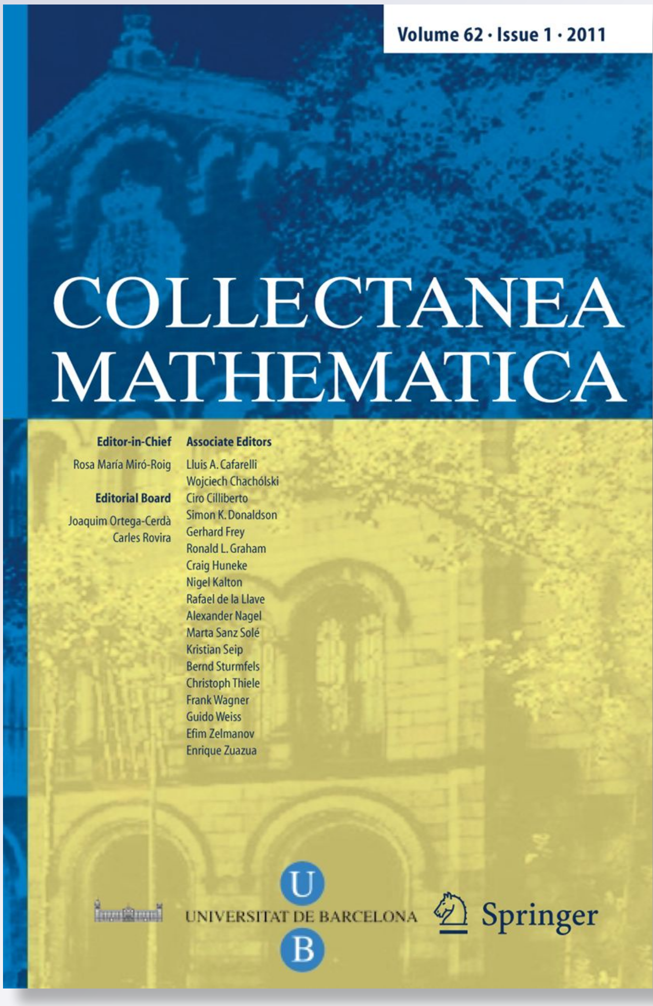
On the spectra of rings of semialgebraic functions

José F. Fernando & J. M. Gamboa

Collectanea Mathematica

ISSN 0010-0757
Volume 63
Number 3

Collect. Math. (2012) 63:299-331
DOI 10.1007/s13348-011-0041-0



Springer

Collect. Math. (2012) 63:299–331
DOI 10.1007/s13348-011-0041-0

On the spectra of rings of semialgebraic functions

José F. Fernando · J. M. Gamboa

Received: 5 June 2010 / Accepted: 25 February 2011 / Published online: 20 March 2011

Abstract In this article we study the most significant algebraic, topological and functorial properties of the Zariski and maximal spectra of rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set.

Keywords Semialgebraic function · Semialgebraic set · Zariski spectrum · Real spectrum · Maximal spectrum · Functoriality · Local compactness · Pieces · Semialgebraic depth · z-ideal

Mathematics Subject Classification (2000) Primary 14P10 · 54C30; Secondary 12D15 · 13E99

1 Introduction

A subset $M \subset \mathbb{R}^n$ is said to be *basic semialgebraic* if it can be written as

$$M = \{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_m(x) > 0\}$$

for some polynomials $f, g_1, \dots, g_m \in \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_n]$. The finite unions of basic semialgebraic sets are called *semialgebraic sets*. A continuous function $f : M \to \mathbb{R}$ is said to be *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{n+1}$. Usually, semialgebraic function just means a function, non necessarily continuous, whose graph is semialgebraic. However, since all semialgebraic functions occurring in this article are continuous we will omit

J. F. Fernando and J. M. Gamboa are supported by Spanish GAAR MTM2008-00272, Proyecto Santander Complutense PR34/07-15813 and GAAR Grupos UCM 910444.

J. F. Fernando · J. M. Gamboa (✉)
Departamento de Álgebra, Facultad de Ciencias Matemáticas,
Universidad Complutense de Madrid, 28040 Madrid, Spain
e-mail: jmgamboa@mat.ucm.es

J. F. Fernando
e-mail: josefer@mat.ucm.es

for simplicity the continuity condition when we refer to them. Likewise, a continuous map $\varphi : N \to M$ between semialgebraic sets whose graph is semialgebraic will be called, simply, a semialgebraic map.

The sum and product of functions, defined pointwise, endow the set $\mathcal{S}(M)$ of semialgebraic functions on M with a natural structure of commutative ring whose unity is the semialgebraic function with constant value 1. In fact $\mathcal{S}(M)$ is an $\mathbb{R}$-algebra, if we identify each real number r with the constant function which just attains this value. The most simple examples of semialgebraic functions on M are the restrictions to M of polynomials in n variables. Other relevant ones are the absolute value of a semialgebraic function, the maximum and the minimum of a finite family of semialgebraic functions, the inverse and the k-root of a semialgebraic function whenever these operations are well-defined.

It is obvious that the subset $\mathcal{S}^*(M)$ of bounded semialgebraic functions on M is a real subalgebra of $\mathcal{S}(M)$. For the time being, we denote by $\mathcal{S}^\diamond(M)$, indistinctly, either $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$ in case the involved statements or arguments are valid for both rings. Moreover, if $p \in M$, we will denote by $\mathfrak{m}_p^\diamond$ the maximal ideal of all functions in $\mathcal{S}^\diamond(M)$ vanishing at p. For each $f \in \mathcal{S}^\diamond(M)$ and each semialgebraic subset $N \subset M$, we denote $Z_N(f) = \{x \in N : f(x) = 0\}$ and $D_N(f) = N \setminus Z_N(f)$. In case $N = M$, we say that $Z_M(f)$ is the *zeroset* of f. Our purpose in this work is to study the algebraic, topological and functorial properties of the Zariski and maximal spectra of the rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set. Since the usual notations for these objects become cumbersome, we replace them by the following ones. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. We denote

$$\operatorname{Spec}_{\mathrm{s}}(M) = \operatorname{Spec}(\mathcal{S}(M)), \quad \operatorname{Spec}_{\mathrm{s}}^*(M) = \operatorname{Spec}(\mathcal{S}^*(M)),$$
$$\beta_{\mathrm{s}} M = \operatorname{Spec}_{\max}(\mathcal{S}(M)), \qquad \beta_{\mathrm{s}}^* M = \operatorname{Spec}_{\max}(\mathcal{S}^*(M)),$$

and we abbreviate $\operatorname{Spec}_{\mathrm{s}}^\diamond(M) = \operatorname{Spec}(\mathcal{S}^\diamond(M))$ and $\beta_{\mathrm{s}}^\diamond M = \operatorname{Spec}_{\max}(\mathcal{S}^\diamond(M))$. As it is well-known the real spectrum and real maximal spectrum of $\mathcal{S}^\diamond(M)$ coincide with its classical Zariski spectrum and maximal spectrum. Consequently, we will be mainly concerned about Zariski spectra. Of course, some of our initial results, that we include in Sect. 3 for the sake of completeness, surely can be obtained using a different approach involving the theory of "Real closed rings" introduced by Schwartz in [16] and [17], and successfully used, for instance, in [4] and [5]. The main reason for our choice is that our approach, that works only over the real numbers and not over an arbitrary real closed field, is based on the celebrated classical theory of rings of continuous functions (see [14]), requires less algebraic background and admits a less involved presentation.

A crucial tool to understand the functorial properties of the operators $\operatorname{Spec}_{\mathrm{s}}(\cdot)$ and $\operatorname{Spec}_{\mathrm{s}}^*(\cdot)$ is the *semialgebraic Tietze–Urysohn Lemma* as stated by Delfs–Knebush in [6], whose scope is determined in 2.9. Of course, the first expected consequence of this result says that if $C \subset M$ is a closed semialgebraic subset of M, then its closure in $\operatorname{Spec}_{\mathrm{s}}^\diamond(M)$ is homeomorphic to $\operatorname{Spec}_{\mathrm{s}}^\diamond(C)$ (see 4.6 and 5.15). This together with the proof of the functoriality of $\operatorname{Spec}_{\mathrm{s}}(\cdot)$ and $\operatorname{Spec}_{\mathrm{s}}^*(\cdot)$ (see 4.1) completes the first part of Sect. 4. The rest of this section is focused on the proof of the following result (see 4.8 and 4.9), which has further applications in other contexts (see [7,9,12]):

(1.1) *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets such that N is open in M and locally compact. Denote $Y = M \setminus N$ and let $j : N \hookrightarrow M$ be the inclusion map. Define $\mathcal{L}(Y) = \bigcup_f \mathcal{Z}(f)$, where $\mathcal{Z}(f) = \{\mathfrak{p} \in \operatorname{Spec}_{\mathrm{s}}(M) : f \in \mathfrak{p}\}$ and f runs over all $f \in \mathcal{S}(M)$ such that $Z_M(f) = Y$.*

Then, the map

$$\operatorname{Spec}_s(j):\operatorname{Spec}_s(N)\to\operatorname{Spec}_s(M),\ \mathfrak{p}\mapsto\operatorname{Spec}_s(j)(\mathfrak{p})=\{g\in\mathcal{S}(M): g\circ j=g|_N\in\mathfrak{p}\}$$

is a homeomorphism onto its image $\operatorname{Spec}_s(M)\setminus\mathcal{L}(Y)$. *Moreover, the preimage of each maximal ideal of* $\mathcal{S}(M)$ *is a maximal ideal of* $\mathcal{S}(N)$, *while the direct image of a maximal ideal of* $\mathcal{S}(N)$ *is not necessarily a maximal ideal of* $\mathcal{S}(M)$. *Even more, if* M *is also locally compact, then* $\mathcal{L}(Y)=\operatorname{Cl}_{\operatorname{Spec}_s(M)}(Y)$.

It is well-known that locally compact semialgebraic sets present a nicer behaviour than arbitrary ones when dealing with their rings of semialgebraic and bounded semialgebraic functions (see for instance [1, Ch. 2] and [10]). The reason is that a locally compact semialgebraic set M is an open subset of each Hausdorff compactification of M. The previous result 1.1, which is a new evidence of the importance of locally compact semialgebraic sets in semialgebraic geometry, is the key to compare the spectra $\operatorname{Spec}_s(M)$ and $\operatorname{Spec}_s(M_{lc})$ for an arbitrary semialgebraic set M, where M_{lc} denotes the largest locally compact and dense subset of M, which turns out to be a semialgebraic set.

As we have shown in [11], it is also useful to compare the spectra $\operatorname{Spec}_s(M)$ and $\operatorname{Spec}_s(X)$ where M is a locally compact semialgebraic set and X is one of its semialgebraic compactifications; recall that a compactification (X, j) of M is a *semialgebraic compactification* of M if $j: M\to X$ is a semialgebraic map. In [11] we compute the Krull dimensions of the rings $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ by comparing them with the Krull dimensions of the rings $\mathcal{S}(X)=\mathcal{S}^*(X)$ for suitable semialgebraic compactifications X of M. Moreover, we see in [7] that these semialgebraic compactifications provide, by using 1.1, further information to study chains of prime ideals in rings of semialgebraic functions.

On the other hand, concerning the spectrum of the ring of bounded semialgebraic functions of a semialgebraic set, the most revealing result in this work, proved in 5.1, is the following:

(1.2) *Let* $N\subset\mathbb{R}^n$ *and* $M\subset\mathbb{R}^m$ *be semialgebraic sets and let* $\varphi: N\to M$ *be a semialgebraic map. Suppose there exists a semialgebraic set* $Y\subset M$ *such that:*

- (i) $M_1=M\setminus Y$ *is locally compact and dense in* M.
- (ii) *The restriction* $\psi=\varphi|_{N_1}: N_1=N\setminus\varphi^{-1}(Y)\to M_1=M\setminus Y$ *is a semialgebraic homeomorphism.*

Let $Z=\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(Y)$. *Then, the map* $\operatorname{Spec}_s^*(\varphi):\operatorname{Spec}_s^*(N)\to\operatorname{Spec}_s^*(M)$ *is surjective and its restriction* $\operatorname{Spec}_s^*(\varphi)|:\operatorname{Spec}_s^*(N)\setminus\operatorname{Spec}_s^*(\varphi)^{-1}(Z)\to\operatorname{Spec}_s^*(M)\setminus Z$ *is a homeomorphism.*

The most typical situation to apply 1.2 concerns the choice $N=N_1=M_1=M_{lc}$ and $\varphi=j: M_{lc}\hookrightarrow M$ the inclusion map. Another typical setting to apply 1.2 is a blowing-up of the Zariski closure of the semialgebraic set M (see [1, 3.5.8]). Namely,

(1.3) *Let* X *be a real affine algebraic set, and let* $Y\subsetneq X$ *be an algebraic subset of* X. *Then, the blowing up* $\sigma: E(X,Y)\to X$ *of* X *with center* Y *is a proper regular map whose restriction* $\sigma|: E(X,Y)\setminus\sigma^{-1}(Y)\to X\setminus Y$ *is a biregular isomorphism.*

Now, let $M\subset\mathbb{R}^n$ be a semialgebraic set and let $Y\subsetneq X$ be an algebraic subset of the Zariski closure X of M such that $M_1=M\setminus Y$ is locally compact and dense in M. Denote $N=\operatorname{Cl}_{E(X,Y)}(\sigma^{-1}(M_1))\cap\sigma^{-1}(M)$ the strict transform of M. Then, 1.2 applies to the restriction $\varphi=\sigma|_N: N\to M$.

Observe that also in this setting of bounded semialgebraic functions, local compactness plays an essential role to compare and understand spectra of rings of bounded semialgebraic functions on semialgebraic sets. On the other hand, we construct in 5.17 a "stratification" of a semialgebraic set M whose strata are ordered in such a way that each of them is maximal with respect to the local compactness and density properties (see 2.4) in the complement in M of the union of the precedent ones (and the first stratum is maximal in M). This construction combined with 1.2 will provide us a better analysis of the spectrum of $\mathcal{S}^*(M)$ when M is not necessarily locally compact (see 5.18). In fact, the local study of the spectrum of the ring of bounded semialgebraic functions on an arbitrary semialgebraic set M is reduced, via 5.19, to the study of the open subsets of the semialgebraic spectrum of $\mathcal{S}^*(\mathbb{R}^m)$ for each $0 \leq m \leq \dim M$.

Next, recall that $\mathcal{S}^*(M)$ is a Gelfand ring (see [11, 3.1(iii)]), and so there exists a continuous retraction $\mathtt{r}_M : \operatorname{Spec}_s^*(M) \to \beta_s^* M$ which maps each prime ideal $\mathfrak{p}$ of $\mathcal{S}^*(M)$ to the unique maximal ideal $\mathfrak{m}^*$ containing $\mathfrak{p}$ (see 3.3). We will use such retraction to transfer in Sect. 6 the statements proved in Sect. 5 for the operator Spec_s^* to the operator β_s^* Here it is worthwhile mentioning that the map $\operatorname{Spec}_s^*(\varphi) : \operatorname{Spec}_s^*(N) \to \operatorname{Spec}_s^*(M)$ induced by a semialgebraic map $\varphi : N \to M$ between semialgebraic sets N and M, maps $\beta_s^* N$ into $\beta_s^* M$, see 5.9. Hence, it makes sense to denote $\beta_s^*\varphi : \beta_s^* N \to \beta_s^* M$ the restriction of $\operatorname{Spec}_s^*(\varphi)$ to $\beta_s^* N$.

The article is organized as follows. In Sects. 2 and 3, which have a preliminary character, we collect basic results and terminology concerning semialgebraic sets and functions and Zariski and maximal spectra of rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set, respectively. We develop in the subsequent sections the main results of this work. In Sect. 4 we approach the study of spectra of rings of semialgebraic functions, while Sect. 5 is devoted to analyze spectra of rings of bounded semialgebraic functions. Finally, in Sect. 6, we transfer the statements proved in Sects. 4 and 5 to maximal spectra of rings of semialgebraic and bounded semialgebraic functions.

2 Preliminaries on semialgebraic sets and functions

As we have announced in Sect. 1, in this section we present some preliminary terminology and results concerning semialgebraic sets and semialgebraic functions that will be useful in the rest of the work. We point out first that sometimes it will be advantageous to assume that the semialgebraic set M we are working with is bounded. Such assumption can be done without loss of generality. Namely,

Remark 2.1 Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathbb{B}_n(0,1) \subset \mathbb{R}^n$ be the open ball of center the origin and radius 1. The semialgebraic homeomorphism

$$\varphi : \mathbb{B}_n(0,1) \to \mathbb{R}^n, \quad x \mapsto \frac{x}{\sqrt{1-\|x\|^2}},$$

induces a ring isomorphism $\mathcal{S}(M) \to \mathcal{S}(N)$, $f \mapsto f \circ \varphi$, where $N = \varphi^{-1}(M)$, that maps $\mathcal{S}^*(M)$ onto $\mathcal{S}^*(N)$. Hence, if necessary, we may always assume that M is bounded.

The following result, which concerns the representation of closed semialgebraic subsets of a semialgebraic set as zerosets of semialgebraic functions, is also well-known and it will be used freely along this work.

Lemma 2.2 *Let Z be a closed semialgebraic subset of the semialgebraic set $M \subset \mathbb{R}^n$. Then, there exists $h \in \mathcal{S}^*(M)$ such that $Z = Z_M(h)$.*

Proof Take for instance $h = \min\{1, \operatorname{dist}(\cdot, Z)\}$. □

Next, we recall some properties of the set of regular points of a semialgebraic set.

(2.3) Set of regular points of a semialgebraic set. Let $M \subset \mathbb{R}^m$ be a d-dimensional semialgebraic set. We denote by $\operatorname{Reg}(M)$ the set of *regular points* of M, that is, those points $x \in M$ which have a neighbourhood V^x in M analytically diffeomorphic to $\mathbb{R}^d$; also denote $\delta(M) = M \setminus \operatorname{Reg}(M)$. Recall that $\operatorname{Reg}(M)$ is a nonempty open semialgebraic subset of M and $\delta(M)$ is a semialgebraic set of dimension $\leq d-1$ (see [18] for further details).

As it is well-known local closedness has been revealed, in the semialgebraic setting and in fact for the purposes of this work, as an important property for the validity of results which are in the core of semialgebraic geometry. Recall that the locally closed subsets of a locally compact topological space coincide with the locally compact ones (see for instance [2, Sect. 9.7. Prop.12–13]). Namely,

Lemma 2.4 *Let X be a Hausdorff and locally compact topological space. Given $M \subset X$, the following conditions are equivalent:*

(i) *M is locally closed.*
(ii) *$M = U \cap \operatorname{Cl}_X(M)$ where $U = X \setminus (\operatorname{Cl}_X(M) \setminus M)$ is an open subset of X.*
(iii) *M is a locally compact space.*

Remark 2.5 Notice that if $M \subset \mathbb{R}^n$ is a semialgebraic set, then also the sets $\operatorname{Cl}_{\mathbb{R}^n}(M)$ and $U = \mathbb{R}^n \setminus (\operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M)$ are semialgebraic. Thus, if $M \subset \mathbb{R}^n$ is a locally compact semialgebraic set, it can be written as the intersection of a closed and an open semialgebraic subsets of $\mathbb{R}^n$.

Next, we recall some of the main properties of the largest locally compact and dense subset M_{lc} of a semialgebraic set M. As we will see later, this set M_{lc} provides very useful information concerning the spectra of the rings of semialgebraic and bounded semialgebraic functions on M. Its construction is the main goal of [7, 3.8].

Theorem 2.6 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Define*

$$\rho_0(M) = \operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M \quad \textit{and} \quad \rho_1(M) = \rho_0(\rho_0(M)) = \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M)) \cap M.$$

Then, the semialgebraic set $M_{\text{lc}} = M \setminus \rho_1(M) = \operatorname{Cl}_{\mathbb{R}^n}(M) \setminus \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M))$ is the largest locally compact and dense subset of M and it coincides with the set of points of M which have a compact neighbourhood in M. Note that M_{lc} is an open subset of M.

In fact, we can go even further showing the local nature of the operator ρ_1. Namely,

Corollary 2.7 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $U \subset \mathbb{R}^n$ be an open semialgebraic set. Then, $\rho_1(M \cap U) = \rho_1(M) \cap U$.*

Proof First, note that

$$\rho_1(M \cap U) = \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M \cap U) \setminus (M \cap U)) \cap M \cap U,$$

while $\rho_1(M) \cap U = \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M)) \cap M \cap U$, and so it suffices to check that

$$\operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M)) \cap U = \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M \cap U) \setminus (M \cap U)) \cap U.$$

Now, U being an open set, we have

$$\begin{aligned}
\operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M)) \cap U &= \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M) \cap U) \cap U = \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M) \cap U \setminus (M \cap U)) \cap U \\
&= \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M \cap U) \cap U \setminus (M \cap U)) \cap U \\
&= \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M \cap U) \setminus (M \cap U)) \cap U,
\end{aligned}$$

and we are done. □

Next, we approach the continuous extension of semialgebraic functions defined on a semialgebraic set N to a larger semialgebraic set M. The most obvious way to do that is to extend bounded semialgebraic functions by zero after multiplying by a semialgebraic function that converges to zero on the semialgebraic set $\operatorname{Cl}_M(N)\setminus N$. Namely,

Lemma 2.8 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets. Write $Y = M\setminus N$ and take $b \in \mathcal{S}^*(N)$. Let $h \in \mathcal{S}^\diamond(M)$ be such that $Y \subset Z_M(h)$. Then, the product $(h|_N)b$ can be continuously extended by 0 to a function $B \in \mathcal{S}^\diamond(M)$.*

Proof Since b is bounded on N and h vanishes identically on Y, the limit $\lim_{x\to p}(h|_N b)(x)$ is 0, for all $p \in Y\cap\operatorname{Cl}_M(N)$. Thus, $(h|_N)b$ can be continuously extended by 0 to the whole M. The graph of such extension B being the union $\operatorname{graph}(h|_N b) \cup (Y \times \{0\})$, is a semialgebraic set, and so $B \in \mathcal{S}^\diamond(M)$. □

On the other hand, we also have continuous extension results for semialgebraic functions in the same vein as the classical Tietze–Urysohn's Lemma (see [6]). In fact, we cannot go much further than to work with closed semialgebraic subsets N of a semialgebraic set M to guarantee the continuous extension to M of any arbitrary semialgebraic function on N. More precisely,

(2.9) Scope of Tietze's extension. *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets. Then,*

(i) *The homomorphism $\phi : \mathcal{S}(M) \to \mathcal{S}(N),\ f \mapsto f|_N$ is surjective if and only if N is a closed subset of M.*
(ii) *The homomorphism $\phi : \mathcal{S}^*(M) \to \mathcal{S}^*(N),\ f \mapsto f|_N$ is surjective if and only if either N is closed in M or, for all $p \in \operatorname{Cl}_M(N)\setminus N$, the local dimension $\dim_p N = 1$ and the germ N_p has just one semialgebraic half-branch set germ.*

Proof We begin by proving (i). If N is closed in M then ϕ is surjective, by the semialgebraic version of Tietze–Urysohn's Lemma [6]. Conversely, if N is not closed in M, then there exists a point $p \in \operatorname{Cl}_M(N)\setminus N$. It is clear that the semialgebraic function on N defined by $f : N \to \mathbb{R},\ x \mapsto 1/\|x - p\|$ cannot be continuously extended to p. Hence, it cannot be continuously extended to M, as wanted.

Next, we proceed to prove (ii). Again, ϕ is surjective for a closed N as follows from the semialgebraic version of Tietze–Urysohn's Lemma. Moreover, if the local dimension $\dim_p N = 1$ and the set germ N_p has just one half-branch for all $p \in \operatorname{Cl}_M(N)\setminus N$, then each bounded semialgebraic function f on N admits a continuous extension to $\operatorname{Cl}_M(N)$. This is so because there exists the limit of f at a point along a half-branch (see for instance [10, 2.6]). Once $f \in \mathcal{S}^*(N)$ is continuously extended to $\operatorname{Cl}_M(N)$, we extend it to a function in $\mathcal{S}^*(M)$ by using again the semialgebraic version of Tietze–Urysohn's Lemma.

Conversely, assume that N is not closed in M and that there exists a point $p \in \operatorname{Cl}_M(N)\setminus N$ such that the set germ N_p contains two different semialgebraic half-branch germs at p. Let $C_0 \subset N$ and $C_1 \subset N$ be representatives of two such half-branch germs, that can be chosen closed in N and disjoint. By the semialgebraic version of Tietze–Urysohn's Lemma, there exists $g \in \mathcal{S}^*(N)$ such that $g|_{C_0} \equiv 0$ and $g|_{C_1} \equiv 1$. Since g cannot be continuously extended to p, it cannot be continuously extended to M. Thus, the germ N_p contains just one semialgebraic half-branch germ at p or, equivalently, as follows from the Curve Selection Lemma (see [1, 2.5.5]), for all $p \in \operatorname{Cl}_M(N)\setminus N$ the local dimension $\dim_p N = 1$ and the set germ N_p has just one semialgebraic half-branch set germ. □

Next, we recall the notion and some remarkable properties of the z-ideals of the ring $\mathcal{S}(M)$ of semialgebraic functions on a semialgebraic set M (see for instance [10, Sect. 3] for further details concerning z-ideals of the ring $\mathcal{S}(M)$).

Definition 2.10 Along this work whenever we consider an ideal of $\mathcal{S}^\diamond(M)$ we mean a proper ideal of $\mathcal{S}^\diamond(M)$. Recall that an ideal $\mathfrak{a}$ of $\mathcal{S}(M)$ is a *z-ideal* if whenever two functions $f, g \in \mathcal{S}(M)$ satisfy $Z_M(f) \subset Z_M(g)$ and $f \in \mathfrak{a}$, then $g \in \mathfrak{a}$.

One of the main properties of z-ideals is that they enjoy a Nullstellensatz (see for instance [10, 3.4]). Namely,

Theorem 2.11 (Nullstellensatz) *Let $M \subset \mathbb{R}^n$ be a locally compact semialgebraic set. Let $\mathfrak{a}$ be an ideal of $\mathcal{S}(M)$. Then, $\mathfrak{a}$ is a z-ideal if and only if $\mathfrak{a}$ is a radical ideal. In particular, if $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}$ is a z-ideal.*

To finish this section, we present the concept and some properties of the semialgebraic depth of a prime ideal (see [11, 4.4]), where it has been fruitfully used to compute the Krull dimension of rings of semialgebraic and bounded semialgebraic functions. In this work, we will provide further applications of such invariant.

(2.12) Semialgebraic depth. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. We define the *semialgebraic depth* of a prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$ as $\mathrm{d}_M(\mathfrak{p}) = \min\{\dim Z_M(f) : f \in \mathfrak{p}\}$.

A basic property of semialgebraic depth is the following one, proved in [11, 4.4].

(2.12.1) *Let $\mathfrak{p}, \mathfrak{q}$ be two prime z-ideals of $\mathcal{S}(M)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$. Then,* $\mathrm{d}_M(\mathfrak{p}) < \mathrm{d}_M(\mathfrak{q})$.

In particular, if $M \subset \mathbb{R}^n$ is a locally compact semialgebraic set, all prime ideals of $\mathcal{S}(M)$ are, by 2.11, z-ideals, and so given prime ideals $\mathfrak{p}, \mathfrak{q}$ of $\mathcal{S}(M)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$, then $\mathrm{d}_M(\mathfrak{p}) < \mathrm{d}_M(\mathfrak{q})$.

3 Generalities about spectra of rings of semialgebraic functions

In this section we study some preliminary algebraic and topological properties concerning spectra of rings of semialgebraic and bounded semialgebraic functions that will be used in the next sections to obtain the main results of this work. Surely, part of the terminology and preliminary results that we present now are well-known, but we include them here to fix the notation and to use them freely in the subsequent sections. We have preferred, because of its simplicity, to adjust the classical approach for rings of continuous functions, which is nicely and rigorously compiled by Gillman–Jerison in [14], instead of the more sophisticated theory of "Real closed rings" created much later by Schwartz in [16] and [17], and employed, among others, by Cherlin and Dickmann (see [4,5]). Although this last theory has been revealed as a very powerful tool, it requires a larger algebraic background that seems to be inessential to approach the study of the algebraic, functorial and topological properties of spectra of rings of semialgebraic functions on a semialgebraic subset of the Euclidean space $\mathbb{R}^n$. Of course, for our approach it is crucial to work over the real numbers and not over an arbitrary real closed field.

(3.1) Zariski spectra versus real spectra. We recall here that the *Zariski spectrum* $\operatorname{Spec}_s^\diamond(M) = \operatorname{Spec}(\mathcal{S}^\diamond(M))$ of $\mathcal{S}^\diamond(M)$ is the collection of all prime ideals of $\mathcal{S}^\diamond(M)$. This set $\operatorname{Spec}_s^\diamond(M)$ is usually endowed with the Zariski topology which has as a basis of open sets the family of sets $\mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f) = \{\mathfrak{p} \in \operatorname{Spec}_s^\diamond(M) : f \notin \mathfrak{p}\}$, where $f \in \mathcal{S}^\diamond(M)$. We denote $\mathcal{Z}_{\operatorname{Spec}_s^\diamond(M)}(f) = \operatorname{Spec}_s^\diamond(M) \setminus \mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f)$.

(3.1.1) We recall first: *For every $\mathfrak{p} \in \operatorname{Spec}_s^\diamond(M)$ the quotient field* $\operatorname{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})$ *admits a unique ordering*. In particular, this implies that $\mathfrak{p}$ is a *real ideal*, that is, if $a_1^2 + \cdots + a_p^2 \in \mathfrak{p}$, then each $a_i \in \mathfrak{p}$. For further details concerning real or orderable fields and the real spectrum of a commutative ring with unity, see [1, Ch.1, Ch.7]. As it is well-known, it is enough to check that each function $f \in \mathcal{S}^\diamond(M)$ is, $\mod \mathfrak{p}$, either a square or the opposite of a square. Indeed, since $(f-|f|)(f+|f|) = f^2-|f|^2 = 0 \in \mathfrak{p}$, we have $f+\mathfrak{p} = \pm(|f|+\mathfrak{p}) = \pm(\sqrt{|f|}+\mathfrak{p})^2$ where $\sqrt{|f|} \in \mathcal{S}^\diamond(M)$.

Thus $\mathrm{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})$ admits a unique ordering $\leq$ whose nonnegative elements are the squares. Hence, the map $\mathfrak{p} \mapsto (\mathfrak{p}, \leq)$ defines a bijection between $\mathrm{Spec}_s^\diamond(M)$ and the *real spectrum* $\mathrm{Spec}_r(\mathcal{S}^\diamond(M))$ of $\mathcal{S}^\diamond(M)$. In what follows both spectra will be denoted by $\mathrm{Spec}_s^\diamond(M)$. Moreover, using the fact that a radical ideal coincides with the intersection of all prime ideals containing it, we deduce that any radical ideal $\mathfrak{a}$ of $\mathcal{S}^\diamond(M)$ is a real ideal.

(3.1.2) Each radical ideal $\mathfrak{a}$ of the ring $\mathcal{S}^\diamond(M)$ satisfies a "convexity condition" which is ubiquitous in Real Geometry. Namely: *Given $f, g \in \mathcal{S}^\diamond(M)$ such that $g \in \mathfrak{a}$ and $0 \leq f(x) \leq g(x)$ for each point $x \in M$, then also $f \in \mathfrak{a}$.*

Indeed, by 2.8, we get a semialgebraic function $h \in \mathcal{S}^\diamond(M)$ defined by

$$h(x) = \begin{cases} \frac{f^2(x)}{g(x)} & \text{if } g(x) \neq 0, \\ 0 & \text{if } g(x) = 0. \end{cases}$$

Since $f^2 = gh \in \mathfrak{a}$, also $f \in \mathfrak{a}$. In particular, if $f, g \in \mathcal{S}^\diamond(M)$ and f is a unit such that $0 < f(x) \leq g(x)$ for each point $x \in M$, then also g is a unit.

(3.1.3) In fact, we can translate the convexity condition to the ordering of the field $\mathrm{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})$. More precisely: *If $0 \leq f+\mathfrak{p} \leq g+\mathfrak{p}$ in the ring $\mathcal{S}^\diamond(M)/\mathfrak{p}$, then we may assume that $0 \leq f(x) \leq g(x)$ for all $x \in M$.*

Indeed, $f+\mathfrak{p} = |f|+\mathfrak{p}$ and so we can substitute f by $|f|$. This guarantees that $f(x) \geq 0$ for all $x \in M$. On the other hand, since $(g-f)+\mathfrak{p} \geq 0$, the difference $h = (g-f)-|g-f| \in \mathfrak{p}$ and $(g-h)-f = |g-f|$ just attains nonnegative values. Thus, substituting g by $g-h$, we are done.

(3.1.4) As another consequence of the convexity, we have: *The set of prime ideals of the ring $\mathcal{S}^\diamond(M)$ containing a fixed prime ideal $\mathfrak{p}$ form a chain.* Otherwise, there would exist two prime ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in $\mathcal{S}^\diamond(M)$ containing $\mathfrak{p}$ such that $\mathfrak{q}_1 \not\subset \mathfrak{q}_2$ and $\mathfrak{q}_2 \not\subset \mathfrak{q}_1$. Thus, there exist $f_1 \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$ and $f_2 \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$. Since $\mathrm{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})$ is a real field, we may assume that $0 \leq f_2^2+\mathfrak{p} \leq f_1^2+\mathfrak{p}$, and in fact we can suppose, by 3.1.3, that $0 \leq f_2^2(x) \leq f_1^2(x)$ for all $x \in M$. This together with $f_1 \in \mathfrak{q}_1$ and 3.1.2 implies that $f_2 \in \mathfrak{q}_1$, a contradiction.

(3.1.5) We also deduce from convexity property for radical ideals that: *The quotient $A = \mathcal{S}^\diamond(M)/\mathfrak{a}$ of the ring $\mathcal{S}^\diamond(M)$ by a radical ideal $\mathfrak{a}$ is an f-ring*, that is, it is a lattice-ordered ring such that for all $\overline{f}, \overline{g}, \overline{h} \geq 0$ in A, $\inf\{\overline{f}, \overline{g}\} = 0$ implies that $\inf\{\overline{fh}, \overline{g}\} = 0$.

Indeed, since $\mathfrak{a}$ is radical, it is a convex ideal and, by [14, 5.2], the quotient A is a lattice-ordered ring. Let $\{\mathfrak{p}_i\}_{i\in I}$ be the collection of all prime ideals of $\mathcal{S}^\diamond(M)$ containing $\mathfrak{a}$. Observe that since $\mathfrak{a} = \bigcap_{i\in I}\mathfrak{p}_i$ the map $\varphi : A = \mathcal{S}^\diamond(M)/\mathfrak{a} \hookrightarrow \prod_{i\in I}\mathcal{S}^\diamond(M)/\mathfrak{p}_i$ is a monomorphism whose compositions $\pi_i \circ \varphi : A \to \mathcal{S}^\diamond(M)/\mathfrak{p}_i$ with the canonical projections π_i are surjective for all $i \in I$, that is, A is the subdirect sum of the totally ordered rings $\{\mathcal{S}^\diamond(M)/\mathfrak{p}_i\}_{i\in I}$. Thus, by [3], A is an f-ring.

(3.1.6) The usual topology in the real spectrum of $\mathcal{S}^\diamond(M)$ is the *spectral topology* which has as a basis of open sets the family of sets

$$\mathcal{U}_{\mathrm{Spec}_s^\diamond(M)}(f_1,\ldots,f_r) = \{\mathfrak{p} \in \mathrm{Spec}_s^\diamond(M) : f_1+\mathfrak{p} > 0,\ldots,f_r+\mathfrak{p} > 0 \text{ in } \mathrm{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})\}$$

where $f_1, \dots, f_r \in \mathcal{S}^\diamond(M)$. In fact, this topology coincides with the Zariski topology because

$$\mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f) = \mathcal{U}_{\operatorname{Spec}_s^\diamond(M)}(f) \cup \mathcal{U}_{\operatorname{Spec}_s^\diamond(M)}(-f)$$
$$\text{and} \quad \mathcal{U}_{\operatorname{Spec}_s^\diamond(M)}(f) = \mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f + |f|).$$

Thus, along the rest of the work we will use indistinctly both basis of open sets according to our convenience.

(3.1.7) Of course, M (endowed with the Euclidean topology) can be embedded in the Zariski spectrum $\operatorname{Spec}_s^\diamond(M)$ as a dense subspace via the map $\phi : M \to \operatorname{Spec}_s^\diamond(M),\ p \mapsto \mathfrak{m}_p^\diamond$. For the time being, we identify M with $\phi(M)$, which provides the equalities: $D_M(f) = \mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f) \cap M$ and $Z_M(f) = \mathcal{Z}_{\operatorname{Spec}_s^\diamond(M)}(f) \cap M$.

The rings $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ differ, in a crucial way, in their respective sets of units. In fact, a function $f \in \mathcal{S}^*(M)$ with empty zeroset is a unit in $\mathcal{S}(M)$, but it is not necessarily a unit in $\mathcal{S}^*(M)$, because $1/f$ needs not to be bounded. The semialgebraic function $f : M \to \mathbb{R},\ x \mapsto 1/(1 + \|x\|)$, where M is an unbounded semialgebraic set, provides an example of such situation. In some sense, this is the main difference between both rings. As we see immediately, the ring $\mathcal{S}(M)$ is a localization of $\mathcal{S}^*(M)$. This provides a nice relation between the prime ideals of both rings. Namely,

Lemma 3.2 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathcal{W}(M) \subset \mathcal{S}^*(M)$ be the multiplicative set of those functions $f \in \mathcal{S}^*(M)$ such that $Z_M(f) = \varnothing$. Then, $\mathcal{S}(M) = \mathcal{S}^*(M)_{\mathcal{W}(M)}$ is the localization of $\mathcal{S}^*(M)$ at the multiplicative set $\mathcal{W}(M)$. Moreover, we denote by $\mathfrak{S}(M) \subset$* $\operatorname{Spec}(\mathcal{S}^*(M))$ *the set of prime ideals of $\mathcal{S}^*(M)$ which do not intersect $\mathcal{W}(M)$. Then:*

(i) $\operatorname{Spec}(\mathcal{S}(M))$ *is in one-to-one correspondence with $\mathfrak{S}(M)$ via the maps*

$$j : \operatorname{Spec}(\mathcal{S}(M)) \to \mathfrak{S}(M), \quad \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^*(M)$$
$$j^{-1} : \mathfrak{S}(M) \to \operatorname{Spec}(\mathcal{S}(M)), \quad \mathfrak{q} \mapsto \mathfrak{q}\mathcal{S}(M).$$

(ii) *Both maps j and j^{-1} preserve inclusions and, in particular, minimal ideals.*

(iii) *The map j is a homeomorphism onto its image $\mathfrak{S}(M)$, and it is moreover a closed map if and only if M is compact; if such is the case $\mathcal{S}^*(M) = \mathcal{S}(M)$ and $j = \mathrm{id}$.*

Proof The equality $\mathcal{S}(M) = \mathcal{S}^*(M)_{\mathcal{W}(M)}$ is pretty evident since each function $f \in \mathcal{S}(M)$ can be written as a quotient $f = g/h$, where $g = f/(1+f^2) \in \mathcal{S}^*(M)$ and $h = 1/(1+f^2) \in \mathcal{W}(M)$. This guarantees all the asserts about the map j except for its not closedness when M is not compact. To check the latter we may assume, by 2.1, that M is bounded. So if it is not compact there exists a point $p \in \operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M$. Consider the bounded semialgebraic function f on M given by $x \mapsto \|x - p\|$. Note that $f \in \mathcal{W}(M)$ but it is not a unit in $\mathcal{S}^*(M)$. Let $\mathfrak{m}^*$ be a maximal ideal in $\mathcal{S}^*(M)$ which contains f. Since $f \in \mathcal{W}(M)$, $\mathfrak{m}^* \notin \operatorname{im} j$ and so the map $j : \operatorname{Spec}_s(M) \to \operatorname{Spec}_s^*(M)$ is not surjective. On the other hand, since M is a dense subset of $\operatorname{Spec}_s^*(M)$ contained in $\operatorname{im} j$, this last is not a closed subset of $\operatorname{Spec}_s^*(M)$. Hence, j is not a closed map. □

Next, we focus our attention in a relevant subspace of $\operatorname{Spec}_s^\diamond(M)$: its maximal spectrum. We begin by exposing some preliminary properties of this space that will be used later.

(3.3) Maximal spectra. The collection $\beta_s^\diamond M$ of all maximal ideals of $\mathcal{S}^\diamond(M)$ is endowed with the topology induced by the Zariski topology (or equivalently the spectral topology) of

$\operatorname{Spec}_s^\diamond(M)$. In what follows, given $f, f_1, \ldots, f_r \in \mathcal{S}^\diamond(M)$, we denote

$$\mathcal{D}_{\beta_s^\diamond M}(f) = \mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f) \cap \beta_s^\diamond M,$$
$$\mathcal{U}_{\beta_s^\diamond M}(f_1, \ldots, f_r) = \mathcal{U}_{\operatorname{Spec}_s^\diamond(M)}(f_1, \ldots, f_r) \cap \beta_s^\diamond M,$$
$$\mathcal{Z}_{\beta_s^\diamond M}(f) = \beta_s^\diamond M \setminus \mathcal{D}_{\beta_s^\diamond M}(f) = \mathcal{Z}_{\operatorname{Spec}_s^\diamond(M)}(f) \cap \beta_s^\diamond M.$$

By [1, 7.1.25(ii)], $\beta_s^\diamond M$ is a compact, Hausdorff space and it contains M as a dense subspace, that is, $\beta_s^\diamond M$ is a Hausdorff compactification of M. Observe that if M is compact, then the injective continuous map $\phi : M \to \beta_s^\diamond M$, $p \mapsto \mathfrak{m}_p^\diamond$ is in fact bijective (because in this case M is dense and closed in $\beta_s^\diamond M$) and so $\beta_s^\diamond M = M$.

(3.3.1) As it happens for rings of continuous functions (see [14, Sect. 7]), the respective maximal spectra $\beta_s M$ and $\beta_s^* M$ of $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ are homeomorphic (see [13, 3.5] for full details). Indeed, $\mathcal{S}^\diamond(M)$ being a Gelfand ring, the map $(\cdot)^* : \operatorname{Spec}_s(M) \to \beta_s^* M, \mathfrak{p} \to \mathfrak{p}^*$, where $\mathfrak{p}^*$ is the only maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{p} \cap \mathcal{S}^*(M)$, is well-defined. In fact, $(\cdot)^* = \mathtt{r}_M \circ j_1$ where $j_1 : \operatorname{Spec}_s(M) \hookrightarrow \operatorname{Spec}_s^*(M), \mathfrak{p} \to \mathfrak{p} \cap \mathcal{S}^*(M)$ and $\mathtt{r}_M : \operatorname{Spec}_s^*(M) \to \beta_s^* M$ is the retraction which maps each prime ideal of $\mathcal{S}^*(M)$ to the only maximal ideal containing it. In addition, the previous map $(\cdot)^*$ is continuous because so are j_1 and $\mathtt{r}_M$ (see [15, 1.2]). Moreover, if $j_2 : \beta_s M \hookrightarrow \operatorname{Spec}_s(M)$ is the inclusion map, also $\Phi = \mathtt{r}_M \circ j_1 \circ j_2 = (\cdot)^* \circ j_2 : \beta_s M \to \beta_s^* M$ is continuous. Furthermore, Φ is proper, because $\beta_s^\diamond M$ is compact and Hausdorff, and in fact it is surjective too, because the closed set $\operatorname{im} \Phi$ contains the dense subset M of $\beta_s^* M$ (see 3.1.7). In fact, as we prove in [13, 3.5], the map Φ is also injective and therefore it is a homeomorphism. More precisely,

(3.3.2) The map $\Phi : \beta_s M \to \beta_s^* M$ which maps each maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$ to the unique maximal ideal $\mathfrak{m}^*$ of $\mathcal{S}^*(M)$ that contains $\mathfrak{m} \cap \mathcal{S}^*(M)$, is a homeomorphism. Moreover, $\Phi(\mathfrak{m}_p) = \mathfrak{m}_p^*$ for all $p \in M$.

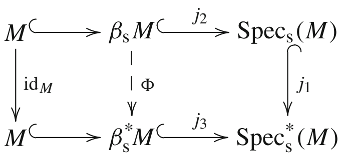
Thus, it is not an abuse of notation to denote $\mathfrak{m}^*$ every maximal ideal of $\mathcal{S}^*(M)$. Moreover, $\mathfrak{m}$ will denote the unique maximal ideal of $\mathcal{S}(M)$ such that $\mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$.

(3.4) Some maximal ideals of $\mathcal{S}^*(M)$ that will be useful in this work are those defined by means of a semialgebraic path (see [10, 2.5]). Namely, given a semialgebraic set $M \subset \mathbb{R}^n$ and a semialgebraic path $\alpha : (0, 1] \to M$, the set $\mathfrak{m}_\alpha^* = \{f \in \mathcal{S}^*(M) : \lim_{t \to 0}(f \circ \alpha)(t) = 0\}$ is a maximal ideal of $\mathcal{S}^*(M)$. Of course, distinct enough semialgebraic paths provide different maximal ideals. More precisely,

Lemma 3.5 *Let $M \subset \mathbb{R}^n$ be a bounded and noncompact semialgebraic set, and let $p \in \operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M$. Let $\alpha_i : [0, 1] \to \mathbb{R}^n$ be two semialgebraic paths such that $\alpha_i(0) = p$, $\alpha_i((0, 1]) \subset M$ and $\alpha_1((0, 1]) \cap \alpha_2((0, 1]) = \varnothing$. Then, $\mathfrak{m}_{\alpha_1}^* \neq \mathfrak{m}_{\alpha_2}^*$.*

Proof Notice that $C_i = \alpha_i((0, 1])$ is a closed subset of M for $i = 1, 2$. By 2.9, there exists $f \in \mathcal{S}^*(M)$ such that $f|_{C_1} \equiv 1$ and $f|_{C_2} \equiv 0$. Hence, $f \in \mathfrak{m}_{\alpha_2}^* \setminus \mathfrak{m}_{\alpha_1}^*$, and so $\mathfrak{m}_{\alpha_1}^* \neq \mathfrak{m}_{\alpha_2}^*$. □

Remark 3.6 Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Then, M is compact if and only if the right square of the following diagram commutes, that is, $j_1 \circ j_2 = j_3 \circ \Phi$. Thus, the behaviour of Φ is not "optimal" in case M is not compact.



Indeed, if M is compact, then $\beta_s^\diamond M = \{\mathfrak{m}_p^\diamond : p \in M\} \equiv M$. Thus, $\mathfrak{m}_p^* = \mathfrak{m}_p \cap \mathcal{S}^*(M)$, that is, $(j_1 \circ j_2)(\mathfrak{m}_p) = (j_3 \circ \Phi)(\mathfrak{m}_p^*)$ for all $p \in M$. On the other hand, if M is not compact, we may assume by 2.1 that M is bounded. Let $p \in \mathrm{Cl}_{\mathbb{R}^n}(M)\setminus M$; by the Curve Selection Lemma [1, 2.5.5], there exists a semialgebraic path $\alpha : [0, 1] \to \mathbb{R}^n$ such that $\alpha(0) = p$ and $\alpha((0, 1]) \subset M$. By 3.4, the set $\mathfrak{m}_\alpha^* = \{f \in \mathcal{S}^*(M) : \lim_{t\to 0}(f \circ \alpha)(t) = 0\}$ is a maximal ideal of $\mathcal{S}^*(M)$. Let $\mathfrak{m} \in \beta_s M$ be the maximal ideal of $\mathcal{S}(M)$ such that $\Phi(\mathfrak{m}) = \mathfrak{m}_\alpha^*$ and consider the bounded semialgebraic function $f : \mathbb{R}^n \to \mathbb{R},\ x \mapsto \|x - p\|/(1 + \|x - p\|)$, whose zeroset in $\mathbb{R}^n$ is $\{p\}$, and so it is a unit in $\mathcal{S}(M)$. Thus, $f \in \mathfrak{m}_\alpha^*\setminus\mathfrak{m}$, which implies $(j_1 \circ j_2)(\mathfrak{m}) = \mathfrak{m} \cap \mathcal{S}^*(M) \neq \mathfrak{m}_\alpha^* = (j_3 \circ \Phi)(\mathfrak{m})$.

4 Functoriality of Spec_s

In this section, we are mainly concerned with two questions: (1) Given a closed semialgebraic subset C of a semialgebraic set M, we will realize the Zariski spectrum of $\mathcal{S}^\diamond(C)$ as the closure of C in $\mathrm{Spec}_s^\diamond(M)$; (2) To compare the spectra of two suitable semialgebraic sets. More precisely, we are led to compare $\mathrm{Spec}_s(N)$ and $\mathrm{Spec}_s(M)$ where M is arbitrary and $N \subset M$ is open in M and locally compact. It is necessary to impose the local compactness condition on N because Łojasiewicz's inequality is not longer true for non locally compact semialgebraic sets (see [10, 3.5]). On the other hand, it is often useful to compare $\mathrm{Spec}_s(M)$ and $\mathrm{Spec}_s(X)$ where X is a semialgebraic compactification of a locally compact semialgebraic set M (see also [11]). Recall also that if M is locally compact, then it is open in X (see 2.4). Hence, both situations are very similar and admit a simultaneous treatment.

We begin by presenting some well-known basic functorial properties for $\mathrm{Spec}_s^\diamond$. From them we achieve straightforwardly our first purpose of realizing the spectrum of $\mathcal{S}^\diamond(C)$ as the closure of C in $\mathrm{Spec}_s^\diamond(M)$ for every closed semialgebraic subset C of M.

Lemma 4.1 *Let $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ be semialgebraic sets and let $\varphi : N \to M$ be a semialgebraic map. Then:*

- (i) *There exists a unique continuous map* $\mathrm{Spec}_s^\diamond(\varphi) : \mathrm{Spec}_s^\diamond(N) \to \mathrm{Spec}_s^\diamond(M)$ *which extends φ.*
- (ii) *Let $\mathfrak{q}$ be a prime z-ideal of $\mathcal{S}(N)$. Then,* $\mathfrak{p} = \mathrm{Spec}_s(\varphi)(\mathfrak{q})$ *is a z-ideal of $\mathcal{S}(M)$.*
- (iii) *Let $\psi : M \to P$ be another semialgebraic map, where $P \subset \mathbb{R}^p$ is a semialgebraic set. Then,* $\mathrm{Spec}_s^\diamond(\psi) \circ \mathrm{Spec}_s^\diamond(\varphi) = \mathrm{Spec}_s^\diamond(\psi \circ \varphi)$.

Proof The homomorphism $\phi : \mathcal{S}^\diamond(M) \to \mathcal{S}^\diamond(N),\ f \mapsto f \circ \varphi$ induces a continuous map $\mathrm{Spec}_s^\diamond(\varphi) : \mathrm{Spec}_s^\diamond(N) \to \mathrm{Spec}_s^\diamond(M),\ \mathfrak{q} \mapsto \phi^{-1}(\mathfrak{q})$, and let us show that $\mathrm{Spec}_s^\diamond(\varphi)$ extends φ.

(i) Let $q \in N$ and $p = \varphi(q) \in M$. Consider the maximal ideal $\mathfrak{n}_q^\diamond$ of $\mathcal{S}^\diamond(N)$ associated to q and the maximal ideal $\mathfrak{m}_p^\diamond$ of $\mathcal{S}^\diamond(M)$ associated to p. To prove that $\mathrm{Spec}_s^\diamond(\varphi)$ extends φ it is enough to check that $\mathrm{Spec}_s^\diamond(\varphi)(\mathfrak{n}_q^\diamond)$ is a maximal ideal, because this together with the obvious inclusion $\mathrm{Spec}_s^\diamond(\varphi)(\mathfrak{n}_q^\diamond) = \phi^{-1}(\mathfrak{n}_q^\diamond) \subset \mathfrak{m}_p^\diamond$ yields the equality $\mathrm{Spec}_s^\diamond(\varphi)(\mathfrak{n}_q^\diamond) = \mathfrak{m}_p^\diamond$. But we have $\mathbb{R} \hookrightarrow \mathcal{S}^\diamond(M)/\phi^{-1}(\mathfrak{n}_q^\diamond) \hookrightarrow \mathcal{S}^\diamond(N)/\,\mathfrak{n}_q^\diamond \cong \mathbb{R}$ and so $\phi^{-1}(\mathfrak{n}_q^\diamond)$ is a maximal ideal of $\mathcal{S}^\diamond(M)$. The uniqueness of $\mathrm{Spec}_s^\diamond(\varphi)$ follows from the density of N in $\mathrm{Spec}_s^\diamond(N)$.

(ii) Let $h \in \mathcal{S}(M)$ and $g \in \mathfrak{p}$ such that $Z_M(g) \subset Z_M(h)$. Thus,

$$Z_N(\phi(g)) = Z_N(g \circ \varphi) \subset Z_N(h \circ \varphi) = Z_N(\phi(h)),$$

and since $\mathfrak{q}$ is a z-ideal and $\phi(g) \in \mathfrak{q}$, we deduce that $\phi(h) \in \mathfrak{q}$, that is, $h \in \mathfrak{p}$.

(iii) It suffices to employ the uniqueness in (i), because the equality

$$(\operatorname{Spec}_s^\diamond(\psi)\circ\operatorname{Spec}_s^\diamond(\varphi))|_N=\operatorname{Spec}_s^\diamond(\psi)|_M\circ\operatorname{Spec}_s^\diamond(\varphi)|_N=\psi\circ\varphi=\operatorname{Spec}_s^\diamond(\psi\circ\varphi)\big|_N$$

implies $\operatorname{Spec}_s^\diamond(\psi)\circ\operatorname{Spec}_s^\diamond(\varphi)=\operatorname{Spec}_s^\diamond(\psi\circ\varphi)$. □

Remarks 4.2 (i) Let $N\subset M\subset\mathbb{R}^n$ be semialgebraic sets, and let $j:N\hookrightarrow M$ be the inclusion map. The induced map $\operatorname{Spec}_s^\diamond(j):\operatorname{Spec}_s^\diamond(N)\to\operatorname{Spec}_s^\diamond(M)$ is defined by $\mathfrak{q}\mapsto\phi^{-1}(\mathfrak{q})$, where $\phi:\mathcal{S}^\diamond(M)\to\mathcal{S}^\diamond(N)$, $g\mapsto g|_N=g\circ j$. By an (intuitive) abuse of notation we will frequently write $\mathfrak{q}\cap\mathcal{S}^\diamond(M)$ instead of $\phi^{-1}(\mathfrak{q})$.

(ii) In the proof of 4.1(ii) we have not used the primality of $\mathfrak{p}$ and $\mathfrak{q}$. Therefore, if $\phi:\mathcal{S}(M)\to\mathcal{S}(N)$, $f\mapsto f\circ\varphi$, then $\phi^{-1}(\mathfrak{a})$ is a z-ideal of $\mathcal{S}(M)$ whenever $\mathfrak{a}$ is a z-ideal of $\mathcal{S}(N)$.

The following results, which are expectable and surely well-known, show the good behaviour of the operator $\operatorname{Spec}_s^\diamond$.

Lemma 4.3 *Let $N\subset M\subset\mathbb{R}^n$ be semialgebraic sets. Consider the homomorphism $\phi:\mathcal{S}^\diamond(M)\to\mathcal{S}^\diamond(N)$, $f\to f|_N$ and a prime ideal $\mathfrak{p}$ of $\mathcal{S}^\diamond(M)$. Then, $\mathfrak{p}\in\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ if and only if* $\ker\phi\subset\mathfrak{p}$. *Moreover, if N is closed in M and $\mathfrak{p}\notin\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ there exists $f\in\mathcal{S}^\diamond(M)\setminus\mathfrak{p}$ such that $N=Z_M(f)$.*

Proof Suppose that $\mathfrak{p}\in\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ and let $f\notin\mathfrak{p}$. Then, $D_N(f)=N\cap\mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f)$ is not empty, that is, there exists a point $p\in N$ such that $f(p)\neq 0$, and so $f\notin\ker\phi$. Thus, $\ker\phi\subset\mathfrak{p}$.

Conversely, suppose that $\ker\phi\subset\mathfrak{p}$ and let $g\notin\mathfrak{p}$, that is, $\mathfrak{p}\in\mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(g)$. Then, $g\notin\ker\phi$, that is, there exists a point $p\in N$ such that $g(p)\neq 0$, or equivalently, $N\cap\mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(g)=D_N(g)\neq\varnothing$. Hence, $\mathfrak{p}\in\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$.

For the second part, there exists, by 2.2, a function $g\in\mathcal{S}^\diamond(M)$ such that $N=Z_M(g)$. If $g\notin\mathfrak{p}$, we choose $f=g$. Hence, suppose that $g\in\mathfrak{p}$. Since $\mathfrak{p}\notin\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ there exists $h\in\ker\phi\setminus\mathfrak{p}$. Then, the function $f=g^2+h^2$ does the job. □

Corollary 4.4 *Let $N\subset M\subset\mathbb{R}^n$ be semialgebraic sets such that M is locally compact. Consider the homomorphism $\phi:\mathcal{S}(M)\to\mathcal{S}(N)$, $f\to f|_N$ and a prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$. Then, $\mathfrak{p}\in\operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$ if and only if there exists $h\in\mathfrak{p}$ such that $Z_M(h)\subset\operatorname{Cl}_M(N)$.*

Proof By 4.3, $\mathfrak{p}\in\operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$ if and only if it contains all the semialgebraic functions on M vanishing identically on N. Suppose first that $\mathfrak{p}\in\operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$, and let $h\in\mathcal{S}(M)$ such that $\operatorname{Cl}_M(N)=Z_M(h)$. Then, $h\in\mathfrak{p}$ because $h|_N\equiv 0$.

Suppose, conversely, that $\mathfrak{p}$ contains a function $h\in\mathcal{S}(M)$ such that $Z_M(h)\subset\operatorname{Cl}_M(N)$, and let $g\in\mathcal{S}(M)$ be a function that vanishes identically on N. Then, $Z_M(h)\subset Z_M(g)$, and since M is locally compact, $\mathfrak{p}$ is, by 2.11, a z-ideal. Therefore $g\in\mathfrak{p}$. □

Corollary 4.5 *Let $C_1,C_2\subset M\subset\mathbb{R}^n$ be semialgebraic sets such that C_1 and C_2 are closed subsets of M. Then, $\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_1\cap C_2)=\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_1)\cap\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_2)$.*

Proof For the nonobvious inclusion let $\mathfrak{q}\in\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_1)\cap\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_2)$. Consider the epimorphisms (see 2.9)

$$\phi:\mathcal{S}^\diamond(M)\to\mathcal{S}^\diamond(C_1\cap C_2),\ f\mapsto f|_{C_1\cap C_2},\qquad \phi_1:\mathcal{S}^\diamond(M)\to\mathcal{S}^\diamond(C_1),\ f\mapsto f|_{C_1},$$
$$\phi_2:\mathcal{S}^\diamond(M)\to\mathcal{S}^\diamond(C_2),\ f\mapsto f|_{C_2},\qquad \theta:\mathcal{S}^\diamond(M)\to\mathcal{S}^\diamond(C_1\cup C_2),\ f\mapsto f|_{C_1\cup C_2}.$$

By 4.3, it suffices to check that $\ker\phi\subset\mathfrak{q}$. The ideal $\mathfrak{q}$ contains $\ker\phi_1+\ker\phi_2$, because $\mathfrak{q}\in\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_1)\cap\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(C_2)$. Thus, it is enough to prove that $\ker\phi\subset\ker\phi_1+\ker\phi_2$.

Indeed, let $f \in \ker\phi$. Since $f|_{C_1\cap C_2} = 0$, there exists $g \in \mathcal{S}^\diamond(C_1 \cup C_2)$ such that $g|_{C_1} = 0$ and $g|_{C_2} = f|_{C_2}$. Since θ is surjective, there exists $h_1 \in \mathcal{S}^\diamond(M)$ such that $\theta(h_1) = g$. Note that $h_1 \in \ker\phi_1$ and $h_2 = f - h_1 \in \ker\phi_2$. So, $f = h_1 + h_2 \in \ker\phi_1 + \ker\phi_2$, and we are done. □

The next result characterizes the surjectivity of the homomorphism $\mathcal{S}^\diamond(M) \to \mathcal{S}^\diamond(N)$ for semialgebraic sets $N \subset M \subset \mathbb{R}^n$ in terms of the corresponding spectra. Namely,

Corollary 4.6 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets and let $j : N \hookrightarrow M$ be the inclusion map. The following statements are equivalent:*

(i) *The homomorphism $\phi : \mathcal{S}^\diamond(M) \to \mathcal{S}^\diamond(N),\ f \mapsto f|_N$ is surjective.*
(ii) $\operatorname{Spec}_s^\diamond(N) \cong \operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ *via* $\operatorname{Spec}_s^\diamond(j)$.

Proof First, we check that (i) $\Longrightarrow$ (ii). Since ϕ is surjective, $\mathcal{S}^\diamond(M)/\ker\phi \cong \mathcal{S}^\diamond(N)$. Hence, there exists a one-to-one correspondence between the prime ideals of $\mathcal{S}^\diamond(N)$ and the prime ideals of $\mathcal{S}^\diamond(M)$ containing $\ker\phi$. That is, by 4.3, the map

$$\gamma : \operatorname{Spec}_s^\diamond(N) \to \operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N) \subset \operatorname{Spec}_s^\diamond(M), \quad \mathfrak{q} \mapsto \operatorname{Spec}_s^\diamond(j)(\mathfrak{q})$$

is bijective and continuous. To prove that it is a homeomorphism it is enough to see that it is an open map. Given $g \in \mathcal{S}^\diamond(N)$ there exists $G \in \mathcal{S}^\diamond(M)$ such that $G|_N = g$ and so $\gamma(\mathcal{D}_{\operatorname{Spec}_s^\diamond(N)}(g)) = \mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(G) \cap \operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ is an open subset of $\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$.

Next, we prove that (ii) $\Longrightarrow$ (i). We distinguish two cases according as we are dealing with $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$. In the first case, suppose by way of contradiction, that ϕ is not surjective or, equivalently, that N is not closed in M (see 2.9(i)). Let $p \in \operatorname{Cl}_M(N)\setminus N$. The maximal ideal $\mathfrak{m}_p \in \operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$ and so $\mathfrak{q} = \operatorname{Spec}_s(j)^{-1}(\mathfrak{m}_p) \in \operatorname{Spec}_s(N)$. Note that $\phi(\mathfrak{m}_p) = \phi(\phi^{-1}(\mathfrak{q})) \subset \mathfrak{q}$. Now, since the semialgebraic function $f(x) = \|x - p\|$ lies in $\mathfrak{m}_p$, then $f|_N \in \mathfrak{q}$. However, this is impossible because $f|_N$ is a unit of the ring $\mathcal{S}(N)$. Thus, N is closed in M, and so ϕ is surjective.

Next, we proceed with the case of bounded functions. If ϕ is not surjective, then, by 2.9, N is not closed in M and there exists a point $p \in \operatorname{Cl}_M(N)\setminus N$ such that the germ N_p contains two different half-branch germs. Let α_1 and α_2 be parametrizations of these half-branch germs. By 3.5, the maximal ideals $\mathfrak{m}^*_{\alpha_i} \in \operatorname{Spec}_s^*(N)$ are different but, as we will see immediately, $\operatorname{Spec}_s^*(j)(\mathfrak{m}^*_{\alpha_i}) = \mathfrak{m}^*_p$ for $i = 1, 2$, which contradicts the injectivity of $\operatorname{Spec}_s^*(j)$. Indeed, $f \in \operatorname{Spec}_s^*(j)(\mathfrak{m}^*_{\alpha_i})$ if and only if $\lim_{t\to 0}(f \circ \alpha_i)(t) = 0$, or equivalently, if $f(p) = 0$; hence, $\operatorname{Spec}_s^*(j)(\mathfrak{m}^*_{\alpha_i}) = \mathfrak{m}^*_p$, as claimed. □

Next, as a consequence of 4.5 and 4.6, we describe the relationship between the connected components of a semialgebraic set M and the ones of $\operatorname{Spec}_s^\diamond(M)$.

Corollary 4.7 *Let $M_1, \ldots, M_k$ be the connected components of the semialgebraic set $M \subset \mathbb{R}^n$. Then, their closures $\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(M_i) \cong \operatorname{Spec}_s^\diamond(M_i)$ are the connected components of $\operatorname{Spec}_s^\diamond(M)$. In particular, $\operatorname{Spec}_s^\diamond(M)$ has a finite number of connected components, and it is connected if and only if M is so.*

Proof First, recall that M has a finite number of connected components ([1, 2.4.5]), say $M_1, \ldots, M_k$, which are closed semialgebraic subsets of M. Note that each $\operatorname{Cl}_{\operatorname{Spec}^\diamond(M)}(M_i)$ is connected and $\operatorname{Spec}_s^\diamond(M) = \bigcup_{i=1}^k \operatorname{Cl}_{\operatorname{Spec}^\diamond(M)}(M_i)$. Thus, the sets $\operatorname{Cl}_{\operatorname{Spec}^\diamond(M)}(M_i)$ being, by 4.5, pairwise disjoint, the closures $\operatorname{Cl}_{\operatorname{Spec}^\diamond(M)}(M_1), \ldots, \operatorname{Cl}_{\operatorname{Spec}^\diamond(M)}(M_k)$ are the connected components of $\operatorname{Spec}_s^\diamond(M)$.

On the other hand, the homomorphism $\phi_i : \mathcal{S}^\diamond(M) \to \mathcal{S}^\diamond(M_i)$, $f \mapsto f|_{M_i}$ is, by 2.9, surjective, because M_i is closed in M. Hence, $\mathrm{Cl}_{\mathrm{Spec}_s^\diamond(M)}(M_i)$ and $\mathrm{Spec}_s^\diamond(M_i)$ are homeomorphic, by 4.6. □

Now, we are ready to focus our efforts in the study of $\mathrm{Spec}_s(M)$ by comparing it with other better known spectra. Given semialgebraic sets $Y \subset M \subset \mathbb{R}^n$ such that Y is closed in M, we denote $\mathcal{E}(Y) = \{f \in \mathcal{S}(M) : Z_M(f) = Y\}$, which is nonempty by 2.2, and define the *spectral envelope of* Y *in* $\mathrm{Spec}_s(M)$ as $\mathcal{L}(Y) = \bigcup_{f \in \mathcal{E}(Y)} \mathcal{Z}_{\mathrm{Spec}_s(M)}(f)$. The main result of this section, which is a precise reformulation of 1.1, is the following.

Theorem 4.8 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that N is open in M and locally compact. Denote $Y = M \setminus N$ and let $j : N \hookrightarrow M$ be the inclusion map. Then,*

- (i) *The map* $\mathrm{Spec}_s(j) : \mathrm{Spec}_s(N) \to \mathrm{Spec}_s(M)$ *is a homeomorphism onto its image* $\mathrm{Spec}_s(M) \setminus \mathcal{L}(Y)$.
- (ii) *Let* $\mathfrak{q} \in \mathrm{Spec}_s(N)$ *such that* $\mathfrak{p} = \mathrm{Spec}_s(j)(\mathfrak{q})$ *is a maximal ideal of* $\mathcal{S}(M)$. *Then,* $\mathfrak{q}$ *is a maximal ideal of* $\mathcal{S}(N)$.
- (iii) *Assume that* $N \subsetneq M$ *is a dense subset of* M.
  - (a) *If* $\mathfrak{p}$ *is a minimal prime ideal of* $\mathcal{S}(M)$, *then* $\mathfrak{p} \notin \mathcal{L}(Y)$.
  - (b) *There exist maximal ideals* $\mathfrak{n}$ *of* $\mathcal{S}(N)$ *whose images* $\mathrm{Spec}_s(j)(\mathfrak{n})$ *are not maximal ideals of* $\mathcal{S}(M)$.

The proof of 4.8 requires some preliminary work that we begin right now. We start with some remarks concerning the spectral envelope $\mathcal{L}(Y)$.

Remarks 4.9 Let $Y \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that Y is closed in M and let $\mathcal{L}(Y)$ be the spectral envelope of Y in $\mathrm{Spec}_s(M)$. Then:

- (i) By 4.3, $Y \subset \mathrm{Cl}_{\mathrm{Spec}_s(M)}(Y) \subset \bigcap_{f \in \mathcal{E}(Y)} \mathcal{Z}_{\mathrm{Spec}_s(M)}(f) \subset \mathcal{L}(Y)$.
- (ii) If M is locally compact then $\mathrm{Cl}_{\mathrm{Spec}_s(M)}(Y) = \mathcal{L}(Y)$.
  Indeed for (ii) it is enough to check, by 4.3, that each $\mathfrak{p} \in \mathcal{L}(Y)$ contains the kernel of the homomorphism $\phi : \mathcal{S}(M) \to \mathcal{S}(Y)$, $g \to g|_Y$. Let $f \in \mathcal{E}(Y)$ such that $f \in \mathfrak{p}$. Observe that $Z_M(f) = Y \subset Z_M(h)$ for each $h \in \ker \phi$ and so, $\mathfrak{p}$ being a z-ideal because M is locally compact, also $h \in \mathfrak{p}$.
- (iii) It is proved in [7, 4.14] that all maximal ideals of $\mathcal{S}(M)$ are z-ideals. Then, arguing as in (ii), it follows that $\mathcal{L}(Y) \cap \beta_s M = \mathrm{Cl}_{\mathrm{Spec}_s(M)}(Y) \cap \beta_s M = \mathrm{Cl}_{\beta_s M}(Y)$. Thus, by 4.8(ii), we deduce that
$$\mathrm{Spec}_s(j)(\mathrm{Spec}_s(N)) \cap \beta_s M = \mathrm{Spec}_s(j)(\beta_s N) \cap \beta_s M = \beta_s M \setminus \mathrm{Cl}_{\beta_s M}(Y).$$
- (iv) If M is not locally compact then the equality $\mathrm{Cl}_{\mathrm{Spec}_s(M)}(Y) = \mathcal{L}(Y)$ is false in general, as we see in the next example.

Example 4.10 Let $M \subset \mathbb{R}^n$ be a semialgebraic set which is not locally compact. Then, there exists a closed semialgebraic subset $Y \subset M$ such that $\mathrm{Cl}_{\mathrm{Spec}_s(M)}(Y) \subsetneq \mathcal{L}(Y)$.

Indeed, there exist, by [10, 2.9], a closed semialgebraic subset $C \subset M$ and a semialgebraic homeomorphism $\psi : C \to T = \{(x, y) \in \mathbb{R}^2 : 0 < y \leq x \leq 1\} \cup \{(0, 0)\}$ and, by [10, 3.5.1], the set

$$\mathfrak{q} = \{f \in \mathcal{S}(T) : \exists \varepsilon > 0 \,|\, f \text{ extends continuously by 0 to } T \cup ((0, \varepsilon] \times \{0\})\}$$

is a fixed prime ideal of $\mathcal{S}(T)$ which is not a z-ideal. Consider the ring epimorphism $\phi : \mathcal{S}(M) \to \mathcal{S}(T)$, $f \mapsto f|_C \circ \psi^{-1}$ and let us check that the prime ideal $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \in \mathcal{L}(Y) \setminus \mathrm{Cl}_{\mathrm{Spec}_s(M)}(Y)$, where $Y = \{p = \psi^{-1}(0, 0)\}$.

By 2.2, there exists $g \in \mathcal{S}(M)$ such that $Z_M(g) = C$. Consider the semialgebraic function $h = y \in \mathcal{S}(T)$. By 2.9, there exists $f \in \mathcal{S}(M)$ such that $f|_C = h \circ \psi$. Then, $h_1 = \sqrt{f^2 + g^2} \in \mathcal{S}(M)$ extends $h \circ \psi$ and $Z_M(h_1) = \{p\}$, that is, $h_1 \in \mathcal{E}(Y)$. Therefore, $\mathfrak{p} \in \mathcal{L}(Y)$, because $h_1 \in \mathfrak{p}$.

On the other hand, by 2.9 there exists $a_1 \in \mathcal{S}(M)$ with $a_1|_T = a \circ \psi$, where $a = x^2 + y^2 \in \mathcal{S}(T)$. Then, $b_1 = \sqrt{a_1^2 + g^2}$ is a semialgebraic extension to M of $a \circ \psi$ and $Z_M(b_1) = \{p\}$, that is, $b_1|_Y \equiv 0$. However, $b_1 \notin \mathfrak{p}$ because $a \notin \mathfrak{q}$, so, by 4.3, $\mathfrak{p} \notin \mathrm{Cl}_{\mathrm{Spec_s}(M)}(Y)$, as wanted. □

Next, we present an algebraic characterization of the prime ideals occurring in the spectral envelope $\mathcal{L}(Y)$.

Lemma 4.11 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that N is open in M and locally compact and denote $Y = M \setminus N$. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$, $f \mapsto f|_N$ and let $\mathfrak{p}$ be a prime ideal of $\mathcal{S}(M)$. Then, the following conditions are equivalent:*

(i) $\phi(\mathfrak{p})\mathcal{S}(N) = \mathcal{S}(N)$.
(ii) *There exists $f \in \mathfrak{p}$ such that $Z_M(f) = Y$, that is, $\mathfrak{p} \in \mathcal{L}(Y)$.*

The proof of 4.11 requires a preliminary result that will be useful also later (see 4.13).

Lemma 4.12 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets such that N is open in M and locally compact and denote $Y = M \setminus N$. Then,*

(i) *$Z = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus N$ is a closed semialgebraic subset of $\mathbb{R}^n$ and $Z \cap M = Y$. In particular, there exists $c \in \mathcal{S}(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(c) = Z$.*
(ii) *For each $f \in \mathcal{S}(N)$ and each $c \in \mathcal{S}(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(c) = Z$, there exist $h \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$ and $k \geq 1$ satisfying $h|_N = (c|_N)^k f$ and $Z_{\mathrm{Cl}_{\mathbb{R}^n}(M)}(h) = Z \cup Z_N(f)$.*

Proof (i) It is pretty obvious that $Z \cap M = (\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus N) \cap M = M \setminus N = Y$. Let us check that Z is closed in $\mathbb{R}^n$. In fact, it is enough to see that N is open in $\mathrm{Cl}_{\mathbb{R}^n}(M)$. Observe that, by 2.6, $N \subset M_{\mathrm{lc}}$ and clearly it is an open subset of M_{lc}. On the other hand, since M_{lc} is dense in $\mathrm{Cl}_{\mathbb{R}^n}(M)$ we deduce, by 2.4, that M_{lc} is an open subset of $\mathrm{Cl}_{\mathbb{R}^n}(M)$; hence, N is open in $\mathrm{Cl}_{\mathbb{R}^n}(M)$, and we are done. Now, the existence of $c \in \mathcal{S}(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(c) = Z$ follows from 2.2.

(ii) By [1, 2.6.4] there exist an integer $k \geq 1$ and $h \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$ which is identically zero outside N satisfying $h|_N = (c|_N)^k f$. Hence, since $Z_N(c) = Z \cap N = \varnothing$,

$$Z_{\mathrm{Cl}_{\mathbb{R}^n}(M)}(h) = (\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus N) \cup Z_N(h) = Z \cup Z_N(f) \cup Z_N(c) = Z \cup Z_N(f),$$

as wanted. □

Proof of Lemma 4.11 (i) $\Longrightarrow$ (ii) If $\phi(\mathfrak{p})\mathcal{S}(N) = \mathcal{S}(N)$ there exist $a_1, \ldots, a_r \in \mathfrak{p}$ and $b_1, \ldots, b_r \in \mathcal{S}(N)$ such that $1 = (a_1|_N)b_1 + \cdots + (a_r|_N)b_r$. By 4.12, there exists $c \in \mathcal{S}(\mathbb{R}^n)$ such that $Z = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus N = Z_{\mathbb{R}^n}(c)$ and $Z \cap M = Y$; in particular $Z_M(c|_M) = Y$. Again by 4.12, there exists an integer $k \geq 1$ such that each product $(c|_N)^k b_i \in \mathcal{S}(N)$ can be continuously extended by zero to the whole $\mathrm{Cl}_{\mathbb{R}^n}(M)$. Denote by f_i such extension and $g_i = f_i|_M$. Then, $(c|_M)^k = a_1 g_1 + \cdots + a_r g_r \in \mathfrak{p}$, and so also $f = c|_M \in \mathfrak{p}$ and $Z_M(f) = Y$.

(ii) $\Longrightarrow$ (i) This is trivial because $(1/f)|_N$ is a unit in $\mathcal{S}(N)$. □

The next result, which has a quite technical formulation explains, among other things, the behaviour of the semialgebraic depth (see 2.12) under suitable extension and contraction of ideals, and it is the key for the proof of 4.8. Anyway, this result has interest by its own and has further consequences (see for instance [7]).

Lemma 4.13 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that N is open in M and locally compact. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(N),\ f \mapsto f|_N$ be the homomorphism induced by the inclusion $j : N \hookrightarrow M$ and denote $Y = M \setminus N$. The following properties hold:*

(i) *Let $\mathfrak{q} \in \operatorname{Spec_s}(N)$ and $\mathfrak{p} = \operatorname{Spec_s}(j)(\mathfrak{q})$. Then, $\mathrm{d}_N(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p})$ and $\mathfrak{p} \notin \mathcal{L}(Y)$.*

(ii) *Let $\mathfrak{p} \in \operatorname{Spec_s}(M) \setminus \mathcal{L}(Y)$. Then, $\mathfrak{p}$ is a z-ideal and $\mathfrak{q} = \phi(\mathfrak{p})\mathcal{S}(N)$ is a prime z-ideal of $\mathcal{S}(N)$. Moreover, $\operatorname{Spec_s}(j)(\mathfrak{q}) = \mathfrak{p}$ and $\mathfrak{q}$ is the unique prime ideal $\mathfrak{a}$ of $\mathcal{S}(N)$ such that $\operatorname{Spec_s}(j)(\mathfrak{a}) = \mathfrak{p}$.*

Proof Let us denote $\phi_1 : \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M)) \hookrightarrow \mathcal{S}(M),\ f \mapsto f|_M$ and $\theta = \phi \circ \phi_1$. By 4.12, there exists $c \in \mathcal{S}(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(c) = Z = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus N$. Let us prove now the items in the statement.

(i) Observe first that the semialgebraic function c has no zero in N and so $c|_M \notin \mathfrak{p} = \operatorname{Spec_s}(j)(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$. Next, let $f \in \mathfrak{q}$; by 4.12(ii) there exist $k \geq 1$ and a semialgebraic function $h \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$ such that $h|_N = (c|_N)^k f$ and $Z_{\mathrm{Cl}_{\mathbb{R}^n}(M)}(h) = Z \cup Z_N(f)$. Thus $h|_N \in \mathfrak{q}$ and so $h \in \theta^{-1}(\mathfrak{q})$. By 2.2, there exists $G \in \mathcal{S}(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(G) = \mathrm{Cl}_{\mathbb{R}^n}(Z_N(f))$. Let us check that the semialgebraic function $g = G|_M \in \mathfrak{p}$ and it satisfies $\dim Z_N(f) \geq \dim Z_M(g)$. First, we see that $g \in \mathfrak{p}$. Indeed,

$$\begin{aligned} Z_{\mathrm{Cl}_{\mathbb{R}^n}(M)}(h) &= Z \cup Z_N(f) = Z_{\mathbb{R}^n}(c) \cup \mathrm{Cl}_{\mathbb{R}^n}(Z_N(f)) \\ &= Z_{\mathbb{R}^n}(c) \cup Z_{\mathbb{R}^n}(G) = Z_{\mathbb{R}^n}(cG) = Z_{\mathrm{Cl}_{\mathbb{R}^n}(M)}(cG). \end{aligned}$$

Therefore, $\mathrm{Cl}_{\mathbb{R}^n}(M)$ being locally compact, we get, by 2.11, $c|_{\mathrm{Cl}_{\mathbb{R}^n}(M)} G|_{\mathrm{Cl}_{\mathbb{R}^n}(M)} \in \theta^{-1}(\mathfrak{q})$. Thus, $(c|_M)g = (c|_M)(G|_M) \in \phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and, since $c|_M \notin \mathfrak{p}$, we conclude that $g \in \mathfrak{p}$. Now we must compare the dimensions of $Z_N(f)$ and $Z_M(g)$:

$$\dim Z_N(f) = \dim \mathrm{Cl}_{\mathbb{R}^n}(Z_N(f)) = \dim Z_{\mathbb{R}^n}(G) \geq \dim Z_M(g).$$

Hence, we conclude that $\mathrm{d}_M(\mathfrak{p}) \leq \mathrm{d}_N(\mathfrak{q})$. The converse inequality is obvious because $\dim Z_M(h) \geq \dim(Z_N(h|_N))$ for each $h \in \mathfrak{p}$, and so $\mathrm{d}_M(\mathfrak{p}) = \mathrm{d}_N(\mathfrak{q})$. Finally, notice that $\phi(\mathfrak{p})\mathcal{S}(N) \subset \mathfrak{q} \neq \mathcal{S}(N)$, that is, $\phi(\mathfrak{p})\mathcal{S}(N) \neq \mathcal{S}(N)$ and, by 4.11, $\mathfrak{p} \notin \mathcal{L}(Y)$.

(ii) First, we check that $\mathfrak{q}$ is a prime ideal. Once this be done, it follows from 2.11 that it is a z-ideal, because N is locally compact. Indeed, let $f, g \in \mathcal{S}(N)$ such that $fg \in \mathfrak{q}$. We write $fg = (a_1|_N)b_1 + \cdots + (a_r|_N)b_r$, where each $a_i \in \mathfrak{p}$ and $b_i \in \mathcal{S}(N)$. By 4.12 there exists an integer $k \geq 1$ such that $(c|_N)^k f$, $(c|_N)^k g$ and $(c|_N)^{2k} b_i$ can be extended by zero, respectively, to semialgebraic functions $F, G, B_i \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$. Therefore, the product $(F|_M)(G|_M)$ satisfies

$$(F|_M)(G|_M) = a_1(B_1|_M) + \cdots + a_r(B_r|_M) \in \mathfrak{p}.$$

Observe that the previous equality holds because

$$(F|_N)(G|_N) = (c|_N)^k f (c|_N)^k g = (c|_N)^{2k} fg = (a_1|_N)(c|_N)^{2k} b_1 + \cdots + (a_r|_N)(c|_N)^{2k} b_r,$$

and outside N both sides of the equality vanish. By the primality of $\mathfrak{p}$, we may assume that $F|_M \in \mathfrak{p}$ and this implies $(c|_N)^k f = F|_N \in \mathfrak{q}$. Since $c|_N$ is a unit in $\mathcal{S}(N)$ because $Z_N(c) = \varnothing$, we conclude that $f \in \mathfrak{q}$.

Let us check now the equality $\mathfrak{p} = \operatorname{Spec_s}(j)(\mathfrak{q})$. Since $\mathfrak{p} \subset \phi^{-1}(\phi(\mathfrak{p})\mathcal{S}(N)) = \operatorname{Spec_s}(j)(\mathfrak{q})$, it suffices to prove the converse inclusion. Fix $h \in \operatorname{Spec_s}(j)(\mathfrak{q})$; hence $h|_N \in \mathfrak{q} = \phi(\mathfrak{p})\mathcal{S}(N)$, and so

$$h|_N = (a_1|_N)b_1 + \cdots + (a_r|_N)b_r, \quad \text{where each } a_i \in \mathfrak{p} \text{ and } b_i \in \mathcal{S}(N).$$

By 4.12, there exists an integer $s \geq 1$ such that $(c|_N)^s(h|_N)$ and $(c|_N)^s b_i$ can be extended by zero, respectively, to semialgebraic functions $H, h_i \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$. Therefore, $H|_M$ satisfies

$$H|_M = a_1(h_1|_M) + \cdots + a_r(h_r|_M) \in \mathfrak{p}.$$

The previous equality holds because

$$H|_N = (c|_N)^s(h|_N) = (a_1|_N)(c|_N)^s b_1 + \cdots + (a_r|_N)(c|_N)^s b_r,$$

and outside N both sides of the equality vanish identically. Thus, $(c|_M)^s h = H|_M \in \mathfrak{p}$, and $c|_M \notin \mathfrak{p}$ because $Z_M(c) = Y$ and $\mathfrak{p} \notin \mathcal{L}(Y)$. Hence $h \in \mathfrak{p}$, and so $\mathfrak{p} = \mathrm{Spec_s}(j)(\mathfrak{q})$.

Next, it follows from 4.1(ii) that $\mathfrak{p} = \mathrm{Spec_s}(j)(\mathfrak{q})$ is a z-ideal since $\mathfrak{q}$ is so. Finally, let $\mathfrak{a}$ be a prime ideal of $\mathcal{S}(N)$ with $\mathrm{Spec_s}(j)(\mathfrak{a}) = \mathfrak{p}$. Thus, $\mathfrak{q} = \phi(\mathfrak{p})\mathcal{S}(N) = \phi(\phi^{-1}(\mathfrak{a}))\mathcal{S}(N) \subset \mathfrak{a}$, and by (i), $\mathrm{d}_N(\mathfrak{a}) = \mathrm{d}_M(\mathfrak{p}) = \mathrm{d}_N(\mathfrak{q})$. Hence, N being locally compact, the equality $\mathfrak{q} = \mathfrak{a}$ follows from 2.11 and 2.12.1. □

Now, we are almost ready to prove 4.8. Just in order to avoid unnecessary repetitions, we recall first an elementary but useful criterion of minimality, whose proof is straightforward.

Lemma 4.14 *Let R be a reduced commutative ring with unity and let $\mathfrak{p}$ be a prime ideal of R. Then, $\mathfrak{p}$ is a minimal prime ideal of R if and only if for every $f \in \mathfrak{p}$ there exists $g \in R \setminus \mathfrak{p}$ such that $fg = 0$.*

Proof of Theorem 4.8 Recall that, by 4.12(ii), there exists a semialgebraic function $c \in \mathcal{S}(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(c) = Z = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus N$.

(4.8.1) Moreover, by 4.12(ii), given $g \in \mathcal{S}(N)$ there exist $G \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$ and $k \geq 1$ such that $G|_N = (c|_N)^k g$ and $Z_{\mathrm{Cl}_{\mathbb{R}^n}(M)}(G) = Z_N(g) \cup Z$.

Let us enter into the proof of the different assertions in the statement of 4.8:

(i) By 4.13, the map $\mathrm{Spec_s}(j) : \mathrm{Spec_s}(N) \to \mathrm{Spec_s}(M) \setminus \mathcal{L}(Y)$ is a continuous bijection. Thus, to prove that it is a homeomorphism, it is enough to check that it is an open map. In fact, it suffices to prove that $\mathrm{Spec_s}(j)(\mathcal{D}_{\mathrm{Spec_s}(N)}(g))$ is an open subset of $\mathrm{Spec_s}(M) \setminus \mathcal{L}(Y)$ for every $g \in \mathcal{S}(N)$. With the notation in 4.8.1, all reduces to check the equality $\mathrm{Spec_s}(j)(\mathcal{D}_{\mathrm{Spec_s}(N)}(g)) = \mathcal{D}_{\mathrm{Spec_s}(M)}(G|_M) \setminus \mathcal{L}(Y)$.

Given $\mathfrak{p} \in \mathcal{D}_{\mathrm{Spec_s}(M)}(G|_M) \setminus \mathcal{L}(Y)$ there exists $\mathfrak{q} \in \mathrm{Spec_s}(N)$ such that $\mathrm{Spec_s}(j)(\mathfrak{q}) = \mathfrak{p}$. Then, $(c|_N)^k g = G|_N \notin \mathfrak{q}$, and so $\mathfrak{q} \in \mathcal{D}_{\mathrm{Spec_s}(N)}(g)$. Therefore $\mathfrak{p} \in \mathrm{Spec_s}(j)(\mathcal{D}_{\mathrm{Spec_s}(N)}(g))$.

Conversely, let $\mathfrak{p} \in \mathrm{Spec_s}(j)(\mathcal{D}_{\mathrm{Spec_s}(N)}(g))$ and let $\mathfrak{q} \in \mathcal{D}_{\mathrm{Spec_s}(N)}(g)$ with $\mathfrak{p} = \mathrm{Spec_s}(j)(\mathfrak{q})$. Notice that $c|_N \notin \mathfrak{q}$, because $Z_N(c) = \varnothing$; hence, $G|_N = (c|_N)^k g \notin \mathfrak{q}$, and so $G|_M \notin \mathfrak{p} = \mathrm{Spec_s}(j)(\mathfrak{q})$.

(ii) Since $Z_M(c) = Y$ and $\mathfrak{p} \notin \mathcal{L}(Y)$, it follows that $c|_M \notin \mathfrak{p}$. Let now $f \in \mathcal{S}(N) \setminus \mathfrak{q}$; by 4.12 there exist an integer $s \geq 1$ and $F \in \mathcal{S}(\mathrm{Cl}_{\mathbb{R}^n}(M))$ such that $F|_N = (c|_N)^s f$. Since both $c|_N, f \notin \mathfrak{q}$, also $F|_N \notin \mathfrak{q}$, that is, $F|_M \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a maximal ideal there exists $H \in \mathcal{S}(M)$ such that $1 - H(F|_M) \in \mathfrak{p}$. Thus, the function $h = H|_N \in \mathcal{S}(N)$ satisfies the equality $1 - h(c|_N)^s f \in \mathfrak{q}$, that is, f is invertible modulo $\mathfrak{q}$; hence, $\mathfrak{q}$ is a maximal ideal.

(iii.a) Assume that a minimal prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$ lies in $\mathcal{L}(Y)$. Thus, $Y = Z_M(f)$ for some $f \in \mathfrak{p}$. By the minimality of $\mathfrak{p}$, there exists $g \in \mathcal{S}(M) \setminus \mathfrak{p}$ such that $fg = 0$ (see 4.14). Hence, $M = \mathrm{Cl}_M(N) = \mathrm{Cl}_M(M \setminus Z_M(f)) \subset Z_M(g)$, that is, $g = 0$, a contradiction.

(iii.b) Suppose, by way of contradiction, that $\mathrm{Spec_s}(j)(\mathfrak{n})$ is a maximal ideal of $\mathcal{S}(M)$ for all $\mathfrak{n} \in \beta_s N$. Hence, since $\beta_s N$ is compact and $\beta_s M$ is Hausdorff, the image of the continuous map $\mathrm{Spec_s}(j)|_{\beta_s N} : \beta_s N \to \beta_s M$ is a closed subset of $\beta_s M$ that contains N.

On the other hand, $\mathrm{Cl}_{\beta_s M}(N) = \beta_s M$, because N is dense in M and M is dense in $\beta_s M$. Therefore, $\mathrm{Spec_s}(j)|_{\beta_s N}$ is surjective. This implies that

$$Y \subset \beta_s M \cap \mathcal{L}(Y) = \mathrm{Spec_s}(j)(\beta_s N) \cap \mathcal{L}(Y) \subset \mathrm{Spec_s}(j)(\mathrm{Spec_s}(N)) \cap \mathcal{L}(Y) = (\mathrm{Spec_s}(M) \setminus \mathcal{L}(Y)) \cap \mathcal{L}(Y) = \varnothing,$$

a contradiction. □

As a consequence of 4.13, and to finish this section, we get sufficient conditions for a prime ideal to be a z-ideal in the general setting.

Corollary 4.15 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{p}$ be a prime ideal of $\mathcal{S}(M)$ such that $Z_M(f) \cap M_{\mathrm{lc}} \neq \varnothing$ for all $f \in \mathfrak{p}$. Then, $\mathfrak{p}$ is a z-ideal.*

Proof Let $Y = M \setminus M_{\mathrm{lc}}$ and $j : M_{\mathrm{lc}} \hookrightarrow M$. By 2.6, M_{lc} is open in M and locally compact. The hypothesis $Z_M(f) \cap M_{\mathrm{lc}} \neq \varnothing$ for all $f \in \mathfrak{p}$ means that $\mathfrak{p} \notin \mathcal{L}(Y)$. Thus, by 4.13(ii), $\mathfrak{p}$ is a z-ideal. □

Example 4.16 The previous condition is sufficient but not necessary for a prime ideal to be a z-ideal. Indeed, let $M \subset \mathbb{R}^n$ be a non locally compact semialgebraic set, and let $p \in M \setminus M_{\mathrm{lc}}$ and $f : M \to \mathbb{R}, x \mapsto \|x - p\|$. Then, the maximal ideal $\mathfrak{m}_p$ is a z-ideal containing f but $Z_M(f) \cap M_{\mathrm{lc}} = \varnothing$.

5 Functoriality of Spec_s^*

In this section we deal with the Zariski spectrum $\mathrm{Spec}_s^*(M)$ of the ring $\mathcal{S}^*(M)$ of bounded semialgebraic functions on a semialgebraic set M. Given a semialgebraic map $\varphi : N \to M$, we will denote by $\mathrm{Spec}_s^*(\varphi) : \mathrm{Spec}_s^*(N) \to \mathrm{Spec}_s^*(M)$ the induced map already defined in 4.1. The main result here is the following (see 1.2), which can be understood as the counterpart of 4.8 for the Zariski spectrum $\mathrm{Spec}_s^*(M)$.

Theorem 5.1 *Let $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ be semialgebraic sets and let $\varphi : N \to M$ be a semialgebraic map. Suppose there exists a semialgebraic set $Y \subset M$ such that:*

(i) *$M_1 = M \setminus Y$ is locally compact and dense in M.*
(ii) *The map $\psi = \varphi|_{N_1} : N_1 = N \setminus \varphi^{-1}(Y) \to M_1 = M \setminus Y$ is a semialgebraic homeomorphism.*

Denote $Z = \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)$. Then, the map $\mathrm{Spec}_s^*(\varphi) : \mathrm{Spec}_s^*(N) \to \mathrm{Spec}_s^*(M)$ is surjective and its restriction $\mathrm{Spec}_s^*(\varphi)\,| : \mathrm{Spec}_s^*(N) \setminus \mathrm{Spec}_s^*(f)^{-1}(Z) \to \mathrm{Spec}_s^*(M) \setminus Z$ is a homeomorphism.

The proof of 5.1 requires some preparation that we present divided into several steps. First, we need to compare the spectra of suitable pairs of semialgebraic sets. More precisely, we are led to compare $\mathrm{Spec}_s^*(N)$ and $\mathrm{Spec}_s^*(M)$ where M is arbitrary and $N \subset M$ is locally compact and dense in M. The local compactness of N is still needed since we will use the results obtained in the previous section about the map $\mathrm{Spec_s}(j) : \mathrm{Spec_s}(N) \to \mathrm{Spec_s}(M)$, where $j : N \hookrightarrow M$ is the inclusion map. Moreover, it is also useful to compare the spectra $\mathrm{Spec}_s^*(M)$ and $\mathrm{Spec}_s^*(X) = \mathrm{Spec_s}(X)$, where X is a semialgebraic compactification of a locally compact semialgebraic set M. As it is well-known, M being locally compact, it is open in X (see 2.4). To avoid unnecessary repetitions we introduce the following definition, which comprises simultaneously the situations described above.

Definition 5.2 A *suitable arranged tuple* is a 5-tuple (M, N, Y, j, i), where:

(i) $N \subsetneq M \subset \mathbb{R}^n$ are semialgebraic sets and N is locally compact and dense in M.
(ii) $Y = M \setminus N$ and $j : N \hookrightarrow M$ and $i : Y \hookrightarrow M$ are the inclusion maps.

Note that the pair (M, N) determines the full tuple.

Remarks 5.3 (i) Recall that N being locally compact and dense in M, it is also an open subset of M, see 2.4. Hence, $Y = M \setminus N = \mathrm{Cl}_M(N) \setminus N$ is a closed subset of M whose dimension is smaller than $\dim N = \dim M$ (see [1, 2.8.13]).

(ii) Notice that both situations described above are particular cases of a suitable arranged tuple. The first one often corresponds to the choice $(M, N) = (M, M_{\mathrm{lc}})$, while the second one corresponds to $(M, N) = (X, M)$.

The first step to approach the proof of 5.1 is the following result, which has interest by its own and will be useful in further contexts (see for instance [9,12]).

Lemma 5.4 *Let (M, N, Y, j, i) be a suitable arranged tuple and let $\mathfrak{p}_0 \subset \mathfrak{p}$ be prime ideals of $\mathcal{S}^*(M)$ such that $\mathfrak{p}_0$ is minimal* [1] *and $\mathfrak{p} \notin Z = \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)$. Then,*

(i) *$\mathfrak{a} = \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(N)$ is a prime ideal of $\mathcal{S}^*(N)$ and $\mathfrak{a} \cap \mathcal{S}^*(M) = \mathfrak{p}_0$.*
(ii) *$\mathfrak{b} = \sqrt{\mathfrak{p}\mathcal{S}^*(N) + \mathfrak{a}}$ is a prime ideal of $\mathcal{S}^*(N)$ and $\mathrm{Spec}_s^*(j)^{-1}(\mathfrak{p}) = \{\mathfrak{b}\}$.*
(iii) *The map $\mathrm{Spec}_s^*(j)| : \mathrm{Spec}_s^*(N) \setminus \mathrm{Spec}_s^*(j)^{-1}(Z) \to \mathrm{Spec}_s^*(M) \setminus Z$ is a homeomorphism.*

Before proving 5.4 we need some preliminary results concerning prime ideals. We begin with a kind of counterpart of 4.11 in our setting. More precisely,

Lemma 5.5 *Let (M, N, Y, j, i) be a suitable arranged tuple and let $\mathfrak{p} \in \mathrm{Spec}_s^*(M)$ be a prime ideal such that $\mathfrak{p}\mathcal{S}^*(N) = \mathcal{S}^*(N)$. Then, $\mathfrak{p} \in \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)$.*

Proof By 4.3 all reduces to prove that $\ker \phi \subset \mathfrak{p}$, where $\phi : \mathcal{S}^*(M) \to \mathcal{S}^*(Y)$, $f \mapsto f|_Y$. Let $h \in \ker \phi$, that is, $Y \subset Z_M(h)$. The hypothesis $\mathfrak{p}\mathcal{S}^*(N) = \mathcal{S}^*(N)$ means that there exist $a_1, \dots, a_r \in \mathfrak{p}$ and $b_1, \dots, b_r \in \mathcal{S}^*(N)$ such that $1 = (a_1|_N)b_1 + \cdots + (a_r|_N)b_r$. By 2.8 there exists, for each index k, an extension $u_k \in \mathcal{S}^*(M)$ of $(h|_N)b_k$. Then, since N is dense in M, we get $h = a_1u_1 + \cdots + a_ru_r \in \mathfrak{p}$, and we have finished. □

Next, we present a useful criterion to determine when a radical ideal of $\mathcal{S}^*(M)$ is a prime ideal; for further results in this direction see also [7, 5.3–4].

Lemma 5.6 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{a}$ be a radical ideal of $\mathcal{S}^*(M)$ which contains a prime ideal $\mathfrak{p}$ of $\mathcal{S}^*(M)$. Then, $\mathfrak{a}$ is a prime ideal.*

Proof Let $f, g \in \mathcal{S}^*(M)$ such that $fg \in \mathfrak{a}$ and let $h = |f| - |g|$. Denote $a_1 = \max\{h, 0\}$ and $a_2 = \min\{h, 0\}$ and observe that $h = a_1 + a_2$, $|h| = a_1 - a_2$ and $a_1a_2 = 0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, we may assume that $a_2 \in \mathfrak{p} \subset \mathfrak{a}$. Note that $h + \mathfrak{m}^* \geq 0$ in the ordered field $\mathcal{S}^*(M)/\mathfrak{m}^*$ for all $\mathfrak{m}^* \in \mathcal{Z}_{\beta_s M}(a_2)$. Indeed,

$$h + \mathfrak{m}^* = a_1 + a_2 + \mathfrak{m}^* = a_1 + \mathfrak{m}^* = a_1 - a_2 + \mathfrak{m}^* = |h| + \mathfrak{m}^* = (\sqrt{|h|} + \mathfrak{m}^*)^2 \geq 0.$$

[1] In fact, for the validity of the statement it is enough to ask that no function in $\mathcal{S}^*(M)$ with empty zeroset occurs in $\mathfrak{p}_0$, and $\mathfrak{p}_0\mathcal{S}(M) \notin \mathcal{L}(Y)$. As we will see in 5.8 each minimal prime ideal $\mathfrak{p}_0$ of $\mathcal{S}^*(M)$ satisfies such properties.

Thus, since $h = |f| - |g|$, we deduce that $\mathcal{Z}_{\beta_s M}(a_2) \cap \mathcal{Z}_{\beta_s M}(f) \subset \mathcal{Z}_{\beta_s M}(g)$ and so

$$\mathcal{Z}_{\beta_s M}(a_2^2 + f^2 g^2) = \mathcal{Z}_{\beta_s M}(a_2^2 + g^2) \subset \mathcal{Z}_{\beta_s M}(g).$$

Hence, by Łojasiewicz inequality for $\mathcal{S}^*(M)$ [10, 3.12], there exist $b \in \mathcal{S}^*(M)$ and $\ell \geq 1$ such that $g^\ell = (a_2^2 + f^2 g^2)b \in \mathfrak{a}$. Therefore $g \in \mathfrak{a}$, because $\mathfrak{a}$ is a radical ideal, and so $\mathfrak{a}$ is a prime ideal. □

Concerning minimal prime ideals we need the following.

Lemma 5.7 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that N is dense in M. Let $\mathfrak{q}$ be a minimal prime ideal of $\mathcal{S}^*(N)$. Then, $\mathfrak{q} \cap \mathcal{S}^*(M)$ is a minimal prime ideal of $\mathcal{S}^*(M)$.*

Proof Notice that, by 2.6, N_{lc} is dense in M because it is dense in N and N is dense in M. Since N_{lc} is a locally compact and dense subset of M, it follows from 2.4 that N_{lc} is open in M. Thus, $Y = M \setminus N_{\text{lc}}$ is closed in M and by 2.2 there exists $h \in \mathcal{S}^*(M)$ such that $Z_M(h) = Y$.

To prove the minimality of $\mathfrak{p} = \mathfrak{q} \cap \mathcal{S}^*(M)$ is equivalent, see 4.14, to check that for each $f \in \mathfrak{p}$ there exists $g \in \mathcal{S}^*(M) \setminus \mathfrak{p}$ such that $fg = 0$. Since $\mathfrak{q}$ is a minimal prime ideal and $f|_N \in \mathfrak{q}$, there exists $g_0 \in \mathcal{S}^*(N) \setminus \mathfrak{q}$ such that $(f|_N)g_0 = 0$. Observe that, by 2.8, there exists an extension $g \in \mathcal{S}^*(M)$ of $(h|_N)g_0$. Notice that $fg = 0$ because N is dense in M, and to finish it is enough to check that $g \notin \mathfrak{p}$. Otherwise, $(h|_N)g_0 = g|_N \in \mathfrak{q}$ and since $g_0 \notin \mathfrak{q}$ we deduce that $h|_N \in \mathfrak{q}$. By the minimality of $\mathfrak{q}$ there exists $b \in \mathcal{S}^*(N) \setminus \mathfrak{q}$ such that $(h|_N)b = 0$. This implies, since $Z_N(h) = N \setminus N_{\text{lc}}$, that $N = \text{Cl}_N(N_{\text{lc}}) \subset Z_N(b)$, that is, $b = 0$, a contradiction. Hence, $g \notin \mathfrak{p}$, and we are done. □

Lemma 5.8 *Let (M, N, Y, j, i) be a suitable arranged tuple and let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{S}^*(M)$. Then, $\mathfrak{p} \notin \text{Cl}_{\text{Spec}_s^*(M)}(Y)$, $\mathfrak{p} \cap \mathcal{W}(M) = \varnothing$, and $\mathfrak{p}\mathcal{S}(M) \notin \mathcal{L}(Y)$, Moreover, $\mathfrak{p}\mathcal{S}(N)$ and $\mathfrak{p}\mathcal{S}(N) \cap \mathcal{S}^*(N)$ are, respectively, minimal prime ideals of $\mathcal{S}(N)$ and $\mathcal{S}^*(N)$.*

Proof First, let us check that $\mathfrak{p} \notin \text{Cl}_{\text{Spec}_s^*(M)}(Y)$. Otherwise, there exists, by 4.3, $f \in \mathfrak{q}$ such that $Z_M(f) = Y$. Since $\mathfrak{p}$ is a minimal prime ideal, there exists, by 4.14, $g \in \mathcal{S}^*(M) \setminus \mathfrak{p}$ such that $fg = 0$. Since $Z_M(f) = Y$, we deduce that $N \subset Z_M(g)$, or equivalently, $g = 0$ because N is dense in M, a contradiction; hence, $\mathfrak{p} \notin \text{Cl}_{\text{Spec}_s^*(N)}(Y)$.

The above argument does not use that $Y \neq \varnothing$. Hence, applied it to the "generalized suitable arranged tuple" $(M, M, \varnothing, i, j)$ we deduce the equality $\mathfrak{p} \cap \mathcal{W}(M) = \varnothing$. Thus, $\mathfrak{p}\mathcal{S}(M)$ is, by 3.2, a minimal prime ideal of $\mathcal{S}(M)$. Moreover, since N is dense in M it follows from 4.8(iii.a) that $\mathfrak{p}\mathcal{S}(M) \notin \mathcal{L}(Y)$. Whence, by 4.13(ii), the map $\text{Spec}_s(j) : \text{Spec}_s(N) \to \text{Spec}_s(M)$ satisfies $\text{Spec}_s(j)(\mathfrak{p}\mathcal{S}(N)) = \mathfrak{p}\mathcal{S}(M)$.

For the last part it suffices to see, using again by 3.2, that $\mathfrak{p}\mathcal{S}(N)$ is a minimal prime ideal of $\mathcal{S}(N)$. Indeed suppose, by way of contradiction, that $\mathfrak{p}\mathcal{S}(N)$ is not a minimal prime ideal of $\mathcal{S}(N)$. Then, there exists a prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}\mathcal{S}(N)$. Observe that $\mathfrak{p}\mathcal{S}(N) \in \text{Cl}_{\text{Spec}_s(N)}(\mathfrak{q})$ and by the continuity of the map $\text{Spec}_s(j) : \text{Spec}_s(N) \to \text{Spec}_s(M)$, it follows that

$$\mathfrak{p}\mathcal{S}(M) = \text{Spec}_s(j)(\mathfrak{p}\mathcal{S}(N)) \in \text{Cl}_{\text{Spec}_s(M)}(\text{Spec}_s(j)(\mathfrak{q})).$$

Thus, $\text{Spec}_s(j)(\mathfrak{q}) \subset \mathfrak{p}\mathcal{S}(M)$ and since $\mathfrak{p}\mathcal{S}(M)$ is a minimal prime ideal of $\mathcal{S}(M)$, we get the equality $\text{Spec}_s(j)(\mathfrak{q}) = \mathfrak{p}\mathcal{S}(M)$, which contradicts the injectivity of $\text{Spec}_s(j)$ proved in 4.8(i). □

Now, we are already prepared to approach the proof of 5.4.

Proof of Lemma 5.4 First, to simplify notation, denote $\phi : \mathcal{S}^*(M) \to \mathcal{S}^*(Y),\ f \mapsto f|_Y$ and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{S}^*(M) & \hookrightarrow & \mathcal{S}^*(N) \\ \downarrow & & \downarrow \\ \mathcal{S}(M) & \hookrightarrow & \mathcal{S}(N) \end{array}$$

Since Y is closed in M and $\mathfrak{p} \notin \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)$ there exists, by 4.3, a function $h \in \mathcal{S}^*(M)\backslash\mathfrak{p}$ such that $Y = Z_M(h)$. We fix this function all along the proof.

(i) The primality of $\mathfrak{a} = \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(N)$ has been just proved in 5.8. Moreover,

$$\mathfrak{a} \cap \mathcal{S}^*(M) = \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(N) \cap \mathcal{S}^*(M) = \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(M) = \mathfrak{p}_0.$$

Just the last equality requires some comment. Of course, it is enough to check the inclusion $\mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(M) \subset \mathfrak{p}_0$. Let $f \in \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(M)$ and let $a_1, \ldots, a_r \in \mathfrak{p}_0$ and $g_1, \ldots, g_r \in \mathcal{S}(N)$ be such that $f|_N = (a_1|_N)g_1 + \cdots + (a_r|_N)g_r$; hence,

$$\frac{f|_N}{1 + \sum_{i=1}^r |g_i|} = \sum_{i=1}^r (a_i|_N)\Big(\frac{g_i}{1 + \sum_{i=1}^r |g_i|}\Big).$$

By 2.8, the bounded semialgebraic functions on N

$$\frac{h|_N}{1 + \sum_{i=1}^r |g_i|} \quad \text{and} \quad \Big(\frac{g_i}{1 + \sum_{i=1}^r |g_i|}\Big) h|_N$$

extend continuously by 0 to semialgebraic functions $G_0, G_i \in \mathcal{S}^*(M)$ and it follows readily $(G_0|_N)(f|_N) = (a_1|_N)(G_1|_N) + \cdots + (a_r|_N)(G_r|_N)$. Moreover, N being dense in M, we have $G_0 f = a_1 G_1 + \cdots + a_r G_r \in \mathfrak{p}_0$. Note that $Y = Z_M(G_0)$ while, by 5.8, $\mathfrak{p}_0\mathcal{S}(M) \notin \mathcal{L}(Y)$, which implies $G_0 \notin \mathfrak{p}_0$; hence, $f \in \mathfrak{p}_0$, as wanted.

(ii) First we check that $\mathfrak{b} \neq \mathcal{S}^*(N)$. Otherwise, there would exist $a_\ell \in \mathfrak{p}$, $b_\ell \in \mathcal{S}^*(N)$ and $c \in \mathfrak{a}$ such that $1 = (a_1|_N)b_1 + \cdots + (a_r|_N)b_r + c$. Note that $(h|_N)c$ admits, by 2.8, an extension $g \in \mathcal{S}^*(M)$ to M. Thus, $g|_N = (h|_N)c \in \mathfrak{a}$, and so, by part (i), $g \in \mathfrak{a} \cap \mathcal{S}^*(M) = \mathfrak{p}_0 \subset \mathfrak{p}$. Also each product $(h|_N)b_\ell$ admits, by 2.8, an extension $g_\ell \in \mathcal{S}^*(M)$. Therefore, the equality

$$h|_N = (a_1|_N)(h|_N)b_1 + \cdots + (a_r|_N)(h|_N)b_r + (h|_N)c$$

can be rewritten as

$$h|_N = (a_1|_N)(g_1|_N) + \cdots + (a_r|_N)(g_r|_N) + g|_N,$$

and, N being dense in M, it follows that $h = a_1 g_1 + \cdots + a_r g_r + g \in \mathfrak{p}$, a contradiction. Thus, $\mathfrak{b}$ is a radical ideal of $\mathcal{S}^*(N)$ containing the prime ideal $\mathfrak{a}$; hence, by 5.6, $\mathfrak{b}$ is a prime ideal too. Moreover, $\mathfrak{p}\mathcal{S}^*(N) \subset \mathfrak{b}$, and so

$$\mathfrak{p} \subset \mathfrak{p}\mathcal{S}^*(N) \cap \mathcal{S}^*(M) \subset \mathfrak{b} \cap \mathcal{S}^*(M).$$

In fact the inclusion $\mathfrak{p} \subset \mathfrak{b} \cap \mathcal{S}^*(M)$ is an equality. Indeed, given $f \in \mathfrak{b} \cap \mathcal{S}^*(M)$ there exist $k \geq 1$ and $a_\ell \in \mathfrak{p}$, $b_\ell \in \mathcal{S}^*(N)$ and $c \in \mathfrak{a}$ such that

$$(f|_N)^k = (a_1|_N)b_1 + \cdots + (a_r|_N)b_r + c.$$

By 2.8, there exists $g \in \mathcal{S}^*(M)$ such that $g|_N = (h|_N)c \in \mathfrak{a}$. Therefore, by part (i), $g \in \mathfrak{a} \cap \mathcal{S}^*(M) = \mathfrak{p}_0 \subset \mathfrak{p}$. Moreover, for each index $1 \leq \ell \leq r$ there exists, by 2.8, a function

$g_\ell \in \mathcal{S}^*(M)$ with $g_\ell|_N = (h|_N)b_\ell$. Hence, N being dense in M,

$$hf^k = a_1g_1 + \cdots + a_rg_r + g \in \mathfrak{p}.$$

Since $h \notin \mathfrak{p}$, we conclude that $f^k \in \mathfrak{p}$ and therefore $f \in \mathfrak{p}$. Thus, $\mathfrak{b} \cap \mathcal{S}^*(M) = \mathfrak{p}$, that is, $\operatorname{Spec}_s^*(j)(\mathfrak{b}) = \mathfrak{p}$.

To finish the proof of (ii), we must show that $\mathfrak{b}$ is the unique prime ideal of $\mathcal{S}^*(N)$ lying over $\mathfrak{p}$. Suppose that $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are two distinct prime ideals of $\mathcal{S}^*(N)$ such that $\mathfrak{b}_i \cap \mathcal{S}^*(M) = \mathfrak{p}$ for $i = 1, 2$. We may assume that there exists a function $q \in \mathfrak{b}_1 \setminus \mathfrak{b}_2$ and, by 2.8, there exists $p \in \mathcal{S}^*(M)$ such that $p|_N = (h|_N)q$. Thus,

$$p \in \mathfrak{b}_1 \cap \mathcal{S}^*(M) = \mathfrak{p} = \mathfrak{b}_2 \cap \mathcal{S}^*(M),$$

and so $(h|_N)q \in \mathfrak{b}_2$, that is, $h|_N \in \mathfrak{b}_2$. Consequently, $h \in \mathfrak{b}_2 \cap \mathcal{S}^*(M) = \mathfrak{p}$, a contradiction.

(iii) By part (ii) the map $\operatorname{Spec}_s^*(j)| : \operatorname{Spec}_s^*(N) \setminus \operatorname{Spec}_s^*(j)^{-1}(Z) \to \operatorname{Spec}_s^*(M) \setminus Z$ is bijective. Since it is continuous, to prove that it is a homeomorphism it is enough to check that it is an open map. For that, it suffices to see that given $g \in \mathcal{S}^*(N)$ the following equality holds true:

$$\operatorname{Spec}_s^*(j)(\mathcal{D}_{\operatorname{Spec}_s^*(N)}(g) \cap (\operatorname{Spec}_s^*(N) \setminus \operatorname{Spec}_s^*(j)^{-1}(Z)))$$
$$= \bigcup_{a \in \ker \phi} \mathcal{D}_{\operatorname{Spec}_s^*(M)}(G_a) \cap (\operatorname{Spec}_s^*(M) \setminus Z),$$

where $G_a \in \mathcal{S}^*(M)$ is the unique extension by 0 of $(a|_N)g$ to the whole M. We check both inclusions:

$\subset$) Let $\mathfrak{q} \in \mathcal{D}_{\operatorname{Spec}_s^*(N)}(g) \cap (\operatorname{Spec}_s^*(N) \setminus \operatorname{Spec}_s^*(j)^{-1}(Z))$. Then, $g \notin \mathfrak{q}$ and there exists $a \in \ker \phi \setminus (\mathfrak{q} \cap \mathcal{S}^*(M))$. Hence $a|_N \notin \mathfrak{q}$, and so $(a|_N)g \notin \mathfrak{q}$. Thus, $G_a \notin \mathfrak{q} \cap \mathcal{S}^*(M)$, that is, $\operatorname{Spec}_s^*(j)(\mathfrak{q}) = \mathfrak{q} \cap \mathcal{S}^*(M) \in \mathcal{D}_{\operatorname{Spec}_s^*(M)}(G_a) \cap (\operatorname{Spec}_s^*(M) \setminus Z)$.

$\supset$) Let $\mathfrak{p} \in \mathcal{D}_{\operatorname{Spec}_s^*(M)}(G_a) \cap (\operatorname{Spec}_s^*(M) \setminus Z)$ for some $a \in \ker \phi$ and let $\mathfrak{q} \in \operatorname{Spec}_s^*(N)$ be the unique prime ideal of $\mathcal{S}^*(N)$ with $\operatorname{Spec}_s^*(j)(\mathfrak{q}) = \mathfrak{p}$. Note that $g \notin \mathfrak{q}$, because $G_a \notin \mathfrak{p}$ and $G_a|_N = (a|_N)g$. Thus, $\mathfrak{q} \in \mathcal{D}_{\operatorname{Spec}_s^*(N)}(g) \cap (\operatorname{Spec}_s^*(N) \setminus \operatorname{Spec}_s^*(j)^{-1}(Z))$, as wanted. □

We have proved in 4.8(iii) that if $N \subsetneq M \subset \mathbb{R}^n$ are semialgebraic sets such that N is dense in M and locally compact, then there exist a maximal ideals $\mathfrak{n}$ of $\mathcal{S}(N)$ whose image $\operatorname{Spec}_s(j)(\mathfrak{n})$ under the map $\operatorname{Spec}_s(j) : \operatorname{Spec}_s(N) \to \operatorname{Spec}_s(M)$ are not maximal ideals of $\mathcal{S}(M)$. The situation changes dramatically in dealing with rings of bounded semialgebraic functions even in the most general possible situation. Namely,

Lemma 5.9 *Let $\varphi : N \to M$ be a semialgebraic map between the semialgebraic sets $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$. Then,* $\operatorname{Spec}_s^*(\varphi) : \operatorname{Spec}_s^*(N) \to \operatorname{Spec}_s^*(M)$ *maps $\beta_s^* N$ into $\beta_s^* M$.*

Proof First, we analyze the quotients of rings of bounded semialgebraic functions by maximal ideals. Let $\mathfrak{n}^*$ be a maximal ideal of $\mathcal{S}^*(N)$. Then, $\mathcal{S}^*(N)/\mathfrak{n}^*$ is (isomorphic to) the field $\mathbb{R}$ of real numbers. Indeed, by 3.1.1 the field $\mathcal{S}^*(N)/\mathfrak{n}^*$ admits a unique ordering, whose cone of nonnegative elements is the subset $\{h + \mathfrak{n}^* \in \mathcal{S}^*(N)/\mathfrak{n}^* : h - |h| \in \mathfrak{n}^*\}$. Thus, the inclusion map $\mathbb{R} \hookrightarrow \mathcal{S}^*(N)/\mathfrak{n}^* : r \mapsto r + \mathfrak{n}^*$, is an (injective) homomorphism of ordered fields and in fact it is an isomorphism, because $\mathcal{S}^*(N)/\mathfrak{n}^*$ is an archimedean extension of $\mathbb{R}$.

Next, consider the homomorphism $\phi : \mathcal{S}^*(M) \to \mathcal{S}^*(N),\ \ f \mapsto f \circ \varphi$. Let $\mathfrak{n}^*$ be a maximal ideal of $\mathcal{S}^*(N)$ and denote $\mathfrak{m}^* = \operatorname{Spec}_s^*(\varphi)(\mathfrak{n}^*) = \phi^{-1}(\mathfrak{n}^*)$. Then, the sequence of homomorphisms

$$\mathbb{R} \hookrightarrow \mathcal{S}^*(M) \xrightarrow{\phi} \mathcal{S}^*(N) \to \mathcal{S}^*(N)/\mathfrak{n}^* \cong \mathbb{R}$$

induce isomorphisms $\mathcal{S}^*(M)/\mathfrak{m}^* \cong \mathcal{S}^*(N)/\mathfrak{n}^* \cong \mathbb{R}$, which implies that $\mathfrak{m}^*$ is a maximal ideal of $\mathcal{S}^*(M)$. □

The next step to approach 5.1 concerns the proof of the fact that if (M, N, Y, j, i) is a suitable arranged tuple, then the map $\operatorname{Spec}_s^*(j) : \operatorname{Spec}_s^*(N) \to \operatorname{Spec}_s^*(M)$ is surjective. In particular, this is so if M is a compactification of a locally compact semialgebraic set N. In fact, we begin by studying this case.

Lemma 5.10 *Let $M \subset \mathbb{R}^n$ be a locally compact semialgebraic set and let (X, j) be a semialgebraic compactification of M. Then:*

(i) *Given a chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in $\mathcal{S}(X)$, there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ in $\mathcal{S}^*(M)$ such that $\mathfrak{q}_k \cap \mathcal{S}(X) = \mathfrak{p}_k$ for $k = 0, \ldots, r$.*
(ii) *The map $\operatorname{Spec}_s^*(j) : \operatorname{Spec}_s^*(M) \to \operatorname{Spec}_s^*(X)$ is surjective.*

Proof Since the second statement is a particular case of (i), it is enough to prove the first one. Without loss of generality we may assume that j is the inclusion map and $X \subset \mathbb{R}^n$. Moreover, we can suppose that $M \subsetneq X$, that is, M is noncompact, since for $M = X$ the result is evident. In fact, we may also assume that the given chain admits no refinement. In particular, this implies that $\mathfrak{p}_0$ is a minimal prime ideal of $\mathcal{S}(X)$ and $\mathfrak{p}_r$ is a maximal ideal of $\mathcal{S}(X)$. Denote also $\mathfrak{p}_{r+1} = \mathcal{S}(X)$.

(5.10.1) We claim that: $\mathfrak{p}_0\mathcal{S}(M)$ *is a prime z-ideal of* $\mathcal{S}(M)$ *and* $\mathfrak{p}_0\mathcal{S}(M) \cap \mathcal{S}(X) = \mathfrak{p}_0$.

Indeed, let $Y = X \setminus M$ and observe that, $\mathfrak{p}_0$ being a minimal prime ideal of $\mathcal{S}(X)$, it follows from 4.8(iii.a) that $\mathfrak{p}_0 \notin \mathcal{L}(Y)$. Now the claim follows from 4.13(ii).

By 5.4(i) and 5.8, $\mathfrak{q}_0 = \mathfrak{p}_0\mathcal{S}(M) \cap \mathcal{S}^*(M)$ is a minimal prime ideal of $\mathcal{S}^*(M)$ such that $\mathfrak{q}_0 \cap \mathcal{S}(X) = \mathfrak{p}_0$. Consider, for each $1 \le k \le r+1$, $\mathfrak{q}_k = \sqrt{\mathfrak{p}_k\mathcal{S}^*(M) + \mathfrak{q}_0}$. Note that each $\mathfrak{q}_k$ is either $\mathcal{S}^*(M)$ or a radical ideal (hence a z-ideal) of $\mathcal{S}^*(M)$ that contains the prime ideal $\mathfrak{q}_0$. Thus, by 5.6, either $\mathfrak{q}_k = \mathcal{S}^*(M)$ or $\mathfrak{q}_k$ is a prime ideal. Fix an index $0 \le k \le r$ and let us see that $\mathfrak{q}_k \subsetneq \mathfrak{q}_{k+1}$. This will prove, in particular, that $\mathfrak{q}_k$ is a prime ideal for $k = 0, \ldots, r$.

Denote $\mathrm{d}_X(\mathfrak{p}_{r+1}) = -1$ and observe that, by 2.12.1, there exists $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$ such that $\dim Z_X(f) = \mathrm{d}_X(\mathfrak{p}_{k+1})$ (for $k = r$, we take $f = 1$, that satisfies $\dim Z_X(f) = -1$). Let us see that $f|_M \in \mathfrak{q}_{k+1} \setminus \mathfrak{q}_k$. Otherwise there exist $a_1, \ldots, a_s \in \mathfrak{p}_k, b_1, \ldots, b_s \in \mathcal{S}^*(M), h \in \mathfrak{q}_0$ and $\ell \ge 1$ such that $(f|_M)^\ell = (a_1|_M)b_1 + \cdots + (a_s|_M)b_s + h$. Since $b_1, \ldots, b_s$ are bounded semialgebraic functions and $f, a_1, \ldots, a_s$ are defined on the whole X there exists, by 2.8, $H \in \mathcal{S}(T)$ such that $H|_M = h$, where $T = M \cup Z_X(a_1^2 + \cdots + a_s^2)$. Hence, we get

(5.10.2) $\mathrm{d}_X(\mathfrak{p}_{k+1}) = \dim Z_X(f^\ell) \ge \dim Z_T(f^\ell) \ge \dim Z_T(H^2 + a_1^2 + \cdots + a_s^2)$.

On the other hand, by 2.2, there exists $g \in \mathcal{S}^*(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(g) = \operatorname{Cl}_X(Z_M(h))$. Note that $Z_M(g) = Z_M(h)$ and, $\mathfrak{p}_0\mathcal{S}(M)$ being a z-ideal, it follows that $g|_M \in \mathfrak{p}_0\mathcal{S}(M)$. Thus, $g|_X \in \mathfrak{p}_0\mathcal{S}(M) \cap \mathcal{S}(X) = \mathfrak{p}_0 \subset \mathfrak{p}_k$ and so $(g|_X)^2 + a_1^2 + \cdots + a_s^2 \in \mathfrak{p}_k$. Moreover, $Z_T(g) = \operatorname{Cl}_X(Z_M(h)) \cap T \subset Z_T(H)$, and consequently

$$Z_T(H^2 + a_1^2 + \cdots + a_s^2) = Z_T(H^2) \cap Z_X(a_1^2 + \cdots + a_s^2) \supset Z_T(g^2) \cap Z_X(a_1^2 + \cdots + a_s^2)$$
$$= T \cap Z_X(g^2) \cap Z_X(a_1^2 + \cdots + a_s^2) = Z_X(g^2 + a_1^2 + \cdots + a_s^2).$$

Therefore, by 5.10.2, we get

$$\mathrm{d}_X(\mathfrak{p}_{k+1}) \ge \dim Z_T\left(H^2 + a_1^2 + \cdots + a_s^2\right) \ge \dim Z_X\left(g^2 + a_1^2 + \cdots + a_s^2\right) \ge \mathrm{d}_X(\mathfrak{p}_k),$$

which contradicts 2.12.1. In this way we have proved that $\mathfrak{q}_k \subsetneq \mathfrak{q}_{k+1}$ for $0 \le k \le r$ and, as observed above, this implies that $\mathfrak{q}_k$ is a prime ideal of $\mathcal{S}^*(M)$ for $k = 0, \ldots, r$.

To finish, we must check the equality $\mathfrak{q}_k \cap \mathcal{S}(X) = \mathfrak{p}_k$. Pick functions $f_{k+1} \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$ with $\dim Z_X(f_{k+1}) = \mathrm{d}_X(\mathfrak{p}_{k+1})$. We have just proved that $f_{k+1}|_M \in \mathfrak{q}_{k+1} \setminus \mathfrak{q}_k$, that is, $f_{k+1} \in \mathfrak{q}_{k+1} \cap \mathcal{S}(X) \setminus \mathfrak{q}_k \cap \mathcal{S}(X)$. In this way, we obtain a chain of prime ideals of $\mathcal{S}(X)$

$$\mathfrak{p}_0 = \mathfrak{q}_0 \cap \mathcal{S}(X) \subsetneq \cdots \subsetneq \mathfrak{q}_r \cap \mathcal{S}(X).$$

Since, by 3.1.4, the unique prime ideals containing $\mathfrak{p}_0$ are the $\mathfrak{p}_k$'s for $k = 0, \dots, r$, it follows that $\mathrm{Spec}_s^*(j)(\mathfrak{q}_k) = \mathfrak{q}_k \cap \mathcal{S}(X) = \mathfrak{p}_k$ for all $k = 0, \dots, r$. □

The next result, that extends the previous one to a more general setting, completes 5.4 in our approach to prove 5.1. Once more, this result have further applications in other contexts (see [7,9]).

Corollary 5.11 *Let (M, N, Y, j, i) be a suitable arranged tuple. Then:*

(i) *Given a chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in $\mathcal{S}^*(M)$, there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ in $\mathcal{S}^*(N)$ such that $\mathfrak{q}_k \cap \mathcal{S}^*(M) = \mathfrak{p}_k$ for $k = 0, \dots, r$.*
(ii) *The map $\mathrm{Spec}_s^*(j) : \mathrm{Spec}_s^*(N) \to \mathrm{Spec}_s^*(M)$ is surjective.*

Proof Again, the second statement follows immediately from (i); hence it is enough to prove the first one. We may assume that the given chain admits no refinement. In particular, this implies that $\mathfrak{p}_0$ is a minimal prime ideal of $\mathcal{S}^*(M)$ and $\mathfrak{p}_r$ is a maximal ideal of $\mathcal{S}^*(M)$. By [11, 4.11], there exists a semialgebraic compactification (X, j_0) of M such that if $\mathfrak{a}_k = \mathfrak{p}_k \cap \mathcal{S}(X)$ then $\mathfrak{a}_k \subsetneq \mathfrak{a}_{k+1}$ for $k = 0, \dots, r-1$.

Notice that $(X, j_1 = j_0 \circ j)$ is a compactification of N because, N being dense in M, it is also dense in X. Thus, by 5.10, there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ in $\mathcal{S}^*(N)$ such that $\mathfrak{q}_k \cap \mathcal{S}(X) = \mathfrak{a}_k$ for $k = 0, \dots, r$. This is the chain we are looking for.

(5.11.1) Let us see first that $\mathfrak{q}_0 \cap \mathcal{S}^*(M) = \mathfrak{p}_0$. Indeed, $\mathfrak{p}_0$ being a minimal prime ideal, it follows from 5.4(i) and 5.8 that $\mathfrak{q} = \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(N)$ is a minimal prime ideal of $\mathcal{S}^*(N)$ such that $\mathfrak{q} \cap \mathcal{S}^*(M) = \mathfrak{p}_0$, and all reduces to check the equality $\mathfrak{q}_0 = \mathfrak{q}$.

But $\mathfrak{a}_0$ is, by 5.7, a minimal prime ideal of $\mathcal{S}(X)$ and so, using again 5.4(i) and 5.8, $\widehat{\mathfrak{q}} = \mathfrak{a}_0\mathcal{S}(N) \cap \mathcal{S}^*(N)$ is a minimal prime ideal of $\mathcal{S}^*(N)$ with $\widehat{\mathfrak{q}} \cap \mathcal{S}(X) = \mathfrak{a}_0$. On the other hand, $\mathfrak{a}_0$ being a minimal prime ideal, it does not occur in $\mathcal{L}(Y) = \mathrm{Cl}_{\mathrm{Spec}_s(X)}(Y)$, see 4.9(ii) and 4.8(iii.a), where $Y = X \setminus N$. Hence, by 5.4(iii), $\widehat{\mathfrak{q}} = \mathfrak{q}_0$ because $\mathrm{Spec}_s(j)(\widehat{\mathfrak{q}}) = \mathrm{Spec}_s(j)(\mathfrak{q}_0) = \mathfrak{a}_0 \notin \mathrm{Cl}_{\mathrm{Spec}_s(X)}(Y)$. Therefore, the inclusion $\mathfrak{a}_0 \subset \mathfrak{p}_0$ implies

$$\mathfrak{q}_0 = \widehat{\mathfrak{q}} = \mathfrak{a}_0\mathcal{S}(N) \cap \mathcal{S}^*(N) \subset \mathfrak{p}_0\mathcal{S}(N) \cap \mathcal{S}^*(N) = \mathfrak{q},$$

and so $\mathfrak{q}_0 = \mathfrak{q}$ because $\mathfrak{q}$ is a minimal prime ideal; hence, $\mathfrak{q}_0 \cap \mathcal{S}^*(M) = \mathfrak{q} \cap \mathcal{S}^*(M) = \mathfrak{p}_0$.

Consider now the chain of prime ideals $\mathfrak{b}_0 \subset \cdots \subset \mathfrak{b}_r$, where $\mathfrak{b}_k = \mathfrak{q}_k \cap \mathcal{S}^*(M)$. We must check that each $\mathfrak{b}_k = \mathfrak{p}_k$, and we have just seen in 5.11.1 that this is so for $k = 0$. In fact $\mathfrak{b}_0 \subset \cdots \subset \mathfrak{b}_r$ is an strict chain because, for $k = 0, \dots, r-1$,

$$\mathfrak{b}_k \cap \mathcal{S}(X) = \mathfrak{q}_k \cap \mathcal{S}(X) = \mathfrak{a}_k \subsetneq \mathfrak{a}_{k+1} = \mathfrak{b}_{k+1} \cap \mathcal{S}(X).$$

By 3.1.4, $\mathfrak{p}_0, \dots, \mathfrak{p}_r$ are all the prime ideals of $\mathcal{S}^*(M)$ containing $\mathfrak{p}_0 = \mathfrak{b}_0$, and so $\mathfrak{b}_k = \mathfrak{p}_k$ for $k = 1, \dots, r$. □

After all this preparatory work, we present now the proof of 5.1 as an almost straightforward consequence of 5.4 and 5.11. More precisely,

Proof of Theorem 5.1 First, consider the inclusion maps $j : N_1 \hookrightarrow N$ and $i : M_1 \hookrightarrow M$, that satisfy $\varphi \circ j = i \circ \psi$. Consequently, $\mathrm{Spec}_s^*(\varphi) \circ \mathrm{Spec}_s^*(j) = \mathrm{Spec}_s^*(i) \circ \mathrm{Spec}_s^*(\psi)$.

Moreover, the map $\operatorname{Spec}_s^*(\psi) : \operatorname{Spec}_s^*(N_1) \to \operatorname{Spec}_s^*(M_1)$ is a homeomorphism because $\psi : N_1 \to M_1$ is so (use 4.1(iii)). By 5.11, the map $\operatorname{Spec}_s^*(i)$ is surjective. Hence, so is the composition $\operatorname{Spec}_s^*(i) \circ \operatorname{Spec}_s^*(\psi) = \operatorname{Spec}_s^*(\varphi) \circ \operatorname{Spec}_s^*(j)$ and so $\operatorname{Spec}_s^*(\varphi)$ is surjective too.

Next, consider the commutative diagrams

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & M \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ N_1 & \xrightarrow{\psi} & M_1 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \operatorname{Spec}_s^*(N) & \xrightarrow{\operatorname{Spec}_s^*(\varphi)} & \operatorname{Spec}_s^*(M) \\ {\scriptstyle \operatorname{Spec}_s^*(j)}\uparrow & & \uparrow{\scriptstyle \operatorname{Spec}_s^*(i)} \\ \operatorname{Spec}_s^*(N_1) & \xrightarrow{\operatorname{Spec}_s^*(\psi)} & \operatorname{Spec}_s^*(M_1) \end{array}$$

As observed above, the map $\operatorname{Spec}_s^*(\psi) : \operatorname{Spec}_s^*(N_1) \to \operatorname{Spec}_s^*(M_1)$ is a homeomorphism. Hence, to achieve the statement it is enough to use the following facts:

(1) $\operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(\varphi^{-1}(Y)) \subset \operatorname{Spec}_s^*(\varphi)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(Y))$,
(2) $\operatorname{Spec}_s^*(\psi)^{-1}(\operatorname{Spec}_s^*(i)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(Y))) = \operatorname{Spec}_s^*(\varphi \circ j)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(Y))$,
(3) Both maps

$$\operatorname{Spec}_s^*(j)| : \operatorname{Spec}_s^*(N_1)\setminus \operatorname{Spec}_s^*(j)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(T)) \to \operatorname{Spec}_s^*(N)\setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(T),$$
$$\operatorname{Spec}_s^*(i)| : \operatorname{Spec}_s^*(M_1)\setminus \operatorname{Spec}_s^*(i)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(T)) \to \operatorname{Spec}_s^*(M)\setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(Y),$$

where $T = \varphi^{-1}(Y)$ are, by 5.4(iv), homeomorphisms.

Next, we present other nice consequences of 5.11. We begin by characterizing under what conditions, given an inclusion map $j : N \hookrightarrow M$ between two semialgebraic sets $N \subset M \subset \mathbb{R}^n$, the spectral map $\operatorname{Spec}_s^*(j) : \operatorname{Spec}_s^*(N) \to \operatorname{Spec}_s^*(M)$ is surjective.

Corollary 5.12 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets and let $j : N \hookrightarrow M$ be the inclusion map. Then, the following statements are equivalent:*

(i) *The homomorphism $\phi : \mathcal{S}(M) \to \mathcal{S}(N),\ f \mapsto f|_N$ is injective.*
(ii) *The set N is dense in M.*
(iii) *The map $\operatorname{Spec}_s^*(j) : \operatorname{Spec}_s^*(N) \to \operatorname{Spec}_s^*(M)$ is surjective.*
(iv) *The image of the restriction $\operatorname{Spec}_s^*(j)| : \beta_s^*N \to \operatorname{Spec}_s^*(M)$ is β_s^*M.*

Proof Let us prove first the equivalence between (i) and (ii). If N is not dense in M, there exists $f \in \mathcal{S}(M)$ with $Z_M(f) = \operatorname{Cl}_M(N) \neq M$. Then, $f \in \ker \phi\setminus\{0\}$, and so ϕ is not injective. Suppose now that N is dense in M, and let $f \in \ker \phi$. Then, $N \subset Z_M(f)$, and so $M = \operatorname{Cl}_M(N) = Z_M(f)$, that is, $f = 0$. Hence, ϕ is injective.

To prove (ii) $\Longrightarrow$ (iii) suppose that N is dense in M and consider the set $N_{\mathrm{lc}} \subset N$, which is locally compact and dense in M. Thus, N_{lc} is an open subset of M (see 2.4) and $M \subset \operatorname{Cl}_{\mathbb{R}^n}(N_{\mathrm{lc}})$. Consider the inclusion maps $j_1 : N_{\mathrm{lc}} \hookrightarrow N, i = j \circ j_1 : N_{\mathrm{lc}} \hookrightarrow M$ and $i_1 : M\setminus N_{\mathrm{lc}} \hookrightarrow M$. Since $(M, N_{\mathrm{lc}}, M\setminus N_{\mathrm{lc}}, i, i_1)$ is a suitable arranged tuple, the composition $\operatorname{Spec}_s^*(i) = \operatorname{Spec}_s^*(j) \circ \operatorname{Spec}_s^*(j_1)$ is, by 5.11, surjective and so $\operatorname{Spec}_s^*(j)$ is surjective too.

Next, we prove (iii) $\Longrightarrow$ (iv). By 5.9, $\operatorname{Spec}_s^*(j)(\beta_s^*N) \subset \beta_s^*M$ and so it only remains to check that $\beta_s^*M \subset \operatorname{Spec}_s^*(j)(\beta_s^*N)$. Indeed, given $\mathfrak{m}^* \in \beta_s^*M$ there exists $\mathfrak{p} \in \operatorname{Spec}_s^*(N)$ such that $\operatorname{Spec}_s^*(j)(\mathfrak{p}) = \mathfrak{m}^*$. Let $\mathfrak{n}^* \in \beta_s^*N$ be the unique maximal ideal of $\mathcal{S}^*(N)$ containing $\mathfrak{p}$. Then, $\mathfrak{m}^* = \operatorname{Spec}_s^*(j)(\mathfrak{p}) \subset \operatorname{Spec}_s^*(j)(\mathfrak{n}^*)$ and, $\mathfrak{m}^*$ being maximal, we get $\mathfrak{m}^* = \operatorname{Spec}_s^*(j)(\mathfrak{n}^*)$.

Finally, to show (iv) $\Longrightarrow$ (ii) we must check that N is dense in M. Otherwise, by 2.7 there exists a nonempty open semialgebraic subset A of $M\setminus\rho_1(M) = M_{\mathrm{lc}}$ such that $A \cap N = \varnothing$.

Moreover, since $\beta_s^* M$ is a Hausdorff compactification of the locally compact space $M_{\rm lc}$, it follows from 2.4 that $M_{\rm lc}$ is an open subset of $\beta_s^* M$. Consequently, A is a nonempty open subset of $\beta_s^* M$ with $A \cap N = \varnothing$, that is, the closed subset $\beta_s^* M \setminus A$ of $\beta_s^* M$ contains N. Therefore,

$$\beta_s^* M = \operatorname{Spec}_s^*(j)(\beta_s^* N) = \operatorname{Spec}_s^*(j)(\operatorname{Cl}_{\beta_s^* N}(N)) \subset \operatorname{Cl}_{\beta_s^* M}(N) \subset \beta_s^* M \setminus A,$$

which contradicts the fact that A is nonempty. □

Corollary 5.13 (Going-up) *Let (M, N, Y, i, j) be a suitable arranged tuple. Then,*

(i) *If $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$ are prime ideals of $\mathcal{S}^*(M)$ and $\mathfrak{q}_0$ is a prime ideal of $\mathcal{S}^*(N)$ such that $\mathfrak{q}_0 \cap \mathcal{S}^*(M) = \mathfrak{p}_0$, there exists a prime ideal $\mathfrak{q}_1$ of $\mathcal{S}^*(N)$ such that $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$ and $\mathfrak{q}_1 \cap \mathcal{S}^*(M) = \mathfrak{p}_1$.*

(ii) *If $\mathfrak{p}$ is a prime ideal of $\mathcal{S}^*(M)$ and $\mathfrak{q} \in \operatorname{Spec}_s^*(j)^{-1}(\mathfrak{p})$, then*

$$\operatorname{Spec}_s^*(j)(\operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(\mathfrak{q})) = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(\mathfrak{p}).$$

Proof Observe first that part (ii) is an straightforward consequence of part (i) and 3.1.4. Thus, let us prove (i). Indeed, let $\mathfrak{a}_0$ be a minimal prime ideal of $\mathcal{S}^*(N)$ contained in $\mathfrak{q}_0$. Observe that $\mathfrak{b}_0 = \mathfrak{a}_0 \cap \mathcal{S}^*(M) \subset \mathfrak{q}_0 \cap \mathcal{S}^*(M) = \mathfrak{p}_0$ is, by 5.7, a minimal prime ideal of $\mathcal{S}^*(M)$. Let $\mathfrak{b}_0 \subsetneq \cdots \subsetneq \mathfrak{b}_r$ be the collection of all the prime ideals of $\mathcal{S}^*(M)$ containing $\mathfrak{b}_0$; of course, $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are two of these prime ideals, and let $1 \le s \le r$ with $\mathfrak{b}_s = \mathfrak{p}_1$. Since $\mathfrak{b}_0$ is a minimal prime ideal, its fiber $\operatorname{Spec}_s^*(j)^{-1}(\mathfrak{b}_0)$ is, by 5.4(iii) and 5.8, a singleton, and so $\operatorname{Spec}_s^*(j)^{-1}(\mathfrak{b}_0) = \{\mathfrak{a}_0\}$. By 5.11, there exists a chain of prime ideals $\mathfrak{a}_0 \subsetneq \cdots \subsetneq \mathfrak{a}_r$ in $\mathcal{S}^*(N)$ such that $\mathfrak{a}_i \cap \mathcal{S}^*(M) = \mathfrak{b}_i$ for $i = 0, \ldots, r$. Hence, $\mathfrak{q}_1 = \mathfrak{a}_s$ is a prime ideal of $\mathcal{S}^*(N)$ and $\mathfrak{q}_1 \cap \mathcal{S}^*(M) = \mathfrak{p}_1$, and all reduces to see that $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$. This follows from 3.1.4, because the prime ideals of $\mathcal{S}^*(N)$ containing $\mathfrak{a}_0$ form a chain. □

Remark 5.14 (Going-down) Let us see now that if (M, N, Y, i, j) is a suitable arranged tuple, then the homomorphism $\phi : \mathcal{S}^*(M) \hookrightarrow \mathcal{S}^*(N),\ f \mapsto f|_N$ enjoys the going-down property if and only if for all $p \in Y$ the dimension $\dim_p N = 1$ and the germ N_p has just one semialgebraic half-branch set germ.

Suppose first that for all $p \in Y = M \setminus N = \operatorname{Cl}_M(N) \setminus N$ the dimension $\dim_p N = 1$ and the germ N_p has just one semialgebraic half-branch set germ. Then, by 2.9, the injective homomorphism ϕ is also surjective. Thus, it trivially enjoys the going-down property. Conversely, assume that there exists $p \in Y$ such that the germ N_p contains two different semialgebraic half-branch set germs. We may assume the existence of two semialgebraic paths $\alpha, \gamma : [0, 1] \to M$ such that $\alpha(0) = \gamma(0) = p, \alpha((0, 1]) \cup \gamma((0, 1]) \subset N$ and $\alpha((0, 1]) \cap \gamma((0, 1]) = \varnothing$. Let $\mathfrak{p}_2 = \mathfrak{m}_p^*$ and consider the prime ideal of $\mathcal{S}^*(M)$ (see [8, 3.5])

$$\mathfrak{p}_1 = \{f \in \mathcal{S}^*(M) : \exists\, \varepsilon > 0 \ \text{ such that } (f \circ \alpha)|_{(0, \varepsilon]} = 0\} \subsetneq \mathfrak{p}_2.$$

The maximal ideal (see 3.4) of $\mathcal{S}^*(N)$ defined by

$$\mathfrak{q}_2 = \{g \in \mathcal{S}^*(N) : \lim_{t \to 0} (g \circ \gamma)(t) = 0\}$$

satisfies $\operatorname{Spec}_s^*(j)(\mathfrak{q}_2) = \mathfrak{p}_2$, and it is enough to prove there is no prime ideal $\mathfrak{q}_1 \subset \mathfrak{q}_2$ of $\mathcal{S}^*(N)$ such that $\operatorname{Spec}_s^*(j)(\mathfrak{q}_1) = \mathfrak{p}_1$. Consider the maximal ideal

$$\mathfrak{n} = \{g \in \mathcal{S}(N) : \exists\, \varepsilon > 0 \ \text{ such that } (g \circ \gamma)|_{(0, \varepsilon]} = 0\}$$

of $\mathcal{S}(N)$, see [8, 3.5]. By [8, 5.17], there is no prime ideal in between $\mathfrak{n} \cap \mathcal{S}^*(N)$ and $\mathfrak{q}_2$. Thus, if $\mathfrak{q}_1 \subset \mathfrak{q}_2$ is a prime ideal of $\mathcal{S}^*(N)$ with $\operatorname{Spec}_s^*(j)(\mathfrak{q}_1) = \mathfrak{p}_1$ then, by [7, 5.1],

$\mathfrak{q}_1 \subset \mathfrak{n} \cap \mathcal{S}^*(N)$. But this is false, because there exists, by 2.2, a function $f \in \mathcal{S}^*(M)$ such that $Z_M(f) = \alpha([0,1])$, and so $f|_N \in \mathfrak{q}_1 \setminus (\mathfrak{n} \cap \mathcal{S}^*(N))$. □

Corollary 5.15 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that N is dense in M and let $j : N \hookrightarrow M$ be the inclusion map. Let $C \subset M$ be a closed semialgebraic subset of M such that $C_1 = C \cap N$ is dense in C. Then,* $\operatorname{Spec}_s^*(j)(\operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(C_1)) = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(C)$.

Proof Consider the inclusion maps $j_1 : C_1 \hookrightarrow N$, $j_2 : C_1 \hookrightarrow C$, $j_3 : C \hookrightarrow M$ and the composition $i = j \circ j_1 = j_3 \circ j_2 : C_1 \hookrightarrow M$. By 5.12, and since C_1 is dense in C, we have $\operatorname{Spec}_s^*(j_2)(\operatorname{Spec}_s^*(C_1)) = \operatorname{Spec}_s^*(C)$. Moreover, since C and C_1 are, respectively, closed semialgebraic subsets of M and N, the following equalities

$$\operatorname{Spec}_s^*(j_1)(\operatorname{Spec}_s^*(C_1)) = \operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(C_1) \quad \text{and} \quad \operatorname{Spec}_s^*(j_3)(\operatorname{Spec}_s^*(C)) = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(C).$$

are the immediate consequence of 2.9 and 4.6. Next, by the functoriality, $\operatorname{Spec}_s^*(i) = \operatorname{Spec}_s^*(j) \circ \operatorname{Spec}_s^*(j_1) = \operatorname{Spec}_s^*(j_3) \circ \operatorname{Spec}_s^*(j_2)$, and we deduce

$$\begin{aligned} \operatorname{Spec}_s^*(j)(\operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(C_1)) &= \operatorname{Spec}_s^*(i)(\operatorname{Spec}_s^*(C_1)) \\ &= \operatorname{Spec}_s^*(j_3)(\operatorname{Spec}_s^*(C)) = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(C), \end{aligned}$$

which concludes the proof. □

(5.16) Behaviour of Spec_s^* under certain "stratifications". The main result 5.1 of this section is useful to analyze the spectrum of $\mathcal{S}^*(M)$ when M is not necessarily locally compact, via the use of different "stratifications" of M. The first one is suggested by the construction of the operator $\rho_1(\cdot)$, see 2.6, and the strata are ordered in such a way that each of them is maximal with respect to the local compactness property in the complement in M of the union of the precedent ones (and the first stratum is maximal in M).

Definition 5.17 Let $M \subset \mathbb{R}^n$ be a semialgebraic set. We define the family $\mathcal{P}_M = \{\mathcal{P}_i(M)\}_{i\geq 1}$ of *maximal locally compact pieces* of M as follows: Consider $N_1 = M$ and $N_{i+1} = \rho_1(N_i)$ for $i \geq 1$ and define $\mathcal{P}_i(M) = N_i \setminus N_{i+1}$ for $i \geq 1$. From [1, 2.8.13] it follows that $\dim N_{i+1} < \dim N_i - 1$. In particular, the family (of nonempty elements of) $\mathcal{P}_M$ is finite. Moreover, $\mathcal{P}_i(M)$ is, by 2.6, the largest locally compact and dense subset of N_i. This together with the equality $\mathcal{P}_1(M) = M \setminus \rho_1(M)$ justify the name of these sets associated to M.

Furthermore, by the definition of $\rho_1(\cdot)$, N_{i+1} is a closed subset of N_i and, by 2.6, $\mathcal{P}_i(M) = N_i \setminus N_{i+1}$ is dense in N_i. Thus, proceeding inductively it follows that each N_i is closed in M and $\operatorname{Cl}_M(\mathcal{P}_i(M)) = N_i$ for $i \geq 1$.

Proposition 5.18 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathcal{P}_M = \{\mathcal{P}_i(M)\}_{i=1}^r$ be the family of maximal locally compact pieces of M. Let $j_i : \mathcal{P}_i(M) \hookrightarrow N_i = \operatorname{Cl}_M(\mathcal{P}_i(M))$ be the inclusion map. Then,* $\operatorname{Spec}_s^*(M)$ *is the disjoint union of subsets homeomorphic to* $\operatorname{Spec}_s^*(\mathcal{P}_i(M)) \setminus \operatorname{Spec}_s^*(j_i)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(N_i)}(\rho_1(N_i)))$ *for $i = 1, \dots, r$.*

Proof Notice that $N_{i+1} = \rho_1(N_i)$ and $\mathcal{P}_i(M) = N_i \setminus N_{i+1}$ for $i = 1, \dots, r$, where $N_{r+1} = \varnothing$. Since $M = \bigcup_{i=1}^r N_i$, we have

$$\begin{aligned} &\bigcup_{i=1}^r (\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N_i) \setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N_{i+1}) \\ &\quad = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N_1) \setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N_{r+1}) = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(M) = \operatorname{Spec}_s^*(M). \end{aligned}$$

By 4.6, the spaces $\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(N_i)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(N_{i+1})$ and $\mathrm{Spec}_s^*(N_i)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(N_i)}(N_{i+1})$ are homeomorphic. Thus, to finish it is enough to check that the spaces

$$\mathrm{Spec}_s^*(N_i)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(N_i)}(N_{i+1}) \quad \text{and} \quad \mathrm{Spec}_s^*(\mathcal{P}_i(M))\setminus \mathrm{Spec}_s^*(j_i)^{-1}(\mathrm{Cl}_{\mathrm{Spec}_s^*(N_i)}(\rho_1(N_i)))$$

are homeomorphic. But this last follows from 5.4(iii) applied to the suitable arranged tuple $(N_k = \mathrm{Cl}_M(\mathcal{P}_k(M)), \mathcal{P}_k(M), N_k\setminus\mathcal{P}_k(M), j_k, i_k)$, where $i_k : N_k\setminus\mathcal{P}_k(M) \hookrightarrow N_k$. □

Our last goal in this section is to reduce the local study of the spectrum of the ring of bounded semialgebraic functions on an arbitrary semialgebraic set M to the study of the semialgebraic spectrum of $\mathcal{S}^*(\mathbb{R}^m)$ for each $0 \le m \le \dim M$.

Proposition 5.19 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Then, there exist semialgebraic sets $A_1, \ldots, A_r \subset M$ and closed semialgebraic subsets $C_1, \ldots, C_r$ of M, such that:*

(i) *Each A_i is Nash diffeomorphic to $\mathbb{R}^{d_i}$ for some $0 \le d_i \le \dim M$, via a Nash diffeomorphism $\varphi_i : \mathbb{R}^{d_i} \to A_i$.*
(ii) *$M = \bigcup_{i=1}^r A_i$.*
(iii) *$\mathrm{Spec}_s^*(M)$ is the disjoint union of open subsets which are homeomorphic via the map $\mathrm{Spec}_s^*(j_i \circ \varphi_i)$ to the open subsets of $\mathrm{Spec}_s^*(\mathbb{R}^{d_i})$*

$$\mathcal{A}_i = \mathrm{Spec}_s^*(\mathbb{R}^{d_i})\setminus \mathrm{Spec}_s^*(j_i \circ \varphi_i)^{-1}(\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(C_i))$$

for $i = 1, \ldots, r$, where $j_i : A_i \hookrightarrow M$ is the inclusion map.

Proof We proceed by induction on the dimension of M. If M has dimension 0 the result is trivially true because $\mathrm{Spec}_s^*(M) = M$. Suppose the result proved for dimension $d-1$ and let us see that it is also true for $d = \dim M$.

Let $B = \mathrm{Reg}(M)$, which is an open semialgebraic subset of M, see 2.3. By [1, 2.9.10], B is the disjoint union of a finite number of Nash submanifolds B_i for $i = 1, \ldots, s$, each of them Nash diffeomorphic to an open hypercube $(0, 1)^{\dim B_i}$. We may assume that for $i = 1, \ldots, \ell$, the Nash manifold B_i has dimension d and that for $i = \ell+1, \ldots, s$ the dimension of B_i is $< d$. Note that $B_0 = \bigcup_{i=1}^{\ell} B_i$ is an open subset of B and so of M, because both B_0 and B are pure dimensional Nash manifolds of dimension d. Let $E_0 = M\setminus B_0$ which is a closed semialgebraic subset of M of dimension $\le d-1$, see 2.3. Let $T_0 = \mathrm{Cl}_M(B_0)$ which is a closed pure dimensional subset of M of dimension d that satisfies $M = T_0 \cup E_0$. This last implies $\mathrm{Spec}_s^*(M) = \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(T_0) \cup \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(E_0)$, and therefore

$$\mathrm{Spec}_s^*(M)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(E_0) = \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(T_0)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(E_0). \qquad (*)$$

Let $j : T_0 \hookrightarrow M$ be the inclusion map. Since T_0 is closed in M it follows from 2.9 and 4.6 that the induced map $\mathrm{Spec}_s^*(j) : \mathrm{Spec}_s^*(T_0) \to \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(T_0)$ is a homeomorphism. By the same reason, $\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(E_0)$ is homeomorphic to $\mathrm{Spec}_s^*(E_0)$ via the map induced by the inclusion map $E_0 \hookrightarrow M$. Thus, the spaces

$$\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(T_0)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(E_0) \quad \text{and} \quad \mathrm{Spec}_s^*(T_0)\setminus \mathrm{Spec}_s^*(j)^{-1}(\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(E_0))$$

are homeomorphic. Let $j' : B_0 \hookrightarrow T_0$ be the inclusion map. Recall that $\dim E_0 \le d-1$ and T_0 is a pure dimensional semialgebraic set of dimension d such that $B_0 = T_0\setminus(E_0 \cap T_0)$ is locally compact, because it is a Nash manifold. Hence, by 5.4, the spaces

$$\mathrm{Spec}_s^*(B_0)\setminus \mathrm{Spec}_s^*(j')^{-1}(\mathrm{Cl}_{\mathrm{Spec}_s^*(T_0)}(E_0 \cap T_0)) \quad \text{and} \quad \mathrm{Spec}_s^*(T_0)\setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(T_0)}(E_0 \cap T_0)$$

are homeomorphic via $\operatorname{Spec}_s^*(j')$. Note also that

$$\operatorname{Cl}_{\operatorname{Spec}_s^*(T_0)}(E_0 \cap T_0) \subset \operatorname{Spec}_s^*(j)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0)).$$

Consequently, the set $\operatorname{Spec}_s^*(B_0)\setminus \operatorname{Spec}_s^*(j\circ j')^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0))$ is homeomorphic, via $\operatorname{Spec}_s^*(j\circ j')$, to $\operatorname{Spec}_s^*(T_0)\setminus \operatorname{Spec}_s^*(j)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0))$ which, as we have seen above in $(*)$, is homeomorphic to $\operatorname{Spec}_s^*(M)\setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0)$.

On the other hand, $B_0 = \bigcup_{i=1}^{\ell} B_i$ where the B_i's are the connected components of B_0. By 4.7, $\operatorname{Spec}_s^*(B_0) \cong \bigsqcup_{i=1}^{\ell} \operatorname{Spec}_s^*(B_i)$ and the subspaces $\operatorname{Cl}_{\operatorname{Spec}_s^*(B_0)}(B_i) \cong \operatorname{Spec}_s^*(B_i)$, for $i = 1,\dots,\ell$ are the connected components of $\operatorname{Spec}_s^*(B_0)$. Thus,

$$\begin{aligned}&\operatorname{Spec}_s^*(B_0)\setminus \operatorname{Spec}_s^*(j\circ j')^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0))\\ &\quad\cong \bigsqcup_{i=1}^{\ell}(\operatorname{Spec}_s^*(B_i)\setminus \operatorname{Spec}_s^*(e_i)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0)))\end{aligned}$$

where $e_i : B_i \hookrightarrow M$ is the inclusion map. Let $\psi_i : \mathbb{R}^d \to B_i$ be a Nash diffeomorphism for $i = 1,\dots,\ell$. Then,

$$\begin{aligned}\operatorname{Spec}_s^*(M)\setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0) &\cong \operatorname{Spec}_s^*(B_0)\setminus \operatorname{Spec}_s^*(j\circ j')^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0))\\ &\cong \bigsqcup_{i=1}^{\ell}(\operatorname{Spec}_s^*(\mathbb{R}^d)\setminus \operatorname{Spec}_s^*(\psi_i\circ e_i)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0))).\end{aligned}$$

Let us denote $A_j = B_j$ and $C_j = E_0$ for $j = 1,\dots,\ell$. Note that each A_j is Nash diffeomorphic to $\mathbb{R}^d$ and each C_j is closed in M. Moreover, $B_0 = M\setminus E_0 = \bigcup_{j=1}^{\ell} A_j$.

Now, since $\dim E_0 \le d-1$ and $\operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(E_0) \cong \operatorname{Spec}_s^*(E_0)$ by 4.6, we apply to E_0 the inductive hypothesis to obtain semialgebraic subsets $A_{\ell+1},\dots,A_r \subset M$ and closed semialgebraic subsets of E_0 (and hence of M), say $C_{\ell+1},\dots,C_r$, such that:

(i) Each A_i is Nash diffeomorphic to $\mathbb{R}^{d_i}$ for some $0 \le d_i \le \dim E_0$, via a Nash diffeomorphism $\varphi_i : \mathbb{R}^{d_i} \to A_i$.
(ii) $E_0 = \bigcup_{i=\ell+1}^{r} A_i$.
(iii) $\operatorname{Spec}_s^*(E_0)$ is the disjoint union of sets which are homeomorphic via the map $\operatorname{Spec}_s^*(j_i\circ\varphi_i)$ to the open subset of $\operatorname{Spec}_s^*(\mathbb{R}^{d_i})$

$$\mathcal{A}_i = \operatorname{Spec}_s^*(\mathbb{R}^{d_i})\setminus \operatorname{Spec}_s^*(j_i\circ\varphi_i)^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(E_0)}(C_i))$$

for $i = \ell+1,\dots,r$, where $j_i : A_i \hookrightarrow E_0$ is the inclusion map.

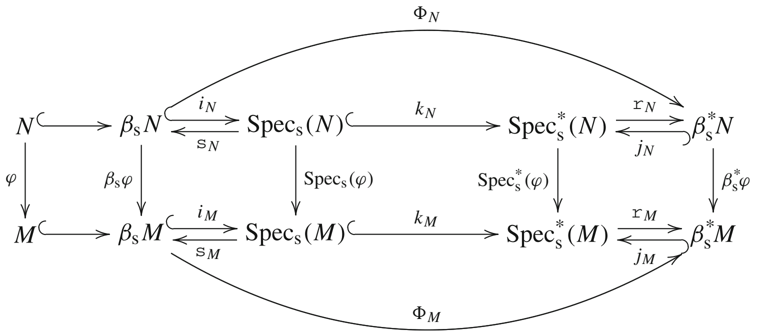
Finally, a straightforward verification shows that the semialgebraic subsets $A_1,\dots,A_r$ and $C_1,\dots,C_r$ of M satisfy all the required conditions. □

6 Functoriality of β_s and β_s^*

We have proved in 4.8 the existence of semialgebraic maps $\varphi : N \to M$ and maximal ideals of $\mathcal{S}(N)$ whose image under the induced map $\operatorname{Spec}_s(\varphi) : \operatorname{Spec}_s(N) \to \operatorname{Spec}_s(M)$ is not a maximal ideal of $\mathcal{S}(M)$. However, by [11, 3.1(iii)] and [15, 1.2], the retraction $\mathtt{s}_M : \operatorname{Spec}_s(M) \to \beta_s M$, which maps each prime ideal of $\mathcal{S}(M)$ to the unique maximal ideal of $\mathcal{S}(M)$ containing it, is a continuous map. In this way, we define $\beta_s\varphi = \mathtt{s}_M \circ \operatorname{Spec}_s(\varphi)|_{\beta_s N} : \beta_s N \to \beta_s M$.

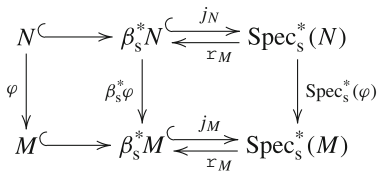
Observe that since N and M are respectively dense in $\beta_s N$ and $\beta_s M$, the map $\beta_s\varphi$ is the unique continuous extension of $\varphi : N \to M$ to $\beta_s N$ taking values in $\beta_s M$.

On the other hand, by 5.9, $\operatorname{Spec}_s^*(\varphi) : \operatorname{Spec}_s^*(N) \to \operatorname{Spec}_s^*(M)$ maps $\beta_s^* N$ into $\beta_s^* M$. Hence, we denote $\beta_s^*\varphi = \operatorname{Spec}_s^*(\varphi)|_{\beta_s^* N} : \beta_s^* N \to \beta_s^* M$. Again by [11, 3.1(iii)] and [15, 1.2], the retraction $\mathtt{r}_M : \operatorname{Spec}_s^*(M) \to \beta_s^* M$ which maps each prime ideal of $\mathcal{S}^*(M)$ to the unique maximal ideal of $\mathcal{S}^*(M)$ containing it, is a continuous map. Consider the inclusion maps $i_M : \beta_s M \hookrightarrow \operatorname{Spec}_s(M)$ and $j_M : \beta_s^* M \hookrightarrow \operatorname{Spec}_s^*(M)$, and let $k_M : \operatorname{Spec}_s(M) \to \operatorname{Spec}_s^*(M)$, $\mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^*(M)$ which is, by 3.2, a homeomorphism onto its image. By 3.3.1&2, the composition $\Phi_M = \mathtt{r}_M \circ k_M \circ i_M : \beta_s M \to \beta_s^* M$ is a homeomorphism. Moreover, we have $\mathtt{s}_M = \Phi_M^{-1} \circ \mathtt{r}_M \circ k_M$ and $k_M \circ \operatorname{Spec}_s(\varphi) = \operatorname{Spec}_s^*(\varphi) \circ k_N$. This, together with the equality $\mathtt{r}_M \circ \operatorname{Spec}_s^*(\varphi) \circ j_N = \operatorname{Spec}_s^*(\varphi) \circ j_N$, provides the following commutative diagram:



Thus, via Φ_N and Φ_M, we can translate the properties of the operator β_s^* to properties of β_s. This is why we focus our attention in the study of the behaviour of β_s^*.

(6.1) In what follows, we will use the retraction $\mathtt{r}_M : \operatorname{Spec}_s^*(M) \to \beta_s^* M$ to transfer to β_s^* many statements proved in Sect. 5 for Spec_s^*. We have the following commutative diagram



We proceed first to establish some direct consequences of the commutativity of the diagram above and the corresponding results for Spec_s^*. Of course, we begin by pointing out (as a direct consequence of 4.1) the functoriality of β_s^*.

Corollary 6.2 *Let $\varphi : N \to M$ and $\psi : M \to P$ be semialgebraic maps between semialgebraic sets $N \subset \mathbb{R}^n$, $M \subset \mathbb{R}^m$ and $P \subset \mathbb{R}^p$. Then $\beta_s^*\psi \circ \beta_s^*\varphi = \beta_s^*(\psi \circ \varphi)$ and $\beta_s^*\varphi : \beta_s^* N \to \beta_s^* M$ is the unique continuous extension of φ to $\beta_s^* N$ taking values in $\beta_s^* M$.*

Next, concerning the closure in $\beta_s^* M$ of a semialgebraic subset of M, we adapt to our context, by means of diagram 6.1, the corresponding results 4.3 and 4.6 for $\operatorname{Spec}_s^*(M)$.

Corollary 6.3 *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets and consider the homomorphism $\phi : \mathcal{S}^*(M) \to \mathcal{S}^*(N)$, $f \to f|_N$ induced by the inclusion map $j : N \hookrightarrow M$.*

(i) *Let $\mathfrak{m}^* \in \beta_s^* M$. Then, $\mathfrak{m}^* \in \operatorname{Cl}_{\beta_s^* M}(N)$ if and only if* $\ker\phi \subset \mathfrak{m}^*$.

(ii) *If ϕ is surjective, then $\beta_s^* N \cong \mathrm{Cl}_{\beta_s^* M}(N) \subset \beta_s^* M$ via $\beta_s^* j$. Moreover, the homeomorphism $\mathrm{Spec}_s^*(j) : \mathrm{Spec}_s^*(N) \to \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(N)$ extends the homeomorphism $\beta_s^* j$.*

Remarks 6.4 (i) In general $\beta_s^* N \not\cong \mathrm{Cl}_{\beta_s^* M}(N)$. Consider, for instance, the open disc $M = \{x^2 + y^2 < 1\} \subset \mathbb{R}^2$ and the punctured circle $N = \{x^2 + y^2 = 1/4\} \setminus \{(1/2, 0)\}$. Then, one can check that $\beta_s^* N$ is homeomorphic to $[0, 1]$ (see for instance [13, 4.9]) while the closure $\mathrm{Cl}_{\beta_s^* M}(N) = \mathrm{Cl}_M(N) = \{x^2 + y^2 = 1/4\}$ is homeomorphic to $\mathbb{S}^1$.

Thus, if we do not impose ϕ to be surjective in 6.3, then the map $\beta_s^* \varphi : \beta_s^* N \to \mathrm{Cl}_{\beta_s^* M}(N)$ is a quotient map, (that is, a proper, surjective and continuous map) which is not in general a homeomorphism. □

Again, as it happens for $\mathrm{Spec}_s^\diamond(M)$ (see 4.5), the closure in $\beta_s^* M$ commutes with finite intersections of closed semialgebraic subsets of M. Namely,

Corollary 6.5 *Let $C_1, C_2 \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that C_1 and C_2 are closed subsets of M. Then, $\mathrm{Cl}_{\beta_s^* M}(C_1 \cap C_2) = \mathrm{Cl}_{\beta_s^* M}(C_1) \cap \mathrm{Cl}_{\beta_s^* M}(C_2)$.*

Concerning the connected components of $\beta_s^* M$, we prove, using 4.7, the following result.

Corollary 6.6 *Let $M_1, \ldots, M_k$ be the connected components of the semialgebraic set $M \subset \mathbb{R}^n$. Then, their closures $\mathrm{Cl}_{\beta_s^* M}(M_i) \cong \beta_s^* M_i$ are the connected components of $\beta_s^* M$. In particular, $\beta_s^* M$ has a finite number of connected components, and it is connected if and only if M is so.*

Proof In 4.7 we proved that the connected components of $\mathrm{Spec}_s^*(M)$ are $\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(M_i) \cong \mathrm{Spec}_s^*(M_i)$. By 6.1, $\mathtt{r}_M(\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(M_i)) = \mathrm{Cl}_{\beta_s^* M}(M_i)$ are connected subsets of $\beta_s^* M$ whose union equals $\beta_s^* M$. Since each $\mathrm{Cl}_{\beta_s^* M}(M_i)$ is closed in $\beta_s^* M$ and they are, by 6.5, finitely many and pairwise disjoint, we conclude that $\mathrm{Cl}_{\beta_s^* M}(M_1), \ldots, \mathrm{Cl}_{\beta_s^* M}(M_k)$ are the connected components of $\beta_s^* M$. Finally, by 6.3, $\mathrm{Cl}_{\beta_s^* M}(M_i) \cong \beta_s^* M_i$ for $i = 1, \ldots, k$.

In the main result of this section, which follows essentially from 4.1 and 6.1, we study the behaviour of the operator β_s^* on a suitable arranged tuple. More precisely,

Theorem 6.7 *Let (M, N, Y, j, i) be a suitable arranged tuple and consider the homomorphism $\phi : \mathcal{S}^*(M) \hookrightarrow \mathcal{S}^*(N)$, $f \mapsto f|_N$ induced by j. Then:*

(i) *The map $\beta_s^* j : \beta_s^* N \to \beta_s^* M$ is surjective, proper and continuous.*
(ii) *The restriction $\beta_s^* j| : \beta_s^* N \setminus (\beta_s^* j)^{-1}(\mathrm{Cl}_{\beta_s^* M}(Y)) \to \beta_s^* M \setminus \mathrm{Cl}_{\beta_s^* M}(Y)$ is a homeomorphism.*
(iii) *The diagram*

$$\begin{array}{ccc} \mathrm{Spec}_s^*(N) \setminus \mathrm{Spec}_s^*(j)^{-1}(\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)) & \longrightarrow & \mathrm{Spec}_s^*(M) \setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y) \\ {\scriptstyle j_N} \uparrow \downarrow {\scriptstyle \mathtt{r}_N} & & {\scriptstyle j_M} \uparrow \downarrow {\scriptstyle \mathtt{r}_M} \\ \beta_s^* N \setminus (\beta_s^* j)^{-1}(\mathrm{Cl}_{\beta_s^* M}(Y)) & \longrightarrow & \beta_s^* M \setminus \mathrm{Cl}_{\beta_s^* M}(Y) \end{array}$$

is commutative. That is, the homeomorphism in the upper row extends the homeomorphism in the bottom row.
(iv) *Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{S}(M)$ and let $\mathfrak{m}^*$ be the maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{p} = \mathfrak{m} \cap \mathcal{S}^*(M)$. Then, $\mathfrak{a} = \phi(\mathfrak{p})\mathcal{S}^*(N) \neq \mathcal{S}^*(N)$. Moreover, each maximal ideal $\mathfrak{n}^*$ of $\mathcal{S}^*(N)$ containing $\mathfrak{a}$ satisfies $\beta_s^* j(\mathfrak{n}^*) = \mathfrak{m}^*$.*

Proof (i) Since $\beta_s^* j$ is a continuous map from the compact space $\beta_s^* N$ to the Hausdorff space $\beta_s^* M$, it is a proper map and in particular its image is a closed subset of $\beta_s^* M$. But im $\beta_s^* j$ contains N, which is a dense subset of $\beta_s^* M$. Hence, $\beta_s^* j$ is also surjective.

Parts (ii) and (iii) follow straightforwardly from 4.1, 5.4 and 6.1.

(iv) Suppose that $\mathfrak{a} = \mathcal{S}^*(N)$. Then, there exist $a_1, \dots, a_r \in \mathcal{S}^*(N)$ and $b_1, \dots, b_r \in \mathfrak{p}$ such that $1 = a_1(b_1|_N) + \cdots + a_r(b_r|_N)$. Since $b_1, \dots, b_r \in \mathfrak{m}$ the zeroset $Z_M(b_1^2 + \cdots + b_r^2) \neq \varnothing$ and we choose a point $p \in Z_M(b_1^2 + \cdots + b_r^2)$. On the other hand, the semialgebraic functions $a_1, \dots, a_r$ being bounded, there exist the limit of each product $a_i b_i$ at the point p, which equals 0, and so $0 = \lim_{x \to p}(a_1 b_1 + \cdots + a_r b_r) = \lim_{x \to p} 1 = 1$, a contradiction. Thus, $\mathfrak{a}$ is a (proper) ideal of $\mathcal{S}^*(N)$. Finally, let $\mathfrak{n}^*$ be a maximal ideal of $\mathcal{S}^*(N)$ which contains $\mathfrak{a}$. Then

$$\mathfrak{p} = \mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{a} \cap \mathcal{S}^*(M) \subset \mathfrak{n}^* \cap \mathcal{S}^*(M) \in \beta_s^* M.$$

But $\mathfrak{m}^*$ is the unique maximal ideal of $\mathcal{S}^*(M)$ that contains the prime ideal $\mathfrak{p}$. Hence, $\beta_s^* j(\mathfrak{n}^*) = \mathfrak{n}^* \cap \mathcal{S}^*(M) = \mathfrak{m}^*$, and we are done. □

The counterpart of 5.1 in our context can be stated as follows.

Corollary 6.8 *Let $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ be semialgebraic sets and let $\varphi : N \to M$ be a semialgebraic map. Suppose that there exists a closed semialgebraic set $Y \subset M$ such that:*

(a) *$M_1 = M \setminus Y$ is locally compact and dense in M.*
(b) *The restriction $\psi = \varphi|_{N_1} : N_1 = N \setminus \varphi^{-1}(Y) \to M_1 = M \setminus Y$ is a semialgebraic homeomorphism.*

Denote $Z = \mathrm{Cl}_{\beta_s^* M}(Y)$. Then, the map $\beta_s^* \varphi : \beta_s^* N \to \beta_s^* M$ is surjective and the restriction $\beta_s^* \varphi| : \beta_s^* N \setminus (\beta_s^* \varphi)^{-1}(Z) \to \beta_s^* M \setminus Z$ is a homeomorphism.

As one can expect, we obtain for the maximal spectrum $\beta_s^\diamond M$ some "stratification" results in the same vein as 5.18 and 5.19.

Corollary 6.9 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathcal{P}_M = \{\mathcal{P}_i(M)\}_{i=1}^r$ be the family of maximal locally compact pieces of M. Let $j_i : \mathcal{P}_i(M) \hookrightarrow N_i = \mathrm{Cl}_M(\mathcal{P}_i(M))$ be the inclusion map. Then, $\beta_s^* M$ is the disjoint union of subsets which are homeomorphic to $\beta_s^* \mathcal{P}_i(M) \setminus (\beta_s^* j_i)^{-1}(\mathrm{Cl}_{\beta_s^* N_i}(\rho_1(N_i)))$ for $i = 1, \dots, r$.*

Finally, as an straightforward consequence of 5.19 and 6.7, the local study of $\beta_s^* M$ for an arbitrary semialgebraic set M can be reduced to analyze the open subsets of $\beta_s^* \mathbb{R}^m$ for all $0 \le m \le \dim M$. Namely,

Corollary 6.10 *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Then, there exist semialgebraic subsets $\{A_i\}_{i=1}^r$ and $\{C_i\}_{i=1}^r$ of M, where each C_i is closed in M, such that $M = \bigcup_{i=1}^r A_i$, and Nash diffeomorphisms $\varphi_i : \mathbb{R}^{d_i} \to A_i$ for some $0 \le d_i \le \dim M$ and each $i = 1, \dots, r$. Moreover, let $j_i : A_i \hookrightarrow M$ be the inclusion map for $i = 1, \dots, r$. Then, $\beta_s^* M$ is the disjoint union of subsets which are homeomorphic via $\beta_s^*(j_i \circ \varphi_i)$ to the open subsets $\mathcal{A}_i = \beta_s^* \mathbb{R}^{d_i} \setminus \beta_s^*(j_i \circ \varphi_i)^{-1}(\mathrm{Cl}_{\beta_s^* M}(C_i))$ of $\beta_s^* \mathbb{R}^{d_i}$.*

References

1. Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Ergeb. Math., vol. 36. Springer, Berlin (1998)
2. Bourbaki, N.: General Topology, chapters 1–4. Elements of Mathematics. Springer, Berlin (1989)

3. Birkhoff, G., Pierce, R.S.: Lattice-ordered rings. An. Acad. Brasil. Ci. **28**, 41–69 (1956)
4. Cherlin, G.-L., Dickmann, M.A.: Real closed rings. I. Residue rings of rings of continuous functions. Fund. Math. **126**(2), 147–183 (1986)
5. Cherlin, G.-L., Dickmann, M.A.: Real closed rings. II. Model theory. Ann. Pure Appl. Log. **25**(3), 213–231 (1983)
6. Delfs, H., Knebusch, M.: Separation, retractions and homotopy extension in semialgebraic spaces. Pac. J. Math. **114**(1), 47–71 (1984)
7. Fernando, J.F.: On chains of prime ideals in rings of semialgebraic functions. http://www.mat.ucm.es/~josefer/pdfs/preprint/chains.pdf (preprint RAAG, 2010)
8. Fernando, J.F.: On distinguished points of the remainder of the semialgebraic Stone–Čech compactification of a semialgebraic set. http://www.mat.ucm.es/~josefer/pdfs/preprint/remainder.pdf (preprint RAAG, 2010)
9. Fernando, J.F.: On the fibers of semialgebraic spectral maps. http://www.mat.ucm.es/~josefer/pdfs/preprint/fibers.pdf (preprint RAAG, 2010)
10. Fernando, J.F., Gamboa, J.M.: On Łojasiewicz's inequality and the Nullstellensatz for rings of semialgebraic functions. http://www.mat.ucm.es/~josefer/pdfs/preprint/null-loj.pdf (preprint RAAG, 2010)
11. Fernando, J.F., Gamboa, J.M.: On the Krull dimension of rings of semialgebraic functions. http://www.mat.ucm.es/~josefer/pdfs/preprint/dim.pdf (preprint RAAG, 2010)
12. Fernando, J.F., Gamboa, J.M.: On Banach-Stone type theorems in the semialgebraic setting. http://www.mat.ucm.es/~josefer/pdfs/preprint/homeo.pdf (preprint RAAG, 2010)
13. Fernando, J.F., Gamboa, J.M.: On the semialgebraic Stone–Čech compactification of a semialgebraic set. Transactions of AMS. http://www.ams.org/cgi-bin/mstrack/accepted_papers?jrnl=tran (2010, accepted)
14. Gillman, L., Jerison, M.: Rings of continuous functions. The University Series in Higher Nathematics, vol. 1. D. Van Nostrand Company, Inc., Princeton (1960)
15. De Marco, G., Orsatti, A.: Commutative rings in which every prime ideal is contained in a unique maximal ideal. Proc. Am. Math. Soc. **30**(3), 459–466 (1971)
16. Schwartz, N.: Real closed spaces. Ordered fields and real algebraic geometry (Boulder, Colo., 1983). Rocky Mt. J. Math. **14**(4), 971–972 (1984)
17. Schwartz, N.: The basic theory of real closed spaces. Mem. Am. Math. Soc. **77**(397) (1989)
18. Stasica, J.: Smooth points of a semialgebraic set. Ann. Polon. Math. **82**(2), 149–153 (2003)