mathcal{T}_1) = \mathcal{S}$ because f is proper and surjective and $\mathcal{S} \setminus A_1$ is dense in $\mathcal{S}$ because $\mathcal{S}$ is pure dimensional. Let us prove:

- $f_1^{-1}(Z)$ is a normal-crossings divisor of Y_1.
- $\mathcal{T}_1 \setminus f_1^{-1}(Z)$ is a union of connected components of $Y_1 \setminus f_1^{-1}(Z)$.
- $\mathcal{T}_1$ is connected.
- $\mathrm{Reg}(\mathcal{T}_1)$ has at most $s-1$ connected components.

Assume that $A_1 \subset Z_k$ exactly for $k = 1, \ldots, \ell$. As A_1 is an irreducible component of $Z_1 \cap \cdots \cap Z_\ell$, the inverse image $f_1^{-1}(Z)$ is a normal-crossings divisor of Y_1. As $X \setminus A_1$ and $Y_1 \setminus f_1^{-1}(A_1)$ are Nash diffeomorphic, also $X \setminus Z$ and $Y_1 \setminus f_1^{-1}(Z)$ are Nash diffeomorphic. As $\mathcal{C}$ is a union of connected components of $X \setminus Z$, the inverse image $f_1^{-1}(\mathcal{C})$ is a union of connected components of $Y_1 \setminus f_1^{-1}(Z)$. Consequently, $f_1^{-1}(\mathcal{C})$ is closed in $Y_1 \setminus f_1^{-1}(Z)$ and $f_1^{-1}(\mathcal{C}) = \mathrm{Cl}(f_1^{-1}(\mathcal{C})) \setminus f_1^{-1}(Z)$. Let us check: $\mathcal{T}_1 = \mathrm{Cl}(f_1^{-1}(\mathcal{C}))$.

We have

$$\mathcal{T}_1 = \mathrm{Cl}(f_1^{-1}(\mathcal{S} \setminus A_1)) = \mathrm{Cl}(f_1^{-1}(\mathcal{C})) \cup \mathrm{Cl}(f_1^{-1}((\mathcal{S} \setminus \mathcal{C}) \setminus A_1)).$$

As $\mathrm{Cl}(f_1^{-1}((\mathcal{S} \setminus \mathcal{C}) \setminus A_1)) \subset f_1^{-1}(Z)$, it holds

$$\mathcal{T}_1 \setminus f_1^{-1}(Z) = \mathrm{Cl}(f_1^{-1}(\mathcal{C})) \setminus f_1^{-1}(Z) = f_1^{-1}(\mathcal{C}).$$

As $f_1^{-1}(\mathcal{S} \setminus A_1)$ and $\mathcal{S} \setminus A_1$ are Nash diffeomorphic and $\mathcal{S} \setminus A_1$ is pure dimensional, also $f_1^{-1}(\mathcal{S} \setminus A_1)$ is pure dimensional, so $\mathcal{T}_1 = \mathrm{Cl}(f_1^{-1}(\mathcal{S} \setminus A_1))$ is pure dimensional. As $\dim(f_1^{-1}(Z)) = d - 1 = \dim(\mathcal{T}_1) - 1$, we deduce $\mathcal{T}_1 = \mathrm{Cl}(\mathcal{T}_1 \setminus f_1^{-1}(Z)) = \mathrm{Cl}(f_1^{-1}(\mathcal{C}))$.

By Lemma 8.2 $\mathcal{S}$ is well-welded, so $\mathcal{T}_1$ is by Lemma 7.16 well-welded and therefore connected. It only remains to check that $\mathrm{Reg}(\mathcal{T}_1)$ has at most $s-1$ connected components.

As $\dim(A_1) \leq \dim(M_i) - 2$, the differences $M_i \setminus A_1$ are connected Nash manifolds, so the same happens with the sets $f_1^{-1}(M_i \setminus A_1)$. Observe that

$$\bigcup_{i=1}^{s} f_1^{-1}(M_i \setminus A_1) \subset \mathrm{Reg}(\mathcal{T}_1) \subset \mathcal{T}_1 = \mathrm{Cl}(f_1^{-1}(\mathcal{C}))$$

$$= \mathrm{Cl}(f_1^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus A_1)) = \mathrm{Cl}\left(\bigcup_{i=1}^{s} f_1^{-1}(M_i \setminus A_1)\right) = \bigcup_{i=1}^{s} \mathrm{Cl}(f_1^{-1}(M_i \setminus A_1)), \quad (8.2)$$

so $\mathrm{Reg}(\mathcal{T}_1)$ has at most s connected components. Let us check that in fact it has at most $s-1$. This follows from equality (8.2) if we prove that $\mathrm{Cl}(f_1^{-1}(M_1 \cup M_2) \setminus A_1)$ is connected.

Recall that A_1 is an irreducible component of $Z_1 \cap \cdots \cap Z_\ell$. Pick a point

$$x \in \mathrm{Reg}(\mathcal{R}) \cap A_1 \setminus \bigcup_{j=\ell+1}^{r} Z_j \subset \mathrm{Reg}(A).$$

We may assume $x \in \mathrm{Cl}(M_1) \cap \mathrm{Cl}(M_2)$. Let $U \subset X \setminus \bigcup_{j=s+1}^{r} Z_j$ be an open semialgebraic neighborhood of x such that $\mathcal{R} \cap U = A \cap U = A_1 \cap U$ that is equipped with a Nash

diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(A_1 \cap U) = \{x_1 = 0, \ldots, x_\ell = 0\}$ and $u(Z \cap U) = \{x_1 \cdots x_\ell = 0\} \subset \mathbb{R}^d$. In particular,

$$\mathrm{Cl}(M_1) \cap \mathrm{Cl}(M_2) \cap U \subset \mathcal{R} \cap U = A_1 \cap U.$$

Observe that $\mathcal{C} \cap U$ is a union of sets of the type $\mathcal{Q}_\epsilon := \{\epsilon_1 x_1 > 0, \ldots, \epsilon_\ell x_\ell > 0\}$ where $\epsilon := (\epsilon_1, \ldots, \epsilon_\ell) \in \{-1, +1\}^\ell$, that is, there exists $\mathfrak{F} \subset \{-1, +1\}^\ell$ such that

$$\mathcal{C} \cap U = \bigcup_{\epsilon \in \mathfrak{F}} \mathcal{Q}_\epsilon.$$

Denote $\overline{\mathcal{Q}}_\epsilon := \{\epsilon_1 x_1 \geq 0, \ldots, \epsilon_\ell x_\ell \geq 0\}$. Observe that $\mathrm{Reg}(\mathcal{S}) \cap U$ is not connected because it has at least two connected components $E_1 := u(M_1 \cap U)$ and $E_2 := u(M_2 \cap U)$. Let $\mathcal{Q}_\epsilon \subset E_1$ and $\mathcal{Q}_{\epsilon'} \subset E_2$. We have

$$\overline{\mathcal{Q}}_\epsilon \cap \overline{\mathcal{Q}}_{\epsilon'} \subset \mathrm{Cl}(E_1) \cap \mathrm{Cl}(E_2) \subset u(\mathcal{R} \cap U) = u(A_1 \cap U).$$

As $\dim(A_1 \cap U) = d - \ell$, we deduce $\epsilon' = -\epsilon$. This means in addition that $u(M_1 \cap U) = \mathcal{Q}_\epsilon$ and $u(M_2 \cap U) = \mathcal{Q}_{-\epsilon}$ and we assume $\epsilon = (1, \ldots, 1)$. In fact, $\mathcal{S} \cap U = (\mathrm{Cl}(M_1) \cup \mathrm{Cl}(M_2)) \cap U$ and

$$u(\mathcal{S} \cap U) = \{x_1 \geq 0, \ldots, x_\ell \geq 0\} \cup \{-x_1 \geq 0, \ldots, -x_\ell \geq 0\}.$$

This is so because if $x \in \mathrm{Cl}(M_j)$ for $j \neq 1$, we have $u(M_j \cap U) = \mathcal{Q}_{\epsilon''}$ for some $\epsilon'' \in \{-1, 1\}^s$. As we have seen $\epsilon'' = -\epsilon$, so $M_j = M_2$.

Let $y \in f_1^{-1}(x)$. There exists an open semialgebraic neighborhood $V \subset Y_1$ of y and a Nash diffeomorphism $v : V \to \mathbb{R}^d$ such that $v(y) = 0$ and $u \circ f_1 \circ v^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$(x_1, \ldots, x_d) \mapsto (x_1, x_1 x_2, \ldots, x_1 x_\ell, x_{s+1}, \ldots, x_d).$$

Consequently,

$$
\begin{aligned}
(u \circ f_1 \circ v^{-1})^{-1}(\mathcal{C} \cap U) &= \{x_1 > 0, x_1 x_2 > 0, \ldots, x_1 x_\ell > 0\} \\
&\quad \cup \{x_1 < 0, x_1 x_2 < 0, \ldots, x_1 x_\ell < 0\} \\
&= \{x_1 > 0, x_2 > 0, \ldots, x_\ell > 0\} \cup \{x_1 < 0, x_2 > 0, \ldots, x_\ell > 0\} \\
&= \{x_1 \neq 0, x_2 > 0, \ldots, x_\ell > 0\},
\end{aligned}
$$

see Fig. 13. Therefore $\mathcal{T}_1$ contains $\mathrm{Cl}(v(\{x_1 \neq 0, x_2 > 0, \ldots, x_\ell > 0\})) = \mathrm{Cl}(v(\{x_2 > 0, \ldots, x_\ell > 0\}))$ and

$$
\begin{aligned}
f_1(v(\{x_1 > 0, x_2 > 0, \ldots, x_\ell > 0\})) &= (M_1 \setminus A_1) \cap U, \\
f_1(v(\{x_1 < 0, x_2 > 0, \ldots, x_\ell > 0\})) &= (M_2 \setminus A_1) \cap U.
\end{aligned}
$$

$(x_1, \ldots, x_d) \longmapsto (x_1, x_1 x_2, \ldots, x_1 x_d)$ $(x_1, \ldots, x_d) \longmapsto (x_1, x_1 x_2, \ldots, x_1 x_d)$

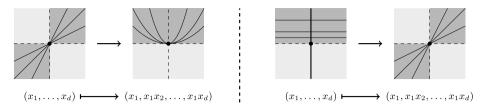**Fig. 13.** Behavior of unions of two quadrants under blow-up.

Thus, $\mathrm{Reg}(\mathrm{Cl}(f_1^{-1}((M_1 \cup M_2) \setminus A_1))) \subset \mathrm{Reg}(\mathfrak{T}_1)$ is connected, so $\mathrm{Reg}(\mathfrak{T}_1)$ has by (8.2) at most $s - 1$ connected components.

8.B.4. We repeat recursively the previous process until we obtain $\mathfrak{T}$ and f satisfying the conditions in the statement, as required. $\quad\square$

Proof of Proposition 8.6 for the general case. Let $\mathcal{S}_0 := \mathrm{Cl}(\mathcal{S}) = \mathrm{Cl}(\mathcal{C})$. By Proposition 8.6 for the closed case there exist a checkerboard set $\mathfrak{T}_0$ and a proper surjective regular map $f_0 : Y_0 := \overline{\mathfrak{T}_0}^{\mathrm{zar}} \to X$ such that $Z_0 := f_0^{-1}(Z)$ is a normal-crossings divisor of Y_0, the restriction $f_0|_{Y_0 \setminus Z_0} : Y_0 \setminus Z_0 \to X \setminus Z$ is a regular diffeomorphism, $\mathfrak{T}_0 = \mathrm{Cl}(f_0^{-1}(\mathcal{C}))$ and $f_0(\mathfrak{T}_0) = \mathcal{S}_0$. Let $\mathfrak{T}_1 := f_0^{-1}(\mathcal{S}) \cap \mathfrak{T}_0$ be the strict transform of $\mathcal{S}$ under f_0. As $\mathcal{S}$ is by Lemma 8.2 well-welded, also $\mathfrak{T}_1$ is by Lemma 7.16 well-welded, so $\mathfrak{T}_1$ is connected. If $\mathrm{Reg}(\mathfrak{T}_1)$ is connected, we are done, so we assume that $\mathrm{Reg}(\mathfrak{T}_1)$ is not connected. Observe that $f_0^{-1}(\mathcal{C})$ is dense in $\mathfrak{T}_1$. Let $N_1, \ldots, N_s$ be the connected components of $\mathrm{Reg}(\mathfrak{T}_1)$. As $\mathfrak{T}_1$ is connected, we suppose $\mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2) \cap \mathfrak{T}_1 \neq \varnothing$.

8.B.5. We may assume: *there exist* $q \in \mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2) \cap \mathfrak{T}_1$ *and an open semialgebraic neighborhood* $U \subset Y_0$ *of* q *equipped with a Nash diffeomorphism* $u : U \to \mathbb{R}^d$ *such that* $u(q) = 0$,

$$\{x_1 > 0, \ldots, x_\ell > 0\} \subset u(N_1 \cap U) \quad and \quad \{x_1 < 0, \ldots, x_\ell < 0\} \subset u(N_2 \cap U).$$

Pick a point $p \in \mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2) \cap \mathfrak{T}_1$ and let $e := \dim(\mathrm{Cl}(N_1)_p \cap \mathrm{Cl}(N_2)_p)$. We distinguish two situations depending on the value of e:

Case 1. $e < d - 1$. Let A_1 be an irreducible component of the Zariski closure A of $\mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2)$ of maximal dimension passing through p. Let $Z_{01}, \ldots, Z_{0\ell}$ be all the irreducible components of Z_0 that contain A_1. Proceeding similarly to the proof 8.B.1 one shows that A_1 is an irreducible component of $Z_{01} \cap \cdots \cap Z_{0\ell}$. Note that $e = d - \ell$. Consider the blow-up (Y_0', f_1) of Y_0 with center A_1. Define $\mathfrak{T}_0' := \mathrm{Cl}(f_1^{-1}(\mathfrak{T}_0 \setminus A_1))$. Proceeding similarly to the proof of 8.B.3 one shows:

(i) $f_1^{-1}(Z_0)$ *is a normal-crossings divisor of* Y_0'.
(ii) $\mathfrak{T}_0' \setminus f_1^{-1}(Z_0)$ *is a union of connected components of* $Y_0' \setminus f_1^{-1}(Z_0)$.
(iii) $\mathfrak{T}_0'$ *is connected.*

In addition, it holds:

(iv) *For each point $z \in f_1^{-1}(p) \cap \mathcal{T}_0'$, we have $\dim(\mathrm{Cl}(f_1^{-1}(N_1))_z \cap \mathrm{Cl}(f_1^{-1}(N_2))_z) = d-1$.*

Denote the union of the irreducible components of Z_0 that do not contain A_1 with Z_0'. To prove (iv) pick

$$x \in \mathrm{Reg}(\mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2)) \cap A_1 \setminus Z_0' \subset \mathrm{Reg}(A)$$

close to p. Let $U \subset Y_0$ be an open semialgebraic neighborhood of x equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(x) = 0$, $u(A \cap U) = \{x_1 = 0, \dots, x_\ell = 0\}$ and $u(Z_0 \cap U) = \{x_1 \cdots x_\ell = 0\}$. We may assume in addition:

- $\mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2) \cap U = A \cap U$ is a connected closed submanifold of U.
- $u(Z_{0i} \cap U) = \{x_i = 0\}$ for $i = 1, \dots, \ell$.

Assume $\mathcal{Q}_1 := \{x_1 > 0, \dots, x_\ell > 0\} \subset u(N_1 \cap U)$ and $\mathcal{Q}_\epsilon := \{\epsilon_1 x_1 > 0, \dots, \epsilon_\ell x_\ell > 0\} \subset u(N_2 \cap U)$ for some $\epsilon := (\epsilon_1, \dots, \epsilon_\ell) \in \{-1, +1\}^\ell$. As

$$\mathrm{Cl}(\mathcal{Q}_1) \cap \mathrm{Cl}(\mathcal{Q}_\epsilon) \subset u(\mathrm{Cl}(N_1) \cap \mathrm{Cl}(N_2) \cap U) = u(A \cap U) = \{x_1 = 0, \dots, x_\ell = 0\},$$

we have $\epsilon = (-1, \dots, -1)$, $\mathcal{Q}_1 = u(N_1 \cap U)$ and $\mathcal{Q}_\epsilon = u(N_2 \cap U)$.

Let $y \in Y_0'$ be a point close to z such that $f_1(y) = x$. There exists an open semialgebraic neighborhood $V \subset Y_0'$ of y and a Nash diffeomorphism $v : V \to \mathbb{R}^d$ such that $u \circ f_1 \circ v^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$(x_1, \dots, x_d) \mapsto (x_1, x_1 x_2, \dots, x_1 x_\ell, x_{\ell+1}, \dots, x_d).$$

We have

$$(u \circ f_1 \circ v^{-1})^{-1}(\mathcal{Q}_1 \cup \mathcal{Q}_\epsilon) = \{x_1 > 0, x_1 x_2 > 0, \dots, x_1 x_\ell > 0\}$$
$$\cup \{x_1 < 0, x_1 x_2 < 0, \dots, x_1 x_\ell < 0\}$$
$$= \{x_1 > 0, x_2 > 0, \dots, x_\ell > 0\} \cup \{x_1 < 0, x_2 > 0, \dots, x_\ell > 0\}$$
$$= \{x_1 \neq 0, x_2 > 0, \dots, x_\ell > 0\}.$$

So $\mathrm{Cl}(f_1^{-1}(N_1)) \cap \mathrm{Cl}(f_1^{-1}(N_2)) \cap V$ contains $v^{-1}(\{x_1 = 0\})$, which has dimension $d-1$. Consequently, $\dim(\mathrm{Cl}(f_1^{-1}(N_1))_z \cap \mathrm{Cl}(f_1^{-1}(N_2))_z) = d-1$.

We assume in what follows $e = d-1$.

Case 2. $e = d-1$. Consider the blow-up (Y_0', f_1) of Y_0 with center $\{p\} \subset \mathcal{T}_1$. Consider the strict transform $\mathcal{T}_1' := \mathrm{Cl}(f_1^{-1}(\mathcal{T}_1 \setminus \{p\})) \cap f_1^{-1}(\mathcal{T}_1)$. We have:

- $f_1^{-1}(Z_0)$ *is a normal-crossings divisor of Y_0'.*

- $\mathcal{T}'_1 \setminus f_1^{-1}(Z_0)$ *is a union of connected components of* $Y'_0 \setminus f_1^{-1}(Z_0)$.
- $\mathcal{T}'_1$ *is connected* (use the argument described above involving Lemmas 8.2 and 7.16).

As $\dim(\mathrm{Cl}(N_1)_p \cap \mathrm{Cl}(N_2)_p) = d - 1$, we assume there exists an open semialgebraic neighborhood U of p in Y_0 and a Nash diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(p) = 0$, $u(Z_0 \cap U) = \{x_1 \cdots x_\ell = 0\}$ and

$$\mathcal{R}_1 := \{x_1 > 0, x_2 > 0 \ldots, x_\ell > 0\} \subset u(N_1 \cap U), \tag{8.3}$$

$$\mathcal{R}_2 := \{x_1 < 0, x_2 > 0 \ldots, x_\ell > 0\} \subset u(N_2 \cap U). \tag{8.4}$$

Consider the Nash path germ $\alpha(t) := u^{-1}(t, t^2, \cdots, t^2) \subset N_1 \cap U$ and observe that

$$q := \lim_{t \to 0^+} f_1^{-1}(\alpha(t)) \in f_1^{-1}(p) \cap \mathrm{Cl}(f_1^{-1}(\mathcal{T}_1 \setminus \{p\})) \subset \mathrm{Cl}(f_1^{-1}(\mathcal{T}_1 \setminus \{p\})) \cap f_1^{-1}(\mathcal{T}_1) = \mathcal{T}'_1. \tag{8.5}$$

Thus, there exists an open semialgebraic neighborhood $V \subset Y'_0$ of q and a Nash diffeomorphism $v : V \to \mathbb{R}^d$ such that $v(q) = 0$ and $u \circ f_1 \circ v^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$(x_1, \ldots, x_d) \mapsto (x_1, x_1 x_2, \ldots, x_1 x_d).$$

We have

$$(u \circ f_1 \circ v^{-1})^{-1}(\mathcal{R}_1 \cup \mathcal{R}_2 \cup \{0\})$$
$$= \{x_1 > 0, x_1 x_2 > 0, \ldots, x_1 x_\ell > 0\} \cup \{x_1 < 0, x_1 x_2 > 0, \ldots, x_1 x_\ell > 0\} \cup \{x_1 = 0\}$$
$$= \{x_1 > 0, \ldots, x_\ell > 0\} \cup \{x_1 < 0, \ldots, x_\ell < 0\} \cup \{x_1 = 0\},$$

see Fig. 13. Consequently, 8.B.5 holds true.

8.B.6. Observe that $q \notin N_i$ for $i = 1, \ldots, s$. Consider the blow-up (Y'_0, f_1) of Y_0 with center $\{q\} \subset \mathcal{T}_1$ and the strict transform $\mathcal{T}'_1 := \mathrm{Cl}(f_1^{-1}(\mathcal{T}_1 \setminus \{q\})) \cap f_1^{-1}(\mathcal{T}_1)$. We have:

- $f_1^{-1}(Z_0)$ is a normal-crossings divisor of Y'_0.
- $\mathcal{T}'_1 \setminus f_1^{-1}(Z_0)$ is a union of connected components of $Y'_0 \setminus f_1^{-1}(Z_0)$.
- $\mathcal{T}'_1$ is connected by Lemmas 7.16 and 8.2.

Let us check that $\mathrm{Reg}(\mathcal{T}'_1)$ has at most $s - 1$ connected components. It holds

$$\bigcup_{i=1}^{s} f_1^{-1}(N_i) \subset \mathrm{Reg}(\mathcal{T}'_1) \subset \mathcal{T}'_1 = \mathrm{Cl}(f_1^{-1}(\mathrm{Reg}(\mathcal{T}_1) \setminus \{q\})) \cap f_1^{-1}(\mathcal{T}_1)$$

$$= \mathrm{Cl}\left(\bigcup_{i=1}^{s} f_1^{-1}(N_i)\right) \cap f_1^{-1}(\mathcal{T}_1) = \bigcup_{i=1}^{s} \mathrm{Cl}(f_1^{-1}(N_i)) \cap f_1^{-1}(\mathcal{T}_1), \tag{8.6}$$

so $\mathrm{Reg}(\mathcal{T}_1')$ has at most s connected components. Let us check that in fact it has at most $s - 1$.

Proceeding similarly to (8.5) we find a point

$$y \in f_1^{-1}(q) \cap \mathrm{Cl}(f_1^{-1}(\mathcal{T}_1 \setminus \{q\})) \subset \mathrm{Cl}(f_1^{-1}(\mathcal{T}_1 \setminus \{p\})) \cap f_1^{-1}(\mathcal{T}_1) = \mathcal{T}_1'.$$

Thus, there exist an open semialgebraic neighborhood $V \subset Y_0'$ of y and a Nash diffeomorphism $v : V \to \mathbb{R}^d$ such that $v(y) = 0$ and $u \circ f_1 \circ v^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$(x_1, \ldots, x_d) \mapsto (x_1, x_1 x_2, \ldots, x_1 x_d).$$

Consequently, if $\mathcal{R}_1$ is defined as in (8.3), then $-\mathcal{R}_1 = \{x_1 < 0, x_2 < 0, \ldots, x_\ell < 0\}$ and

$$
\begin{aligned}
(u \circ f_1 \circ v^{-1})^{-1}((\mathcal{R}_1 \cup -\mathcal{R}_1 \cup \{0\}) &= \{x_1 > 0, x_1 x_2 > 0, \ldots, x_1 x_\ell > 0\} \\
&\cup \{x_1 < 0, x_1 x_2 < 0 \ldots, x_1 x_\ell < 0\} \cup \{x_1 = 0\} \\
&= \{x_2 > 0, \ldots, x_\ell > 0\} \cup \{x_1 = 0\},
\end{aligned}
$$

see Fig. 13. So $\mathcal{T}_1'$ contains $v(\{x_2 > 0, \ldots, x_\ell > 0\})$ and

$$
\begin{aligned}
f_1(v(\{x_1 > 0, x_2 > 0 \ldots, x_\ell > 0\})) &\subset N_1 \cap U, \\
f_1(v(\{x_1 < 0, x_2 > 0 \ldots, x_\ell > 0\})) &\subset N_2 \cap U.
\end{aligned}
$$

Thus, $\mathrm{Reg}(\mathrm{Cl}(f_1^{-1}(N_1 \cup N_2)) \cap f_1^{-1}(\mathcal{T}_1)) \subset \mathrm{Reg}(\mathcal{T}_1')$ is connected, so $\mathrm{Reg}(\mathcal{T}_1')$ has by (8.6) at most $s - 1$ connected components.

8.B.7. We repeat recursively the previous process until we obtain $\mathcal{T}$ and f satisfying the conditions in the statement, as required. $\quad\square$

We are ready to prove Theorem 8.4.

Proof of Theorem 8.4. Let X be the Zariski closure of $\mathcal{S}$. By Theorem 2.5 there exist a non-singular algebraic set X' and a proper regular map $f : X' \to X$ such that the restriction $f|_{X' \setminus f^{-1}(\mathrm{Sing}(X))} : X' \setminus f^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}(X)$ is a biregular diffeomorphism. The strict transform $\mathcal{S}' := \mathrm{Cl}(f^{-1}(\mathcal{S} \setminus \mathrm{Sing}(X))) \cap f^{-1}(\mathcal{S})$ is by Lemma 7.16 well-welded and $f(\mathcal{S}') = \mathcal{S}$. So we may assume from the beginning that X is non-singular.

Let Z be the Zariski closure of $\mathcal{R} := \mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})$. By Theorem 2.7 there exist a non-singular algebraic set X' and a proper surjective regular map $f : X' \to X$ such that $f^{-1}(Z)$ is a normal-crossings divisor of X' and the restriction $f|_{X' \setminus f^{-1}(Z)} : X' \setminus f^{-1}(Z) \to X \setminus Z$ is a biregular diffeomorphism. The strict transform $\mathcal{S}' := f^{-1}(\mathcal{S}) \cap \mathrm{Cl}(f^{-1}(\mathcal{S} \setminus Z))$ is by Lemma 7.16 well-welded and $f(\mathcal{S}') = \mathcal{S}$. Even more $\mathcal{S} \setminus Z = \mathrm{Cl}(\mathcal{S}) \setminus Z = \mathrm{Reg}(\mathcal{S}) \setminus Z$

is a closed and open subset of $X \setminus Z$, so $\mathcal{C} := f^{-1}(\mathcal{S} \setminus Z)$ is a closed and open subset of $X' \setminus f^{-1}(Z)$. Consequently, $\mathcal{C}$ is a union of connected components of $X' \setminus f^{-1}(Z)$.

Thus, we may assume from the beginning:

- The Zariski closure of $\mathcal{S}$ is non-singular.
- The Zariski closure of $\mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})$ is contained in a normal-crossings divisor Z.
- $\mathcal{S} \setminus Z$ is a union of connected components of $X \setminus Z$.
- $\mathcal{S}$ is connected (and so well-welded by Lemma 8.2).

By Proposition 8.6 there exist a checkerboard set $\mathcal{T} \subset \mathbb{R}^m$ and a proper surjective regular map $f : Y := \overline{\mathcal{T}}^{\mathrm{zar}} \to X$ such that $f(\mathcal{T}) = \mathcal{S}$, as required. $\square$

8.C. Proof of Theorem 1.4

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) are straightforward. As quoted in the Introduction, only the proof of the non-completely trivial implication (ii) $\Longrightarrow$ '$\mathcal{S}$ is pure dimensional' requires a comment and it is shown in Corollary 6.3. The implication (iii) $\Longrightarrow$ (vii) is proved in Lemma 7.1 whereas (v) $\Longrightarrow$ (vii) follows from Lemma 7.2. In addition, (iv) $\Longrightarrow$ (vii) is shown in Lemma 7.14 and (vi) $\Longrightarrow$ (vii) in Lemma 7.15. To finish we prove (vii) $\Longrightarrow$ (i), that is, *a well-welded semialgebraic set of dimension d is a Nash image of $\mathbb{R}^d$*. By Theorem 8.4 we may assume: $\mathcal{S}$ *is a checkerboard set*. Let $M_0 := \overline{\mathcal{S}}^{\mathrm{zar}}$, which is a Nash manifold, and let Z be the smallest Nash subset of M_0 that contains $\partial \mathcal{S}$, which is a Nash normal crossings divisor of M_0 (because its irreducible components as a Nash set are connected components of the irreducible components of $\overline{\partial \mathcal{S}}^{\mathrm{zar}}$ as an algebraic set). By Remark 8.1 the difference $\mathcal{S} \setminus \overline{\partial \mathcal{S}}^{\mathrm{zar}} = \mathrm{Reg}(\mathcal{S}) \setminus \overline{\partial \mathcal{S}}^{\mathrm{zar}}$ is a union of connected components of $M_0 \setminus \overline{\partial \mathcal{S}}^{\mathrm{zar}}$. As $\overline{\partial \mathcal{S}}^{\mathrm{zar}}$ is a normal crossings divisor of M_0, we conclude that $\partial \mathcal{S} = \mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})$ is a pure dimensional semialgebraic set of dimension $d-1$. To get a general idea on how this proof works see Fig. 15. The proof is conducted in several steps.

8.C.1. Step 1. Initial preparation. Let $Z_1, \ldots, Z_s$ be the irreducible components of Z as a Nash subset of M_0. Let $\mathcal{B} \subset \partial \mathcal{S} \subset Z$ be a semialgebraic set. For each $i = 1, \ldots, s$ let $\mathcal{B}_i$ be the closure of the set of points of dimension $< \dim(\mathcal{B})$ of the intersection $\mathcal{B} \cap Z_i$. Denote $\mathcal{B}^* := \mathcal{B} \setminus \bigcup_{i=1}^s \mathcal{B}_i$ and observe that $\mathcal{B}^*$ is pure dimensional open semialgebraic subset of $\mathcal{B}$ and $\dim(\mathcal{B} \setminus \mathtt{Sth}(\mathcal{B}^*)) < \dim(\mathcal{B})$. We claim: *if $\mathcal{A}$ is a connected component of $\mathtt{Sth}(\mathcal{B}^*)$, then for each $i = 1, \ldots, s$ either $\mathcal{A} \cap Z_i = \varnothing$ or $\mathcal{A} \subset Z_i$.*

Fix $i = 1, \ldots, s$ such that $\mathcal{A} \cap Z_i \neq \varnothing$ and pick a point $p \in \mathcal{A} \cap Z_i$. Then $p \in (\mathcal{B} \cap Z_i) \setminus \mathcal{B}_i$, so p is a point of dimension $\dim(\mathcal{B})$ of $\mathcal{B} \cap Z_i$. As $\dim(\mathcal{B} \setminus \mathtt{Sth}(\mathcal{B}^*)) < \dim(\mathcal{B})$, we deduce $p \in \mathtt{Sth}(\mathcal{B}^*)$ is a point of dimension $\dim(\mathtt{Sth}(\mathcal{B}^*)) = \dim(\mathcal{B})$ of $\mathtt{Sth}(\mathcal{B}^*) \cap Z_i$. As $\mathcal{A}$ is the connected component of $\mathtt{Sth}(\mathcal{B}^*)$ that contains p, we have $\dim(\mathcal{A} \cap Z_i) = \dim(\mathtt{Sth}(\mathcal{B}^*) \cap Z_i) = \dim(\mathtt{Sth}(\mathcal{B}^*)) = \dim(\mathcal{A})$. As $\mathcal{A}$ is a connected Nash manifold and Z_i a Nash subset of M_0, we conclude by the identity principle $\mathcal{A} \subset Z_i$.

8.C.2. *Step* 2. *Construction of the bad subset* $\mathcal{T}$ *of* $\mathcal{S}$ *and a suitable partition of* $\mathcal{T}$ *into Nash manifolds of different dimensions.* As $\overline{\mathcal{S}}^{\mathrm{zar}}$ is a non-singular real algebraic set and $\mathcal{S}$ is pure dimensional, $\mathrm{Reg}(\mathcal{S}) = \mathtt{Sth}(\mathcal{S})$ and $\mathrm{Sing}(\mathcal{S}) = \mathtt{NSth}(\mathcal{S})$ by Remark 2.2. In particular, $\partial\mathcal{S} = \mathrm{Cl}(\mathcal{S}) \setminus \mathtt{Sth}(\mathcal{S})$. Let $\mathcal{G}$ be the set of points of $\mathrm{Sing}(\mathcal{S})$ of local dimension $< d-1$. Define the *bad subset* of $\mathcal{S}$ as:

$$\mathcal{T} := \mathtt{NSth}(\mathtt{NSth}(\mathcal{S})) \cup (\mathrm{Cl}(\mathcal{G}) \cap \mathcal{S}) \subset Z,$$

which is a semialgebraic set of dimension $< d-1$. Define $\mathcal{R}_0 := \mathcal{T}$, $\mathcal{R}_k := \mathcal{R}_{k-1} \setminus \mathcal{T}_k$ and $\mathcal{T}_k := \mathtt{Sth}(\mathcal{R}_{k-1}^*) \subset \mathcal{R}_{k-1}$ for $k \geq 1$ (see 8.C.1 for the definition of $\mathcal{R}_{k-1}^*$ from $\mathcal{R}_{k-1}$). Each semialgebraic set $\mathcal{T}_k$ is a Nash manifold (and an open subset of $\mathcal{R}_{k-1}$) and each semialgebraic set $\mathcal{R}_k$ is a closed subset of $\mathcal{T}$. In addition, if $1 \leq k < j$,

$$\mathrm{Cl}(\mathcal{T}_j) \cap \mathcal{T}_k = \mathrm{Cl}(\mathcal{T}_j) \cap \mathcal{T} \cap \mathcal{T}_k \subset \mathcal{R}_{j-1} \cap \mathcal{T}_k \subset \mathcal{R}_k \cap \mathcal{T}_k = \varnothing. \tag{8.7}$$

Observe that $\dim(\mathcal{R}_{k+1}) < \dim(\mathcal{R}_k)$ for $k \geq 0$ (if $\mathcal{R}_k \neq \varnothing$), so $\mathcal{R}_{d-1} = \varnothing$. Consequently,

$$\mathcal{T} = \mathcal{T}_1 \sqcup \mathcal{R}_1 = \mathcal{T}_1 \sqcup \mathcal{T}_2 \sqcup \mathcal{R}_2 = \cdots = \bigsqcup_{k=1}^{d-1} \mathcal{T}_k,$$

$$\mathcal{S} = \mathtt{Sth}(\mathcal{S}) \sqcup (\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}) \sqcup \mathcal{T}.$$

8.C.3. *Step* 3. *The difference* $\mathcal{S} \setminus \mathcal{T}$ *is a Nash manifold with boundary the difference* $\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}$ *and interior* $\mathtt{Sth}(\mathcal{S})$.

It holds that $\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}$ is either empty or a Nash manifold of dimension $d-1$. Assume $\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T} \neq \varnothing$ and pick a point $x \in \mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}$. As $\mathtt{NSth}(\mathcal{S}) \subset Z$ and Z is a normal-crossings divisor, there exists an open semialgebraic neighborhood U of x in M_0 equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(x) = 0$ and $u(Z \cap U) = \{x_1 \cdots x_r = 0\}$ for some $1 \leq r \leq d$. As $\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}$ is a Nash manifold of dimension $d-1$, we may assume $u((\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}) \cap U) = \{x_1 = 0\}$. As $\mathtt{Sth}(\mathcal{S}) \setminus Z$ is a union of connected components of $M_0 \setminus Z$, the Nash manifold $\mathtt{Sth}(\mathcal{S})$ is connected and $\mathcal{S}$ is pure dimensional, we may assume $u(\mathcal{S} \cap U) = \{x_1 \geq 0\}$. Consequently, $\mathcal{S} \setminus \mathcal{T}$ is a Nash manifold with boundary $\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}$ and interior $\mathtt{Sth}(\mathcal{S})$.

8.C.4. *Step* 4. Define $\mathfrak{G} := \{1 \leq k \leq d-1 : \mathcal{T}_k \neq \varnothing\}$ and $\ell(\mathcal{S}) := \#\mathfrak{G}$. We prove next by induction on $\ell(\mathcal{S})$ the following: *There exist a connected Nash manifold H with boundary ∂H and a surjective Nash map $f : H \to \mathcal{S}$ such that $\mathrm{Int}(H)$ is Nash diffeomorphic to $\mathtt{Sth}(\mathcal{S})$.* Once this will be done, recall that H is by Theorem 1.5 a Nash image of $\mathbb{R}^d$, so also $\mathcal{S}$ will be a Nash image of $\mathbb{R}^d$ and the proof will be finished.

If $\ell(\mathcal{S}) = 0$ (or equivalently $\mathcal{T} = \varnothing$), then $\mathcal{S} = \mathcal{S} \setminus \mathcal{T}$ is by 8.C.3 a Nash manifold with boundary and it is enough to take $H := \mathcal{S}$ and $f := \mathrm{id}_{\mathcal{S}}$. Assume statement 8.C.4 true for $\ell(\mathcal{S}) - 1$ and let us check that it is also true for $\ell(\mathcal{S})$.

8.C.5.　Write $\ell := \ell(\mathcal{S})$ and observe that $\mathcal{T}_k = \varnothing$ if and only if $k \geq \ell+1$. As $\mathcal{T}_\ell$ is locally compact (because it is a Nash manifold), the semialgebraic set $\mathcal{C} := \mathrm{Cl}(\mathcal{T}_\ell) \setminus \mathcal{T}_\ell$ is closed. By (8.7) $\mathcal{C}$ does not meet $\mathcal{T}_j$ for $1 \leq j \leq \ell$, so $\mathcal{C} \cap \mathcal{T} = \varnothing$. As $\mathcal{T}$ is a closed subset of $\mathcal{S}$ and $\mathcal{C} \subset \mathrm{Cl}(\mathcal{T}) \setminus \mathcal{T}$, we have $\mathcal{S} \cap \mathcal{C} = \varnothing$. Let $M := M_0 \setminus \mathcal{C}$, which is a Nash manifold of dimension d that contains $\mathcal{S}$. Observe that $\mathcal{T}_\ell$ is a closed Nash submanifold of M of dimension $e < d$. Let $(\widetilde{M}_+, \pi_+)$ be the drilling blow-up of M with center $\mathcal{T}_\ell$ and let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of $(\widetilde{M}_+, \pi_+)$. By 5.B.1 $\pi_+^{-1}(\mathcal{T}_\ell)$ is a closed non-singular Nash hypersurface of the Nash manifold $\widehat{M}$. Denote $\mathcal{E} := \mathrm{Cl}(\pi_+^{-1}(Z \setminus \mathcal{T}_\ell)) \cap \widetilde{M}_+$ and $\mathcal{K} := \pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{E}$, which are closed semialgebraic subsets of $\widehat{M}$. Observe that $\mathcal{E} = \pi_+^{-1}(Z \setminus \mathcal{T}_\ell) \sqcup \mathcal{K}$. Define

$$\mathcal{S}_1' := \pi_+^{-1}(\mathcal{S}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell)) \quad \text{and} \quad \mathcal{S}_1 := \mathcal{S}_1' \setminus \mathcal{K}.$$

It holds $\mathcal{S}_1 \setminus \pi_+^{-1}(Z) = \pi_+^{-1}(\mathcal{S} \setminus Z)$ and $(\mathcal{S}_1 \setminus \mathcal{E}) \cap \pi_+^{-1}(\mathcal{T}_\ell) = \mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell)$. Denote $M' := \widehat{M} \setminus \mathcal{K}$, which is a Nash manifold, and $Z' := (\mathcal{E} \setminus \mathcal{K}) \sqcup (\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{S}_1)$.

8.C.6.　We claim: *Z' is the smallest Nash subset of M' that contains the semialgebraic set $\partial \mathcal{S}_1 := (\mathrm{Cl}(\mathcal{S}_1) \cap M') \setminus \mathrm{NSth}(\mathcal{S}_1)$ and it is a Nash normal crossings divisor of M'.* In addition, $\mathrm{Sth}(\mathcal{S}_1)$ *is a connected Nash manifold of dimension d.* Once this is proved, we deduce by Lemma 8.3 that, up to a suitable Nash embedding of M', the semialgebraic set $\mathcal{S}_1$ is a checkerboard set such that $\mathrm{Reg}(\mathcal{S}_1) = \mathrm{Sth}(\mathcal{S}_1)$ and $\mathrm{Sing}(\mathcal{S}_1) = \mathrm{NSth}(\mathcal{S}_1)$.

As $\pi_+|_{\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)} : \widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell) \to M \setminus \mathcal{T}_\ell$ is a Nash diffeomorphism, we have:

- $\mathcal{E} \setminus \mathcal{K} = \pi_+^{-1}(Z \setminus \mathcal{T}_\ell)$ is the smallest Nash subset of $\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)$ that contains the semialgebraic set $\partial \mathcal{S}_1 \setminus \pi_+^{-1}(\mathcal{T}_\ell) = \pi_+^{-1}(\partial \mathcal{S} \setminus \mathcal{T}_\ell)$, because $Z \setminus \mathcal{T}_\ell$ is the smallest Nash subset of $M \setminus \mathcal{T}_\ell$ that contains $\partial \mathcal{S} \setminus \mathcal{T}_\ell$. To prove this last fact recall that $\partial \mathcal{S}$ is pure dimensional of dimension $d-1$ and $\dim(\mathcal{T}_\ell) < d-1$.
- $\mathcal{E} \setminus \mathcal{K} = \pi_+^{-1}(Z \setminus \mathcal{T}_\ell)$ is a Nash normal crossings divisor of $\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)$ because $Z \setminus \mathcal{T}_\ell$ is a Nash normal crossings divisor of $M \setminus \mathcal{T}_\ell$.

As $\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)$ is an open and closed semialgebraic subset of $M' \setminus \pi_+^{-1}(\mathcal{T}_\ell)$, we deduce that $\mathcal{E} \setminus \mathcal{K}$ is the smallest Nash subset of $M' \setminus \pi_+^{-1}(\mathcal{T}_\ell)$ that contains the semialgebraic set $\partial \mathcal{S}_1 \setminus \pi_+^{-1}(\mathcal{T}_\ell)$ and it is a Nash normal crossings divisor of $M' \setminus \pi_+^{-1}(\mathcal{T}_\ell)$. As $\mathcal{E} \setminus \mathcal{K}$ is a closed semialgebraic subset of M', we conclude by [34, Prop.II.5.3] that $\mathcal{E} \setminus \mathcal{K}$ is the smallest Nash subset of M' that contains the semialgebraic set $\partial \mathcal{S}_1 \setminus \pi_+^{-1}(\mathcal{T}_\ell)$ and a Nash normal crossings divisor of M'.

8.C.7.　Let us check: *$(\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{S}_1)$ is a closed non-singular Nash hypersurface of M'.* As $(\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{S}_1) \cap (\mathcal{E} \setminus \mathcal{K}) = \varnothing$, this will show that Z' is the smallest Nash subset of M' that contains $\partial \mathcal{S}_1 = \pi_+^{-1}(\partial \mathcal{S} \setminus \mathcal{T}_\ell) \sqcup (\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{S}_1)$ and it is a Nash normal crossings divisor of M'.

We have $\mathtt{Sth}(\mathcal{S}) \subset \mathcal{S} \setminus \mathcal{T}_\ell$, so $\mathtt{Sth}(\mathcal{S}) = \mathtt{Sth}(\mathcal{S} \setminus \mathcal{T}_\ell)$. As $\pi_+^{-1}(\mathcal{T}_\ell) = \partial \widetilde{M}_+$ and $\dim(\mathcal{S}_1) = \dim(\widetilde{M}_+)$, we have $\mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell) \subset \mathtt{NSth}(\mathcal{S}_1)$ and $\mathtt{Sth}(\mathcal{S}_1) = \mathtt{Sth}(\mathcal{S}_1 \setminus \pi_+^{-1}(\mathcal{T}_\ell))$. As $\pi_+|_{\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)}$ is a Nash diffeomorphism,

$$\mathtt{Sth}(\mathcal{S}_1) = \mathtt{Sth}(\pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell)) = \pi_+^{-1}(\mathtt{Sth}(\mathcal{S} \setminus \mathcal{T}_\ell)) = \pi_+^{-1}(\mathtt{Sth}(\mathcal{S})),$$

which is a connected Nash manifold of dimension d. Consequently, $\mathtt{NSth}(\mathcal{S}_1) = \pi_+^{-1}(\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}_\ell) \cup (\mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell))$ and $\partial \mathcal{S}_1 = \pi_+^{-1}(\partial \mathcal{S} \setminus \mathcal{T}_\ell) \sqcup (\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{S}_1)$. As $\widetilde{M}_+$ is a Nash manifold with boundary $\pi_+^{-1}(\mathcal{T}_\ell)$, the difference $\widetilde{M}_+ \setminus \mathcal{E}$ is a Nash manifold with boundary $\pi_+^{-1}(\mathcal{T}_\ell) \setminus \mathcal{E} = \pi_+^{-1}(\mathcal{T}_\ell) \setminus \mathcal{K}$.

Observe that $\mathcal{S} \setminus Z = \mathrm{Cl}(\mathcal{S}) \setminus Z = \mathtt{Sth}(\mathcal{S}) \setminus Z$ is a closed an open subset of $M_0 \setminus Z = M \setminus Z$ (see Remark 8.1). As $\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathtt{Sth}(\mathcal{S}_1) = \varnothing$,

$$\pi_+^{-1}(\mathcal{S} \setminus Z) = \pi_+^{-1}(\mathtt{Sth}(\mathcal{S}) \setminus Z) = \mathtt{Sth}(\mathcal{S}_1) \setminus \pi_+^{-1}(Z) = \mathtt{Sth}(\mathcal{S}_1) \setminus \mathcal{E}$$

is a union of connected components of $\pi_+^{-1}(M \setminus Z) = (\widetilde{M}_+ \setminus \mathcal{E}) \setminus \pi_+^{-1}(\mathcal{T}_\ell)$, that is, $\mathtt{Sth}(\mathcal{S}_1) \setminus \mathcal{E}$ is a union of connected components of the interior $\mathrm{Int}(\widetilde{M}_+ \setminus \mathcal{E})$ of the Nash manifold with boundary $\widetilde{M}_+ \setminus \mathcal{E}$. Thus, the closure $\mathcal{D}$ of $\mathtt{Sth}(\mathcal{S}_1) \setminus \mathcal{E}$ in $\widetilde{M}_+ \setminus \mathcal{E}$ is a Nash manifold with boundary $\mathcal{D} \cap (\pi_+^{-1}(\mathcal{T}_\ell) \setminus \mathcal{K})$.

8.C.8. We claim: $\mathcal{S}_1 \setminus \mathcal{E}$ *is the closure of* $\mathtt{Sth}(\mathcal{S}_1) \setminus \mathcal{E}$ *in* $\widetilde{M}_+ \setminus \mathcal{E}$.

As $\mathcal{S} \setminus Z = \mathrm{Cl}(\mathcal{S}) \setminus Z$ and $\mathcal{T}_\ell \subset Z \cap \mathcal{S}$, we have $\mathrm{Cl}(\mathcal{S}) \setminus (Z \setminus \mathcal{T}_\ell) = \mathcal{S} \setminus (Z \setminus \mathcal{T}_\ell) = (\mathcal{S} \setminus Z) \cup \mathcal{T}_\ell$. As $\pi_+^{-1}(\mathrm{Cl}(\mathcal{S}) \setminus (Z \setminus \mathcal{T}_\ell)) \setminus \mathcal{K}$ is a closed subset of $\widetilde{M}_+ \setminus \mathcal{E} = (\pi_+^{-1}(M \setminus (Z \setminus \mathcal{T}_\ell)) \setminus \mathcal{K}$ that contains $\pi_+^{-1}(\mathcal{S} \setminus Z)$, we deduce

$$\begin{aligned} \mathcal{S}_1 \setminus \mathcal{E} = \mathcal{S}_1' \setminus \mathcal{E} &= (\pi_+^{-1}(\mathcal{S} \setminus (Z \setminus \mathcal{T}_\ell)) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell))) \setminus \mathcal{K} \\ &= (\pi_+^{-1}(\mathrm{Cl}(\mathcal{S}) \setminus (Z \setminus \mathcal{T}_\ell)) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus Z))) \setminus \mathcal{K} \subset (\widetilde{M}_+ \setminus \mathcal{E}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus Z)) \end{aligned}$$

is the closure of $\pi_+^{-1}(\mathcal{S} \setminus Z)$ in $\widetilde{M}_+ \setminus \mathcal{E}$.

8.C.9. Thus, $\mathcal{S}_1 \setminus \mathcal{E}$ is a Nash manifold with boundary $\mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell)$. As $\mathcal{S}_1 \setminus \mathcal{E}$ is a closed subset of $\widetilde{M}_+ \setminus \mathcal{E}$, the intersection $(\mathcal{S}_1 \setminus \mathcal{E}) \cap \pi_+^{-1}(\mathcal{T}_\ell) = \mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell)$ is a closed subset of $(\widetilde{M}_+ \setminus \mathcal{E}) \cap \pi_+^{-1}(\mathcal{T}_\ell) = \pi_+^{-1}(\mathcal{T}_\ell) \setminus \mathcal{K}$, which is itself a closed subset of $M' = \widehat{M} \setminus \mathcal{K}$. Consequently, $\mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell)$ is a closed non-singular Nash hypersurface of M'.

8.C.10. We check next: $\ell(\mathcal{S}_1) = \ell(\mathcal{S}) - 1$.

As $\mathtt{NSth}(\mathcal{S}_1) = \pi_+^{-1}(\mathtt{NSth}(\mathcal{S}) \setminus \mathcal{T}_\ell) \cup (\mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell))$, the restriction map $\pi_+|_{\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)} : \widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell) \to M \setminus \mathcal{T}_\ell$ is a Nash diffeomorphism and $\mathcal{S}_1 \cap \pi_+^{-1}(\mathcal{T}_\ell)$ is a Nash manifold of dimension $d - 1$, we deduce $\mathtt{NSth}(\mathtt{NSth}(\mathcal{S}_1)) = \pi_+^{-1}(\mathtt{NSth}(\mathtt{NSth}(\mathcal{S})) \setminus \mathcal{T}_\ell)$ and the set $\mathcal{G}'$ of points of $\mathtt{NSth}(\mathcal{S}_1)$ of dimension $< d - 1$ is $\pi_+^{-1}(\mathcal{G} \setminus \mathcal{T}_\ell)$. We claim: *the bad subset* $\mathcal{T}' := \mathtt{NSth}(\mathtt{NSth}(\mathcal{S}_1)) \cup (\mathrm{Cl}(\mathcal{G}') \cap \mathcal{S}_1)$ *of* $\mathcal{S}_1$ *equals* $\pi_+^{-1}(\mathcal{T} \setminus \mathcal{T}_\ell) = \bigcup_{k=1}^{s-1} \pi_+^{-1}(\mathcal{T}_k)$. We have to prove: $\mathrm{Cl}(\mathcal{G}') \cap \mathcal{S}_1 = \pi_+^{-1}((\mathrm{Cl}(\mathcal{G}) \cap \mathcal{S}) \setminus \mathcal{T}_\ell)$.
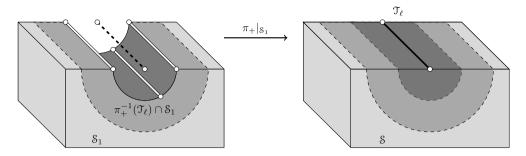
Fig. 14. Behavior of the surjective Nash map $\pi_+|_{\mathcal{S}_1} : \mathcal{S}_1 \to \mathcal{S}$.

As $\mathcal{G}' = \pi_+^{-1}(\mathcal{G} \setminus \mathcal{T}_\ell) \subset \mathrm{Cl}(\pi_+^{-1}(Z \setminus \mathcal{T}_\ell)) \cap \widetilde{M}_+ = \mathcal{E}$, also $\mathrm{Cl}(\mathcal{G}') \cap \widetilde{M}_+ \subset \mathcal{E}$. Thus,

$$\mathrm{Cl}(\mathcal{G}') \cap \mathcal{S}_1 = (\mathrm{Cl}(\mathcal{G}') \cap \pi_+^{-1}(\mathcal{S}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell))) \setminus (\pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{E})$$
$$= \mathrm{Cl}(\mathcal{G}') \cap \mathcal{E} \cap (\pi_+^{-1}(\mathcal{S}) \setminus \pi_+^{-1}(\mathcal{T}_\ell)) = \mathrm{Cl}(\mathcal{G}') \cap \pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell).$$

As $\pi_+|_{\widetilde{M}_+ \setminus \pi_+^{-1}(\mathcal{T}_\ell)}$ is a Nash diffeomorphism and $\mathcal{G}' = \pi_+^{-1}(\mathcal{G} \setminus \mathcal{T}_\ell)$, it holds

$$\mathrm{Cl}(\mathcal{G}') \cap \mathcal{S}_1 = \mathrm{Cl}(\mathcal{G}') \cap \pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell) = \pi_+^{-1}(\mathrm{Cl}(\mathcal{G}) \cap \mathcal{S} \setminus \mathcal{T}_\ell),$$

as claimed. Therefore, $\mathcal{T}' = \pi_+^{-1}(\mathcal{T} \setminus \mathcal{T}_\ell)$.

Decompose $\mathcal{T}' = \bigsqcup_{k=1}^{d-1} \mathcal{T}'_k$ following the algorithm proposed in 8.C.2. As $\pi_+|_{M \setminus \mathcal{T}_\ell}$ is a Nash diffeomorphism and $\mathcal{T}_k \cap \mathcal{T}_\ell = \varnothing$ if $k \neq \ell$, we deduce $\mathcal{T}'_k = \pi_+^{-1}(\mathcal{T}_k)$ for $k = 1, \dots, \ell - 1$ and $\mathcal{T}'_k = \varnothing$ for $k = \ell, \dots, d-1$. As $\mathcal{T}'_{\ell-1} \neq \varnothing$, we conclude

$$\ell(\mathcal{S}_1) = \#\{1 \leq k \leq d-1 : \ \mathcal{T}'_k \neq \varnothing\} = \ell(\mathcal{S}) - 1.$$

8.C.11.   We claim: $\pi_+(\mathcal{S}_1) = \mathcal{S}$, see Fig. 14.

As $\mathcal{S}_1 = (\pi_+^{-1}(\mathcal{S}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus \mathcal{T}_\ell))) \setminus \mathcal{K}$ and $\mathcal{K} = \pi_+^{-1}(\mathcal{T}_\ell) \cap \mathcal{E}$, we have $\mathcal{S} \setminus \mathcal{T}_\ell \subset \pi_+(\mathcal{S}_1) \subset \mathcal{S}$. Thus, to prove $\pi_+(\mathcal{S}_1) = \mathcal{S}$, it is enough to show $\mathcal{T}_\ell \subset \pi_+(\mathcal{S}_1)$. Pick a point $a \in \mathcal{T}_\ell$. As $Z \setminus \mathcal{C}$ is a Nash normal crossing divisor of M, there exists an open semialgebraic neighborhood $U \subset M$ of a equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(a) = 0$ and $u(U \cap Z) = \{x_t \cdots x_d = 0\}$ for some $t \leq d$. By 8.C.1 and the definition of $\mathcal{T}_\ell$ (see 8.C.2) the connected component $\mathcal{T}_\ell^a$ of $\mathcal{T}_\ell$ that contains a is contained in the irreducible components of Z that contain a. Thus, $u(\mathcal{T}_\ell^a \cap U) \subset \{x_t = 0, \dots, x_d = 0\}$. Shrinking U is necessary, we may assume $\mathcal{T}_\ell^a \cap U = \mathcal{T}_\ell \cap U$ is a closed subset of U and by 2.C.3 we may modify u in order to have $u(\mathcal{T}_\ell \cap U) = \{x_{e+1} = 0, \dots, x_d = 0\}$ for some $1 \leq e + 1 \leq t$. Consider coordinates $(x_{e+1}, \dots, x_d)$ in $\mathbb{R}^{d-e}$. By 5.A.5 there exists a Nash diffeomorphism $\Phi : \mathbb{R}^e \times [0, +\infty) \times \mathbb{S}^{d-e-1} \to V := \pi_+^{-1}(U)$ such that

$$u \circ \pi_+ \circ \Phi : \mathbb{R}^e \times [0, +\infty) \times \mathbb{S}^{d-e-1} \to \mathbb{R}^d, \ (y, \rho, w) \mapsto (y, \rho w).$$

Observe that $\Phi^{-1}(\mathcal{E} \cap V) = \mathbb{R}^e \times [0, +\infty) \times (\mathbb{S}^{d-e-1} \cap \{x_t \cdots x_d = 0\})$. We may assume
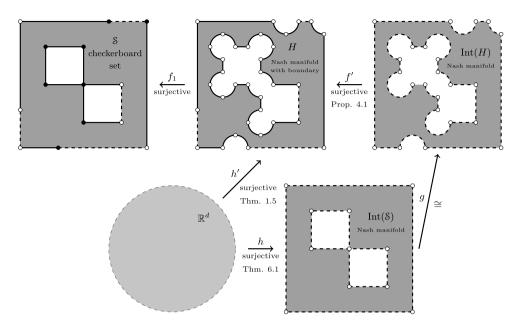
Fig. 15. Sketch of proof of the implication (vii) $\Longrightarrow$ (i) of Theorem 1.4.

$$\{x_t > 0, \ldots, x_d > 0\} \subset u((\mathcal{S} \setminus Z) \cap U).$$

Let $w_0 \in \{x_t > 0, \ldots, x_d > 0\} \subset \mathbb{R}^{d-e}$ be a unitary vector. We have $(u \circ \pi_+ \circ \Phi)(0, \rho, w_0) = \rho w_0 \in u((\mathcal{S} \setminus Z) \cap U)$ for each $\rho \geq 0$, so in particular $(u \circ \pi_+ \circ \Phi)(0, 0, w) = 0$. Thus,

$$\Phi(0, 0, w) \in (\pi_+^{-1}((\mathcal{S} \setminus (Z \setminus \mathcal{T}_\ell)) \cap U) \cap \mathrm{Cl}(\pi_+^{-1}((\mathcal{S} \setminus Z) \cap U))) \setminus \mathcal{E} \subset \mathcal{S}_1,$$

so $a \in \pi_+(\mathcal{S}_1)$. Consequently, $\mathcal{T}_\ell \subset \pi_+(\mathcal{S}_1)$.

8.C.12. We have proved that $\mathcal{S}_1$ is (up to a suitable Nash embedding of M') a checkerboard set such that $\ell(\mathcal{S}_1) = \ell(\mathcal{S}) - 1$ and $\mathtt{Sth}(\mathcal{S}_1)$ is Nash diffeomorphic to $\mathtt{Sth}(\mathcal{S})$ via π_+. By induction hypothesis there exist a connected Nash manifold H with boundary and a surjective Nash map $f_1 : H \to \mathcal{S}_1$ such that $\mathrm{Int}(H)$ is Nash diffeomorphic to $\mathtt{Sth}(\mathcal{S}_1)$, which is itself Nash diffeomorphic to $\mathtt{Sth}(\mathcal{S})$ via π_+. Thus, $f := \pi_+ \circ f_1 : H \to \mathcal{S}$ is a surjective Nash map and $\mathrm{Int}(H)$ is Nash diffeomorphic to $\mathtt{Sth}(\mathcal{S})$, as required. $\quad\square$

9. Nash path-connected components of a semialgebraic set

To take advantage of the full strength of Theorem 1.4 applied to an arbitrary semialgebraic set $\mathcal{S}$ we introduce the Nash path-components of a semialgebraic set. Recall that by Theorem 1.4 Nash path-connected and well-welded semialgebraic sets coincide.

Definition 9.1. A semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ admits a decomposition into *Nash path-connected components* if there exist semialgebraic sets $\mathcal{S}_1, \ldots, \mathcal{S}_r \subset \mathcal{S}$ such that:

(1) Each $\mathcal{S}_i$ is Nash path-connected.
(2) If $\mathcal{T} \subset \mathcal{S}$ is a Nash path-connected semialgebraic set that contains $\mathcal{S}_i$, then $\mathcal{S}_i = \mathcal{T}$.
(3) $\mathcal{S}_i \not\subset \bigcup_{j \neq i} \mathcal{S}_j$.
(4) $\mathcal{S} = \bigcup_{i=1}^{r} \mathcal{S}_i$.

Theorem 9.2. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semialgebraic set. Then $\mathcal{S}$ admits a decomposition into Nash path-connected components and this decomposition is unique. In addition, the Nash path-connected components of a semialgebraic set are closed in $\mathcal{S}$.*

Before proving Theorem 9.2 we need a preliminary result.

Lemma 9.3. *Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$ be two well-welded semialgebraic sets of dimension d such that $\dim(\mathcal{S}_1 \cap \mathcal{S}_2) = d$. Then $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ is well-welded.*

Proof. Let $x_k \in \mathcal{S}_k$ for $k = 1, 2$ and let $y \in (\mathcal{S}_1 \cap \mathcal{S}_2) \setminus \mathrm{Sing}(\mathcal{S})$. As each $\mathcal{S}_k$ is well-welded, there exist by Corollary 7.9 continuous semialgebraic paths $\alpha_k : [0, 1] \to \mathcal{S}_k$ such that $\alpha_k(0) = x_k, \alpha_k(1) = y$ and $\eta(\alpha_k) \subset \mathrm{Reg}(\mathcal{S}_k) \setminus \overline{\mathrm{Sing}(\mathcal{S})}^{\mathrm{zar}}$. The continuous semialgebraic path $\alpha := \alpha_1 * \alpha_2^{-1}$ connects the points x_1 and x_2 and satisfies

$$\eta(\alpha) \subset \eta(\alpha_1) \cup \eta(\alpha_2) \cup \{y\} \subset \mathrm{Reg}(\mathcal{S}).$$

Consequently, $\mathcal{S}$ is well-welded, as required. $\square$

Proof of Theorem 9.2. We divide the proof into two parts:

Existence. We proceed by induction on the dimension d of $\mathcal{S}$. If $d = 0$, the Nash path-connected components of $\mathcal{S}$ coincide with its connected components. Suppose the result true for dimension $\leq d - 1$ and let us see that it is also true for dimension d.

Let $\mathcal{S}_0$ be the (semialgebraic) set of points of dimension d of $\mathcal{S}$. Write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{T}$ where $\mathcal{T} := \mathcal{S} \cap \overline{\mathcal{S} \setminus \mathcal{S}_0}^{\mathrm{zar}}$. Observe that $\mathcal{S}_0$ and $\mathcal{T}$ are closed subsets of $\mathcal{S}$ and $\dim(\mathcal{T}) \leq d - 1$. In addition, $\overline{\mathcal{T}}^{\mathrm{zar}} = \overline{\mathcal{S} \setminus \mathcal{S}_0}^{\mathrm{zar}}$. Let $M_1, \ldots, M_r$ be the connected components of $\mathrm{Reg}(\mathcal{S}_0)$. Observe that $\mathrm{Cl}(M_i) \cap \mathcal{S}$ is Nash path-connected for $i = 1, \ldots, r$.

9.2.1.   We claim: *There exists a partition $I_1, \ldots, I_\ell$ of the set $\{1, \ldots, r\}$ such that*

$$\mathcal{S}_k := \bigcup_{i \in I_k} \mathrm{Cl}(M_i) \cap \mathcal{S}$$

is Nash path-connected for $k = 1, \ldots, \ell$ and for each $J \subset \{1, \ldots, \ell\}$ of cardinal ≥ 2 the semialgebraic set $\bigcup_{j \in J} \mathcal{S}_j$ is not Nash path-connected.
Define

$$\mathfrak{F}_1 := \left\{ I \subset \{1, \ldots, r\} : 1 \in I \quad \text{and} \quad \bigcup_{i \in I} \mathrm{Cl}(M_i) \cap \mathcal{S} \quad \text{is Nash path-connected} \right\}.$$

Observe that $\mathfrak{F}_1 \neq \varnothing$. If $J_1, J_2 \subset \mathfrak{F}_1$, then by Lemma 9.3 $J_1 \cup J_2 \in \mathfrak{F}_1$. Let I_1 be the maximum of $\mathfrak{F}_1$ ordered with respect to the inclusion. Denote $K := \{1, \ldots, r\} \setminus I_1$. By induction there exists a partition $I_2, \ldots, I_\ell$ of K such that $\mathcal{S}_k := \bigcup_{i \in I_k} \mathrm{Cl}(M_i) \cap \mathcal{S}$ is Nash path-connected for $k = 2, \ldots, \ell$ and for each $J \subset \{2, \ldots, \ell\}$ of cardinal ≥ 2 the semialgebraic set $\bigcup_{j \in J} \mathcal{S}_j$ is not Nash path-connected. It can be checked that the sets $I_1, \ldots, I_\ell$ satisfy the required properties.

9.2.2. Observe that $\mathcal{S}_0 = \bigcup_{i=1}^\ell \mathcal{S}_i$ and by Lemma 9.3 each intersection $\mathcal{S}_i \cap \mathcal{S}_j$ has dimension $\leq d - 1$ if $i \neq j$.

9.2.3. By induction hypothesis there exists a family $\mathcal{T}_1, \ldots, \mathcal{T}_p$ of Nash path-connected components of $\mathcal{T}$. We may assume, after eliminating redundant $\mathcal{T}_j$, that $\mathcal{T}_1, \ldots, \mathcal{T}_m$ satisfy the following: $\mathcal{S} = \mathcal{S}_0 \cup \bigcup_{j=1}^m \mathcal{T}_j$ and $\mathcal{T}_j \not\subset \mathcal{S}_0 \cup \bigcup_{k \neq j} \mathcal{T}_k$ for $j = 1, \ldots, m$.

9.2.4. Let us check that $\mathcal{S}_1, \ldots, \mathcal{S}_\ell, \mathcal{T}_1, \ldots, \mathcal{T}_m$ are a family of Nash path-connected components of $\mathcal{S}$. By construction they satisfy conditions (1), (3) and (4) of Definition 9.1. Let us check that they also satisfy condition (2). In particular, by Lemma 7.4 a Nash path-connected component of $\mathcal{S}$ is a closed subset of $\mathcal{S}$. We distinguish two possibilities accordingly to the different dimensions of the semialgebraic sets $\mathcal{S}_i$ and $\mathcal{T}_j$.

9.2.5. Let $\mathcal{R} \subset \mathcal{S}$ be a Nash path-connected semialgebraic set that contains $\mathcal{S}_1$. As $\mathcal{S}_1$ has dimension d, also $\mathcal{R}$ has dimension d. As $\mathcal{R}$ is pure dimensional, it is contained in $\mathcal{S}_0 = \bigcup_{i=1}^\ell \mathcal{S}_i$. We may assume that $\dim(\mathcal{R} \cap \mathcal{S}_i) = d$ exactly for $1 \leq i \leq s \leq \ell$. We claim: $s = 1$.

Otherwise, each union $\mathcal{S}_j \cup \mathcal{R}$ is Nash path-connected by Lemma 9.3 for $j = 1, \ldots, s$. Consequently, by Lemma 9.3 $\mathcal{R}' := \mathcal{R} \cup \bigcup_{j=1}^s \mathcal{S}_j$ is Nash path-connected. On the other hand, $\dim(\mathcal{R} \cap \mathcal{S}_j) < d$ for $j = s + 1, \ldots, \ell$, so the semialgebraic set $\mathcal{C}' := \bigcup_{j=s+1}^\ell \mathcal{R} \cap \mathcal{S}_j$ has dimension $< d - 1$ and satisfies

$$\mathcal{R}' = \mathcal{R} \cup \bigcup_{j=1}^s \mathcal{S}_j = \bigcup_{j=1}^s \mathcal{S}_j \cup \mathcal{C}'$$

because $\mathcal{R} \subset \mathcal{S}_0 = \bigcup_{i=1}^\ell \mathcal{S}_i$. As $\mathcal{R}'$ is pure dimensional,

$$\mathrm{Cl}(\mathcal{R}') \cap \mathcal{S} = \mathrm{Cl}(\mathcal{R}' \setminus \mathcal{C}') \cap \mathcal{S} = \mathrm{Cl}\left(\bigcup_{j=1}^s \mathcal{S}_j\right) \cap \mathcal{S} = \bigcup_{j=1}^s \mathcal{S}_j.$$

Consequently, $\bigcup_{j=1}^s \mathcal{S}_j$ is Nash path-connected, which contradicts 9.2.1.

As $s = 1$, it holds $\dim(\mathcal{R} \cap \mathcal{S}_j) < d$ for $2 \leq j \leq \ell$. Thus, the semialgebraic set $\mathcal{C} := \bigcup_{j=2}^\ell \mathcal{R} \cap \mathcal{S}_j$ has dimension $< d$ and satisfies $\mathcal{R} = \mathcal{S}_1 \cup \mathcal{C}$. As $\mathcal{R}$ is pure dimensional,

$$\mathcal{S}_1 \subset \mathcal{R} \subset \mathrm{Cl}(\mathcal{R}) \cap \mathcal{S} = \mathrm{Cl}(\mathcal{R} \setminus \mathcal{C}) \cap \mathcal{S} = \mathrm{Cl}(\mathcal{S}_1 \setminus \mathcal{C}) \cap \mathcal{S} = \mathcal{S}_1.$$

Consequently, $\mathcal{S}_1 = \mathcal{R}$.

9.2.6. Let $\mathcal{R} \subset \mathcal{S}$ be a Nash path-connected semialgebraic set that contains $\mathcal{T}_j$. We claim: $\mathcal{R} \subset \mathcal{T}$. Assume this proved for a while. As $\mathcal{T}_j$ is a Nash path-connected component of $\mathcal{T}$, we have $\mathcal{R} = \mathcal{T}_j$. Thus, the semialgebraic sets $\mathcal{S}_1, \ldots, \mathcal{S}_\ell, \mathcal{T}_1, \ldots, \mathcal{T}_m$ satisfy condition (2) of Definition 9.1. Consequently, it is enough to prove: $\mathcal{R} \subset \mathcal{T}$.

As $\mathcal{R}$ is Nash path-connected, it is pure dimensional. As $\mathcal{R} \not\subset \mathcal{S}_0$ (because $\mathcal{T}_j \not\subset \mathcal{S}_0$) and $\mathcal{S}_0$ is a closed subset of $\mathcal{S}$,

$$\dim(\mathcal{R}) = \dim(\mathcal{R} \setminus \mathcal{S}_0) = \dim(\mathcal{R} \cap (\mathcal{S} \setminus \mathcal{S}_0)) \leq \dim(\mathcal{R} \cap \mathcal{T}) \leq \dim(\mathcal{R}).$$

As $\mathcal{R}$ is Nash path-connected, it is irreducible. As $\dim(\mathcal{R} \cap \mathcal{T}) = \dim(\mathcal{R})$, we have $\overline{\mathcal{R}}^{\mathrm{zar}} = \overline{\mathcal{R} \cap \mathcal{T}}^{\mathrm{zar}} \subset \overline{\mathcal{T}}^{\mathrm{zar}} = \overline{\mathcal{S} \setminus \mathcal{S}_0}^{\mathrm{zar}}$, so

$$\mathcal{R} \subset \mathcal{S} \cap \overline{\mathcal{R}}^{\mathrm{zar}} \subset \mathcal{S} \cap \overline{\mathcal{T}}^{\mathrm{zar}} = \mathcal{S} \cap \overline{\mathcal{S} \setminus \mathcal{S}_0}^{\mathrm{zar}} =: \mathcal{T}.$$

Uniqueness. Let $\{\mathcal{S}_i\}_{i=1}^r$ and $\{\mathcal{R}_j\}_{j=1}^s$ be two families of semialgebraic sets satisfying the conditions of Definition 9.1. Assume $s \leq r$.

Let $x \in \mathcal{R}_1 \not\subset \bigcup_{j=2}^s \mathcal{R}_j$. As each $\mathcal{R}_j$ is a closed subset of $\mathcal{S}$, the difference $\mathcal{S} \setminus \bigcup_{j \neq 1} \mathcal{R}_j = \mathcal{R}_1 \setminus \bigcup_{j \neq 1} \mathcal{R}_j$ is an open neighborhood of x in $\mathcal{S}$, so $\mathcal{S}_x = \mathcal{R}_{1,x}$. In particular, $\mathcal{S}$ and $\mathcal{R}_1$ have the same dimension at x. As $\mathcal{S}_x = \mathcal{R}_{1,x}$ and $\dim(\mathcal{S}_x) = \max\{\dim(\mathcal{S}_{i,x}) : i = 1, \ldots, r\}$, we assume $\dim(\mathcal{S}_{1,x}) = \dim(\mathcal{S}_x) = \dim(\mathcal{R}_{1,x})$. As $\mathcal{S}_{1,x} = \mathcal{S}_{1,x} \cap \mathcal{S}_x = \mathcal{S}_{1,x} \cap \mathcal{R}_{1,x}$, we have $\dim(\mathcal{S}_{1,x}) = \dim(\mathcal{R}_{1,x}) = \dim(\mathcal{S}_{1,x} \cap \mathcal{R}_{1,x})$. As $\mathcal{S}_1, \mathcal{R}_1$ are pure dimensional, $\dim(\mathcal{S}_1) = \dim(\mathcal{R}_1) = \dim(\mathcal{R}_1 \cap \mathcal{S}_1)$. By Lemma 9.3 $\mathcal{S}_1 \cup \mathcal{R}_1$ is Nash path-connected, so by condition (2) $\mathcal{S}_1 = \mathcal{S}_1 \cup \mathcal{R}_1 = \mathcal{R}_1$. Proceeding similarly with $\mathcal{R}_2, \ldots, \mathcal{R}_s$, we assume $\mathcal{R}_i = \mathcal{S}_i$ for $i = 1, \ldots, s$. As $\mathcal{S} = \bigcup_{i=1}^s \mathcal{R}_i = \bigcup_{i=1}^s \mathcal{S}_i$, we deduce by condition (3) that $r = s$. Consequently, the families $\{\mathcal{S}_i\}_{i=1}^r$ and $\{\mathcal{R}_j\}_{j=1}^s$ coincide, as required. $\quad\square$

Examples 9.4. (i) Let $\mathcal{S} := \{x^2 z - y^2 = 0\} \subset \mathbb{R}^3$. The Nash path-connected components of $\mathcal{S}$ are $\mathcal{S}_1 := \mathcal{S} \cap \{z \geq 0\}$ and $\mathcal{S}_2 := \{x = 0, y = 0\}$.

(ii) Let $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \subset \mathbb{R}^3$, where

$$\mathcal{S}_1 := [-1, 1] \times [-2, 2] \times \{0\}, \quad \mathcal{S}_2 := [-2, -1] \times \{-1, 1\} \times [-1, 1],$$

$$\& \quad \mathcal{S}_3 := [1, 2] \times \{-1, 1\} \times [-1, 1].$$

The Nash path-connected components of $\mathcal{S}$ are $\mathcal{S}_1$, $\mathcal{S}_2 \cap \{y = 1\}$, $\mathcal{S}_2 \cap \{y = -1\}$, $\mathcal{S}_3 \cap \{y = 1\}$ and $\mathcal{S}_3 \cap \{y = -1\}$. In contrast, X has three irreducible components [11, Rmk.4.4(iii)], which are $\mathcal{S} \cap \{z = 0\}$, $\mathcal{S} \cap \{y = 1\}$ and $\mathcal{S} \cap \{y = -1\}$.

10. Two relevant consequences of Main Theorem 1.4

In this section we prove Corollaries 1.7 and 1.8.

10.A. Proof of Corollary 1.7

By Theorem 1.4 it is enough to prove that $\mathcal{S}$ is well-welded. The proof is conducted in several steps.

10.2.1. As $\mathcal{S}$ is an irreducible semialgebraic set, $\mathrm{Cl}(\mathcal{S})$ is an irreducible arc-symmetric semialgebraic set. Let X be the Zariski closure of $\mathcal{S}$ and let $\pi : \widetilde{X} \to X$ be a resolution of the singularities of X (Theorem 2.5). By [28, Thm.2.6] applied to the irreducible arc-symmetric set $\mathrm{Cl}(\mathcal{S})$ there exists a connected component E of $\widetilde{X}$ such that $\pi(E) = \mathrm{Cl}(\mathrm{Reg}(\mathrm{Cl}(\mathcal{S}))) = \mathrm{Cl}(\mathcal{S})$ (recall that $\mathrm{Cl}(\mathcal{S})$ is pure dimensional). As $\pi|_{\widetilde{X} \setminus \pi^{-1}(\mathrm{Sing}(X))} :$ $\widetilde{X} \setminus \pi^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}(X)$ is a regular diffeomorphism and $\mathcal{S} \setminus \mathrm{Sing}(X)$ is dense in $\mathrm{Cl}(\mathcal{S})$, we have $\mathrm{Cl}(\pi^{-1}(\mathcal{S} \setminus \mathrm{Sing}(X)) = E$. Let $\widetilde{\mathcal{S}} := \pi^{-1}(\mathcal{S}) \cap \mathrm{Cl}(\pi^{-1}(\mathcal{S} \setminus \mathrm{Sing}(X))) = \pi^{-1}(\mathcal{S}) \cap E$ be the strict transform of $\mathcal{S}$ under π. We claim: $\widetilde{\mathcal{S}}$ *is connected.*

Otherwise, there exists a closed semialgebraic subset $\mathcal{C} \subset E$ such that $\widetilde{\mathcal{S}} \subset E \setminus \mathcal{C}$ and $E \setminus \mathcal{C}$ is not connected. As π is proper, $\mathcal{C}' := \pi(\mathcal{C})$ is a closed subset of X. Observe that $\mathcal{S} \cap \mathcal{C}' = \varnothing$. Let Y be the smallest Nash subset of $\mathbb{R}^n \setminus \mathcal{C}'$ that contains $\mathcal{S}$. As $\mathcal{S}$ is irreducible, also Y is irreducible. Observe that Y is an irreducible component of the Nash subset $X \setminus \mathcal{C}'$ of $\mathbb{R}^n \setminus \mathcal{C}'$ and $(\widetilde{Y} := \pi^{-1}(Y) \cap E, \pi|_{\widetilde{Y}})$ is a resolution of the singularities of Y. As Y is irreducible, also $\widetilde{Y}$ is irreducible, so $\widetilde{Y} \subset E \setminus \mathcal{C}$ is connected. As $\widetilde{\mathcal{S}}$ is dense in E, we conclude that $\widetilde{Y}$ meets all the connected components of $E \setminus \mathcal{C}$, which is a contradiction because $\widetilde{Y}$ is connected but $E \setminus \mathcal{C}$ is not. Thus, $\widetilde{\mathcal{S}}$ is connected.

10.2.2. We prove next: $\widetilde{\mathcal{S}}$ *is well-welded.* Once this is done, $\mathcal{S} = \pi(\widetilde{\mathcal{S}})$ is by Lemma 7.11 well-welded. As $\widetilde{\mathcal{S}}$ is a connected dense semialgebraic subset of the Nash manifold E, it is enough to check: *If* $\mathcal{T}$ *is a connected dense semialgebraic subset of a Nash manifold* M, *then* $\mathcal{T}$ *is well-welded.*

Let $p \in \mathcal{T} \setminus \mathrm{Int}_M(\mathcal{T})$, where $\mathrm{Int}_M(\mathcal{T})$ denotes the interior of $\mathcal{T}$ in M. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the connected components of $\mathrm{Int}_M(\mathcal{T})$ whose closures contain p. Let $L \subset M$ be a compact semialgebraic neighborhood of p. By [4, Thm.9.2.1 & Rmk.9.2.3] there exists a finite simplicial complex K and a semialgebraic homeomorphism $\Phi : |K| \to L$ such that $\{p\}$ and each semialgebraic set $\mathcal{C}_k \cap L$ is a finite union of images $\Phi(\sigma^0)$ where $\sigma \in K$ and the restriction $\Phi|_{\sigma^0} : \sigma^0 \to L \subset M$ is a Nash embedding for each $\sigma \in K$.

It holds that given two connected components $\mathcal{C}_i$ and $\mathcal{C}_j$ there exist finitely many connected components $\mathcal{C}_{i_1}, \ldots, \mathcal{C}_{i_s}$ such that $\mathcal{C}_i = \mathcal{C}_{i_1}$, $\mathcal{C}_j = \mathcal{C}_{i_s}$ and $\mathrm{Cl}(\mathcal{C}_{i_k}) \cap \mathrm{Cl}(\mathcal{C}_{i_{k+1}}) \cap L$ contains the image under Φ of a simplex of dimension $d - 1$ that contains $\{p\}$ as a vertex.

10.2.3. To prove that $\mathcal{T}$ is well-welded, it is enough by Corollary 7.6 to show the following: *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two connected components of $\mathrm{Int}_M(\mathcal{T})$ that contain p in their closure and $\mathrm{Cl}(\mathcal{C}_1) \cap \mathrm{Cl}(\mathcal{C}_2) \cap L$ contains the image under Φ of a simplex σ of dimension $\dim(\mathcal{S}) - 1$ that contains $\{p\}$ as a vertex. Then there exists an analytic path $\alpha : [-1,1] \to \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{p\}$ such that $\alpha((-1,0)) \subset \mathcal{C}_1$, $\alpha((0,1)) \subset \mathcal{C}_2$ and $\alpha(0) = \{p\}$.*

Denote $\mathcal{A}_0 := \Phi(\sigma^0)$. By the Nash curve selection lemma there exists a Nash path $\beta : (-1,1) \to M$ such that $\beta(0) = p$ and $\beta((0,1)) \subset \mathcal{A}_0$. After a change of coordinates, we may assume by [4, Cor.9.3.10] that the projection $\rho : \mathbb{R}^d := \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^{d-1} \times \{0\}$ induces a Nash diffeomorphism $\rho|_{\mathcal{A}} : \mathcal{A} \to \rho(\mathcal{A})$ where $\mathcal{A}$ is an open semialgebraic subset of $\mathcal{A}_0$ that contains p in its closure. Let $\phi : \rho(\mathcal{A}) \to \mathbb{R}$ be a Nash function such that $\mathcal{A} = \mathrm{graph}(\phi)$. The Nash map

$$\varphi : \Omega := \rho(\mathcal{A}) \times \mathbb{R} \to \rho(\mathcal{A}) \times \mathbb{R}, \ (x', x_d) \mapsto (x', x_d - \phi(x'))$$

induces a Nash diffeomorphism that maps $\mathcal{A}$ onto $\rho(\mathcal{A}) \times \{0\}$. In addition, we may assume that $\varphi(\mathcal{C}_1 \cap \mathrm{Int}(L) \cap \Omega)$ is an open semialgebraic subset of $\rho(\mathcal{A}) \times (-\infty, 0)$ whose boundary contains $\rho(\mathcal{A}) \times \{0\}$ and $\varphi(\mathcal{C}_2 \cap \mathrm{Int}(L) \cap \Omega)$ is an open semialgebraic subset of $\rho(\mathcal{A}) \times (0, +\infty)$ whose boundary contains $\rho(\mathcal{A}) \times \{0\}$. For each $\ell \geq 1$ consider the Nash path

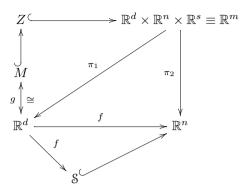$$\gamma_\ell : (-1,1) \to \mathbb{R}^d, \ t \mapsto ((\rho \circ \beta)(t^2), t^{2\ell+1}).$$

Observe that for $\ell \geq 1$ large enough, we may assume $\gamma_\ell((-1,0)) \subset \varphi(\mathcal{C}_1 \cap \mathrm{Int}(L) \cap \Omega)$, $\gamma_\ell(0) = (\rho(p), 0)$ and $\gamma_\ell((0,1)) \subset \varphi(\mathcal{C}_2 \cap \mathrm{Int}(L) \cap \Omega)$. Thus, for $\ell \geq 1$ large enough the Nash path

$$\alpha_\ell : (-1,1) \to \mathbb{R}^d, \ t \mapsto \beta(t^2) + (0, \dots, 0, t^{2\ell+1})$$

satisfies $\alpha_\ell((-1,0)) \subset \mathcal{C}_1$, $\alpha_\ell(0) = p$ and $\alpha_\ell((0,1)) \subset \mathcal{C}_2$ because $\alpha_\ell|_{(-1,1)\setminus\{0\}} = \varphi^{-1} \circ \gamma_\ell|_{(-1,1)\setminus\{0\}}$, as required. $\square$

10.B. Proof of Corollary 1.8

(i) By Theorem 1.4 there exists a Nash map $f : \mathbb{R}^d \to \mathbb{R}^n$ whose image is $\mathcal{S}$. By Artin–Mazur's description [4, Thm.8.4.4] of Nash maps there exist $s \geq 1$ and a non-singular irreducible algebraic set $Z \subset \mathbb{R}^{d+n+s}$ of dimension d, a connected component M of Z and a Nash diffeomorphism $g : \mathbb{R}^d \to M$ such that the following diagram is commutative.

We denote the projection of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^s$ onto the first space $\mathbb{R}^d$ with π_1 and the projection of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^s$ onto the second space $\mathbb{R}^n$ with π_2. Write $m := d + n + s$. By [29] applied to M and the union of the remaining connected components of Z there exist finitely many polynomials $P_1, \ldots, P_\ell, Q_1, \ldots, Q_\ell \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_m]$ such that each Q_j is strictly positive on $\mathbb{R}^m$ and

$$M = Z \cap \Big\{ \sum_{j=1}^{\ell} P_j \sqrt{Q_j} > 0 \Big\}.$$

Observe that M is the projection of the algebraic set

$$Y := \Big\{ (x, y, t) \in Z \times \mathbb{R}^\ell \times \mathbb{R} : \ \Big(\sum_{j=1}^{\ell} P_j y_j^2 \Big) t^2 - 1 = 0, \ y_j^4 - Q_j = 0 \text{ for } j = 1, \ldots, \ell \Big\}$$

under $\pi : \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}^m, \ (x, y, t) \mapsto x$.

Fix $\epsilon := (\epsilon_1, \ldots, \epsilon_\ell, \epsilon_{\ell+1}) \in \{-1, 1\}^{\ell+1}$ and let $M_\epsilon := Y \cap \{\epsilon_1 y_1 > 0, \ldots, \epsilon_\ell y_\ell > 0, \epsilon_{\ell+1} t > 0\}$. Consider the Nash diffeomorphism

$$\varphi_\epsilon : M \to M_\epsilon, \ x \mapsto \Big(x, \epsilon_1 \sqrt[4]{Q_1(x)}, \ldots, \epsilon_\ell \sqrt[4]{Q_\ell(x)}, \epsilon_{\ell+1} \frac{1}{\sqrt{\sum_{j=1}^{\ell} P_j(x) \sqrt{Q_j(x)}}} \Big)$$

whose inverse map is the restriction to M_ϵ of the projection π.

Observe that $\{M_\epsilon\}_{\epsilon \in \{-1,1\}^{\ell+1}}$ is the collection of the connected components of Y. As $\pi(M_\epsilon) = M$ and using the diagram above, we deduce

$$(\pi_2 \circ \pi)(M_\epsilon) = \pi_2(M) = (f \circ \pi_1)(M) = f(\mathbb{R}^d) = \mathcal{S}.$$

In addition, each M_ϵ is Nash diffeomorphic to $\mathbb{R}^d$ and for $\varepsilon \neq \varepsilon'$ the polynomial map

$$\phi_{\epsilon, \epsilon'} : \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R}, \ (x, y, t) \mapsto (x, (\epsilon \cdot \epsilon') \cdot (y, t))$$

induces an involution of Y such that $\phi_{\epsilon, \epsilon'}(M_\epsilon) = M_{\epsilon'}$. We have denoted

$$(\epsilon \cdot \epsilon') \cdot (y, t) := (\epsilon_1 \epsilon_1' y_1, \ldots, \epsilon_\ell \epsilon_\ell' y_\ell, \epsilon_{\ell+1} \epsilon_{\ell+1}' t).$$

As Z is non-singular, also Y is non-singular. Let X be the irreducible component of Y that contains $M_{(1,\ldots,1)}$. Then $k := m + \ell + 1 - n$ and the non-singular algebraic set X satisfy the requirements in the statement.

In addition, each connected component of X is Nash diffeomorphic to $\mathbb{R}^d$ and it has finitely many. Thus, X is Nash diffeomorphic to $\mathbb{R}^d \times \{1, \ldots, s\}$, where s is the number of connected components of X.

(ii) Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be the Nash path-components of $\mathcal{S}$, which satisfy $\mathcal{S} = \bigsqcup_{i=1}^r \mathcal{S}_i$. By (i) there exist $m \geq 1$ and for each $i = 1, \ldots, r$ a non-singular algebraic set $X_i \subset \mathbb{R}^m$ that is Nash diffeomorphic to a disjoint union of affine subspaces of $\mathbb{R}^{d+1}$ (all of them affinely equivalent to $\mathbb{R}^{d_i}$ where $d_i := \dim(\mathcal{S}_i) \leq d = \dim(\mathcal{S})$) and satisfies $\pi(X_i) = \mathcal{S}_i$, where $\pi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$, $(x, y) \mapsto x$ is the projection onto the first n coordinates. Consider the pairwise disjoint union $X := \bigsqcup_{i=1}^r X_i \times \{i\} \subset \mathbb{R}^{m+1}$ and the projection $\pi' : \mathbb{R}^n \times \mathbb{R}^{m+1-n} \times \mathbb{R} \to \mathbb{R}^n$, $(x, y, t) \mapsto x$. Then X is a non-singular algebraic set, which is Nash diffeomorphic to a pairwise disjoint union of affine subspaces of $\mathbb{R}^{d+1}$ and satisfies $\pi(X) = \mathcal{S}$, as required. $\square$

The following example shows that Corollary 1.8 is somehow sharp.

Example 10.1. Let $X \subset \mathbb{R}^n$ be a real algebraic curve Nash diffeomorphic to $\mathbb{R}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a linear projection. Then $\pi(X)$ is not a proper open interval of $\mathbb{R}$.

Notice that $Y := \mathrm{Cl}_{\mathbb{RP}^n}(X) = X \cup \{p_\infty\}$ where p_∞ is a certain point of the hyperplane of infinity of $\mathbb{RP}^n$. Observe that π is the restriction to $\mathbb{R}^n$ of a central projection $\Pi : \mathbb{RP}^n \dashrightarrow \mathbb{RP}^1$ with center a projective subspace L of $H_\infty(\mathbb{R})$ of dimension $n - 2$.

If $p_\infty \notin L$, then $\Pi(Y)$ is a compact subset of $\mathbb{RP}^1$ and $\Pi(p_\infty)$ is the point at infinity of $\mathbb{RP}^1$. Thus, $\pi(X)$ is a closed semialgebraic subset of $\mathbb{R}$.

If $p_\infty \in L$, we assume by contradiction that $\pi(X)$ is a proper open interval of $\mathbb{R}$. Then Y has at least two different tangents at p_∞. However, as X is Nash diffeomorphic to $\mathbb{R}$, the analytic germ Y_{p_∞} has only one branch, which is a contradiction. Thus, $\pi(X)$ is not a proper open interval of $\mathbb{R}$.

Remark 10.2. Let $\mathcal{S} := (0, 1) \subset \mathbb{R}$. By Corollary 1.8 there exist $n > 0$ and an algebraic set $X \subset \mathbb{R}^{n+1}$ whose connected components are Nash diffeomorphic to $\mathbb{R}$ and a projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\pi(X) = (0, 1)$. By Example 10.1 we know that X is not connected.

Appendix A. Miscellanea of $\mathcal{C}^2$ semialgebraic diffeomorphisms between intervals

We present examples of $\mathcal{S}^2$ diffeomorphisms between intervals required in Section 4 and 5.
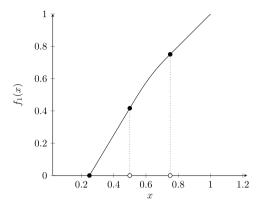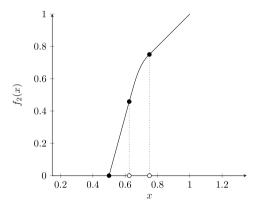
Fig. A.16. Graph of f_1.

Fig. A.17. Graph of f_2.

Examples A.1. (i) The function $f_1 : [\frac{1}{4}, 1) \to [0, 1)$ given by

$$
f_1(x) := \begin{cases} \frac{5}{12}(4x - 1) & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ \frac{1}{3}(64x^4 - 160x^3 + 144x^2) - 17x + \frac{9}{4} & \text{if } \frac{1}{2} \le x \le \frac{3}{4}, \\ x & \text{if } \frac{3}{4} \le x \le 1 \end{cases}
$$

is an $\mathcal{S}^2$ diffeomorphism such that $f_1|_{[\frac{3}{4},1)} = \mathrm{id}_{[\frac{3}{4},1)}$, see Fig. A.16.

(ii) The function $f_2 : [\frac{1}{2}, 1) \to [0, 1)$ given by

$$
f_2(x) := \begin{cases} \frac{11}{6}(2x - 1) & \text{if } \frac{1}{2} \le x \le \frac{5}{8}, \\ 2048(\frac{x^4}{3} - \frac{11x^3}{12} + \frac{15x^2}{16}) - 863x + 144 & \text{if } \frac{5}{8} \le x \le \frac{3}{4}, \\ x & \text{if } \frac{3}{4} \le x \le 1 \end{cases}
$$

is an $\mathcal{S}^2$ diffeomorphism such that $f_2|_{[\frac{3}{4},1)} = \mathrm{id}_{[\frac{3}{4},1)}$, see Fig. A.17.
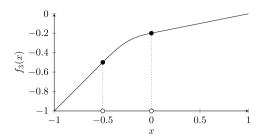
Fig. A.18. Graph of f_3.

(iii) The function $f_3 : (-1, 1) \to (-1, 0)$ given by

$$
f_3(x) := \begin{cases} x & \text{if } -1 < x \leq -\frac{1}{2}, \\ \frac{1}{5}(16x^4 + 16x^3 + x - 1) & \text{if } -\frac{1}{2} \leq x \leq 0, \\ \frac{1}{5}(x - 1) & \text{if } 0 \leq x < 1 \end{cases}
$$

is an $\mathcal{S}^2$ diffeomorphism such that $f_3|_{(-1, -\frac{1}{2}]} = \mathrm{id}_{(-1, -\frac{1}{2}]}$, see Fig. A.18.

Appendix B. Strict transform of Nash arcs under blow-up

We recall here that the strict transform of an irreducible Nash curve germ under a sequence of blow-ups is again an irreducible Nash curve germ. We analyze first the images under Nash parameterization germs.

Lemma B.1. *Let $g := (g_1, \ldots, g_n) : \mathbb{R}_0 \to \mathbb{R}_0^n$ be an analytic map germ such that the germs $g(\{t > 0\}_0)$ and $g(\{t < 0\}_0)$ are different. Then $\mathrm{Im}(g)$ is an irreducible analytic curve germ. In addition, if g is Nash, then $\mathrm{Im}(g)$ is an irreducible Nash curve germ.*

Proof. Let $G : \mathbb{C}_0 \to \mathbb{C}_0^n$ be the complex analytic map germ induced by g. As G is not identically zero, $G^{-1}(0) = \{0\}$. By [5, Thm.3.4.24] G is a finite analytic map germ. Using a finite representative of $G : \mathbb{C}_0 \to \mathbb{C}_0^n$ and Remmert's Theorem [32, Thm.VII.2.2] we deduce that $Z_0 := G(\mathbb{C}_0)$ is a 1-dimensional analytic germ. Consequently, $X_0 := Z_0 \cap \mathbb{R}_0^n$ is an irreducible analytic curve germ. Let $\theta : \mathbb{C}_0 \to Z_0$ be a normalization of Z_0 that is invariant under conjugation. As X_0 is coherent, $\theta(\mathbb{R}_0) = X_0$. There exists an analytic map germ $F : \mathbb{C}_0 \to \mathbb{C}_0$ such that $G = \theta \circ F$. As G and θ are invariant under conjugation, also F is invariant under conjugation. Then F is a univariate analytic function germ of order $k \geq 1$ with real coefficients. As $g(\{t > 0\}_0)$ and $g(\{t < 0\}_0)$ are different, k is odd and $g(\mathbb{R}_0) = X_0$ is an irreducible analytic curve germ. The Nash case is now straightforward. $\square$

Lemma B.2. *Let $\Gamma_0 \subset \mathbb{R}_0^n$ be an irreducible Nash curve germ and let Γ be a representative. Let $f : M \to N$ be a Nash map between Nash manifolds and let $Z \subset M$ and $Y \subset N \subset$*

$\mathbb{R}^n$ *be Nash subsets such that* $0 \in Y$. *Assume that* $\Gamma_0 \subset N$, $\Gamma_0 \not\subset Y$, *the restriction* $f|_{f^{-1}(\Gamma) \cap Z} : f^{-1}(\Gamma) \cap Z \to \Gamma$ *is proper and the restriction* $f|_{f^{-1}(\Gamma \setminus Y) \cap Z} : f^{-1}(\Gamma \setminus Y) \cap Z \to \Gamma \setminus Y$ *is bijective. Denote* $\Lambda := \mathrm{Cl}(f^{-1}(\Gamma \setminus Y) \cap Z) \cap f^{-1}(\Gamma) \cap Z$. *Then there exists a point* $p \in f^{-1}(0) \cap Z$ *such that* Λ_p *is an irreducible Nash curve germ,* $\Lambda_p \cap f^{-1}(Y) = \{p\}$ *and* $f(\Lambda_p) = \Gamma_0$.

Proof. We may assume $\Gamma \cap Y = \{0\}$ and $\Gamma = \mathrm{Im}(\gamma)$ where $\gamma : (-1,1) \to N$ is a Nash arc such that $\gamma(0) = 0$ and γ is a homeomorphism onto its image. We claim: $\Lambda \cap f^{-1}(Y) \cap Z$ is a singleton $\{p\}$ and Λ_p is an irreducible Nash curve germ.

The map $g := f|_\Lambda : \Lambda \to \Gamma$ is proper. Thus, the restriction $g|_{\Lambda \setminus g^{-1}(0)} : \Lambda \setminus g^{-1}(0) \to \Gamma \setminus \{0\}$ is proper and bijective, so it is a semialgebraic homeomorphism. As g is proper, $g^{-1}(0) \neq \varnothing$. Let $p \in g^{-1}(0)$ and observe that $p \in \mathrm{Cl}(g^{-1}(\gamma((0,1)))) \cup \mathrm{Cl}(g^{-1}(\gamma((-1,0))))$.

Assume $p \in \mathcal{T} := \mathrm{Cl}(g^{-1}(\gamma((0,1))))$. Let $X_{1,p}$ be the smallest Nash germ that contains $\mathcal{T}_p$. Observe that $X_{1,p} \cap f^{-1}(Y)_p = \{p\}$ and $\gamma((0,1))_0 \subset f(X_{1,p}) \subset \Gamma_0$. As $f|_{f^{-1}(\Gamma \setminus Y)} : f^{-1}(\Gamma \setminus Y) \to \Gamma \setminus Y$ is bijective, $f(X_{1,p})$ is by Lemma B.1 a Nash germ. Thus, $f(X_{1,p}) = \Gamma_0$ and $p \in \mathrm{Cl}(g^{-1}(\gamma((-1,0))))$. As Γ is semialgebraically homeomorphic to $(-1,1)$, we conclude $\Lambda \setminus g^{-1}(0)$ is semialgebraically homeomorphic to $(-1,1) \setminus \{0\}$. Consequently, $\Lambda \cap f^{-1}(Y) = \{p\}$ is a singleton and $\Lambda_p = X_{1,p}$ is an irreducible Nash curve germ, as required. $\square$

Appendix C. Algebraic structure of a Nash normal crossing divisor

As an application of classical algebraization Artin–Mazur's result for Nash functions [4, §8.4], we show that if Y is a Nash normal crossing divisor of a Nash manifold $M \subset \mathbb{R}^n$, there exists a Nash immersion of M in some affine space $\mathbb{R}^m$ such that M is a union of some connected components of its Zariski closure V in $\mathbb{R}^m$ (a non-singular real algebraic subset of $\mathbb{R}^m$), the Zariski closure X of Y in $\mathbb{R}^m$ is a normal crossings divisor of V and $M \cap X = Y$.

Lemma C.1. *Let* $M \subset \mathbb{R}^n$ *be a Nash manifold of dimension* d *and let* $Y \subset M$ *be a Nash normal crossings divisor of* M. *Then up to a suitable Nash embedding of* M *in some affine space* $\mathbb{R}^m$ *we may assume:*

(i) *M is a union of connected components of its Zariski closure* V *in* $\mathbb{R}^m$, *which is in addition a non-singular real algebraic subset of* $\mathbb{R}^m$ *of pure dimension* d.
(ii) *The Zariski closure* X *of* Y *in* $\mathbb{R}^m$ *is a normal crossings divisor of* V *and* $M \cap X = Y$.

Proof. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y as a Nash subset of M, which are non-singular Nash hypersurfaces of M in general position. As $\mathcal{N}(M)$ is a noetherian ring, there exist finitely many Nash functions $g_{ij} \in \mathcal{N}(M)$ such that $I(Y_i) = (g_{i1}, \ldots, g_{i\ell}) \mathcal{N}(M)$. Let $y \in Y$ and assume (after reordering the indices i if necessary) that $y \in Y_i$ exactly for $i = 1, \ldots, e$. As Y is a Nash normal crossings

divisor of M, we may assume (after reordering the indices j if necessary) that the linear forms $d_y g_{11}, \ldots, d_y g_{e1} : T_y M \to \mathbb{R}$ are linearly independent and Y coincides with $\{g_{11} \cdots g_{e1} = 0\}$ in some neighborhood of y in M, see [17, Prop.5.1].

By Artin–Mazur's description of Nash functions [4, §8.4] M is up to a Nash diffeomorphism an open and closed subset of a non-singular real algebraic set $V_0 \subset \mathbb{R}^m$ of pure dimension d, which is in addition the Zariski closure of M, and the Nash functions g_{ik} are restrictions to M of polynomial functions G_{ik} on V_0. Consider the real algebraic sets

$$X_{0i} := \{x \in V_0 : \ G_{i1}(x) = 0, \ldots, G_{i\ell}(x) = 0\} \quad \text{and} \quad X_0 := X_{01} \cup \cdots \cup X_{0r}.$$

The obstructions for X_0 to be a Nash normal crossings divisor of V_0 concentrate outside M, because $X_{0i} \cap M = Y_i$ for $i = 1, \ldots, r$ and Y is a Nash normal crossings divisor. In addition, $X_0 \cap M = Y_1 \cup \cdots \cup Y_r = Y$.

C.1.1. We claim: *There exists an algebraic subset Z of V_0 satisfying the following:*

(1) $M \cap Z = \varnothing$.
(2) *Pick $x \in X_0 \setminus Z$ and suppose (after reordering the indices i if necessary) that $x \in X_{0i}$ exactly for $i = 1, \ldots, e$. Then we may assume (after reordering the indices j if necessary) that the linear forms $d_x G_{11}, \ldots, d_x G_{e1} : T_y V_0 \to \mathbb{R}$ are linearly independent and there exists an open Zariski neighborhood U of x in $V_0 \setminus Z$ such that $X_0 \cap U = \{G_{11} \cdots G_{e1} = 0\} \cap U$.*

Assume statement C.1.1 proved for a while and let $h \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_m]$ be a polynomial equation of Z in $\mathbb{R}^m$. Consider the algebraic sets

$$V := \{(x, y) \in V_0 \times \mathbb{R} : \ yh(x) = 1\} \quad \text{and} \quad X := \{(x, y) \in X_0 \times \mathbb{R} : \ yh(x) = 1\},$$

which are biregularly equivalent to the constructible sets $V_0 \setminus Z$ and $X_0 \setminus Z$ via the projection onto the first m coordinates. Using [5, Cor.4.3.18] the reader can check readily that V is a non-singular algebraic subset of $\mathbb{R}^{m+1}$ that, up to a Nash diffeomorphism, contains M as an open and closed subset and X is a normal crossing divisor of V such that $X \cap M = Y$.

C.1.2. Let Z_0 be the union of the singular sets $\mathrm{Sing}(\overline{Y_i}^{\mathrm{zar}})$ for $i = 1, \ldots, r$ and all the irreducible components of X_0 different from $\overline{Y_1}^{\mathrm{zar}}, \ldots, \overline{Y_r}^{\mathrm{zar}}$ (which are irreducible algebraic sets because $Y_1, \ldots, Y_r$ are irreducible Nash sets). The irreducible components of X_0 different from $\overline{Y_1}^{\mathrm{zar}}, \ldots, \overline{Y_r}^{\mathrm{zar}}$ do not meet M because: $X_0 \cap M = Y$ and for each $y \in Y$ we assume $y \in Y_i$ exactly for $i = 1, \ldots, e$ (after reordering the indices i if necessary), the linear forms $d_y g_{11}, \ldots, d_y g_{e1} : T_y M \to \mathbb{R}$ are linearly independent (after reordering the indices j if necessary) and Y coincides with $\{g_{11} \cdots g_{e1} = 0\}$ in some neighborhood of y in M (recall that $g_{ij} = G_{ij}|_M$ for each pair of indices i, j). We have

$$Y_i \subset \overline{Y_i}^{\mathrm{zar}} \cap M \subset X_{0i} \cap M = Y_i \quad \text{and} \quad \mathrm{Sing}(\overline{Y_i}^{\mathrm{zar}}) \cap M \subset \mathrm{Sing}(X_{0i}) \cap M = \varnothing.$$

Consequently, $Z_0 \cap M = \varnothing$.

For each $i = 1, \ldots, r$ let $G_{i1}, \ldots, G_{i\ell}, G_{i,\ell+1} \ldots, G_{is}$ be a system of generators of the ideal $I_{\mathbb{R}}(Y_i) = I_{\mathbb{R}}(\overline{Y}^{\mathrm{zar}})$ of polynomials in $\mathbb{R}[\mathbf{x}]$ vanishing identically on Y_i. Let $P_1, \ldots, P_s$ be a system of generators of the ideal $I_{\mathbb{R}}(M) = I_{\mathbb{R}}(V_0)$ of polynomials in $\mathbb{R}[\mathbf{x}]$ vanishing identically on M. As V_0 is non-singular and has dimension d, the rank

$$\mathrm{rk}\{d_x P_1, \ldots, d_x P_s\} = m - d$$

for each $x \in V_0$. Denote $I_r := \{1, \ldots, r\}$ and for each non-empty subset $I \subset I_r$ define

$$Z_I := \left\{ x \in \bigcap_{i \in I} \overline{Y_i}^{\mathrm{zar}} : \mathrm{rk}\{d_x G_{ij}, d_x P_j : T_x \mathbb{R}^m \to \mathbb{R} : i \in I, \ j = 1, \ldots, s\} < m - d + \#I \right\}$$
$$\subset X_0,$$

where $\#I$ denotes the cardinal of I, and observe that $Y \cap Z_I = \varnothing$. Thus, $Z := Z_0 \cup \bigcup_{\varnothing \neq I \subset I_r} Z_I$ is an algebraic subset of $X_0 \subset V_0$. If $x \in X_0 \setminus Z$, we define $I_x := \{i \in I_r : x \in \overline{Y_i}^{\mathrm{zar}}\}$. Let us check: $Z \cap M = \varnothing$ and for each $x \in X_0 \setminus Z$ the rank

$$\mathrm{rk}\{d_x G_{ij}, d_x P_j(x) : T_x \mathbb{R}^m \to \mathbb{R} : \ i \in I_x, \ j = 1, \ldots, s\} = m - d + \#I_x$$

and there exists an open Zariski neighborhood U of x in V_0 such that $X_0 \cap U = \{G_{1j_1} \cdots G_{rj_r} = 0\} \cap U$ for some indices $1 \le j_1, \ldots, j_r \le s$ (depending on x).

Observe first that

$$M \cap Z = M \cap \bigcup_{\varnothing \neq I \subset I_r} Z_I = M \cap X_0 \cap \bigcup_{\varnothing \neq I \subset I_r} Z_I = Y \cap \bigcup_{\varnothing \neq I \subset I_r} Z_I = \varnothing.$$

Fix a point $x \in X_0 \setminus Z$. As $x \notin Z_{I_x}$, we have

$$\mathrm{rk}\{d_x G_{ij}, d_x P_j : T_x \mathbb{R}^m \to \mathbb{R} : \ i \in I_x, \ j = 1, \ldots, s\} = (m - d) + \#I_x.$$

We may assume (after reordering the indices i, j if necessary) that $I_x := \{1, \ldots, e\}$ and

$$\mathrm{rk}\{d_x G_{11}, \ldots, d_x G_{e1}, d_x P_1, \ldots, d_x P_{m-d} : T_x \mathbb{R}^m \to \mathbb{R}\} = (m - d) + e. \tag{C.1}$$

Let Z_x be the union of the irreducible components of the real algebraic set $\{G_{11} \cdots G_{e1} = 0\}$ different from $\overline{Y_1}^{\mathrm{zar}}, \ldots, \overline{Y_e}^{\mathrm{zar}}$ and observe that $\{G_{11} \cdots G_{e1} = 0\} \setminus Z_x = (\bigcup_{i=1}^{e} \overline{Y_i}^{\mathrm{zar}}) \setminus Z_x$. Condition (C.1) guarantees that $x \notin Z_x$. Consider the open Zariski neighborhood

$$U := \{y \in V_0 : \ \mathrm{rk}\{d_y G_{11}, \ldots, d_y G_{e1}, d_y P_1, \ldots, d_y P_{m-d} : T_y \mathbb{R}^m \to \mathbb{R}\} = (m - d) + e\}$$
$$\setminus \left(Z_x \cup Z \cup \bigcup_{i=e+1}^{r} \overline{Y_i}^{\mathrm{zar}} \right)$$

of x in $V_0 \setminus Z$. Observe that $X_0 \cap U = (\bigcup_{i=1}^e \overline{Y_i}^{\text{zar}}) \cap U = \{G_{11} \cdots G_{e1} = 0\} \cap U$, as required. $\square$

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