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ON THE FINITENESS OF PYTHAGORAS NUMBERS OF REAL MEROMORPHIC FUNCTIONS

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ABSTRACT. — We consider the 17th Hilbert Problem for global real analytic functions in a modified form that involves infinite sums of squares. Then we prove a local-global principle for a real global analytic function to be a sum of squares of global real meromorphic functions. We deduce that an affirmative solution to the 17th Hilbert Problem for global real analytic functions implies the finiteness of the Pythagoras number of the field of global real meromorphic functions, hence that of the field of real meromorphic power series. This measures the difficulty of the 17th Hilbert problem in the analytic case.

RÉSUMÉ (*Sur la finitude des nombres de Pythagore des fonctions méromorphes réelles*)

Nous considérons le 17^e problème de Hilbert pour les fonctions analytiques réelles globales sous une forme modifiée faisant intervenir des sommes infinies de carrés. Nous

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démontrons alors un principe local-global pour qu'une fonction analytique réelle globale soit une somme de carrés de fonctions méromorphes réelles globales. Nous déduisons qu'une solution affirmative au 17^e problème de Hilbert pour les fonctions analytiques réelles globales entraîne la finitude du nombre de Pythagore du corps des fonctions méromorphes réelles globales et donc celle du corps des séries méromorphes réelles. Cela donne une mesure de la difficulté du 17^e problème de Hilbert dans le cas analytique.

1. Introduction

Of all possible versions of the famous 17th Hilbert Problem, that for global real analytic functions is the one that has resisted any substantial progress. As is well known, the problem is whether

Every positive semidefinite global real analytic function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is a sum of squares of global real meromorphic functions.

In this formulation, sums are *finite*. The best result we can state today goes back to the early 1980s: *a positive semidefinite global real analytic function whose zero set is discrete off a compact set is a sum of squares of global real meromorphic functions* ([4] and [14], [11], see also [13] and [2]). Thus the non-compact case remains wide open, except for surfaces ([3], [1], [9]). In this paper we explore some remarkable features that make the non-compact case quite different from the compact one. Recall that the *Pythagoras number* of a ring is the smallest integer p (or $+\infty$) such that any sum of squares in the ring is a sum of p squares. We will prove:

PROPOSITION 1.1. — *Suppose that every positive semidefinite global real analytic function on \mathbb{R}^m is a finite sum of squares of global real meromorphic functions. Then the field of global real meromorphic functions on \mathbb{R}^m has finite Pythagoras number.*

And the same conclusion holds for the field of real meromorphic function germs. Thus, if we can represent every positive semidefinite function as a finite sum of squares (a qualitative matter), we will not encounter sums of arbitrary length (a quantitative matter). This kind of surprise will come out after the consideration of *infinite sums of squares*. Indeed, dealing with analytic functions, convergent infinite sums have a meaning, and they are a more subtle way to produce positive semidefinite functions. In this setting of infinite sums of squares, we localize the obstruction for a function to be a sum of squares at the germ of its zero set. After this sketchy preamble, let us now be more precise.

In what follows, we consider a real analytic manifold $M \subset \mathbb{R}^n$ (which we can suppose embedded as a closed set). This embedding dimension n will appear in various bounds in our results; the dimension of M will be denoted by m .

(1.2) Germs at a non-empty closed set $Z \subset M$. — Germs at Z are defined exactly as germs at a point, through neighborhoods of Z in M ; we will denote by f_Z the germ at Z of an analytic function f defined in some neighborhood of Z . We have the ring $\mathcal{O}(M_Z)$ of analytic function germs at Z , and its total ring of fractions $\mathcal{M}(M_Z)$, which is the ring of meromorphic function germs at Z . Note that for $Z = M$ we get nothing but global analytic and global meromorphic functions on M , hence definitions and results for germs apply in particular to global functions. If Z is connected, then $\mathcal{O}(M_Z)$ is a domain and $\mathcal{M}(M_Z)$ a field.

As usual, a germ f_Z is *positive semidefinite* when some representative f is positive semidefinite on some neighborhood of Z .

Next, we define infinite sums of squares. The first attempt to use convergent, even uniformly convergent, series of squares cannot work, as, in the real case, uniform convergence does not guarantee analyticity. As we must operate freely with these infinite sums, we must resort to complexification, which on the other hand is customary in real analytic geometry. Thus, we are led to the following:

DEFINITION 1.3. — Let $Z \subset M$ be a non-empty closed set. An *infinite sum of squares of analytic function germs at Z* is a series $\sum_{k \geq 1} f_k^2$, where all $f_k \in \mathcal{O}(M_Z)$, such that:

- (i) in some complexification \widetilde{M} of M there is a neighborhood \mathcal{V} of Z on which each f_k extends to a holomorphic function F_k , and
- (ii) for every compact set $L \subset \mathcal{V}$, $\sum_{k \geq 1} \sup_L |F_k|^2 < +\infty$.

The condition (ii) is the standard bound one uses to check that a function series is absolutely and uniformly convergent on compact sets. Accordingly, on the neighborhood $V = \mathcal{V} \cap M$ of Z , the infinite sum $\sum_{k \geq 1} f_k^2$ converges and defines a real analytic function f , and hence we have an analytic function germ f_Z ; we write $f_Z = \sum_{k \geq 1} f_k^2 \in \mathcal{O}(M_Z)$. Hence, it makes sense to say that an element of the ring $\mathcal{O}(M_Z)$ is a sum of p squares in $\mathcal{O}(M_Z)$, even for $p = +\infty$.

Next, we consider meromorphic functions:

DEFINITION 1.4. — Let $Z \subset M$ be a non-empty closed set. An analytic function germ f_Z is a *sum of $p \leq +\infty$ squares of real meromorphic function germs at Z* if there is $g_Z \in \mathcal{O}(M_Z)$ such that $g_Z^2 f_Z$ is a sum of p squares of real analytic function germs at Z . The zero set $\{g_Z = 0\}$ is called the *bad set* of the sum of squares.

The above notion of bad set mimics the terminology introduced in [7], but notice that here it refers to each given sum of squares, not to the function it represents.

We will say that the bad set of a particular representation of a germ f_Z as a sum of squares of real meromorphic function germs is *controlled* if it is contained in the zero set germ $\{f_Z = 0\}$. We will often seek representations with controlled bad set.

There is little need to remark here that the preceding definitions are restrictive in many ways. But this only means that any positive result on the representation of a function as a sum of squares will be stronger than one would have stated naively.

The central result in this paper is the following local-global principle:

THEOREM 1.5. — *Let $f : M \rightarrow \mathbb{R}$ be a positive semidefinite global real analytic function, and $Z = \{f = 0\} \neq \emptyset$ its zero set. Suppose that f_Z is a sum of $p \leq +\infty$ squares of real meromorphic function germs. Then, f is a sum of $2^{n-1}p + 1$ squares of global real meromorphic functions with controlled bad set.*

REMARK 1.6. — To check that f_Z is a sum of $p \leq +\infty$ squares it is enough to check that for every connected component Y of Z the germ f_Y is a sum of p squares.

Indeed, those connected components Y form a locally finite family of disjoint closed sets of M , hence of any given complexification \widetilde{M} of M . Thus, we can find a locally finite family of disjoint sets $\mathcal{V}_Y \supset Y$, open in \widetilde{M} , whose union \mathcal{V} is an open neighborhood of Z in \widetilde{M} . The functions on that union \mathcal{V} are defined through their restrictions to the \mathcal{V}_Y 's, and convergence on compact sets works fine. Indeed, every compact set $L \subset \mathcal{V}$ is the union of the sets $L \cap \mathcal{V}_Y \neq \emptyset$, which are finite in number and compact; then we have the bound

$$\sup_L |F|^2 \leq \sum_{L \cap \mathcal{V}_Y \neq \emptyset} \sup_{L \cap \mathcal{V}_Y} |F|^2$$

for any function F . □

Our central result, Theorem 1.5, splits into two separate parts. Firstly, what concerns bad sets:

PROPOSITION 1.7. — *Let $Z \subset M$ be a non-empty closed set, and f_Z an analytic function germ which is a sum of $p \leq +\infty$ squares of meromorphic function germs. Then f_Z is a sum of $2^n p$ squares of meromorphic function germs with controlled bad set. The number of squares can be lowered to $2^{n-1} p$ if f_Z vanishes on Z .*

Let us mention here that this control of bad sets is crucial in the study of irreducible factors of positive semidefinite global analytic functions [8].

Secondly, what concerns global sums of squares representations:

PROPOSITION 1.8. — *Let $f : M \rightarrow \mathbb{R}$ be a positive semidefinite global real analytic function, and $Z = \{f = 0\} \neq \emptyset$ its zero set. Suppose that the germ f_Z is a sum of $q \leq +\infty$ squares of real meromorphic function germs with controlled bad set. Then f is a sum of $q + 1$ squares of global real analytic functions with controlled bad set.*

Coming back to Theorem 1.5, the only positive result we know so far is that every positive semidefinite analytic function germ on a compact set is a finite sum of squares of real meromorphic function germs [2]. By Remark 1.6, we deduce:

COROLLARY 1.9. — *Let $f : M \rightarrow \mathbb{R}$ be a positive semidefinite analytic function such that all connected components of its zero set $\{f = 0\}$ are compact. Then f is a sum of squares of global real meromorphic functions with controlled bad set.*

But notice that the sum here might well be infinite, since we have no common bound on the number of squares needed to represent the germs f_Y . In one case we do know such a bound: when Y is a singleton, f_Y is a sum of $p = 2^m + m$ squares (because a suitable modification of the germ is algebraic, see [4, 418-420]). In view of this, the result from the early 1980s stated at the very beginning of this introduction follows readily:

COROLLARY 1.10. — *Let $f : M \rightarrow \mathbb{R}$ be a positive semidefinite analytic function such that the set $\{f = 0, \|x\| \geq \rho\}$ is discrete for some $\rho > 0$. Then f is a finite sum of squares of global real meromorphic functions with controlled bad set.*

As was roughly explained at the beginning, from Theorem 1.5 we will deduce this quantitative conclusion:

PROPOSITION 1.11. — *Suppose that every infinite sum of squares of meromorphic functions on \mathbb{R}^m is a finite sum. Then the Pythagoras number of the field $\mathcal{M}(\mathbb{R}^m)$ of global meromorphic functions on \mathbb{R}^m is finite.*

Since infinite sums of squares are positive semidefinite, this implies Proposition 1.1.

The same follows readily for the field $\mathbb{R}(\{x\}) = \mathbb{R}(\{x_1, \dots, x_m\})$ of meromorphic power series in m variables, because the Pythagoras number of $\mathcal{M}(\mathbb{R}^m)$ bounds that of $\mathbb{R}(\{x\})$. In fact, every finite sum of squares in $\mathbb{R}(\{x\})$ can be

represented by a finite sum of squares of meromorphic functions on an open ball, which is analytically equivalent to \mathbb{R}^m .

We remark that the computation of the Pythagoras number of $\mathbb{R}(\{x\})$ is an important old problem in the theory of quadratic forms: the only bounds known are 1, 2, 8 for $m = 1, 2, 3$ ([6, Problem 6 in §9], [12]), and even finiteness remains open for larger m . This we consider a measure of the very great difficulty of the 17th Hilbert Problem for global analytic functions.

The paper is organized as follows. In Section 2 we prove two key lemmas concerning the extension of holomorphic functions and sums of squares with fixed values on a given zero set; these lemmas are also needed in the study of irreducible factors of positive semidefinite global analytic functions [8]. Section 3 is devoted to the proof of Proposition 1.8. In Section 5 we prove Proposition 1.7, and Theorem 1.5 follows from this. Finally, Section 6 is devoted to the finiteness implications of the 17th Hilbert Problem (Proposition 1.11).

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2. Preliminaries on holomorphic functions

Although our problem concerns real analytic functions, we will of course use some complex analysis. For holomorphic functions we refer the reader to the classical [10], for real analytic functions and complexification, to [5].

(2.1) General terminology. — In what follows we denote the coordinates in \mathbb{C}^n by $z = (z_1, \dots, z_n)$, with $z_i = x_i + \sqrt{-1}y_i$, where $x_i = \text{Re}(z_i)$ and $y_i = \text{Im}(z_i)$ are respectively the real and the imaginary parts of z_i . Also, we consider the usual conjugation $\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n : z \mapsto \bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$, whose fixed points are \mathbb{R}^n . We say that a subset $\mathcal{J} \subset \mathbb{C}^n$ is (σ) -invariant if $\sigma(\mathcal{J}) = \mathcal{J}$; clearly, $\mathcal{J} \cap \sigma(\mathcal{J})$ is the biggest invariant subset of \mathcal{J} . Thus, we see real spaces as subsets of complex spaces. We will use the notations Int and Cl to denote topological interiors and closures, respectively, with subscripts to specify the ambient space if necessary.

Let $\mathcal{U} \subset \mathbb{C}^n$ be an invariant open set and let $F : \mathcal{U} \rightarrow \mathbb{C}$ be a holomorphic function. We say that F is (σ) -invariant if $F(z) = \overline{F(\bar{z})}$. This implies that F restricts to a real analytic function on $U = \mathcal{U} \cap \mathbb{R}^n$. In general, we denote by:

$$\begin{aligned} \Re(F) : \mathcal{U} &\rightarrow \mathbb{C} & \Im(F) : \mathcal{U} &\rightarrow \mathbb{C} \\ z &\mapsto \frac{F(z) + \overline{F(\bar{z})}}{2} & z &\mapsto \frac{F(z) - \overline{F(\bar{z})}}{2\sqrt{-1}} \end{aligned}$$

the *real* and the *imaginary* parts of F , which satisfy $F = \Re(F) + \sqrt{-1}\Im(F)$. Note that both are invariant holomorphic functions.

As a kind of converse, every real analytic function f defined on an open set $V \subset \mathbb{R}^n$ extends to an invariant holomorphic function on some invariant open set $\mathcal{V} \subset \mathbb{C}^n$ with $V = \mathcal{V} \cap \mathbb{R}^n$.

We also recall that \mathbb{R}^n has in \mathbb{C}^n a neighborhood basis consisting of *invariant open Stein neighborhoods*. □

We next see how to extend a holomorphic function modulo another with some control on its behaviour.

LEMMA 2.2. — *Let \mathcal{U} be an invariant open Stein neighborhood of \mathbb{R}^n in \mathbb{C}^n and let $\Phi : \mathcal{U} \rightarrow \mathbb{C}$ be an invariant holomorphic function. Let $\mathcal{V} \subset \mathcal{U}$ be an invariant (non-empty) open set such that $\{\Phi = 0\} \cap \mathcal{V}$ is closed in \mathcal{U} . Let $K \subset \mathcal{U}$ be an invariant compact set. Then there exist a real constant $\mu > 0$ and an invariant compact set $L \subset \mathcal{V}$ for which the following property holds:*

(*) *for every invariant holomorphic function $C : \mathcal{V} \rightarrow \mathbb{C}$ there exists an invariant holomorphic function $A : \mathcal{U} \rightarrow \mathbb{C}$ such that $\Phi|_{\mathcal{V}}$ divides $A|_{\mathcal{V}} - C$ and*

$$\sup_K |A| \leq \mu \sup_L |C|.$$

Proof. — First, consider the coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{\mathcal{U}}^{\mathbb{C}}$ generated by Φ , and the exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{O}_{\mathcal{U}}/\mathcal{J} \rightarrow 0.$$

Now, we have a corresponding diagram of cross sections:

$$\begin{array}{ccccc} \mathcal{J}(\mathcal{U}) & \longrightarrow & \mathcal{O}(\mathcal{U}) & \longrightarrow & \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}/\mathcal{J}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}(\mathcal{V}) & \longrightarrow & \mathcal{O}(\mathcal{V}) & \longrightarrow & \Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{U}}/\mathcal{J}) \end{array}$$

Here, the upper right arrow is onto because \mathcal{U} is Stein. Furthermore, the right vertical arrow is onto too. Indeed, each cross section of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ on \mathcal{V} can be extended by zero to \mathcal{U} , because the support of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ in \mathcal{V} is closed in \mathcal{U} . Hence we have a linear surjective homomorphism

$$\varphi : \mathcal{O}(\mathcal{U}) \longrightarrow \Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{U}}/\mathcal{J}) \equiv \mathcal{O}(\mathcal{V})/\mathcal{J}(\mathcal{V}).$$

We equip these vector spaces with their natural topologies. As is well known, $\mathcal{O}(\mathcal{U})$ and $\mathcal{O}(\mathcal{V})$ are Fréchet spaces with the topology of uniform convergence on compact sets. Also, by the closure of modules theorem, we know that $\mathcal{J}(\mathcal{V})$ is a closed subspace of $\mathcal{O}(\mathcal{V})$, and $\mathcal{O}(\mathcal{V})/\mathcal{J}(\mathcal{V})$ is also a Fréchet space with the quotient topology. Summing up, φ is a continuous surjective homomorphism

of Fréchet spaces, consequently open [15, III.2.1]. In order to make use of this, we describe explicitly the topologies involved.

Let $\{K_i\}_i$ and $\{L_i\}_i$ be families of invariant compact sets in \mathcal{U} and \mathcal{V} , respectively, such that:

- $\text{Int}_{\mathbb{C}^n}(L_1) \neq \emptyset$,
- $L_i \subset K_i$ for all i ,
- $L_i \subset L_{i+1}$ and $K_i \subset K_{i+1}$ for all i , and
- $\bigcup_i \text{Int}_{\mathbb{C}^n}(L_i) = \mathcal{V}$ and $\bigcup_i \text{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$.

Then the topologies of $\mathcal{O}(\mathcal{U})$ and $\mathcal{O}(\mathcal{V})$ are defined, respectively, by the pseudonorms:

$$\|F\| = \sum_i \frac{1}{2^i} \frac{\sup_{K_i} |F|}{1 + \sup_{K_i} |F|} \quad \text{for } F \in \mathcal{O}(\mathcal{U}), \text{ and}$$

$$\|G\|' = \sum_i \frac{1}{2^i} \frac{\sup_{L_i} |G|}{1 + \sup_{L_i} |G|} \quad \text{for } G \in \mathcal{O}(\mathcal{V}).$$

Moreover, by [15, I.6.3], the quotient topology of $\mathcal{O}(\mathcal{V})/\mathcal{J}(\mathcal{V})$ is given by the following third pseudonorm:

$$\|\xi\|^* = \inf_G \{\|G\|' : \xi = G + \mathcal{J}(\mathcal{V})\} \quad \text{for } \xi \in \mathcal{O}(\mathcal{V})/\mathcal{J}(\mathcal{V}).$$

Next, given the compact set $K \subset \mathcal{U}$, we have the following open set in $\mathcal{O}(\mathcal{U})$:

$$W = \left\{ H \in \mathcal{O}(\mathcal{U}) : \sup_K |H| < 1 \right\}.$$

Since φ is open, $\varphi(W)$ is an open neighborhood of 0 in $\mathcal{O}(\mathcal{V})/\mathcal{J}(\mathcal{V})$, and there exists $\varepsilon > 0$ such that

$$W^* = \{\xi : \|\xi\|^* < \varepsilon\} \subset \varphi(W).$$

Then, we pick $\mu > \frac{2}{\varepsilon}$, and $L = L_i$ with i such that $\sum_{j>i} \frac{1}{2^j} < \frac{\varepsilon}{2}$. We will prove the condition (*) in the statement for such $\mu > 0$ and $L \subset V$, which by construction depend only on Φ , \mathcal{V} and K .

Let $C \in \mathcal{O}(\mathcal{V})$ be a nonzero holomorphic function (for $C \equiv 0$, just take $A \equiv 0$). Since the interior of L in \mathbb{C}^n is not empty, $a = \sup_L |C| > 0$, and we denote $G = \frac{1}{a\mu} C \in \mathcal{O}(\mathcal{V})$. Then $\sup_L |G| = \frac{1}{\mu} < \frac{\varepsilon}{2}$, and we have:

$$\begin{aligned} \|G\|' &= \sum_j \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} = \sum_{j=1}^i \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} + \sum_{j>i} \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} \\ &< \sup_L |G| \sum_{j=1}^i \frac{1}{2^j} + \sum_{j>i} \frac{1}{2^j} < \sup_L |G| \sum_{j=1}^i \frac{1}{2^j} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

From this, setting $\xi = G + \mathcal{J}(\mathcal{V})$, we get

$$\|\xi\|^* \leq \|G\|' < \varepsilon,$$

and $\xi \in W^* \subset \varphi(W)$. Consequently, there exists $H \in W$ such that $\varphi(H) = \xi$, that is, $H|_{\mathcal{V}} - G \in \mathcal{J}(\mathcal{V})$, and the holomorphic function $F = a\mu H \in \mathcal{O}(\mathcal{U})$ satisfies the required conditions. For, $F|_{\mathcal{V}} - C = a\mu(H|_{\mathcal{V}} - G) \in \mathcal{J}(\mathcal{V})$, and since $\sup_K |H| < 1$:

$$\sup_K |F| = a\mu \sup_K |H| \leq a\mu = \mu \sup_L |C|.$$

Finally, if C is invariant we take $A = \Re(F)$, and A satisfies the same conditions. First,

$$A|_{\mathcal{V}} - C = \Re(F|_{\mathcal{V}}) - C = \Re(F|_{\mathcal{V}} - C) = \Re(\Lambda\Phi|_{\mathcal{V}}) = \Re(\Lambda)\Phi|_{\mathcal{V}} \in \mathcal{J}(\mathcal{V})$$

for some $\Lambda \in \mathcal{O}(\mathcal{V})$. Secondly, as K is invariant:

$$\sup_K |A| = \sup_K |\Re(F)| = \sup_K \left| \frac{F + \overline{F \circ \sigma}}{2} \right| \leq \sup_K \frac{|F| + |\overline{F \circ \sigma}|}{2} \leq \sup_K |F| \leq \mu \sup_L |C|,$$

and the proof is complete. □

Now we apply this to infinite sums of squares:

PROPOSITION 2.3. — *Let \mathcal{U} be an invariant open Stein neighborhood of \mathbb{R}^n in \mathbb{C}^n and let $\Phi : \mathcal{U} \rightarrow \mathbb{C}$ be an invariant holomorphic function. Let $\mathcal{V} \subset \mathcal{U}$ be an invariant (non-empty) open set such that $\{\Phi = 0\} \cap \mathcal{V}$ is closed in \mathcal{U} . Let $C_k : \mathcal{V} \rightarrow \mathbb{C}$ be invariant holomorphic functions such that $\sum_k \sup_L |C_k|^2 < +\infty$ for every compact set $L \subset \mathcal{V}$. Then there exist invariant holomorphic functions $A_k : \mathcal{U} \rightarrow \mathbb{C}$, such that $\sum_k \sup_K |A_k|^2 < +\infty$ for every compact set $K \subset \mathcal{U}$ and $\Phi|_{\mathcal{V}}$ divides all the differences $A_k|_{\mathcal{V}} - C_k$.*

Furthermore, if C_1 is divisible on \mathcal{V} by some holomorphic function $H : \mathcal{U} \rightarrow \mathbb{C}$, the function A_1 can be chosen divisible on \mathcal{U} by H .

Proof. — Let $\{K_i\}$ be a family of invariant compact sets such that

- $K_i \subset K_{i+1}$ for all i , and
- $\bigcup_i \text{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$.

By 2.2, for each i there exists $\mu_i > 0$ and a compact set $L_i \subset \mathcal{V}$ such that if $C \in \mathcal{O}(\mathcal{V})$ is invariant, there exists an invariant $A \in \mathcal{O}(\mathcal{U})$ such that $A|_{\mathcal{V}} - C = \Phi|_{\mathcal{V}}B$ for some $B \in \mathcal{O}(\mathcal{V})$ and $\sup_{K_i} |A| \leq \mu_i \sup_{L_i} |C|$. We may assume that $L_i \subset L_{i+1}$ for all i .

Since $\sum_k \sup_{L_i} |C_k|^2 < +\infty$ for all i , there exists a strictly increasing sequence (k_i) of positive integers such that

$$\sum_{k \geq k_i} \sup_{L_i} |C_k|^2 \leq \frac{1}{2^i \mu_i^2}.$$

For each k such that $k_i \leq k < k_{i+1}$ there exists a holomorphic function $A_k : \mathcal{U} \rightarrow \mathbb{C}$ such that $\sup_{K_i} |A_k| \leq \mu_i \sup_{L_i} |C_k|$ and $\Phi|_{\mathcal{V}}$ divides $A_k|_{\mathcal{V}} - C_k$. Let us see that for every compact set $K \subset \mathcal{U}$ the series $\sum_k \sup_K |A_k|^2 < +\infty$. Since $\bigcup_i \text{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$, it is enough to check that $\sum_k \sup_{K_i} |A_k|^2 < +\infty$ for all i . But

$$\begin{aligned} \sum_k \sup_{K_i} |A_k|^2 &= \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \left(\sum_{k_j \leq k < k_{j+1}} \sup_{K_i} |A_k|^2 \right) \\ &\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \left(\sum_{k_j \leq k < k_{j+1}} \sup_{K_j} |A_k|^2 \right) \\ &\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \left(\sum_{k \geq k_j} \mu_j^2 \sup_{L_j} |C_k|^2 \right) \\ &\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \mu_j^2 \frac{1}{2^j \mu_j^2} \leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + 1 < +\infty. \end{aligned}$$

This concludes the proof of the statement, except for its last assertion. To see that, notice that the convergence bound does not depend on the choice of a single term of the series. Then, we write $C_1 = C_1^* H|_{\mathcal{V}}$, where C_1^* is holomorphic on \mathcal{V} , and by 2.2 there is a holomorphic function A_1^* such that $\Phi|_{\mathcal{V}}$ divides $A_1^*|_{\mathcal{V}} - C_1^*$. We conclude by taking $A_1 = A_1^* H$. \square

3. Globalization of sums of squares

The purpose of this section is to prove Proposition 1.8. We start with the following:

PROPOSITION 3.1. — *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a global real analytic function, and $Z = \{f = 0\} \neq \emptyset$ its zero set. Suppose that the germ f_Z is a sum of $q \leq +\infty$ squares of analytic function germs. Then f divides a sum $\sum_{k=1}^q a_k^2$ of q squares of global real analytic functions on \mathbb{R}^n , and the global real analytic function $\sum_{k=1}^q a_k^2 / f$ is strictly positive on Z .*

Furthermore, if (the germ at Z) of a global real analytic function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ divides the first addend of the representation of f_Z as a sum of squares, then a_1 can be chosen divisible by h .

Proof. — Consider an invariant open Stein neighborhood \mathcal{U}_1 of \mathbb{R}^n in \mathbb{C}^n on which f has an invariant holomorphic extension F . By hypothesis, there are q invariant holomorphic functions $C_k : \mathcal{V}_1 \rightarrow \mathbb{C}$, defined on an invariant open neighborhood $\mathcal{V}_1 \subset \mathcal{U}_1$ of Z in \mathbb{C}^n , such that $F|_{\mathcal{V}_1} = \sum_k C_k^2$, where the series converges in the strong sense of 1.3(ii).

Now, $\mathcal{V}_1 \cup \{F \neq 0\} \subset \mathcal{U}_1$ is an open neighborhood of \mathbb{R}^n in \mathbb{C}^n , hence \mathbb{R}^n has an invariant open Stein neighborhood $\mathcal{U} \subset \mathcal{V}_1 \cup \{F \neq 0\}$; set $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{U}$. One

checks easily that $\{F = 0\} \cap \mathcal{V}$ is closed in \mathcal{U} , and henceforth we substitute \mathcal{U} and \mathcal{V} for \mathcal{U}_1 and \mathcal{V}_1 . By 2.3, applied to $\Phi = F^2$, there exist invariant holomorphic functions $A_k : \mathcal{U} \rightarrow \mathbb{C}$ such that $\sum_k \sup_K |A_k|^2 < +\infty$ for all compact set $K \subset \mathcal{U}$ and F^2 divides $A_k|_{\mathcal{V}} - C_k$ on \mathcal{V} (and if h is given, its complexification H is defined on \mathcal{U} and divides A_1).

On \mathcal{V} we have:

$$\sum_k A_k^2 - F = \sum_k A_k^2 - \sum_k C_k^2 = \sum_k (A_k^2 - C_k^2),$$

and this series is convergent on compact sets, as $\sum_k A_k^2$ and $\sum_k C_k^2$ are so. By construction, F^2 divides on \mathcal{V} each term $A_k^2 - C_k^2 = (A_k + C_k)(A_k - C_k)$, hence it divides their sum $\sum_k A_k^2 - F$. Thus there is a holomorphic function $\Psi : \mathcal{V} \rightarrow \mathbb{C}$ such that on \mathcal{V} we have:

$$\sum_k A_k^2 = F + \Psi F^2 = uF, \quad \text{where } u = 1 + \Psi F.$$

Clearly, u has no zeros in the zero set of F , hence u is a holomorphic unit in a perhaps smaller \mathcal{V} . To conclude, take $a_k = A_k|_{\mathbb{R}^n}$. □

We are ready for the:

Proof of PROPOSITION 1.8. — Recall that M is embedded as a closed set in \mathbb{R}^n , hence there is an analytic function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with zero set $\{h = 0\} = M$. Consider an open tubular neighborhood Ω of M in \mathbb{R}^n , endowed with the corresponding analytic retraction $\pi : \Omega \rightarrow M$. By composition with π , all functions extend from M to Ω ; we denote those extensions with bars.

By hypothesis, on an open neighborhood V of $Z = \{f = 0\}$ in M we have a representation as a sum of q squares:

$$g^2 f = \sum_k b_k^2,$$

with $\{g = 0\} \subset Z$; multiplying by g^2 , we can assume $g \geq 0$.

Step I: Global denominator. Consider the function $g' = h^2 + \bar{g}$. It is analytic on the neighborhood $U = \pi^{-1}(V)$ of Z in \mathbb{R}^n and $\{g' = 0\} = \{g = 0\} \subset Z$ by our choice of g . Thus, g' is an analytic function on U and its zero set is closed in \mathbb{R}^n . Consequently we can consider the locally principal coherent analytic sheaf of ideals defined by:

$$\mathcal{J}_x = \begin{cases} g'_x \mathcal{O}_{\mathbb{R}^n, x} & \text{if } x \in U, \\ \mathcal{O}_{\mathbb{R}^n, x} & \text{if } x \notin Z. \end{cases}$$

As $H^1(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}^*) = H^1(\mathbb{R}^n, \mathbb{Z}_2) = 0$, this sheaf is globally principal, say generated by g'' . The zero set of g'' is that of g' and contained in Z , and $v = g''/g'$

is an analytic unit on U . On $V = U \cap M$ we can write:

$$g''^2 f = (vg')^2 f = \sum_k (vb_k)^2 = \sum_k c_k^2.$$

Thus g'' is an analytic function on \mathbb{R}^n , its zero set is contained in Z , and it is a denominator for a sum of squares representation of f .

Step II. Global sum of squares. After Step I, we only care for the sum of squares $\gamma = \sum_k c_k^2$. By composition with π , this sum of squares extends to $\sum_k \bar{c}_k^2$ on U (π respects convergence, as one easily sees by complexification), and we consider

$$\gamma' = h^2 + \bar{\gamma} = h^2 + \sum_{k=1}^q \bar{c}_k^2.$$

This analytic function is defined on U , but as done before for g and g' , we find a analytic function γ'' defined on \mathbb{R}^n with the same zero set Z as γ and γ' , and such that γ''/γ' is a positive analytic function on U . Now γ'' is a sum of $q + 1$ squares of analytic functions on U , which is a neighborhood of its zero set, and the first square is h^2 . Hence by Proposition 3.1, γ'' divides a sum $\sum_k a_k^2$ of $q + 1$ squares of analytic functions on \mathbb{R}^n , and the quotient w is strictly positive on Z ; furthermore we can choose a_1 divisible by h .

Step III. Additional square. The function

$$\alpha = \frac{\gamma''^2 + \sum_k a_k^2}{\gamma''} = \gamma'' + w$$

is a strictly positive analytic function on \mathbb{R}^n : both addends on the right hand side are ≥ 0 , the first one does not vanish off Z , and the second one does not vanish on Z . Thus, letting β stand for the positive square root of $1/\alpha$, we have the following sum of $q + 2$ squares of analytic functions on \mathbb{R}^n :

$$\gamma'' = (\beta\gamma'')^2 + \sum_k (\beta a_k)^2.$$

As h divides a_1 , when we restrict this to M , the square $(\beta a_1)^2$ disappears, and $\gamma''|_M$ is a sum of $q + 1$ squares of analytic functions on M .

Step IV. Conclusion. By construction, $g''^2 f$ and $\gamma''|_M$ have the same zero set Z , and $g''^2 f/\gamma''|_M$ is a positive unit on a neighborhood of it. Hence that quotient is a positive unit on M , hence has an analytic square root u . We conclude that $g''^2 f = u^2 \gamma''|_M$ is a sum of $q + 1$ squares of analytic functions on M , as $\gamma''|_M$ is. □

4. Bad sets

The purpose of this section is to show how to control the bad set of a sum of squares of meromorphic functions, which is the content of Proposition 1.7. This control is essential to apply Proposition 1.8. First of all, we can always reduce to the case when $M \subset \mathbb{R}^n$ is an open set $\Omega \subset \mathbb{R}^n$ using a tubular neighborhood Ω of M .

After this remark, it is clear that the following statement implies Proposition 1.7:

LEMMA 4.1. — *Let $f : \Omega \rightarrow \mathbb{R}$ be an analytic function defined on a connected open set $\Omega \subset \mathbb{R}^n$. Let $h : \Omega \rightarrow \mathbb{R}$ be a nonzero analytic function such that $h^2 f$ is a sum of $p \leq +\infty$ squares of analytic functions. Set $d = \dim(\{h = 0, f \neq 0\})$. Then, there exists an analytic function $g : \Omega \rightarrow \mathbb{R}$ such that $g^2 f$ is a sum of $q \leq 2^{d+1}p$ squares of analytic functions on Ω , and $\{g = 0\} \subset \{f = 0\}$. Moreover, on a smaller neighborhood of $\{f = 0\}$ we can assume $q \leq 2^d p$.*

Proof. — Consider the global analytic set $Y = \{h = 0\}$. We pick a point y_i in each irreducible component Y_i of Y that is not contained in $\{f = 0\}$. Clearly, we can suppose $f(y_i) \neq 0$ and that the y_i 's form a discrete set. By a small diffeotopy around each y_i we can move y_i off Y , to obtain a smooth diffeomorphism $\psi : \Omega \rightarrow \Omega$ which maps each y_i to $y'_i \notin Y$ and is the identity on a neighborhood of $\{f = 0\}$. By the latter condition, f^2 divides the map $\psi - \text{Id}$, hence $\psi = \text{Id} + f^2 \mu$ for a smooth map $\mu : \Omega \rightarrow \mathbb{R}^n$. Now, let $\eta : \Omega \rightarrow \mathbb{R}^n$ be an analytic mapping close to μ . Then $\varphi = \text{Id} + f^2 \eta$ is close to ψ , and consequently φ is an analytic diffeomorphism of Ω . Note that φ is the identity on $\{f = 0\}$, and so f and $f \circ \varphi$ have the same zeros. Also, by looking at Taylor expansions, one sees that $f \circ \varphi = f + f^2 \alpha$ for some analytic map $\alpha : \Omega \rightarrow \mathbb{R}$. Thus we can write $f \circ \varphi = v f$, where $v = 1 + f \alpha$ has no zero: a zero x of v would be a zero of $f \circ \varphi$, hence one of f , and $v(x) = 1 + f(x)\alpha(x) = 1!$ Moreover, as f is positive semidefinite, so is v , and $u = +\sqrt{v}$ is a strictly positive analytic function such that $f \circ \varphi = u^2 f$. By hypothesis, $h^2 f = \sum_j h_j^2$, which gives:

$$u^2 (h \circ \varphi)^2 f = (h \circ \varphi)^2 (f \circ \varphi) = \sum_j (h_j \circ \varphi)^2$$

(note that if the sum is infinite, it converges in the sense of 1.3(ii)). Hence,

$$(h^2 + u^2 (h \circ \varphi)^2) f = \sum_j (h_j^2 + (h_j \circ \varphi)^2).$$

Now, we multiply both sides times $h^2 + u^2 (h \circ \varphi)^2$ to get

$$\delta^2 f = \sum_j (h_j^2 + (h_j \circ \varphi)^2) (h^2 + u^2 (h \circ \varphi)^2),$$

with $\delta = h^2 + u^2(h \circ \varphi)^2$. If the sum is infinite, we have another infinite sum. In case the sum is finite, then we recall that the product of two sums of two squares is again a sum of two squares, and we get twice the number of squares. Finally, the bad set now is:

$$\{\delta = 0\} = \{h = 0\} \cap \{h \circ \varphi = 0\} = \{h = 0\} \cap \varphi^{-1}(Y),$$

so that

$$\{\delta = 0\} \setminus \{f = 0\} \subset \bigcup_i Y_i \cap \varphi^{-1}(Y).$$

But no irreducible component Y_i is contained in $\varphi^{-1}(Y)$, because $\varphi(y_i) \notin Y$, hence $\dim(Y_i \cap \varphi^{-1}(Y)) < \dim(Y_i) \leq d$.

Thus the dimension of the bad set drops off $\{f = 0\}$, and after $d + 1$ repetitions we get the first assertion of the statement. If, instead, we stop after d times, then

$$\dim(\{g = 0, f \neq 0\}) \leq 0.$$

This means that $D = \{g = 0, f \neq 0\}$ is a discrete closed subset of Ω , and this latter can be replaced by $\Omega \setminus D$ to get the second assertion. □

Consequently, Propositions 1.7 and 1.8 are proved, and together they imply Theorem 1.5. In the next section we use the latter to reveal the quantitative content of the 17th Hilbert problem in the non-compact case.

5. The finiteness implications

To start with, we prove the following reformulation of the half of Proposition 1.11 that concerns global meromorphic functions on \mathbb{R}^m .

PROPOSITION 5.1. — *Suppose that the Pythagoras number of $\mathcal{M}(\mathbb{R}^m)$ is $+\infty$. Then there is a global real analytic function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ that is a sum of squares of global real meromorphic functions (and hence is positive semidefinite), but not a finite sum of such squares.*

Proof. — By hypothesis, for each $p \geq 1$ there is an analytic function $f_p : \mathbb{R}^m \rightarrow \mathbb{R}$ ($m > 1$) that is a sum of squares of meromorphic functions, but not of p such squares. We may suppose that the zero set Z_p of f_p has codimension ≥ 2 .

Indeed, set $g = f_p$. At each zero x of g , we have a unique factorization of analytic germs $g_x = \zeta_x^2 \eta_x$, η_x without multiple factors. The germ $\{\eta_x = 0\}$ has codimension ≥ 2 , since otherwise some irreducible factor ξ_x of η_x would change sign at x , and g_x too. Now, the ζ_x 's generate a locally principal coherent analytic sheaf of ideals, \mathcal{J} , which is globally principal (once again we use that on \mathbb{R}^m locally principal sheaves are all globally principal). Let h be a global generator of \mathcal{J} , so that $g = h^2 g'$. Each germ g'_x coincides with η_x up to a unit,

hence its zero set has codimension ≥ 2 and g'_x does not change sign. Replacing g by g' we may suppose its zero set has codimension ≥ 2 as claimed.

Back to our f_p 's, assume for a moment that Z_p can be moved into the open cylinder

$$V_p = \{x = (x', x_m) \in \mathbb{R}^m : \|x' - a'_p\| < \frac{1}{4}\},$$

where $a'_p = (p, 0, \dots, 0) \in \mathbb{R}^{m-1}$. Then the Z_p 's form a locally finite family, and $Z = \bigcup_p Z_p$ is a closed analytic subset of \mathbb{R}^m . Consequently, we can define the following locally principal sheaf:

$$\mathcal{J}_x = \begin{cases} f_p \mathcal{O}_{\mathbb{R}^m, x} & \text{if } x \in Z_p, \\ \mathcal{O}_{\mathbb{R}^m, x} & \text{if } x \notin Z. \end{cases}$$

Again, we know that \mathcal{J} has a global generator f . Thus, on each V_p there is an analytic unit v_p such that $f = v_p f_p$. Note also that the zero set of f is Z , which does not disconnect \mathbb{R}^m , because its codimension is ≥ 2 . Hence, f has constant sign on \mathbb{R}^m and we may assume $f \geq 0$, and $v_p > 0$. In particular, $+\sqrt{v_p}$ is analytic on V_p , so that f and f_p behave the same concerning sums of squares. Since, by construction, the connected components Y of Z are those of the Z_p 's, we deduce that each germ f_Y is a sum of squares, and by Remark 1.6, so is the germ f_Z . Thus, by Theorem 1.5, f is a sum of squares of meromorphic functions. However, this sum cannot be finite, say of q squares, because f_q is not a sum of q squares.

To complete the proof it only remains to move each Z_p by a suitable analytic diffeomorphism of \mathbb{R}^m . This we do now.

Since Z_p has codimension ≥ 2 , many lines do not meet Z_p , and after a linear change of coordinates, we may assume this is the case for the x_m -axis. Then, picking an analytic function $\delta(x_m)$ such that $0 < \delta(x_m) < \text{dist}(Z_p, (0, \dots, 0, x_m))$, the analytic diffeomorphism of \mathbb{R}^m

$$(x', x_m) \mapsto \left(\frac{\sqrt{1 + x_m^2}}{\delta(x_m)} x', x_m \right)$$

moves Z_p off $\{\|x'\|^2 < 1 + x_m^2\}$. Next, we define $(x', x_m) \mapsto (y', y_m)$ by:

$$y' - a'_p = \frac{x'}{4(1 + y_m^2)}, \quad y_m = 2\|x'\|^2 - x_m.$$

We claim that this analytic diffeomorphism moves the set $\{\|x'\|^2 \geq 1 + x_m^2\}$ inside the cylinder $\|y' - a'_p\| < \frac{1}{4}$, and we are done.

Indeed, if $\|x'\|^2 \geq 1 + x_m^2 (\geq 1)$, then

$$y_m = 2\|x'\|^2 - x_m \geq \|x'\|^2 + 1 + x_m^2 - x_m > \|x'\|.$$

Consequently:

$$\|y' - a'_p\| = \frac{\|x'\|}{4(1 + y_m^2)} < \frac{\|x'\|}{4(1 + \|x'\|^2)} < \frac{1}{4}. \quad \square$$

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