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On the positive extension property and Hilbert's 17th problem for real analytic sets

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Abstract. In this work we study the Positive Extension ($\mathscr{PE}$) property and Hilbert's 17th problem for real analytic germs and sets. A real analytic germ X_0 of $\mathbb{R}^n_0$ has the $\mathscr{PE}$ property if every positive semidefinite analytic function germ on X_0 has a positive semidefinite analytic extension to $\mathbb{R}^n_0$; analogously one states the $\mathscr{PE}$ property for a global real analytic set X in an open set Ω of $\mathbb{R}^n$. These $\mathscr{PE}$ properties are natural variations of Hilbert's 17th problem. Here, we prove that: (1) A real analytic germ $X_0 \subsetneqq \mathbb{R}^3_0$ has the $\mathscr{PE}$ property if and only if every positive semidefinite *analytic* function germ on X_0 is a sum of squares of *analytic* function germs on X_0; and (2) a global real analytic set X of dimension $\leqq 2$ and local embedding dimension $\leqq 3$ has the $\mathscr{PE}$ property if and only if it is coherent and all its germs have the $\mathscr{PE}$ property. If that is the case, every positive semidefinite analytic function on X is a sum of squares of analytic functions on X. Moreover, we classify the singularities with the $\mathscr{PE}$ property.

1. Introduction and statement of the main results

In the study of positive semidefinite functions and sums of squares one main problem is whether or not every positive semidefinite function is a sum of squares of functions of the same class. As is well known, the interest on these questions comes from Hilbert's 17th problem, and has been one streamline of research in real algebra and geometry. The history of the topic is long and rich, and we refer the reader to [BCR], [ChDLR] and [Sch]. In the relevant case of analytic functions, we refer to [BKS], [Rz1] and [Jw2] for classical results, and for more recent progress, to [ADR], [ABFR1], [Fe5], [ABFR2] and [ABFR3].

In the analytic setting, there are always two complementary viewpoints: (a) the local one, *germs*, which involves real algebra and real spectra in essential ways; and (b) the global one, *sets*, for which complex classical analysis ([GuRo]) and Cartan's Theorems A and B ([Ca]) play a crucial role.

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Local approach.

(1.1) Analytic germs. We recall some notation and terminology. Let $X_0 \subset \mathbb{R}^n_0$ be a real analytic (set) germ (at the origin of $\mathbb{R}^n$, to simplify notations); we denote by $\mathcal{O}(X_0)$ the ring of germs of analytic functions on X_0. For instance, $\mathcal{O}(\mathbb{R}^n_0)$ is the ring $\mathbb{R}\{x\}$ of convergent power series in the variables $x = (x_1, \dots, x_n)$. As $X_0 \subset \mathbb{R}^n_0$, we have $\mathcal{O}(X_0) = \mathbb{R}\{x\}/\mathcal{J}(X_0)$, where $\mathcal{J}(X_0)$ is the ideal of all analytic function germs vanishing on X_0. The *embedding dimension of* X_0 is the minimum number of generators of the maximal ideal of the local ring $\mathcal{O}(X_0)$. A germ $f_0 \in \mathcal{O}(X_0)$ is *positive semidefinite* if it is $\geqq 0$ on X_0; and $\mathcal{P}(X_0)$ is the set of all positive semidefinite analytic function germs on X_0. We denote by $\Sigma(X_0)$ (resp. $\Sigma_p(X_0)$) the set of all sums of (resp. p) squares of elements of $\mathcal{O}(X_0)$. Recall also that an analytic germ X_0 is *unmixed* if all its irreducible components have the same dimension and it is *mixed* otherwise.

Clearly, $\Sigma(X_0) \subset \mathcal{P}(X_0)$ and the question, commonly known as Hilbert's 17th problem for the analytic ring $\mathcal{O}(X_0)$, consists of determining whether the equality $\mathcal{P}(X_0) = \Sigma(X_0)$ holds. If $\mathcal{P}(X_0) = \Sigma(X_0)$ for a germ X_0, we will say that X_0 has $\mathcal{P} = \Sigma$. Referring to this, in [Fe3] we proved that if $\mathcal{P}(X_0) = \Sigma(X_0)$, then $\dim X_0 \leqq 2$. Moreover, in [Sch], 3.9, the author characterizes the 1-dimensional analytic germs for which $\mathcal{P} = \Sigma$; and, in [Fe2] we determined the full list of all the analytic surface germs X_0 of $\mathbb{R}^3_0$ with $\mathcal{P} = \Sigma$. Notice also that if we consider *meromorphic* instead of analytic function germs, $\mathcal{P} = \Sigma$ holds always true, see [ABR], VIII.2.9.

On the other hand, recall that the analytic function germs on an analytic germ X_0 of $\mathbb{R}^n_0$ are the restrictions of the analytic function germs of $\mathbb{R}^n_0$. Thus, it could be $\mathcal{P}(X_0) \neq \Sigma(X_0)$ because there exist positive semidefinite analytic function germs on $\mathbb{R}^n_0$ which are not sums of squares in $\mathcal{O}(\mathbb{R}^n_0)$. Hence, to avoid this disturbance we look at a more general property. Namely, an analytic germ $X_0 \subset \mathbb{R}^n_0$ has the *Positive Extension* ($\mathcal{PE}$) *property*, if the following assertion holds true:

Local $\mathcal{PE}$ property. Every positive semidefinite analytic function germ f_0 on X_0 is the restriction to X_0 of a positive semidefinite analytic function germ on $\mathbb{R}^n_0$.

In relation with this, see [BP], §5. Although the $\mathcal{PE}$ property actually refers to the embedding $X_0 \subset \mathbb{R}^n_0$, it is in fact intrinsic and does not depend on how the analytic germ is embedded in the Euclidean space. On the other hand, clearly $\mathcal{P} = \Sigma$ implies $\mathcal{PE}$, hence what matters is the converse. Now, for dimension $d \geqq 3$ it is easy to find analytic germs with the $\mathcal{PE}$ property, for which $\mathcal{P} \neq \Sigma$ ([Fe3]). An immediate example is $X_0 = \mathbb{R}^3_0, \mathbb{R}^4_0, \dots$, but there are also singular examples. For instance, $X_0 = \{x_{d+1} = 0\} \cup \{x_1 = \cdots = x_d = 0\}$ in $\mathbb{R}^{d+1}_0$ for $d \geqq 3$. Thus, the interesting dimensions are 1 and 2. Our main result concerning this is the following:

Theorem 1.2. *Let $X_0 \subsetneqq \mathbb{R}^3_0$ be a real analytic germ. If X_0 has the $\mathcal{PE}$ property, then X_0 is (equivalent to) one among*:

Curve germs of $\mathbb{R}^3_0$ with the $\mathcal{PE}$ property	
(i)	$x = 0,\ y = 0$ (a line)
(ii)	$xy = 0,\ z = 0$ (two tranversal lines)
(iii)	$xy = 0,\ xz = 0,\ yz = 0$ (three independent lines)

Unmixed surface germs of $\mathbb{R}^3_0$ with the $\mathscr{PE}$ property	
(iv) $z = 0$ (plane)	
(v) $z^2 - x^3 - y^5 = 0$ (Brieskorn's singularity)	
(vi) $z^2 - x^3 - xy^3 = 0$	
(vii) $z^2 - x^3 - y^4 = 0$	
(viii) $z^2 - x^2 = 0$ (two transversal planes)	
(ix) $z^2 - x^2 - y^2 = 0$ (cone)	
(x) $z^2 - x^2 - y^k = 0$, $k \geqq 3$ (deformations of two planes)	
(xi) $z^2 - x^2y = 0$ (Whitney's umbrella)	(non-coherent)
(xii) $z^2 - x^2y + y^3 = 0$	
(xiii) $z^2 - x^2y - (-1)^k y^k = 0$, $k \geqq 4$ (deformations of Whitney's umbrella)	
Mixed surface germs of $\mathbb{R}^3_0$ with the $\mathscr{PE}$ property	
(xiv) $zx = 0$, $zy = 0$ (union of a plane and a transversal line)	

In what follows, this table of analytic germs will be called *the List*.

As we have pointed out above the analytic germs $X_0 \subset \mathbb{R}^3_0$ with $\mathscr{P} = \Sigma$ have been already characterized. More precisely:

(1) In [Sch], 3.9, the author determines the analytic curves germs $X_0 \subset \mathbb{R}^n$ for which $\mathscr{P} = \Sigma$ and proves that it is enough one square to represent a positive semidefinite analytic function on such an X_0. For $n = 3$, the curve germs with $\mathscr{P} = \Sigma$ are those in the List.

(2) In [Rz3], it is proved that if an unmixed analytic surface germ $X_0 \subset \mathbb{R}^3_0$ has $\mathscr{P}(X_0) = \Sigma_2(X_0)$, then it is (equivalent to) one of the germs in the List. Using this, in [Fe2] and [FR] it is shown that an unmixed analytic surface germ $X_0 \subset \mathbb{R}^3_0$ has $\mathscr{P} = \Sigma$, and in fact $\mathscr{P} = \Sigma_2$, if and only if X_0 belongs to the List.

(3) Finally, by [Fe4], 3.1, a mixed analytic surface germ with $\mathscr{P} = \Sigma$, and in fact with the $\mathscr{P} = \Sigma_2$ property, is the union of a plane and a tranversal line.

Putting all together we conclude the following:

Theorem 1.3. *Let $X_0 \subsetneqq \mathbb{R}^3_0$ be a real analytic germ. Then the following assertions are equivalent*:

(a) $\mathscr{P}(X_0) = \Sigma_2(X_0)$.

(b) $\mathscr{P}(X_0) = \Sigma(X_0)$.

(c) *X_0 has the $\mathscr{PE}$ property.*

(d) *X_0 belongs to the List.*

Global approach.

(1.4) Global analytic sets. Let $\Omega \subset \mathbb{R}^n$ be an open set and $\mathcal{O}_\Omega$ the (coherent) sheaf of analytic function germs on Ω. We denote by $\mathcal{O}(\Omega) = H^0(\Omega, \mathcal{O}_\Omega)$ the ring of global analytic functions on Ω. A set $X \subset \Omega$ is *global analytic*, if there exist global analytic functions $f_1, \dots, f_r : \Omega \to \mathbb{R}$ such that $X = \{f_1 = 0, \dots, f_r = 0\}$, or equivalently, if X is the zero set of a coherent sheaf of ideals on Ω ([Ca]). For such sets we consider the coherent sheaf of ideals $\mathcal{J}_X = \mathcal{J}(X)\mathcal{O}_\Omega$, generated by the ideal $\mathcal{J}(X) \subset \mathcal{O}(\Omega)$ of all global analytic functions on Ω vanishing on X. This sheaf $\mathcal{J}_X$ is the biggest coherent sheaf of ideals with zero set X (see [Ca]). But $\mathcal{J}_X$ may well be smaller than the sheaf of function germs vanishing on X. When both sheafs are equal, that is, $\mathcal{J}_{X,x} = \mathcal{J}(X_x)$ for all $x \in X$, the set X is called *coherent*.

In any case, $\mathcal{O}_X = \mathcal{O}_\Omega / \mathcal{J}_X$ is the sheaf of global analytic function germs on X and $\mathcal{O}(X) = H^0(X, \mathcal{O}_X) = \mathcal{O}(\Omega)/\mathcal{J}(X)$ is the ring of global analytic functions on X. A *positive semidefinite* (*global*) *analytic function* on X is an element $f \in \mathcal{O}(X)$ such that $f(x) \geqq 0$ for all $x \in X$. We denote by $\mathcal{P}(X)$ the set of all the analytic functions which are positive semidefinite on X and by $\Sigma(X)$ (resp. $\Sigma_p(X)$) the set of all sums of (resp. p) squares of the ring $\mathcal{O}(X)$. Similarly to the local case, we will say that X has $\mathcal{P} = \Sigma$ if the equality $\mathcal{P}(X) = \Sigma(X)$ holds. Moreover, X has the *Positive Extension* ($\mathcal{PE}$) *property*, if the following assertion holds true:

Global $\mathcal{PE}$ property. Every positive semidefinite analytic function f on X is the restriction to X of a positive semidefinite analytic function on Ω.

Again, if $\mathcal{P} = \Sigma$ for X, then X has the $\mathcal{PE}$ property and what matters is the converse. First of all, we prove:

Theorem 1.5. *Let X be a global analytic set in an open set $\Omega \subset \mathbb{R}^n$ with $\mathcal{P}(X) = \Sigma(X)$. Then*:

(a) *X has dimension $\leqq 2$ and it is coherent.*

(b) *The germs X_x have $\mathcal{P} = \Sigma$ for all $x \in X$.*

Once more, for dimension $d \geqq 3$ it is easy to find global analytic sets, even singular, with the $\mathcal{PE}$ property for which clearly $\mathcal{P} \neq \Sigma$. To progress further, recall that the *local embedding dimension of an analytic set X* is the number

$$\sup\{\operatorname{emb}\dim(X_x) : x \in X\}.$$

Then we will prove:

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let X be a global analytic set in Ω of dimension $\leqq 2$ and local embedding dimension $\leqq 3$. Then the following assertions are equivalent*:

(a) $\mathcal{P}(X) = \Sigma_6(X)$.

(b) $\mathcal{P}(X) = \Sigma(X)$.

(c) *X has the $\mathscr{P}\mathscr{E}$ property.*

(d) *X is coherent and all the germs X_x belong to the List.*

The previous result gives in fact a criterion to determine if a global analytic set X in an open set $\Omega \subset \mathbb{R}^3$ has $\mathscr{P} = \Sigma$ and/or the $\mathscr{P}\mathscr{E}$ property. One just checks that its singularities are in the List, Whitney's umbrella excepted (because it is non-coherent).

Actually, one can show (see Section 2) that for Whitney's umbrella the positive semidefinite analytic function $f(x, y, z) = x^2 - x + (y + 1)^2$ cannot be positively extended to $\mathbb{R}^3$. However, the $\mathscr{P} = \Sigma$ and $\mathscr{P}\mathscr{E}$ properties are *almost* local for a real coherent analytic set X of dimension $\leqq 2$ and local embedding dimension $\leqq 3$, that is, both properties are local for those analytic sets which do not have singularities equivalent to Whitney's umbrella.

The article is organized as follows. In Section 2 we get several local consequences of the $\mathscr{P} = \Sigma$ and/or the $\mathscr{P}\mathscr{E}$ property for the germs at the points of a real analytic set having such properties. In Sections 3 and 4 we respectively prove local results for dimensions 1 and 2 from which it follows 1.2. The next step is to study what happens with respect to both properties around the set of non-isolated singular points of an analytic set. This is approached in Section 5. The next section is devoted to prove 1.6. Finally, in Section 7, we formulate two conjectures (one local and the other one global) for analytic curves and propose some open questions referring the $\mathscr{P} = \Sigma$ and $\mathscr{P}\mathscr{E}$ properties.

2. Local consequences of the global properties

The purpose of this section is to show that if a global analytic set X has either $\mathscr{P} = \Sigma$ and/or the $\mathscr{P}\mathscr{E}$ property, then the germs at all its points have *almost* such properties. We begin with the $\mathscr{P} = \Sigma$ property whose behaviour is, as we have stated in 1.5, the expectable one.

(2.1) Local consequences of the global $\mathscr{P} = \Sigma$ property. Before proving 1.5 we need some preliminary results:

Lemma 2.2. *Let $X_0 \subset \mathbb{R}^n_0$ be an analytic germ of dimension $\geqq 1$. An analytic function germ $h \in \mathcal{O}(X_0)$ is positive semidefinite on X_0 if and only if for every half-branch curve germ $Y_0 \subset X_0$, with parametrization $\alpha : \{t > 0\} \to Y_0$, we have $h \circ \alpha \geqq 0$.*

Proof. Indeed, the only if condition is clearly true. To prove the converse we proceed as follows. Suppose that $h \notin \mathscr{P}(X_0)$. Then the germ $\{h < 0\} \cap X_0$ is non empty and open in X_0; hence, it has dimension $\geqq 1$. Thus, by the curve selection lema [ABR], VII.4, there exists a half-branch curve germ Y_0 through the origin such that $Y_0 \subset \{h < 0\} \cap X_0$, against our hypothesis. Therefore, $h \in \mathscr{P}(X_0)$, as wanted. □

Lemma 2.3. *Let X be an analytic set in an open set $\Omega \subset \mathbb{R}^n$. Fix a point $a \in X$ and let $f_a \in \mathcal{O}(\mathbb{R}^n_a)$ be an analytic function germ which is positive semidefinite on X_a. Then, for every* *integer $m \geqq 1$ there exists a polynomial $g_m \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ such that $g_m \in \mathscr{P}(X)$ and the order at a of the function germ $f_a - g_{m,a} \in \mathcal{O}(\mathbb{R}^n_a)$ is $\geqq m$.*

Proof. First, to simplify notations we assume $a = 0$. Fix an integer $m \geqq 1$. We set

$$\overline{g_m} = j^{2m}(f_0) + (x_1^2 + \cdots + x_n^2)^m$$

where $j^{2m}(f_0)$ is the jet of degree $2m$ of f_0. We claim that: $\overline{g_m} \in \mathscr{P}(X_0)$.

Indeed, let $\alpha : \{t > 0\} \to X_0$ be a half-branch curve germ and consider the analytic series

$$\overline{g_m} \circ \alpha = j^{2m}(f_0) \circ \alpha + (\alpha_1^2 + \cdots + \alpha_n^2)^m \in \mathbb{R}\{t\}.$$

We can write $f_0 = j^{2m}(f_0) + h_{2m+1}$ where $h_{2m+1} \in \mathbb{R}\{x\}$ is an analytic series of order $\geqq 2m + 1$. Thus,

$$f_0 \circ \alpha = j^{2m}(f_0) \circ \alpha + h_{2m+1} \circ \alpha$$

where $\omega(h_{2m+1} \circ \alpha) \geqq (2m+1)\omega(\|\alpha\|)$ and $\|\alpha\| = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2} \in \mathbb{R}\{t\}$. Recall that $\omega(.)$ gives the order of the involved series. Next, we distinguish two cases:

(a) $r = \omega(f_0 \circ \alpha) \leqq 2m\omega(\|\alpha\|)$. Since $f_0 \circ \alpha > 0$ for $t > 0$, we get that

$$f_0 \circ \alpha = a_r t^r + a_{r+1} t^{r+1} + \cdots, \quad a_r > 0.$$

Hence, $j^{2m}(f_0) \circ \alpha = f_0 \circ \alpha - h_{2m+1} \circ \alpha = a_r t^r + \cdots > 0$ because

$$\omega(h_{2m+1} \circ \alpha) \geqq (2m+1)\omega(\|\alpha\|) > 2m\omega(\|\alpha\|) \geqq r.$$

Therefore, we conclude that

$$\overline{g_m} \circ \alpha = j^{2m}(f_0) \circ \alpha + (\alpha_1^2 + \cdots + \alpha_n^2)^m > 0.$$

(b) $r = \omega\big(j^{2m}(f_0) \circ \alpha\big) = \omega(f_0 \circ \alpha) > 2m\omega(\|\alpha\|)$. Since

$$\omega\big((\alpha_1^2 + \cdots + \alpha_n^2)^m\big) = 2m\omega(\|\alpha\|),$$

we have that

$$\overline{g_m} \circ \alpha = j^{2m}(f_0) \circ \alpha + (\alpha_1^2 + \cdots + \alpha_n^2)^m > 0.$$

Thus, by 2.2, the polynomial germ g_m is positive semidefinite on X_0. Hence, there exists $\varepsilon > 0$ such that $\overline{g_m} \geqq 0$ on $X \cap B_\varepsilon(0)$. We write $\overline{g_m} = \sum_{|v| \leqq 2m} a_v x^v$. If $\|x\| \geqq \varepsilon$ we have $\frac{\|x\|}{\varepsilon} \geqq 1$ and therefore

$$
\begin{aligned}
|\overline{g_m}(x)| &= \Big|\sum_{|\nu|\leqq 2m} a_\nu x^\nu\Big| \leqq \sum_{|\nu|\leqq 2m} |a_\nu|\,|x^\nu| \\
&\leqq \sum_{|\nu|\leqq 2m} |a_\nu|\,\|x\|^{\nu} \leqq \sum_{|\nu|\leqq 2m} |a_\nu|\,\|x\|^{2m}\frac{1}{\varepsilon^{2m-|\nu|}} \\
&= \|x\|^{2m}\sum_{|\nu|\leqq 2m} |a_\nu|\frac{1}{\varepsilon^{2m-|\nu|}} = M_m\|x\|^{2m}
\end{aligned}
$$

for certain real number $M_m > 0$. Hence, the polynomial

$$g_m(x) = \overline{g_m}(x) + M_m(x_1^2 + \cdots + x_n^2)^m \in \mathbb{R}[x]$$

has the desired properties:

(i) $\omega(f_0 - g_m) = \omega\big(f_0 - j^{2m}(f_0) - (M_m + 1)(x_1^2 + \cdots + x_n^2)^m\big) \geqq 2m$, and

(ii) $g_m \in \mathscr{P}(X)$.

Indeed, if $p \in X \cap B_\varepsilon(0)$ it is clear that $g_m(p) \geqq 0$ and if $p \in \mathbb{R}^n \backslash B_\varepsilon(0)$ we conclude that $g_m(p) = \overline{g_m}(p) + M_m\|p\|^{2m} \geqq -|\overline{g_m}(p)| + M_m\|p\|^{2m} \geqq 0$, as wanted. $\square$

Lemma 2.4. *Let X be a global analytic set of dimension $\geqq 3$ in an open set $\Omega \subset \mathbb{R}^n$. Then $\mathscr{P}(X) \neq \Sigma(X)$.*

Proof. Indeed, let $a \in X$ be a non-singular point such that $\dim(X_a) \geqq 3$. To simplify the notation we suppose $a = 0$. By the Jacobian Criterion ([JP], 4.3.10) there exist analytic function germs $f_1, \ldots, f_{n-3} \in \mathbb{R}\{x\}$ such that $\mathscr{J}(X_0) = (f_1, \ldots, f_{n-3})$ and $\mathrm{rk}\left(\frac{\partial f_i}{\partial x_j}(0)\right) = n - 3$. Thus, after a linear change of coordinates (which is an analytic global change), we can assume that $f_i = x_i + g_i(x_{n-2}, x_{n-1}, x_n)$ where $g_i \in \mathfrak{m}_n^2 \cap \mathbb{R}\{x_{n-2}, x_{n-1}, x_n\}$, for $i = 1, \ldots, n-3$. Now, we choose a positive semidefinite homogeneous polynomial $h \in \mathbb{R}[x_{n-2}, x_{n-1}, x_n]$ which is not a sum of squares. We can take, for instance, Motzkin's polynomial

$$h(x_{n-2}, x_{n-1}, x_n) = x_{n-2}^6 + x_{n-1}^4 x_n^2 + x_{n-1}^2 x_n^4 - 3x_{n-2}^2 x_{n-1}^2 x_n^2$$

(see [BCR], 6.4.20). One can check that $h \in \mathscr{P}(X)$ but $h_0 \in \mathscr{P}(X_0)\backslash\Sigma(X_0)$. Hence, if $\mathscr{P}(X) = \Sigma(X)$ we would deduce that $h_0 \in \Sigma(X_0)$, a contradiction. Thus, we conclude that $\mathscr{P}(X) \neq \Sigma(X)$. $\square$

Now, we are ready to prove 1.5.

Proof of Theorem 1.5. First, by 2.4, we have $\dim X \leqq 2$. Next, we claim that: *If $a \in X$ and $f_a \in \mathcal{O}(X_a)$ is positive semidefinite on X_a, then f_a is a sum of squares in the ring $A = \mathcal{O}(\mathbb{R}_a^n)/\mathscr{J}(X)\mathcal{O}(\mathbb{R}_a^n)$.*

Indeed, we may assume, to simplify notation, that $a = 0$. Since $\mathscr{J}(X)\mathbb{R}\{x\}$ is a finitely generated ideal, there exist analytic functions $h_1, \ldots, h_r \in \mathscr{J}(X)$ which generate $\mathscr{J}(X)\mathbb{R}\{x\}$. By 2.3, for each integer $m \geqq 1$ there exists $g_m \in \mathbb{R}[x]$ such that $g_m \in \mathscr{P}(X)$ and

$g_m - f_0 \in \mathfrak{m}_n^m$, where $\mathfrak{m}_n$ denotes the maximal of $\mathbb{R}\{x\}$. Since $\mathscr{P}(X) = \Sigma(X)$, for each $m \geqq 0$ there exist $\alpha_{1m}, \dots, \alpha_{r_m m} \in \mathcal{O}(\Omega)$ such that

$$g_m = \alpha_{1m}^2 + \cdots + \alpha_{r_m m}^2 \mod \mathscr{J}(X).$$

Considering germs at the origin in the previous equality, we get that

$$g_{m,0} = \alpha_{1m,0}^2 + \cdots + \alpha_{r_m m,0}^2 \mod \mathscr{J}(X)\mathbb{R}\{x\}.$$

Since $X_0 = \mathscr{Z}\big(\mathscr{J}(X)\mathbb{R}\{x\}\big)$ and $\dim X_0 \leqq 2$, by [Fe1], 1.4, and [Fe3], 1.1, there exists an integer $p \geqq 1$ such that each sum of squares in $A = \mathbb{R}\{x\}/\mathscr{J}(X)\mathbb{R}\{x\}$ can be written as a sum of p squares in A. Hence, for every $m \geqq 1$ there exist analytic function germs $\beta_{1m,0}, \dots, \beta_{pm,0}, \lambda_{1m,0}, \dots, \lambda_{rm,0} \in \mathbb{R}\{x\}$ such that

$$g_{m,0} = \beta_{1m,0}^2 + \cdots + \beta_{pm,0}^2 + \lambda_{1m,0}h_1 + \cdots + \lambda_{rm,0}h_r.$$

That is, the equation

$$f_0 = Y_1^2 + \cdots + Y_p^2 + Z_1 h_1 + \cdots + Z_\ell h_\ell$$

has a solution $\operatorname{mod} \mathfrak{m}_n^m$ for all $m \geqq 1$. By M. Artin's Approximation Theorem ([Ar], [Ku et al.]), we conclude that f_0 is a sum of squares in A.

In particular, we have $\mathscr{P}(X_a) = \Sigma(X_a)$ for every $a \in X$, that is, the statement (b) holds. To end, it remains to check that X is coherent.

Indeed, suppose that X is not coherent. Then, there exists a point $a \in X$, which may be assumed to be the origin, and an analytic function germ $h_0 \in \mathscr{J}(X_0)\backslash\mathscr{J}(X)\mathbb{R}\{x\}$. Next, we will show that $h_0 \in \bigcap_{k \in \mathbb{N}} \mathfrak{m}_A^k = \{0\}$ where $\mathfrak{m}_A$ is the maximal ideal of A, against the condition $h_0 \in \mathscr{J}(X_0)\backslash\mathscr{J}(X)\mathbb{R}\{x\}$.

Since $h_0 \in \mathscr{P}(X_0)$, by the previous claim, $h_0 = h_{1,0}^2 + \cdots + h_{s,0}^2$ in A. Thus, $h_{i,0} \in \mathfrak{m}_A$ and so $h_0 \in \mathfrak{m}_A^2$. Furthermore, since $h_0 \in \mathscr{J}(X_0)$ which is a real radical ideal, the function germ $h_{i,0} \in \mathscr{J}(X_0)$, hence $h_{i,0} \in \mathscr{P}(X_0)$. Again, $h_{i,0} = h_{i1,0}^2 + \cdots + h_{ir_i,0}^2$ in A where $h_{ij,0} \in \mathfrak{m}_A$ and $h_{ij,0} \in \mathscr{J}(X_0)$, thus $h \in \mathfrak{m}_A^4$. Repeating this, we conclude that $h_0 \in \bigcap_{k \in \mathbb{N}} \mathfrak{m}_A^k$. □

Next, we proceed with the $\mathscr{PE}$ property. Before that we need to introduce an additional property for analytic germs. We say that an analytic germ $X_0 \subset \mathbb{R}_0^n$ has the $\mathscr{PE}^+$ *property* if every analytic function germ which is strictly positive on $X_0\backslash\{0\}$ has a positive semidefinite analytic extension to $\mathbb{R}_0^n$. Clearly, an analytic germ which has the $\mathscr{PE}$ property also has the $\mathscr{PE}^+$ property.

(2.5) Local consequences of the global $\mathscr{PE}$ property. *Let X be a global analytic set in an open set $\Omega \subset \mathbb{R}^n$. If X has the $\mathscr{PE}$ property, then the analytic germs X_x have the $\mathscr{PE}^+$ property for all $x \in X$.*

Before proving this, we would like to justify the introduction of the $\mathscr{PE}^+$ property. The kind of statement one expects to have is the following:

$(\star)$ $\mathscr{PE}$ *for* X $\Rightarrow$ $\mathscr{PE}$ *for all the germs* X_x.

The natural strategy to prove $(\star)$ should be the following. Let $x \in X$ and assume that x is a singular point of X. If the germ X_x is not singular, it trivially has the $\mathscr{PE}$ property. Next, take a positive semidefinite analytic function germ f_x on X_x which vanishes at x; otherwise there is nothing to prove. Since X has the $\mathscr{PE}$ property, we should extend f_x to a positive semidefinite analytic function on X. In general a representant of f_x cannot be extended even if X is a curve. For instance,

Example 2.6. Let $X : x^2 - y^2 - y^3 = 0$ which is an analytic curve in $\mathbb{R}^2$. The analytic function germ $f_0 = y(x + y\sqrt{1+y})$ is positive semidefinite on X_0. However, it cannot be extended analytically to X because f_0 is identically 0 on one of the branches of X_0 and these branches form part of a *loop* of X.

Thus, we should extend positively to X a suitable *modification* g_x of one of the representatives of f_x. Clearly, the zero set of g_x must be the germ at x of a global analytic subset of X, and the most natural choice is to ask that $\{g_x = 0\} = \{x\}$. Next, using that X has the $\mathscr{PE}$ property we conclude that g_x can be extended to a positive semidefinite analytic function germ on $\mathbb{R}^n_x$.

Now, we would like to use this to prove that f_x can also be extended positively to $\mathbb{R}^n_x$. However, if f_x and g_x do not generate the same ideal of $\mathcal{O}(X_x)$, it seems a difficult matter to determine if f_x can be extended positively to $\mathbb{R}^n_x$. This is essentially because two extensions $\hat{f}_{1,x}$ and $\hat{f}_{2,x}$ to $\mathbb{R}^n_x$ of two positive semidefinite analytic germs $f_{1,x}$ and $f_{2,x}$ on X_x have no relation, even if $f_{1,x} - f_{2,x} \in \mathfrak{m}_x^r$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}(X_x)$ and r is a large integer. Note that if $\hat{f}_x$ is a positive semidefinite analytic extension to $\mathbb{R}^n_x$ of a positive semidefinite analytic germ f_x on X_x and $g_{1,x}, \dots, g_{s,x} \in \mathscr{J}(X_x)$, then $\hat{f}_x + g_{1,x}^2 + \cdots + g_{s,x}^2$ is also a positive semidefinite analytic extension of f_x to $\mathbb{R}^n_x$.

After all these considerations, we prove 2.5:

Proof of 2.5. Let $a \in X$ be a point and $f_a \in \mathcal{O}(X_a)$ be an analytic function germ such that f_a is strictly positive on $X_a \setminus \{a\}$. To simplify the notation we assume that $a = 0$. Choose a representative of f_0 in $\mathbb{R}\{x\}$ and denote it again by f_0. Since f_0 is strictly positive on $X_0 \setminus \{0\}$ there exist analytic functions $h_{1,0}, \dots, h_{r,0} \in \mathscr{J}(X_0)$ such that $\{f_0 = 0, h_{1,0} = 0, \dots, h_{r,0} = 0\} = \{0\}$. Hence, if

$$\eta_0 = f_0^2 + h_{1,0}^2 + \cdots + h_{r,0}^2 \in \mathbb{R}\{x\}$$

we have $\{\eta_0 = 0\} = \{0\}$. By Łojasiewicz's inequality ([To], V.4), there exists an integer $m \geqq 1$ such that

$$\eta_0(x) > (x_1^2 + \cdots + x_n^2)^m$$

on $\mathbb{R}^n_0 \setminus \{0\}$. By 2.3, there exists $g \in \mathbb{R}[x]$ such that $g \in \mathscr{P}(X)$ and $\omega(g_0 - f_0) \geqq 2m + 2$. Since $g \in \mathscr{P}(X)$ and X has the $\mathscr{PE}$ property, there exists a positive semidefinite analytic function $\hat{g} : \Omega \to \mathbb{R}$ such that $\hat{g}|_X = g$. In particular, there exists an analytic function germ $h_0 \in \mathscr{J}(X_0)$ such that $\hat{g}_0 = g_0 + h_0$. Let us see that $F_0 = f_0 + h_0 + \eta_0$ is a positive semidefinite analytic function germ on $\mathbb{R}^n_0$. For that, by 2.2, it is enough to check that if $\alpha : \mathbb{R}_0 \to \mathbb{R}^n_0$ is a parametrization then $F_0 \circ \alpha \geqq 0$.

Indeed, we distinguish two cases:

(1) $\omega\big((f_0+h_0)\circ\alpha\big) \geqq (2m+2)\omega(\|\alpha\|)$ where $\|\alpha\| = \sqrt{\alpha_1^2+\cdots+\alpha_n^2} \in \mathbb{R}\{t\}$. Then

$$F_0\circ\alpha = (f_0+h_0)\circ\alpha + \eta_0\circ\alpha \geqq (f_0+h_0)\circ\alpha + \|\alpha\|^{2m} > 0.$$

(2) $\omega\big((f_0+h_0)\circ\alpha\big) < (2m+2)\omega(\|\alpha\|)$. Since $\omega(g_0-f_0) \geqq 2m+2$, there exists $\zeta_0 \in \mathfrak{m}_n^{2m+2}$ such that $f_0 = g_0+\zeta_0$. Hence,

$$(f_0+h_0)\circ\alpha = \zeta_0\circ\alpha + (g_0+h_0)\circ\alpha = \zeta_0\circ\alpha + \hat{g}_0\circ\alpha > 0,$$

because $\hat{g}_0\circ\alpha \geqq 0$, $\omega(\zeta_0\circ\alpha) \geqq (2m+2)\omega(\|\alpha\|)$ and $\omega\big((f_0+h_0)\circ\alpha\big) < (2m+2)\omega(\|\alpha\|)$. Thus,

$$F_0\circ\alpha = (f_0+h_0)\circ\alpha + \eta_0\circ\alpha > 0.$$

Therefore, we deduce that F_0 is positive semidefinite on $\mathbb{R}^n_0$. Finally, note that $F_0|_{X_0} = f_0+f_0^2 = f_0(1+f_0)$. Hence,

$$\hat{f}_0 = \frac{F_0}{1+f_0} \in \mathcal{O}(\mathbb{R}^n_0)$$

is a positive semidefinite analytic function germ such that $\hat{f}_0|_{X_0} = f_0$. Thus, X_a has the $\mathscr{PE}^+$ property for all $a \in X$. $\square$

(2.7) Examples of non coherent surfaces. We finish this section with several examples of non coherent global analytic sets which do not have any of the two properties we are studying, but whose germs at all their points have both properties. We begin with Whitney's umbrella.

Examples 2.8. (a) Let $X: z^2 - x^2y = 0$ be Whitney's umbrella. By 1.5, we have $\mathscr{P}(X) \neq \Sigma(X)$, since X is not coherent. Moreover, for each point $x \in X$ we have $\mathscr{P}(X_x) = \Sigma(X_x)$; because for each $x \in X$ the germ X_x is equivalent to one of the following analytic germs of $\mathbb{R}^3$ at the origin:

(i) $z = 0$,

(ii) $x = 0,\ y = 0$,

(iii) $z^2 - x^2 = 0$,

(iv) $z^2 - x^2y = 0$.

Recall that all of them appear in the List and therefore, have $\mathscr{P} = \Sigma$. Hence, the $\mathscr{PE}$ property also holds for X_x for all $x \in X$.

Next, let us see that X does not have the $\mathscr{PE}$ property. In fact, we see that for every $\varepsilon > 0$ the analytic surface $Y = X \cap B_\varepsilon(0)$ does not have the $\mathscr{PE}$ property.

Indeed, consider the analytic function

$$f:\mathbb{R}^3\to\mathbb{R},\quad (x,y,z)\mapsto\left(x-\frac{\varepsilon}{4}\right)^2+\left(y+\frac{\varepsilon}{2}\right)^2-\frac{\varepsilon^2}{16}=x^2-\frac{\varepsilon}{2}x+\left(y+\frac{\varepsilon}{2}\right)^2$$

which is $\geqq 0$ on Y. Moreover, $\mathscr{J}(Y)$ is generated by the function z^2-x^2y which has order 2 at the point $p_0=(0,-\varepsilon/2,0)$ while f has order 1 at such point. Thus, we conclude that f cannot be extended to a positive semidefinite analytic function on $B_\varepsilon(0)$ because if such extension existed, it would have even order at all its points, which is impossible for any analytic extension of f.

(b) In fact, proceeding analogously, one can check that if S is an analytic surface (in an open set $\Omega\subset\mathbb{R}^n$) which has a singularity equivalent to Whitney's umbrella, then neither $\mathscr{P}=\Sigma$ nor $\mathscr{PE}$ hold for S. $\square$

Moreover, for each embedding dimension there exist non coherent analytic surface germs with the $\mathscr{P}=\Sigma$ and $\mathscr{PE}$ properties. However, any of its representatives have none of them. For that, we recall certain examples already introduced in [Fe4].

Examples 2.9. The *generalized Whitney's umbrellas* $Y_{n,0}\subset\mathbb{R}^{n+1}$, $n\geqq 2$, are the analytic closures of the set germs parametrized by

$$\varphi_n:(s,t)\mapsto(s,st,\dots,st^{n-1},t^n)=(x_0,x_1,\dots,x_{n-1},x_n).$$

It can be checked that the ideal of $Y_{n,0}$ is generated by the polynomials

$$x_ix_j-x_0x_\ell x_n^q:\quad i+j=qn+\ell\quad\text{and}\quad\begin{cases}1\leqq i\leqq j\leqq n-1,\\ 0\leqq\ell\leqq n-1.\end{cases}$$

Moreover, $Y_{n,0}$ consists of the union of the image of φ_n and the x_n-axis. Hence, $Y_{n,0}$ is a non coherent germ for all $n\geqq 2$ (see [N], §V. Prop. 7). We find that the multiplicity of $Y_{n,0}$ is n and its embedding dimension $n+1$.

These analytic surface germs have $\mathscr{P}=\Sigma$ ([Fe4], 4.4), hence the $\mathscr{PE}$ property. The first umbrella $Y_2\subset\mathbb{R}^3$ is the classical Whitney umbrella $x_1^2=x_0^2x_2$.

Again we have, proceeding similarly to example 2.8 (a), that each representative of the germ $Y_{n,0}$ does not have the $\mathscr{PE}$ property for all $n\geqq 2$. $\square$

3. Local results for dimension one

In this section we study both properties for analytic curve germs. In [Sch], 3.9, the author characterizes the analytic curve germs in $\mathbb{R}^n_0$ with $\mathscr{P}=\Sigma$, which are those equivalent to a union of independent lines through the origin. As we will see along this section the approach to the $\mathscr{PE}$ property is quite more delicate. Our main result here is the following:

Theorem 3.1. *Let $X_0\subset\mathbb{R}^3_0$ be an analytic curve germ. The following assertions are equivalent*:

(a) $\mathscr{P}(X_0) = \Sigma_1(X_0)$.

(b) $\mathscr{P}(X_0) = \Sigma(X_0)$.

(c) *X_0 has the $\mathscr{P}\mathscr{E}$ property.*

(d) *X_0 has the $\mathscr{P}\mathscr{E}^+$ property.*

(e) *X_0 is equivalent to a union of independent lines in $\mathbb{R}^3_0$.*

The key result (3.9) to prove 3.1, that will be introduced later, implies the following one for arbitrary embedding dimension:

Theorem 3.2. *Let $X_0 \subset \mathbb{R}^n_0$ be an analytic curve germ with the $\mathscr{P}\mathscr{E}^+$ property and embedding dimension n. Then X_0 is either regular or a reducible curve germ whose tangent cone is the union of $r \leqq n$ independent lines. Moreover, if $r = n$ then X_0 is equivalent to the union of n independent lines through the origin and we may assume that $\mathscr{J}(X_0) = \{x_i x_j : 1 \leqq i < j \leqq n\}$.*

We need to introduce here several preliminary results.

Lemma 3.3. *Let $f \in \mathbb{R}\{x\} = \mathbb{R}\{x_1, \dots, x_n\}$ be an analytic series of order $s \geqq 1$. Then, there exists $M > 0$ such that $|f| < M\|x\|^s$.*

Proof. Indeed, since $\omega(f^2) = 2s$ we can write $f^2 = \sum_{|\nu|=2s} a_\nu(x) x^\nu$ for some analytic series $a_\nu \in \mathbb{R}\{x\}$, and so, near the origin we have

$$|f(x)|^2 \leqq \sum_{|\nu|=2s} c_\nu |x|^\nu = \sum_{|\nu|=2s} c_\nu \|x\|^{2s} |y|^\nu = \|x\|^{2s} \sum_{|\nu|=2s} c_\nu |y|^\nu$$

where $c_\nu = 1 + |a_\nu(0)|$ and $y = x/\|x\|$. The function $\sum_{|\nu|=2r} c_\nu |y|^\nu$ is bounded on the compact set $\|y\| = 1$, say by $M > 0$, and we conclude $|f|^2 < M\|x\|^{2s}$, hence $|f| < M\|x\|^s$, as wanted. □

Lemma 3.4. *Let $X_0 \subset \mathbb{R}^n_0$ be an analytic germ of the dimension d. Then, after a linear change of coordinates, the analytic function germs $g_{k,i}(x) = k(x_1^2 + \cdots + x_d^2) - x_i^2$ are positive semidefinite on X_0 for $i = d+1, \dots, n$ and k large enough.*

Proof. First, by Rückert's parametrization (see [Rz2], 3.4) we may assume, after a linear change of coordinates, that there exist Weierstrass polynomials $P_{d+1}, \dots, P_n \in \mathbb{R}\{x_1, \dots, x_d\}[T] = \mathbb{R}\{x'\}[T]$ such that $P_i(x_i) \in \mathscr{J}(X_0)$. Recall that a polynomial $F \in \mathbb{R}\{x'\}[T]$ is a *Weierstrass polynomial* if it is monic and its degree with respect to the variable T is equal to its order as a series.

Fix $i = d+1, \dots, n$ and let $r_i > 0$ be the degree of the polynomial P_i. We write

$$P_i(x_i) = x_i^{r_i} + a_{i,r_i-1} x_i^{r_i-1} + \cdots + a_{i1} x_i + a_{i0},$$

where each $a_{ij} \in \mathbb{R}\{x'\} = \mathbb{R}\{x_1, \dots, x_d\}$ and $\omega(a_{ij}) \geqq r_i - j$ for $0 \leqq j \leqq r_i$. By 3.3, there is $M > 0$ such that $|a_{ij}| < M\|x'\|^{r_i - j}$ for all i, j.

Now, for each integer $k \geqq 1$ we consider the quadratic form

$$g_{k,i} = k^2(x_1^2 + \cdots + x_d^2) - x_i^2 = k^2\|x'\|^2 - x_i^2,$$

and will prove that for k large enough the series $g_{k,i}$ is positive semidefinite on X_0.

In fact, otherwise, X_0 would contain a sequence $x^{(k)} = (x'^{(k)}, z^{(k)}) \to 0$ such that $g_k(x^{(k)}) < 0$, that is,

$$0 \leqq k\rho_k < |x_i^{(k)}|, \quad \text{where } \rho_k = \|x'^{(k)}\|.$$

Since $P_i \in \mathscr{J}(X_0)$, we have $P_i(x^{(k)}) = 0$, and consequently

$$\begin{aligned} |x_i^{(k)}|^{r_i} &= \left|\sum_{j=0}^{r_i-1} a_{ij}(x'^{(k)})(x_i^{(k)})^j\right| \leqq \sum_{j=0}^{r_i-1} |a_{ij}(x'^{(k)})|\,|x_i^{(k)}|^j \\ &\leqq M\sum_{j=0}^{r_i-1} \rho_k^{r_i-j}|x_i^{(k)}|^j < M\sum_{j=0}^{r_i-1}\frac{|x_i^{(k)}|^{r_i}}{k^{r_i-j}} = M|x_i^{(k)}|^{r_i}\sum_{j=0}^{r_i-1}\frac{1}{k^{r_i-j}}. \end{aligned}$$

But $|x_i^{(k)}|^{r_i} > 0$, and we get $1 < M\left(\frac{1}{k} + \cdots + \frac{1}{k^{r_i}}\right)$, a contradiction. Thus, $g_{i,k}$ is positive semidefinite on X_0 for k large enough, as wanted. □

Remark 3.5. In particular, if $X_0 \subset \mathbb{R}_0^n$ is an analytic germ with the $\mathscr{PE}^+$ property, then the analytic function germ $h = k(x_1^2 + \cdots + x_d^2) - x_n^2 + (x_1^2 + \cdots + x_n^2)^2$, is strictly positive on $X_0\backslash\{0\}$ for k large enough. Thus, we deduce, that

$$\omega\big(\mathscr{J}(X_0)\big) = \min\{\omega(f) : f \in \mathscr{J}(X_0)\} \leqq 2.$$

Otherwise, $\omega\big(\mathscr{J}(X_0)\big) \geqq 3$ and there exists an analytic function germ $f \in \mathscr{J}(X_0)$ such that $h + f$ is positive semidefinite in $\mathbb{R}_0^n$ and $\omega(f) \geqq 3$. In particular, its initial form $\mathrm{In}(h + f) = \mathrm{In}(h) = k(x_1^2 + \cdots + x_d^2) - x_n^2$ is a positive semidefinite quadratic form, a contradiction. □

In what follows, we focus our attention on curve germs. Let $X_0 \subset \mathbb{R}_0^n$ be an irreducible curve germ. Recall that a parametrization $\varphi : \mathbb{R}_0 \to X_0$ of X_0 is *primitive* if there do not exist another parametrization $\psi : \mathbb{R}_0 \to X_0$ of X_0 and an integer $p \geqq 2$ such that $\varphi(t) = \psi(t^p)$. If $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{R}_0 \to X_0$ is primitive, we define the *multiplicity* $m(X_0)$ of X_0 by

$$m = m(X_0) = \min\{\omega(\varphi_i) : i = 1, \ldots, n\}.$$

The *tangent line to* X_0 is the straight line parametrized by $t \mapsto tv$, where $v = \psi(0)$ and $\psi = \varphi/t^m$. A rutinary checking shows that the multiplicity of X_0 and its tangent line do not depend on the chosen primitive parametrization of X_0.

If X_0 is an analytic curve germ we define its *tangent cone* as the union of the tangent lines to all the irreducible components of X_0.

Lemma 3.6. *Let X_0 be an analytic curve germ. If $f \in \mathscr{J}(X_0)\setminus\{0\}$ then the initial form* $\mathrm{In}(f)$ *of f is identically* 0 *on the tangent cone to X_0.*

Proof. First, we write $X_0 = X_{1,0} \cup \cdots \cup X_{r,0}$ as the union of its irreducible components.

Fix $i = 1, \ldots, r$ and let $\alpha_i : \mathbb{R}_0 \to X_{i,0}$ be a primitive parametrization of $X_{i,0}$. Let m_i be the multiplicity of $X_{i,0}$, $\beta_i = \alpha_i / t^{m_i}$ and $v_i = \beta_i(0) \in \mathbb{R}^n$. Recall that v_i generates the tangent line to $X_{i,0}$. Let us check that $\mathrm{In}(f)(v_i) = 0$.

We write $f = \sum_{k \geqq k_0} f_k$, where each f_k is either 0 or a homogeneous polynomial of degree k and $f_{k_0} = \mathrm{In}(f)$. Since $f \circ \alpha_i = 0$, we get

$$0 = f \circ \alpha_i = f(t^{m_i}\beta_i) = \sum_{k \geqq k_0} f_k(t^{m_i}\beta_i) = \sum_{k \geqq k_0} t^{km_i} f_k(v_i + \gamma_i)$$

where $\gamma_i \in (t)\mathbb{R}\{t\}^n$. In particular, this shows that $\mathrm{In}(f)(v_i) = f_{k_0}(v_i) = 0$. Therefore $\mathrm{In}(f)$ vanishes on the tangent cone to X_0, as wanted. $\square$

Next, we see a method to construct analytic function germs on X_0 which are strictly positive on $X_0\setminus\{0\}$. In fact, we denote by $\mathscr{P}^+(X_0) \subset \mathscr{O}(X_0)$ the set of all the strictly positive function germs on $X_0\setminus\{0\}$.

Lemma 3.7. *Let X_0 be an analytic curve germ and let $v_1, \ldots, v_r \in \mathbb{R}^n$ be generators of the tangent lines to the irreducible components $X_{1,0}, \ldots, X_{r,0}$ of X_0. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of even degree such that $f(v_i) > 0$ for $i = 1, \ldots, n$. Then, $f \in \mathscr{P}^+(X_0)$.*

Proof. First, note that $\mathscr{P}^+(X_0) = \bigcap_{i=1}^{r} \mathscr{P}^+(X_{i,0})$. Thus, it is enough to check that $f \in \mathscr{P}^+(X_{i,0})$ for $i = 1, \ldots, r$. Note that if $\alpha_i : \mathbb{R}_0 \to X_{i,0}$ is a primitive parametrization of $X_{i,0}$, then $f \in \mathscr{P}(X_{i,0})^+$ if and only if $f \circ \alpha_i \in \mathbb{R}\{t\}\setminus\{0\}$ is a positive semidefinite series, that is, $f \circ \alpha_i(t) = a_{2r}t^{2r} + \cdots$, for some $a_{2r} > 0$.

Next, we write $m_i = m(X_{i,0})$ and $\beta_i = \alpha_i / t^{m_i}$. Since f is homogeneous of even degree, say 2ℓ, we get

$$f \circ \alpha_i(t) = t^{2\ell m_i} f \circ \beta_i = t^{2\ell m_i}\big(f(v_i) + t\xi_i(t)\big)$$

for certain analytic series $\xi_i \in \mathbb{R}\{t\}$. Since $f(v_i) > 0$, we conclude that $f \circ \alpha_i \in \mathbb{R}\{t\}\setminus\{0\}$ is positive semidefinite, and we are done. $\square$

Proposition 3.8. *Let $X_0 \subset \mathbb{R}^n$ be an irreducible curve germ with the $\mathscr{P}\mathscr{E}^+$ property. Then X_0 is a regular curve germ.*

Proof. If X_0 is regular, there is nothing to prove. Thus, we may assume that X_0 is singular and that its embedding dimension is n. Thus, in particular $\omega\big(\mathscr{J}(X_0)\big) \geqq 2$. After a change of coordinates we may assume that the tangent line to X_0 is $L : x_2 = 0, \ldots, x_n = 0$. Let α be a primitive parametrization of X_0. After a new change of coordinates that keeps invariant the line L, we may assume that

$$\alpha(t) = (t^{k_1}u_1, t^{k_2}u_2, \dots, t^{k_n}u_n),$$

where $u_1, \dots, u_n \in \mathbb{R}\{t\}$ are units and $2 \leqq k_1 < k_2 < \cdots < k_n$ are positive integers.

If some k_i is even, then $x_i u_i(0)$ is strictly positive on $X_0 \setminus \{0\}$ but cannot be extended positively to $\mathbb{R}^n_0$. Thus, we may assume that every k_i is odd. Then the function germ $f = x_1 x_2 u_1(0) u_2(0)$ is strictly positive on $X_0 \setminus \{0\}$. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property, there exists an analytic function germ $g \in \mathscr{J}(X_0)$ such that $f + g$ is positive semidefinite on $\mathbb{R}^n_0$. In particular, its initial form $\mathrm{In}(f + g)$ is positive semidefinite on $\mathbb{R}^n$. We have two possibilities:

(i) $\mathrm{In}(g)(x_1, x_2, 0, \dots, 0) = -x_1 x_2 u_1(0) u_2(0)$. We claim that $k_1 + k_2 \geqq 3k_1$.

Indeed, $g = \mathrm{In}(g) + h$, where $h \in \mathbb{R}\{x\} = \mathbb{R}\{x_1, \dots, x_n\}$ is an analytic function germ of order $\geqq 3$. Since $k_1 < k_2 < \cdots < k_n$ and $\mathrm{In}(g)(x_1, x_2, 0, \dots, 0) = -x_1 x_2 u_1(0) u_2(0)$, we have that $\omega\big(\mathrm{In}(g) \circ \alpha\big) = k_1 + k_2$. On the other hand,

$$\omega(h \circ \alpha) \geqq \omega(h) \cdot \min\{\omega(\alpha_i), i = 1, \dots, n\} \geqq 3k_1.$$

Therefore, since $g \circ \alpha \equiv 0$, we deduce that $k_1 + k_2 = \omega\big(\mathrm{In}(g) \circ \alpha\big) = \omega(h \circ \alpha) \geqq 3k_1$.

Thus, since $k_1 + k_2 \geqq 3k_1$, we have $k_2 \geqq 2k_1$ and for $M > 0$ large enough we deduce that $f_1 = x_2 + Mx_1^2 \in \mathscr{P}^+(X_0)$ is strictly positive on $X_0 \setminus \{0\}$, but cannot be extended positively to $\mathbb{R}^n_0$, a contradiction.

(ii) $\mathrm{In}(g)(x_1, 0, \dots, 0) = \alpha^2 x_1^2$ for some $\alpha > 0$. If this is the case, g cannot vanish on X_0, a contradiction.

Thus, we conclude that X_0 is a regular curve germ. $\square$

Now, we are to prove the technical result announced at the beginning of the section, which summarizes all the information we know about a curve germ $X_0 \subset \mathbb{R}^n_0$ with the $\mathscr{P}\mathscr{E}^+$ property. The full statement of this result, and not only 3.2, will be crucial to prove 3.1.

Theorem 3.9. *Let $X_0 \subset \mathbb{R}^n_0$ be an analytic curve germ with the $\mathscr{P}\mathscr{E}^+$ property and embedding dimension n. Then*:

(a) *There exists a quadratic form $q \in \mathscr{J}(X_0)$ of rank n and signature $n - 1$, that is, q is equivalent to $x_1^2 + \cdots + x_{n-1}^2 - x_n^2$.*

(b) *The tangent cone to X_0 is the union of $s \leqq n$ independent lines $L_1, \dots, L_s$. Moreover, after a change of coordinates, we may assume that L_i is generated by the vector $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ whose ith coordinate is 1 and all the others are 0. Then, for each $1 \leqq i < j \leqq s$ there exists an analytic series $\phi_{ij} \in \mathscr{J}(X_0)$ whose initial form is $\mathrm{In}(\phi_{ij}) = x_i x_j + \Phi_{ij}$ where Φ_{ij} is a quadratic form identically 0 on the set $\{x_{s+1} = 0, \dots, x_n = 0\}$ (which is the linear subspace of $\mathbb{R}^n$ generated by the lines $L_1, \dots, L_s$).*

(c) *If $s = n$, X_0 is equivalent to the union of n independent lines and we may assume that $\mathscr{J}(X_0) = \{x_i x_j : 1 \leqq i < j \leqq n\}$.*

(d) *If $s \leqq n-1$ we have the following extra information*:

(i) *If the initial form $q = \mathrm{In}(f)$ of a series $f \in \mathscr{J}(X_0)$ is a positive semidefinite quadratic form q, then it is identically* 0 *on the vectorial subspace of $\mathbb{R}^n$ generated by the tangent cone to X_0 (hence, q has rank $\leqq n-s$) and $g = f - q$ is an analytic series of order* 3.

(ii) *There exists an analytic series $f \in \mathscr{J}(X_0)$ whose initial form is a positive semidefinite quadratic form.*

The general strategy to prove 3.9 consists, roughly speaking, of finding for an X_0 not satisfying any of such conditions a positive semidefinite analytic function germ $f \in \mathcal{O}(X_0)$ of order 1. Obviously such an f cannot be extended positively to $\mathbb{R}^n_0$ and X_0 cannot have the $\mathscr{P}\mathscr{E}^+$ property.

Proof of Theorem 3.9. First, we write $X_0 = X_{1,0} \cup \cdots \cup X_{r,0}$ as the union of its irreducible components. For each $f \in \mathbb{R}\{x\}$ of order $\geqq 2$, let $\mathrm{In}(f)$ stand for the initial form of f and

$$\mathrm{q}(f) = \begin{cases} \mathrm{In}(f) & \text{if } \omega(f) = 2, \\ 0 & \text{if } \omega(f) \geqq 3. \end{cases}$$

The proof runs in several steps:

Step 1. *After a change of coordinates, we may assume that*

$$x_1^2 + \cdots + x_{n-1}^2 - x_n^2 \in \mathscr{J}(X_0).$$

Indeed, by 3.4, we may assume that the function germs $kx_n^2 - x_i^2 \in \mathscr{P}(X_0)$ for $k > 0$ large enough and $i = 1, \ldots, n-1$. Thus,

$$(n-1)kx_n^2 - (x_1^2 + \cdots + x_{n-1}^2) = \sum_{i=1}^{n-1} (kx_n^2 - x_i^2) \in \mathscr{P}(X_0)$$

for k large enough. Hence,

$$f = mx_n^2 - (x_1^2 + \cdots + x_{n-1}^2) + (x_1^2 + \cdots + x_n^2)^2 \quad \text{where } m = (n-1)k,$$

is strictly positive on $X_0 \backslash \{0\}$. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property there exists $g \in \mathscr{J}(X_0)$ such that $f + g$ is positive semidefinite on $\mathbb{R}^n_0$. In particular, if $q_0 = \mathrm{q}(g)$, then the quadratic form

$$mx_n^2 - (x_1^2 + \cdots + x_{n-1}^2) + q_0$$

is positive semidefinite on $\mathbb{R}^n$. Thus,

$$q_1(x_1, \ldots, x_{n-1}) = -(x_1^2 + \cdots + x_{n-1}^2) + q_0(x_1, \ldots, x_{n-1}, 0)$$

is positive semidefinite on $\mathbb{R}^{n-1}$. By the spectral theorem, there exists an orthogonal basis for the usual scalar product of $\mathbb{R}^{n-1}$ that diagonalizes $q_0(x_1, \ldots, x_{n-1}, 0)$. Thus, after a suitable linear change of coordinates, we may assume that

$$q_1(x_1,\dots,x_{n-1}) = -(x_1^2+\cdots+x_{n-1}^2)+(a_1x_1^2+\cdots+a_{n-1}x_{n-1}^2),$$

for certain real numbers $a_1,\dots,a_{n-1} \geqq 1$. Since X_0 has embedding dimension n, we have $\mathscr{J}(X_0) \subset \mathfrak{m}_n^2$. Hence, by classification of singularities (see [JP], 9.2.12), we may assume, after a new change of coordinates, that

$$g = x_1^2+\cdots+x_{n-1}^2 - x_n^k \in \mathscr{J}(X_0),$$

for some $k \geqq 2$; recall that X_0 has dimension 1. Now, if $k = 2\ell+1$ is odd, then

$$x_1^2+\cdots+x_{n-1}^2 = x_n^{2\ell+1}.$$

Hence, $x_n \in \mathscr{P}(X_0)$ and $x_n + (x_1^2+\cdots+x_n^2)$ is strictly positive on $X_0\backslash\{0\}$ but it cannot be extended positively to $\mathbb{R}_0^n$, a contradiction.

On the other hand, if $k = 2\ell \geqq 4$ is even, then

$$|x_1| \leqq \sqrt{x_1^2+\cdots+x_{n-1}^2} = |x_n|^\ell \leqq x_n^2.$$

Hence, $x_1 + x_n^2 \in \mathscr{P}(X_0)$ and $x_1 + x_n^2 + (x_1^2+\cdots+x_n^2)$ is strictly positive on $X_0\backslash\{0\}$ but it cannot be extended positively to $\mathbb{R}_0^n$, a contradiction.

Therefore, $k = 2$ and we deduce $x_1^2+\cdots+x_{n-1}^2 - x_n^2 \in \mathscr{J}(X_0)$, which proves (a). $\square$

Step 2. Consider $\mathfrak{Q} = \{\mathrm{q}(f) : f \in \mathscr{J}(X_0)\}$ and $C = \{x \in \mathbb{R}^n : q(x) = 0\ \forall q \in \mathfrak{Q}\}$. We claim that: *$C$ is the tangent cone to X_0*.

Indeed, let L_i be the tangent line to the curve germ $X_{i,0}$ for $i = 1,\dots,r$. By 3.6, it is clear that the tangent cone $\bigcup_{i=1}^{r} L_i$ to X_0 is contained in C.

Next, we check that $C = \bigcup_{i=1}^{r} L_i$. Since $x_1^2+\cdots+x_{n-1}^2 - x_n^2 \in \mathscr{J}(X_0)$, we have $x_1^2+\cdots+x_{n-1}^2 - x_n^2 \in \mathfrak{Q}$. Hence, the set C, which is the zero set of a family of homogeneous equations, is the cone over $C_1 = C \cap \{x_n = 1\}$. Let $\{p_i\} = L_i \cap \{x_n = 1\}$ be the intersection point of the tangent line to $X_{i,0}$ with the hyperplane $x_n = 1$.

Suppose that there exists a point $p_0 \in C_1 \subset \{x_1^2+\cdots+x_{n-1}^2 = 1, x_n = 1\} = \mathbb{S}^{n-2}$ different from $p_1,\dots,p_r$. After a linear change of coordinates that preserves $\mathbb{S}^{n-2}$ we may assume that $p_0 = (1,0,\dots,0,1)$.

Note that $x_1 = 1$, $x_n = 1$ defines the tangent affine subspace to $\mathbb{S}^{n-2}$ at the point $p_0 = (1,0,\dots,0,1)$. Moreover, the function $(1-x_1)(2+x_1)$ is positive semidefinite on $\mathbb{S}^{n-2}$ and only vanishes at the point p_0. Let $\delta > 0$ be small enough such that $f = (1-\delta-x_1)(2+x_1)$ is strictly positive at the points $p_1,\dots,p_r$. However, note that $f(p_0) < 0$.

Therefore, f is not positive semidefinite on C_1, but it is strictly positive at the points $p_1,\dots,p_r$. Thus, $F = \big((1-\delta)x_n - x_1\big)(2x_n + x_1)$ is a quadratic form strictly positive, outside the origin, on the lines $L_1, \dots, L_r$ through the origin determined by the points $p_1, \dots, p_r$. Recall that the tangent cone to X_0 is the union of these lines. Hence, by 3.7, F defines a strictly positive analytic function germ on $X_0 \setminus \{0\}$. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property, there exists an analytic function $g \in \mathscr{J}(X_0)$ such that $F + g$ is positive semidefinite on $\mathbb{R}^n_0$. Thus, $\mathrm{q}(F+g) = F + \mathrm{q}(g)$ is a positive semidefinite quadratic form on $\mathbb{R}^n$. However, since $p_0 \in C$ we get

$$\mathrm{q}(F+g)(p_0) = F(p_0) + \mathrm{q}(g)(p_0) = F(p_0) < 0,$$

a contradiction. Thus, $C = \bigcup_{i=1}^{r} L_i$.

Remark. In the following step, we will perform new linear changes of coordinates, that will transform the quadratic form $x_1^2 + \cdots + x_{n-1}^2 - x_n^2 \in \mathscr{J}(X_0)$ into a quadratic form q of rank n and signature $n-1$, as stated in (i). We point out here that the quadratic form $x_1^2 + \cdots + x_{n-1}^2 - x_n^2$ obtained in Step 1 has been used just to prove that C is the tangent cone to X_0 and will not be used any more along the proof.

Step 3. *The tangent cone C to X_0 is the union of $s \leqq r$ independent lines.*

Let W be the vectorial subspace of $\mathbb{R}^n$ generated by C and $s = \dim W$. We may assume, after a new linear change of coordinates, that the tangent lines $L_1, \dots, L_s$ generate W and that L_i is generated by the vector $e_i = (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0)$ for $1 \leqq i \leqq s$.

Let $\mathfrak{Q}_0 = \{\mathrm{q}(f)(x_1, \dots, x_s, 0, \dots, 0) : f \in \mathscr{J}(X_0)\}$. We claim that: *$\mathfrak{Q}_0$ is a vectorial space of dimension $d = s(s-1)/2$, hence, $\{x_j x_k : 1 \leqq j < k \leqq s\}$ is a basis of $\mathfrak{Q}_0$.*

Assume this claim true for a moment. Since $C = \bigcup_{i=1}^{r} L_i \subset W = \{x_{s+1} = 0, \dots, x_n = 0\}$ and C is the zero set of $\mathfrak{Q}$, we deduce that C is the intersection of the zero set of $\mathfrak{Q}_0$ and W. On the other hand, by the claim, such intersection is equal to $\bigcup_{i=1}^{s} L_i$. Hence, $s = r$ and C is the union of s independent lines, which proves statement (b). Thus, we turn to prove our claim.

First, note that $\dim \mathfrak{Q}_0 \leqq s(s-1)/2$. Suppose that $d = \dim \mathfrak{Q}_0 < s(s-1)/2$ and consider a basis $\{q_1, \dots, q_d\}$ of $\mathfrak{Q}_0$. Since each quadratic form q_ℓ vanishes on the lines $L_i = L[e_i]$, we deduce that the correspondent coefficients of q_ℓ to the monomials $x_1^2, \dots, x_s^2$ are all zero. Thus, we write

$$q_\ell = \sum_{j<k} \lambda_{jk}^{\ell} x_j x_k$$

for some $\lambda_{jk}^{\ell} \in \mathbb{R}$. For each $1 \leqq j < k \leqq s$ consider the linear form

$$\Lambda_{jk} = \sum_{\ell=1}^{d} \lambda_{jk}^{\ell} \mu_\ell$$

in the variables $\mu = (\mu_1, \dots, \mu_d)$. Note that since $d < \frac{s(s-1)}{2}$ the previous linear forms are dependent. Thus, there exist $1 \leqq j_0 < k_0 \leqq s$ such that

$$\Lambda_{j_0 k_0} = \sum_{\substack{j<k,\\ (j,k) \neq (j_0,k_0)}} \alpha_{jk} \Lambda_{jk},$$

for certain $\alpha_{jk} \in \mathbb{R}$.

Next, for each $i = 1, \dots, r$ choose a vector v_i that generates the tangent line L_i to $X_{i,0}$. Consider the quadratic form

$$f_{j_0 k_0} = N^2(2x_{k_0} + M^2 x_{j_0})x_{j_0} + \sum_{\substack{1 \leqq j \leqq s,\\ j \neq j_0}} x_j^2$$

for $N, M > 1$. Note that if M is large enough $f_{j_0 k_0}(v_i) > 0$ for $i = 1, \dots, r$ and all $N > 1$. Thus, by 3.7, the quadratic form $f_{j_0 k_0}$ is strictly positive on $X_0 \setminus \{0\}$. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property, there exists $g_N \in \mathscr{J}(X_0)$ such that $F_{j_0 k_0} = f_{j_0 k_0} + g_N$ is positive semidefinite on $\mathbb{R}^n_0$. Thus, $\mathrm{q}(F_{j_0 k_0}) = f_{j_0 k_0} + \mathrm{q}(g_N)$ is a positive semidefinite quadratic form. Substituting $x_{s+1} = 0, \dots, x_n = 0$, we deduce that

$$f_{j_0 k_0} + \mathrm{q}(g_N)(x_1, \dots, x_s, 0, \dots, 0)$$

is a positive semidefinite quadratic form. Note that $\mathrm{q}(g_N)(x_1, \dots, x_s, 0, \dots, 0) \in \mathfrak{Q}_0$. Hence, there exists $m = (m_1, \dots, m_d) \in \mathbb{R}^d$ such that

$$\mathrm{q}(g_N)(x_1, \dots, x_s, 0, \dots, 0) = \sum_{\ell=1}^{d} 2m_\ell q_\ell$$

and therefore

$$\begin{aligned} Q = f_{j_0 k_0} + \sum_{\ell=1}^{d} 2m_\ell q_\ell &= f_{j_0 k_0} + \sum_{\ell=1}^{d} 2m_\ell \sum_{j<k} \lambda^\ell_{jk} x_j x_k = f_{j_0 k_0} + \sum_{j<k} 2\left(\sum_{\ell=1}^{d} m_\ell \lambda^\ell_{jk}\right) x_j x_k \\ &= N^2(2x_{k_0} + M^2 x_{j_0})x_{j_0} + \sum_{j \neq j_0} x_j^2 + \sum_{j<k} 2\Lambda_{jk}(m) x_j x_k \end{aligned}$$

is a positive semidefinite quadratic form. Let $A = (a_{jk})_{1 \leqq j,k \leqq s}$ be the real symmetric matrix such that

$$Q(x_1, \dots, x_s) = (x_1, \dots, x_s) A \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}.$$

Since Q is positive semidefinite, all the ordered 2 minors $a_{jj}a_{kk} - a_{jk}^2$ are $\geqq 0$ for $1 \leqq j < k \leqq s$. Thus,

$$0 < a_{jj}a_{kk} - a_{jk}^2 = \begin{cases} 1 - \Lambda_{jk}(m)^2 & \text{if } j_0 \neq j < k, \\ N^2 M^2 - \Lambda_{j_0 k}(m)^2 & \text{if } j_0 = j < k \neq k_0, \\ N^2 M^2 - \left(N^2 + \Lambda_{j_0 k_0}(m)\right)^2 & \text{if } j = j_0 \text{ and } k = k_0. \end{cases}$$

Note that if $(j,k) \neq (j_0,k_0)$, then $\Lambda_{jk}(m)^2 < N^2M^2$; recall that $N, M > 1$. Hence, if $(j,k) \neq (j_0,k_0)$, we get that $|\Lambda_{jk}(m)| < NM$. Thus,

$$|\Lambda_{j_0k_0}(m)| = \left| \sum_{\substack{j<k,\\ (j,k)\neq(j_0,k_0)}} \alpha_{jk}\Lambda_{jk}(m) \right| \leqq \sum_{\substack{j<k,\\ (j,k)\neq(j_0,k_0)}} |\alpha_{jk}|\,|\Lambda_{jk}(m)| \leqq PNM$$

for $P = \sum_{\substack{j<k,\\ (j,k)\neq(j_0,k_0)}} |\alpha_{jk}| \in \mathbb{R}$. On the other hand,

$$|N^2 + \Lambda_{j_0k_0}(m)| \geqq |N^2| - |\Lambda_{j_0k_0}(m)| \geqq N^2 - PNM > NM$$

if $N > 1$ is large enough. Therefore, for such N we have

$$0 < a_{j_0j_0}a_{k_0k_0} - a_{j_0k_0}^2 = N^2M^2 - \left(N^2 + \Lambda_{j_0k_0}(m)\right)^2 < 0,$$

a contradiction. Thus, $d = s(s-1)/2$, as claimed. $\square$

Step 4. *If $s = n$, X_0 is equivalent to the union of n independent lines.*

We keep all the conditions obtained at the end of Step 3, even the same coordinate system. Thus, by the previous step, we may assume that $\mathfrak{Q} = \mathfrak{Q}_0$ is generated by the quadratic forms x_jx_k where $1 \leqq j \leqq k \leqq n$. Now, for each $1 \leqq j < k \leqq n$ there exists an analytic series $g_{jk} \in \mathscr{J}(X_0)$ whose initial form is x_jx_k, that is, $g_{jk} = x_jx_k + h_{jk}$, where $h_{jk} \in \mathbb{R}\{x_1, \ldots, x_n\}$ is an analytic series of order $\geqq 3$. Thus, if $\mathfrak{m}_n$ denotes the maximal ideal of $\mathbb{R}\{x\}$, the vectorial space

$$\mathfrak{m}_n^\ell + \mathscr{J}(X_0)/\left(\mathfrak{m}_n^{\ell+1} + \mathscr{J}(X_0)\right)$$

is generated by the vectors $x_j^\ell + \left(\mathfrak{m}_n^{\ell+1} + \mathscr{J}(X_0)\right)$ where $1 \leqq j \leqq n$; hence, it has dimension $\leqq n$ for all $\ell \leqq 1$. Therefore, the Hilbert-Samuel function of X_0,

$$\mathrm{HS}(\ell) = \sum_{k=0}^{\ell} \dim\left(\mathfrak{m}_n^k + \mathscr{J}(X_0)/\left(\mathfrak{m}_n^{k+1} + \mathscr{J}(X_0)\right)\right),$$

which is equal to a polynomial $\mathrm{HSP} \in \mathbb{Q}[\ell]$ for ℓ large enough, is $\leqq n\ell + 1$. Since X_0 has dimension 1, HSP is a polynomial of degree 1 whose principal coefficient is equal to the multiplicity $m(X_0)$. Since $\mathrm{HSP}(\ell) \leqq n\ell + 1$, we deduce that $\sum_{i=1}^{r} m(X_{i,0}) = m(X_0) \leqq n$ (for more details, see [JP], 4.2). On the other hand, since $r \geqq s = n$ and $m(X_{i,0}) \geqq 1$, we also have that

$$n \leqq \sum_{i=1}^{r} m(X_{i,0}) = m(X_0) \leqq n.$$

Hence, $m(X_0) = n$, and this necessarily means that $r = s = n$ and $m(X_{i,0}) = 1$ for $1 \leqq i \leqq n$. Thus, each $X_{i,0}$ is a regular curve germ and the tangent lines are linearly independent. Therefore, X_0 is equivalent to the union of n independent lines, which proves (c). $\square$

Step 5. *If $s < n$ and the initial form $q = \mathrm{In}(f)$ of a series $f \in \mathscr{J}(X_0)$ is a positive semidefinite quadratic form, then it is identically* 0 *on the vectorial subspace of $\mathbb{R}^n$ generated by the tangent cone to X_0 and $g = f - q \in \mathbb{R}\{x_1, \dots, x_n\}$ is an analytic series of order* 3.

By 3.6, the quadratic form $q = \mathrm{In}(f)$ is identically zero on the tangent cone to X_0. Since q is moreover a positive semidefinite quadratic form, its zero set is a vectorial subspace of $\mathbb{R}^n$. Hence q is identically 0 on the vectorial subspace of $\mathbb{R}^n$ generated by the tangent cone to X_0.

Let us see now that $g = f - q$ is an analytic series of order 3. It is clear that g has order $\geqq 3$, and we have to show that it cannot have order $\geqq 4$. If g has order $\geqq 4$, by 3.3, there exists $M > 0$ such that

$$|g| \leqq M^2(x_1^2 + \cdots + x_n^2)^2.$$

Since q is a nonzero positive semidefinite quadratic form, there exist nonzero linear forms $a_1, \dots, a_p$ such that $q = a_1^2 + \cdots + a_p^2$. Hence,

$$|a_1| \leqq \sqrt{a_1^2 + \cdots + a_p^2} = \sqrt{|q|} = \sqrt{|g|} \leqq M(x_1^2 + \cdots + x_n^2)$$

on X_0, because $f = q + g \in \mathscr{J}(X_0)$. Thus, $a_1 + (M+1)(x_1^2 + \cdots + x_n^2)$ is strictly positive on $X_0 \backslash \{0\}$ but cannot be extended positively to $\mathbb{R}_0^n$, a contradiction. Hence, $f - q$ has order 3, and the statement (d)(i) is proved. $\square$

Step 6. *If $s < n$, there exists an analytic series $f \in \mathscr{J}(X_0)$ whose initial form is a positive semidefinite quadratic form.*

We keep the coordinate system fixed in Step 3. Let $\mathscr{S}_n$ be the set of all the symmetric matrices of order n. Note that $\mathfrak{Q} = \{\mathrm{q}(f) : f \in \mathscr{J}(X_0)\}$ can be canonically embedded in $\mathscr{S}_n$ as a vectorial subspace, identifying each quadratic form q with its associated symmetric matrix. Note that if $A = (a_{ij}) \in \mathfrak{Q}$, then by 3.6 and the special coordinate system we have fixed, we have $a_{11} = 0, \dots, a_{ss} = 0$. We consider the norm

$$\|.\| : \mathscr{S}_n \to \mathbb{R}, \quad A \mapsto +\sqrt{\sum_{j,k} a_{jk}^2}$$

on $\mathscr{S}_n$ and the sphere $\mathbb{S} = \{A \in \mathscr{S}_n : \|A\| = 1\}$. Note that if $A \in \mathscr{S}_n \backslash \{0\}$ then $A/\|A\| \in \mathbb{S}$. Moreover, if $q \equiv A \in \mathscr{S}_n$, then

$$|q(x)| \leqq n\|q\|\,\|x\|^2 \quad \text{for all } x \in \mathbb{R}^n.$$

Indeed, if $A = (a_{ij})$, then

$$|q(x)| = \left|\sum_{i,j=1}^n a_{ij}x_ix_j\right| \leqq \sum_{i,j=1}^n |a_{ij}|\,|x_i|\,|x_j| \leqq \|x\|^2 \sum_{i,j=1}^n |a_{ij}| \leqq \|x\|^2\|q\|\sqrt{n^2} = n\|q\|\,\|x\|^2.$$

Next, consider the compact set

$$\mathfrak{U} = \mathfrak{Q} \cap \mathbb{S} \subset \{a_{11} = 0, \dots, a_{ss} = 0\}.$$

If there exists $q \in \mathfrak{U}$ positive semidefinite we are done. Thus, we suppose that every $q \in \mathfrak{U}$ is a non definite quadratic form, and we will achieve a contradiction.

First, we claim that: *For each* $q \in \mathfrak{U}$ *there exist* $\varepsilon > 0$ *and* $M_0 > 0$ *such that if* $q' \in \mathscr{S}$, $M > M_0$ *and* $\|q - q'\| < \varepsilon$, *then* $Mq' + (x_1^2 + \cdots + x_s^2)$ *is non definite.*

Indeed, since $q \in \mathfrak{U}$ is a non definite quadratic form, there exist $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that $q(u) > 0$ and $q(v) < 0$. Take

$$\varepsilon = \frac{1}{3n} \min\left\{\frac{|q(u)|}{\|u\|^2}, \frac{|q(v)|}{\|v\|^2}\right\} > 0 \quad \text{and} \quad M_0 = \frac{3(1 + v_1^2 + \cdots + v_s^2)}{|q(v)|}.$$

If $\|q - q'\| < \varepsilon$ and $M > M_0$ then

$$\begin{aligned} q'(u) + \frac{1}{M}(u_1^2 + \cdots + u_s^2) &\geqq q(u) + q'(u) - q(u) \geqq q(u) - |q(u) - q'(u)| \\ &\geqq q(u) - n\|q - q'\| \, \|u\|^2 > q(u) - n\|u\|^2\varepsilon \\ &\geqq q(u) - \frac{|q(u)|}{3} = \frac{2q(u)}{3} > 0, \end{aligned}$$

$$\begin{aligned} q'(v) + \frac{1}{M}(v_1^2 + \cdots + v_s^2) &\leqq q'(v) + \frac{1}{M_0}(v_1^2 + \cdots + v_s^2) \leqq q'(v) + \frac{|q(v)|}{3} \\ &= q'(v) - \frac{q(v)}{3} = \frac{2q(v)}{3} + q'(v) - q(v) \leqq \frac{2q(v)}{3} + |q(v) - q'(v)| \\ &\leqq \frac{2q(v)}{3} + n\|q - q'\| \, \|v\|^2 < \frac{2q(v)}{3} + n\|v\|^2\varepsilon \\ &\leqq \frac{2q(v)}{3} + \frac{|q(v)|}{3} = \frac{q(v)}{3} < 0, \end{aligned}$$

hence, $Mq' + (x_1^2 + \cdots + x_s^2)$ is a non definite quadratic form. The claim is proved.

Next, since $\mathfrak{U}$ is compact, there exists $M_0 > 0$ such that for all $q \in \mathfrak{U}$ and all $M > M_0$ the quadratic form $Mq + (x_1^2 + \cdots + x_s^2)$ is non definite.

Consider now the quadratic form $f = -2M_0(x_{s+1}^2 + \cdots + x_n^2) + x_1^2 + \cdots + x_s^2$ which is strictly positive over the vectors $e_1, \ldots, e_s$. Then, by 3.7, f is strictly positive on $X_0 \setminus \{0\}$. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property, there exist $g \in \mathscr{J}(X_0)$ such that $f + g$ is positive semidefinite on $\mathbb{R}_0^n$. In particular, its initial form $f + \mathrm{q}(g)$ is a positive semidefinite quadratic form. Note that $q = \mathrm{q}(g) \neq 0$.

Now, we distinguish two cases:

(i) If $\|q\| > M_0$, then

$$q_0 = q + x_1^2 + \cdots + x_s^2 = \|q\| \frac{q}{\|q\|} + x_1^2 + \cdots + x_s^2$$

is a non definite quadratic form, because $q/\|q\| \in \mathfrak{U} = \mathfrak{Q} \cap \mathbb{S}$. Let $v \in \mathbb{R}^n$ such that $q_0(v) < 0$. Then

$$\begin{aligned}(f + \mathrm{q}(g))(v) &= f(v) + q(v) = -2M_0(v_{s+1}^2 + \cdots + v_n^2) + v_1^2 + \cdots + v_s^2 + q(v)\\ &= -2M_0(v_{s+1}^2 + \cdots + v_n^2) + q_0(v) < 0,\end{aligned}$$

which is impossible, because $f + \mathrm{q}(g)$ is a positive semidefinite quadratic form.

(ii) If $\|q\| \leqq M_0$, we get that

$$(f + \mathrm{q}(g))(e_{s+1}) = f(e_{s+1}) + q(e_{s+1}) \leqq -2M_0 + |q(e_{s+1})| \leqq -2M_0 + \|q\| \leqq -M_0 < 0,$$

a contradiction.

Therefore there exists a positive semidefinite quadratic form $q \in \mathfrak{U}$, and statement (d)(ii) holds true. □

As a nice application of the previous result we have the following:

Corollary 3.10. *Let $X_0 \subset \mathbb{R}_0^n$ be an analytic curve whose irreducible components are all regular. The following assertions are equivalent*:

(a) $\mathscr{P}(X_0) = \Sigma(X_0)$.

(b) *X_0 has the $\mathscr{P}\mathscr{E}$ property.*

(c) *X_0 has the $\mathscr{P}\mathscr{E}^+$ property.*

(d) *X_0 is equivalent to a union of independent lines.*

Proof. First, note that we may assume that the embedding dimension of X_0 is equal to n. Moreover, by [Sch], 3.9, statements (a) and (d) are equivalent. Note also that obviously, (a) implies (b) and (b) implies (c). Thus, it is enough to prove that (c) implies (d).

By 3.9 (b), the tangent cone to X_0 is the union of s independent lines $L_1, \ldots, L_s$. After a linear change of coordinates, we may assume that $L_1, \ldots, L_s$ are respectively generated by the vectors $e_1, \ldots, e_s$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. If $s = n$, by 3.9 (c), X_0 is equivalent to a union of independent lines, and we are done.

If $s < n$, let us see that X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property. Indeed, write $X_0 = X_{1,0} \cup \cdots \cup X_{r,0}$ as the union of its irreducible components and $f_j = x_j$ for $j = 1, \ldots, n$.

Fix $1 \leqq i \leqq r$ and let α_i be a primitive parametrization of $X_{i,0}$. Say that L_j is the tangent line to $X_{i,0}$. We have that $\omega(f_j \circ \alpha_i) = 1$ and $\omega(f_n \circ \alpha_i) \geqq 2$ (because $j < n$). Thus, the function germ $x_n + Mx_j^2$ is strictly positive on $X_{i,0}$ for $M > 0$ large enough.

Hence, the function germ $x_n + M(x_1^2 + \cdots + x_s^2)$ is strictly positive on $X_0 \backslash \{0\}$ for $M > 0$ large enough, but it cannot be extended positively to $\mathbb{R}_0^3$. Thus, X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property. $\square$

Now, we are ready to prove 3.1.

Proof of Theorem 3.1. Again, by [Sch], 3.9, statements (a), (b) and (e) are equivalent. Moreover, (b) implies (c) and (c) implies (d). Thus, it is enough to prove that (d) implies (e).

Indeed, let $n \leqq 3$ be the embedding dimension of X_0. We distinguish several cases:

Case 1. If $n = 1$ there is nothing to prove, since X_0 is a straight-line.

Case 2. If $n = 2$, by 3.9 (a), after an additional linear change of coordinates, we may assume that $(xy)\mathbb{R}\{x, y\} \subset \mathscr{J}(X_0)$. Thus, X_0 is contained in the union of two transversal lines. But since the embedding dimension of X_0 is 2, we deduce that $\mathscr{J}(X_0) = (xy)\mathbb{R}\{x, y\}$ and X_0 is the union of two transversal lines.

Case 3. If $n = 3$, by 3.9 (b), the tangent cone to X_0 is the union of $s \leqq 3$ lines. By 3.9 (c), if $s = 3$, then X_0 is equivalent to the union of three independent lines. Thus, we have to show that $s = 3$. $\square$

(3.11) *If $s = 2$, X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property.*

Proof. We write $X_0 = X_{1,0} \cup \cdots \cup X_{r,0}$ as the union of its irreducible components. After a linear change of coordinates, we may assume that the tangent cone to X_0 is the union of the lines $y = 0$, $z = 0$ and $x = 0$, $z = 0$. By 3.9, there exist

$$g_1 = xy + axz + byz + cz^2 + h_1, \quad g_2 = z^2 + h_2 \in \mathscr{J}(X_0)$$

where $a, b, c \in \mathbb{R}$ and $h_1, h_2 \in \mathbb{R}\{x, y, z\}$ have order $\geqq 3$. Let $\mu \in \mathbb{R}$ be a real number such that the initial form of $g_1 + \mu g_2$ is a quadratic form of rank 3; we denote again by g_1 the analytic series $g_1 + \mu g_2$. By classification of singularities ([JP], 9.2.12), we may assume that

$$g_1 = xy - z^2, \quad g_2 = z^2 + h_2 \in \mathscr{J}(X_0)$$

where $h_2 \in \mathbb{R}\{x, y, z\}$ is an analytic series of order $\geqq 3$. Note that the tangent cone to X_0 is still the union of the lines $x = 0$, $z = 0$ and $y = 0$, $z = 0$. Let us see that $h_2(x, 0, 0)$ or $h_2(0, y, 0)$ are analytic series of order 3. Otherwise, we can write

$$h_2(x, y, z) = zh_{21}(x, y, z) + xyh_{22}(x, y) + x^4h_{23}(x) + y^4h_{24}(y)$$

where $h_{2j} \in \mathbb{R}\{x, y, z\}$ for $1 \leqq j \leqq 4$, h_{21} has order $\geqq 2$ and h_{22} has order $\geqq 1$. Consider

$$\begin{aligned} g_2' &= g_2 - h_{22}g_1 = z^2(1 + h_{22}) + zh_{21} + x^4h_{23} + y^4h_{24} \\ &= \left(z\sqrt{1 + h_{22}} + \frac{h_{21}}{2\sqrt{1 + h_{22}}}\right)^2 + x^4h_{23} + y^4h_{24} - \frac{h_{21}^2}{4(1 + h_{22})}. \end{aligned}$$

Notice that since h_{21} has order $\geqq 2$ and h_{22} has order $\geqq 1$, then, after the change of coordinates

$$\left(x, y, z\sqrt{1+h_{22}} + \frac{h_{21}}{2\sqrt{1+h_{22}}}\right) \mapsto (x, y, z),$$

we deduce that the tangent cone to X_0 is still the union of the lines $x = 0$, $z = 0$ and $y = 0$, $z = 0$, and $g_2' = z^2 - h_4$, for certain analytic series $h_4 \in \mathbb{R}\{x, y, z\}$ of order $\geqq 4$. Thus, by 3.9 (d), X_0 does not have the $\mathscr{PE}^+$ property. Hence, we may assume that $h_2(x, 0, 0)$ has order 3, and, in fact, that $h_2(x, 0, 0) = a^2x^3 + \cdots$ for some $a > 0$.

Fix $1 \leqq i \leqq r$ and let α_i be a primitive parametrization of the irreducible curve $X_{i,0}$. We have that:

(a) If the tangent line to $X_{i,0}$ is the line $y = 0$, $z = 0$, then using that $g_1 = xy - z^2 \in \mathscr{J}(X_0)$ we can write

$$\alpha_i(t) = (\varepsilon t^k, \varepsilon t^{k+2\ell}u^2, t^{k+\ell}u),$$

where $u \in \mathbb{R}\{t\}$ is a unit, $k, \ell \geqq 1$ are positive integers and $\varepsilon = \pm 1$. Moreover, since $g_2 = z^2 + h_2 \in \mathscr{J}(X_0)$, where $h_2(x, 0, 0) = a^2x^3 + \cdots$ for some $a > 0$, we have

$$0 = g_2 \circ \alpha_i = t^{2k+2\ell}u^2 + t^{3k}\varepsilon v^2(t),$$

where $v \in \mathbb{R}\{t\}$ is a unit. Thus $k = 2\ell$ is even and $\varepsilon = -1$. Therefore, the function germ $-x$ is strictly positive on $X_{i,0} \setminus \{0\}$.

(b) If the tangent line to $X_{i,0}$ is the line $x = 0$, $z = 0$, then using that $g_1 = xy - z^2 \in \mathscr{J}(X_0)$ we can write

$$\alpha_i(t) = (\varepsilon t^{k+2\ell}u^2, \varepsilon t^k, t^{k+\ell}u),$$

where $u \in \mathbb{R}\{t\}$ is a unit, $k, \ell \geqq 1$ are positive integers and $\varepsilon = \pm 1$. Moreover, since $g_2 = z^2 + h_2 \in \mathscr{J}(X_0)$,

$$0 = g_2 \circ \alpha_i = t^{2k+2\ell}u^2 + t^{3k}\varepsilon\xi(t),$$

where $\xi \in \mathbb{R}\{t\}$. Thus, $2\ell \geqq k$ and the function germ $-x + My^2$ is strictly positive on $X_{i,0} \setminus \{0\}$ for $M > 0$ large enough.

Hence, if $M > 0$ is large enough, then $-x + My^2$ is strictly positive on $X_0 \setminus \{0\}$ but it cannot be extended positively to $\mathbb{R}^3_0$. Thus, X_0 does not have the $\mathscr{PE}^+$ property. $\square$

(3.12) *If $s = 1$, X_0 does not have the $\mathscr{PE}^+$ property.*

Proof. We write $X_0 = X_{1,0} \cup \cdots \cup X_{r,0}$ as the union of its irreducible components. After a linear change of coordinates, we may assume, by 3.9 (a), that $y^2 - xz \in \mathscr{J}(X_0)$. After an additional linear change that keeps the equation $y^2 - xz$ invariant, we may assume that the tangent cone to X_0 is the line $y = 0$, $z = 0$.

Fix $1 \leqq i \leqq r$ and let α_i be a primitive parametrization of the irreducible component $X_{i,0}$. We may assume, after a reparametrization, that,

$$\alpha_i(t) = (\varepsilon t^k, t^{k+\ell} u, \varepsilon t^{k+2\ell} u^2)$$

where $u \in \mathbb{R}\{t\}$ is a unit, $\varepsilon = \pm 1$ and k, ℓ are positive integers. Note that all the previous elements depend on i.

Now, by 3.9 (d), there exists an analytic series $f \in \mathscr{J}(X_0)$ whose initial form is a nonzero positive semidefinite quadratic form of rank $\rho \leqq 2$ which vanishes on the line $y = 0$, $z = 0$. If $\rho = 2$, by classification of singularities and 3.9 (d), we may assume (after a new change of coordinates) that $f = y^2 + z^2 - x^3$. Since $y^2 + z^2 = x^3$ on X_0, we have $x \in \mathscr{P}(X_0)$ and $x + (x^2 + y^2 + z^2)$ is strictly positive on $X_0 \setminus \{0\}$ but it cannot be extended positively to $\mathbb{R}^3_0$. Thus, X_0 does not have the $\mathscr{PE}^+$ property.

Hence, in what follows, we assume that $\rho = 1$ and, by 3.9 (d), there exists $g \in \mathscr{J}(X_0)$ of the type

$$\begin{aligned} g = {} & (ay + bz)^2 + a_0 x^3 + a_1 x^2 y + a_2 x^2 z + a_3 x y^2 \\ & + a_4 x z^2 + a_5 x y z + a_6 y^2 z + a_7 z^2 y + a_8 y^3 + a_9 z^3 + h, \end{aligned}$$

where $a_j \in \mathbb{R}$ and $h \in \mathbb{R}\{x, y, z\}$ has order $\geqq 4$. Using that $y^2 - xz \in \mathscr{J}(X_0)$ we may assume that a_3, a_6, a_8 are 0, that is,

$$g = (ay + bz)^2 + a_0 x^3 + a_1 x^2 y + a_2 x^2 z + a_4 x z^2 + a_5 x y z + a_7 z^2 y + a_9 z^3 + h,$$

where $h \in \mathbb{R}\{x, y, z\}$ has order $\geqq 4$. We have that $g \circ \alpha_i = 0$, that is

$$(*) \qquad \begin{aligned} & (a t^{k+\ell} u + b \varepsilon t^{k+2\ell} u^2)^2 + a_0 \varepsilon t^{3k} + a_1 u t^{3k+\ell} + a_2 \varepsilon t^{3k+2\ell} u^2 + a_4 \varepsilon t^{3k+4\ell} u^4 \\ & + a_5 u^3 t^{3k+3\ell} + a_7 u^5 t^{3k+5\ell} + a_9 \varepsilon u^6 t^{3k+6\ell} + t^{4k} \xi(t) = 0, \end{aligned}$$

where $\xi(t) \in \mathbb{R}\{t\}$. We distinguish several cases:

(3.12.1) $a_0 \neq 0$. Then (from the equality $(*)$) we deduce that k is even and $-a_0 \varepsilon > 0$. Thus, the function germ $-a_0 x$ is strictly positive on $X_{i,0} \setminus \{0\}$. Since this happens for all i, we have $-a_0 x$ is strictly positive on $X_0 \setminus \{0\}$. Hence, X_0 does not have the $\mathscr{PE}^+$ property. □

(3.12.2) $a_0 = 0$, $a \neq 0$. Then, we deduce that $\ell \geqq k$. Thus, the function germ $f = y + M x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$ for $M > 0$ large enough. Since this happens for all i, the function germ f is strictly positive on $X_0 \setminus \{0\}$ for $M > 0$ large enough. Hence, X_0 does not have the $\mathscr{PE}^+$ property. □

(3.12.3) $a_0 = 0$, $a = 0$. Then $b \neq 0$ and we may assume that $b = 1$. Thus, we have

$$(**) \qquad \begin{aligned} & t^{2k+4\ell} u^4 + a_1 u t^{3k+\ell} + a_2 \varepsilon t^{3k+2\ell} u^2 + a_4 \varepsilon t^{3k+4\ell} u^4 \\ & + a_5 u^3 t^{3k+3\ell} + a_7 u^5 t^{3k+5\ell} + a_9 \varepsilon u^6 t^{3k+6\ell} + t^{4k} \xi(t) = 0, \end{aligned}$$

and we distinguish several subcases:

(a) $a_1 \neq 0$. Then $2k + 4\ell = 3k + \ell$ or $3k + \ell \geqq 4k$. If $2k + 4\ell = 3k + \ell$, we have $k = 3\ell$. Hence

$$\alpha_i = (\varepsilon t^{3\ell}, t^{4\ell} u, \varepsilon t^{5\ell} u^2).$$

Moreover, since $2k + 4\ell = 10\ell < 12\ell = 4k$, we have $-a_1 u(0) = u(0)^4 > 0$. Thus, the function $-a_1 y$ is strictly positive on $X_{i,0} \setminus \{0\}$.

If $3k + \ell \geqq 4k$ then $\ell \geqq k$, that is, $\ell = k + j$ for an integer $j \geqq 0$. Hence,

$$\alpha_i = (\varepsilon t^k, t^{2k+j} u, \varepsilon t^{3k+2j} u^2),$$

and the function $-a_1 y + \big(|a_1 u(0)| + 1\big) x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$.

Thus, for $M > 0$ large enough, the function $f = -a_1 y + M x^2$ is strictly positive on $X_0 \setminus \{0\}$. Whence, X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property.

(b) $a_1 = 0$ and $a_2 \neq 0$. Then $2k + 4\ell = 3k + 2\ell$ or $3k + 2\ell \geqq 4k$. If

$$2k + 4\ell = 3k + 2\ell,$$

we have $k = 2\ell$. Hence

$$\alpha_i = (\varepsilon t^{2\ell}, t^{3\ell} u, \varepsilon t^{4\ell} u^2).$$

Thus, the function $z + \big(1 + u(0)^2\big) x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$.

If $3k + 2\ell \geqq 4k$ then $2\ell \geqq k$, that is, $2\ell = k + j$ for an integer $j \geqq 0$. Hence,

$$\alpha_i = (\varepsilon t^k, t^{k+\ell} u, \varepsilon t^{2k+j} u^2),$$

and the function $z + \big(1 + u(0)^2\big) x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$.

Thus, for $M > 0$ large enough, the function $f = z + M x^2$ is strictly positive on $X_0 \setminus \{0\}$. Hence, X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property.

(c) $a_1, a_2 = 0$ and $a_5 \neq 0$. Then $2k + 4\ell = 3k + 3\ell$ or $2k + 4\ell \geqq 4k$. If

$$2k + 4\ell = 3k + 3\ell,$$

we have $k = \ell$. Hence,

$$\alpha_i = (\varepsilon t^k, t^{2k} u, \varepsilon t^{3k} u^2).$$

Thus, the function $z + x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$.

If $2k + 4\ell \geqq 4k$ then $2\ell \geqq 2k$, that is, $2\ell = k + j$ for an integer $j \geqq 0$. Hence,

$$\alpha_i = (\varepsilon t^k, t^{k+\ell} u, \varepsilon t^{2k+j} u^2),$$

and the function $z + \big(1 + u(0)^2\big) x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$.

Thus, for $M > 0$ large enough, the function $f = z + Mx^2$ is strictly positive on $X_0 \setminus \{0\}$. Therefore, X_0 does not have the $\mathscr{PE}^+$ property.

(d) $a_1, a_2, a_5 = 0$. Then $2k + 4\ell \geqq 4k$ and $2\ell \geqq k$, that is, $2\ell = k + j$ for an integer $j \geqq 0$. Hence,

$$\alpha_i = (\varepsilon t^k, t^{k+\ell} u, \varepsilon t^{2k+j} u^2),$$

and the function $z + \big(1 + u(0)^2\big)x^2$ is strictly positive on $X_{i,0} \setminus \{0\}$.

Thus, for $M > 0$ large enough, the function $f = z + Mx^2$ is strictly positive on $X_0 \setminus \{0\}$. Therefore, X_0 does not have the $\mathscr{PE}^+$ property.

In this way, we conclude that if $s = 1$, X_0 does not have the $\mathscr{PE}^+$ property, as wanted. $\square$

4. Local results for dimension two

The purpose of this section is to prove 1.2. To make easier the proof of 1.6 that will be done in the next section we prove the following slightly stronger result:

Theorem 4.1. *Let $X_0 \subsetneqq \mathbb{R}^3_0$ be a real analytic germ. If X_0 has the $\mathscr{PE}^+$ property, then X_0 is one among the analytic germs of the List.*

Proof. First, note that if X_0 has dimension 1 then, by 3.1, the germ X_0 is an analytic curve germ in the List. Thus, we assume that X_0 has dimension 2. We distinguish two cases:

Case 1. *X_0 has irreducible components of dimension* 1. We have to check that X_0 is equivalent to the union of a plane and a transversal line.

Since X_0 has the $\mathscr{PE}^+$ property, by 3.5, $\omega\big(\mathscr{J}(X_0)\big) = 2$. Let I (resp. J) be the ideal of the union of the components of X of dimension 2 (resp. 1). Then $\mathscr{J}(X_0) = I \cap J$. Moreover, since the ideal $I \subset \mathbb{R}\{x, y, z\}$ has height 1, it is principal; and we write $I = (\varphi)$ with $\varphi \in \mathbb{R}\{x, y, z\}$. One can check that $\mathscr{J}(X_0) = I \cdot J$; hence, $2 = \omega\big(\mathscr{J}(X_0)\big) = \omega(I) + \omega(J)$. Thus, $\omega(I) = \omega(J) = 1$ and we may assume that $I = (z)$ and $J = (\psi_1, \psi_2)$ where $\psi_j \in \mathbb{R}\{x, y, z\}$ and $1 = \omega(\psi_1) \leqq \omega(\psi_2)$. Let us see that we may assume that $\psi_1 = x$.

Otherwise, we can suppose that the initial form of ψ_1 is equal to z and, after an analytic change of coordinates, that $\psi_1 = z + 2F(x, y)$ for certain analytic series $F(x, y) \in \mathfrak{m}_2^2$. We have that $\mathscr{J}(X_0) = \big(z(z + 2F(x, y)), z\psi_2\big)$. Note that since

$$z\big(z + 2F(x, y)\big) = \big(z + F(x, y)\big)^2 - F^2(x, y) \in \mathscr{J}(X),$$

the following equality holds for X_0:

$$|z + F(x, y)| = |F(x, y)|.$$

On the other hand, since $\omega(F) \geqq 2$, by 3.3, there exists $c > 0$ such that $|F(x, y)| \leqq c^2(x^2 + y^2)$. Thus, we get that

$$|z + F(x, y)| = |F(x, y)| \leqq c^2(x^2 + y^2)$$

and the analytic function germ $h_0 = (c+1)^2(x^2 + y^2) + F(x, y) + z$ is positive semidefinite on X_0, hence $h = h_0 + (x^2 + y^2 + z^2)^2$ is strictly positive on $X_0 \setminus \{0\}$, but does not admit a positive semidefinite extension to $\mathbb{R}^3_0$; impossible, because X_0 has the $\mathscr{P}\mathscr{E}^+$ property.

Thus, in what follows, we may assume that $J = (x, \psi_2)$ where $\psi_2 \in \mathbb{R}\{y, z\}$ is a series of order $\geqq 1$. We are to prove that after a new change of coordinates $\psi_2(y, z) = y$, hence $J = (x, y)$, which means that X_0 is (equivalent to) the union of a plane and a transversal line. To that end, we begin by proving that the curve germ Y_0 of $\mathbb{R}^2_0$ given by $Y_0 : z\psi_2(y, z) = 0$ has the $\mathscr{P}\mathscr{E}^+$ property.

Indeed, note that $\mathscr{J}(Y_0) = (z\psi_2)\mathbb{R}\{y, z\}$. Let $f(y, z) \in \mathscr{P}(Y_0)$ be such that f is strictly positive on $Y_0 \setminus \{0\}$. Then $f(y, z) + x^2$ is strictly positive on $X_0 \setminus \{0\}$. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property there exist analytic functions $g \in \mathscr{P}(\mathbb{R}^3_0)$ and $b_1, b_2 \in \mathbb{R}\{x, y, z\}$ such that

$$f(y, z) + x^2 = g(x, y, z) + zxb_1(x, y, z) + z\psi_2(y, z)b_2(x, y, z).$$

Making $x = 0$ in the previous equation we get that

$$f(y, z) = g(0, y, z) + z\psi_2(y, z)b_2(0, y, z),$$

where $g(0, y, z) \in \mathscr{P}(\mathbb{R}^2_0)$. Thus, $Y_0 \subset \mathbb{R}^2_0$ has the $\mathscr{P}\mathscr{E}^+$ property.

Next, since $\omega(\psi_2) \geqq 1$, by 3.1, the germ Y_0 is equivalent to the union of two transversal lines. Hence, after a change of coordinates, we may assume that $\psi_2 = y$, and we are done.

Case 2. X_0 *does not have irreducible components of dimension* 1. If X_0 is regular, then X_0 is equivalent to a plane, which belongs to the List. Thus, we may assume that X_0 is singular. Note that since X_0 does not have irreducible components of dimension 1, the ideal $\mathscr{J}(X_0)$ is principal. Since X_0 has the $\mathscr{P}\mathscr{E}^+$ property and it is singular, by 3.5, $\omega(\mathscr{J}(X_0)) = 2$. Thus, after a change of coordinates, we may assume that $\mathscr{J}(X_0) = (z^2 - F(x, y))\mathbb{R}\{x, y, z\}$ for certain analytic series $F \in \mathbb{R}\{x, y\}$ of order $\geqq 2$. Note that the ring $\mathscr{O}(X_0)$ is a free $\mathbb{R}\{x, y\}$-module of rank 2 with basis $\{1, z\}$. By [Rz3], 4.3, we have

$$(\bullet) \qquad \mathscr{P}(X_0) = \{f + zg : f, g \in \mathbb{R}\{x, y\}, f \in \mathscr{P}(F \geqq 0), f^2 - Fg^2 \in \mathscr{P}(\mathbb{R}^2_0)\},$$

where $\mathscr{P}(F \geqq 0)$ denotes the set of the analytic series of $\mathbb{R}\{x, y\}$ which are positive semidefinite on the germ $\{F \geqq 0\}$. In what follows, we obtain successive restrictions on the series F. To start with, we get rid of order $\geqq 4$ series:

(4.2) First restriction. $\omega(F) \leqq 3$.

Proof. Indeed, suppose $\omega(F) \geqq 4$. By 3.3, there exists $c > 0$ such that

$$|F(x, y)| \leqq c^2(x^2 + y^2)^2.$$

Now, we set $h = c(x^2 + y^2)$. Thus, h and $h^2 - F$ are positive semidefinite, and from (•), we have $h + z \in \mathscr{P}(X_0)$; hence, we deduce that $h + z + (x^2 + y^2 + z^2)^2$ is strictly positive on $X_0 \setminus \{0\}$, but does not admit a positive semidefinite extension to $\mathbb{R}^3_0$. Consequently, X_0 does not have the $\mathscr{PE}^+$ property. □

This completes the proof of (4.2), and we can assume henceforth $\omega(F) \leqq 3$. Concerning order 2 series we have:

(4.3) Second restriction. *If* $\omega(F) = 2$, *then* X_0 *is equivalent to* $z^2 - x^2 = 0$ *or* $z^2 - x^2 - y^k = 0$ *for some* $k \geqq 2$.

Proof. After a change of coordinates, we can suppose that the equation of X_0 is $z^2 - x^2 = 0$ or of the type $z^2 + \varepsilon x^2 - y^k$ with $\varepsilon = \pm 1$, $k \geqq 2$. If $k = 2$, $z^2 + \varepsilon x^2 - y^2 = 0$ is equivalent to $z^2 - x^2 - y^2 = 0$. Now, we prove that ε must be -1 for $k \geqq 3$. This is discussed as follows.

On the surface $X_0 : z^2 + x^2 = y^{2k}$, $k \geqq 2$, we have $|y^k| = \sqrt{z^2 + x^2} \geqq |x|$. If k is even, we deduce that $x + y^k$ is positive semidefinite on X_0, and $h = x + y^k + (x^2 + y^2 + z^2)^k$ is strictly positive on $X_0 \setminus \{0\}$, but cannot be extended positively to $\mathbb{R}^3_0$. If k is odd, we get that $y^{k+1} = |y^{k+1}| \geqq |xy|$. Thus $y(x + y^k)$ is positive semidefinite on X_0, and $g = y(x + y^k) + (x^2 + y^2 + z^2)^k$ is strictly positive on $X_0 \setminus \{0\}$. If g could be extended positively to $\mathbb{R}^3_0$, there would exist an equation of the type

$$g = G + (z^2 + x^2 - y^{2k})a$$

for certain $G \in \mathscr{P}(\mathbb{R}^3_0)$ and $a \in \mathbb{R}\{x, y, z\}$. Looking at the initial forms we find a positive semidefinite quadratic form q and a constant $c \in \mathbb{R}$ such that $yx = q - c(x^2 + z^2)$, which implies that $yx + cx^2 + cz^2$ is positive semidefinite, a contradiction.

Finally, we exclude $X_0 : z^2 + x^2 = y^{2k+1}$, because the analytic series

$$y + (x^2 + y^2 + z^2)$$

is strictly positive on $X_0 \setminus \{0\}$ but cannot be extended positively to $\mathbb{R}^3_0$. □

Next we look at order 3 series and get:

(4.4) Third restriction. *If* $\omega(F) = 3$, *then* X_0 *is equivalent to one of the following*:

$$\begin{cases} z^2 - x^2 y - (-1)^k y^k = 0 \quad (k \geqq 3), \quad z^2 - x^2 y; \\ z^2 - x^3 + xy^3 = 0, \quad z^2 - x^3 - y^4 = 0 \quad \textit{or} \quad z^2 - x^3 - y^5 = 0. \end{cases}$$

Proof. After a linear change, the initial form of F is $x^2 y$, $x^2 y \pm y^3$ or x^3. We study two cases:

(4.4.1) *If* $\mathrm{In}(F) = x^2 y$ *or* $x^2 y \pm y^3$, *then* X_0 *is equivalent either to* $z^2 - x^2 y - (-1)^k y^k = 0$ $(k \geqq 3)$ *or* $z^2 - x^2 y$.

After a change of coordinates (classification of singularities), we can suppose that F is one of the following power series: $x^2 y$, $x^2 y \pm y^k$, $k \geqq 3$. If $F = x^2 y - (-1)^k y^k$

we show that there exists a strictly positive analytic function on $X_{k,0}\setminus\{0\}$ (where $X_{k,0}: z^2 = x^2y - (-1)^k y^k$ and $k \geqq 3$) which cannot be extended positively to $\mathbb{R}^3_0$.

Indeed, if k is even, we have $z^2 + y^k = x^2y$. Thus, $y \in \mathscr{P}(X_{k,0})$ and $h = y + (x^2 + y^2 + z^2)$ is strictly positive on $X_{k,0}\setminus\{0\}$, but cannot be extended positively to $\mathbb{R}^3_0$. If k is odd, we have $z^2 = (x^2 + y^{k-1})y$. Thus, $y \in \mathscr{P}(X_{k,0})$ and $h = y + (x^2 + y^2 + z^2)$ is strictly positive on $X_{k,0}\setminus\{0\}$, but cannot be extended positively to $\mathbb{R}^3_0$.

After this, we see that:

(4.4.2) *If* $\mathrm{In}(F) = x^3$ *then* X_0 *is equivalent to* $z^2 - x^3 + xy^3 = 0$, $z^2 - x^3 - y^4 = 0$ *or* $z^2 - x^3 - y^5 = 0$.

Changing x by $-x$ if necessary, there exist a Weierstrass polynomial

$$P = x^3 + p_1(y)y^2x^2 + p_2(y)y^3x + p_3(y)y^4 \quad (p_i \in \mathbb{R}\{y\})$$

and a unit $U \in \mathbb{R}\{x, y\}$ such that $U(0,0) > 0$ and $F = PU$. After the change of coordinates $(x, y, z) \mapsto \left(x - p_1(y)y^2/3, y, \sqrt{U}z\right)$, we can suppose that the equation of X_0 is of the type $z^2 - F(x, y)$ where $F(x, y) = x^3 + a(y)y^3x + b(y)y^4$ for some $a, b \in \mathbb{R}\{y\}$. After this preparation we proceed in several steps:

(a) *If* $\omega(a) \geqq 1$ *and* $\omega(b) \geqq 2$ *then* X_0 *does not have the* $\mathscr{P}\mathscr{E}^+$ *property.*

We claim that: *The analytic function germ* $x + cy^2$ *is positive semidefinite on* X_0 *for* $c > 0$ *large enough.* Thus, $h = x + cy^2 + (x^2 + y^2 + z^2)^4$ is strictly positive on $X_0\setminus\{0\}$, but cannot be extended positively to $\mathbb{R}^3_0$.

Let us see now our claim. In view of the equality ($\bullet$) above, we have to show that for $c > 0$ large enough $x + cy^2 \in \mathscr{P}(F \geqq 0)$, that is, $\{x + cy^2 \geqq 0\} \supset \{F \geqq 0\}$, or equivalently, $\{x + cy^2 < 0\} \subset \{F < 0\}$. Thus, we have to check that for a fixed $c > 0$ large enough, $0 > F(-ct^2 - v, t)$ for $v > 0$. Indeed,

$$\begin{aligned} F(-ct^2 - v, t) &= (-ct^2 - v)^3 + a(t)t^3(-ct^2 - v) + b(t)t^4 \\ &= -c^3t^6 - 3c^2vt^4 - 3ct^2v^2 - v^3 - ca(t)t^5 - va(t)t^3 + b(t)t^4 \\ &= -t^6\left(c^3 + c\frac{a(t)}{t} + \frac{b(t)}{t^2}\right) - vt^4\left(3c^2 + \frac{a(t)}{t}\right) - 3ct^2v^2 - v^3. \end{aligned}$$

Since $\omega(a) \geqq 1$ and $\omega(b) \geqq 2$, the series $a_1(t) = a(t)/t$ and $b_1(t) = b(t)/t^2$ belong to $\mathbb{R}\{t\}$. Thus, if $c > 0$ is such that $c^3 + ca_1(0) + b_1(0) > 0$ and $3c^2 + a_1(0) > 0$, then $F(-ct^2 - v, t) < 0$ for $v > 0$, and we are done.

Next, we discuss the factorization of $F = x^3 + a(y)y^3x + b(y)y^4$:

(b) *If* F *is the product of three* (*possibly equal*) *irreducible factors, then* X_0 *does not have the* $\mathscr{P}\mathscr{E}^+$ *property.*

Suppose $F = f_1 f_2 f_3$, where some or all the factors may coincide. Since the initial form of F is x^3, we can write $f_k = x + \lambda_k(x, y)$ where $\omega(\lambda_k) \geqq 2$ and then

$$\begin{aligned} F &= (x+\lambda_1)(x+\lambda_2)(x+\lambda_3) \\ &= x^3 + x^2(\lambda_1+\lambda_2+\lambda_3) + x(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) + (\lambda_1\lambda_2\lambda_3) \\ &= x^3 + a(y)y^3x + b(y)y^4. \end{aligned}$$

From this equality we deduce that

$$b(y)y^4 = F(0, y) = \lambda_1(0, y)\lambda_2(0, y)\lambda_3(0, y) \text{ has order} \geqq 6,$$

$$a(y)y^3 = \frac{\partial F}{\partial x}(0, y) = \sum_{\substack{1 \leqq i<j \leqq 3, \\ 1 \leqq k \leqq 3, k \neq i, j}} \lambda_i(0, y)\lambda_j(0, y)\left(1 + \frac{\partial \lambda_k}{\partial x}(0, y)\right) \text{ has order} \geqq 4.$$

Hence, $\omega(a) \geqq 1$, $\omega(b) \geqq 2$ and, by (4.4.2)(a), X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property.

(c) *If F is reducible, then $F = x^3 - xy^3$.*

By the previous remark, $F = fg$ and f, g must be irreducible, say $\omega(f) = 2$, $\omega(g) = 1$ and we can suppose $\mathrm{In}(f) = x^2$, $\mathrm{In}(g) = x$. If f is semidefinite, it is a sum of two squares with initial form x^2. Choosing a suitable representation of f as a sum of two squares, we can suppose $f = \big(x + \mu_1(x, y)\big)^2 + \big(\mu_2(x, y)\big)^2$ and $g = x + \mu_3(x, y)$ with $\omega(\mu_k) \geqq 2$. Thus,

$$F = \big(x + \mu_1(x, y) + i\mu_2(x, y)\big)\big(x + \mu_1(x, y) - i\mu_2(x, y)\big)\big(x + \mu_3(x, y)\big).$$

Proceeding similarly to (4.4.2)(b) (we have again three irreducible factors although two of them are complex) we are in the hypothesis of (4.4.2)(a) and X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property.

Hence, if X_0 has the $\mathscr{P}\mathscr{E}^+$ property, f should be irreducible and real. Thus, we can assume $F = (x^2 - y^k)\big(x + \mu(x, y)\big)$, $k \geqq 3$, $\omega(\mu) \geqq 2$. By the Weierstrass Preparation Theorem there exist a series $\alpha \in \mathbb{R}\{y\}$ and a unit $U \in \mathbb{R}\{x, y\}$ such that $x + \mu(x, y) = \big(x + \alpha(y)y^2\big)U(x, y)$. Changing x by $-x$ (if necessary) we can suppose $U(0,0) > 0$ and after a change $(x, y, z) \mapsto \big(x, y, \sqrt{U(x, y)}z\big)$, the equation of our germ is $z^2 - (x^2 - y^k)\big(x + \alpha(y)y^2\big)$.

For $k \geqq 4$, $F = x^3 + \alpha(y)x^2y^2 - y^kx - y^{k+2}\alpha(y)$. After the change

$$x \mapsto x - \alpha(y)y^2/3,$$

we are again in the conditions of (4.4.2)(a). Hence, X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property.

Finally, for $k = 3$ we get $F = (x^2 - y^3)(x + \cdots)$ and by classification of singularities F is equivalent to $x^3 - xy^3$.

(d) *If F is irreducible, then $F = x^3 + y^4$ or $x^3 + y^5$.*

Suppose F irreducible. By classification of singularities we can transform F into $x^3 \pm y^4$ or $F = x^3 + xy^4a'(y) + y^5b'(y)$. Suppose first $F = x^3 + xy^4a'(y) + y^5b'(y)$. If

$b'(0) = 0$, by (4.4.2)(a), X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property. If $b'(0) \neq 0$ then another change makes $F = x^3 + y^5$.

For $F = x^3 - y^4$ we see that X_0 does not have the $\mathscr{P}\mathscr{E}^+$ property. Since $z^2 + y^4 = x^3$ we have that $x \in \mathscr{P}(X_0)$ and $h = x + (x^2 + y^2 + z^2)$ is strictly positive on $X_0 \setminus \{0\}$, but cannot be extended positively to $\mathbb{R}^3_0$. $\square$

Thus, we have proved (4.4.2). Summing up, (4.2) says that $\omega(F) \leqq 3$, (4.3) that if $\omega(F) = 2$, the germ $z^2 - F = 0$ is one among (viii)–(x) in the List, and (4.4) that if $\omega(F) = 3$, the germ $z^2 - F = 0$ is one among (v)–(vii) or (xi)–(xiii) in the List. All together we conclude that an unmixed surface germ $X_0 \subset \mathbb{R}^3_0$ with the $\mathscr{P}\mathscr{E}^+$ property belongs to the List, as wanted. $\square$

Thus, we conclude the following:

Corollary 4.5. *Let $X_0 \subsetneqq \mathbb{R}^3_0$ be an analytic germ. Then the following assertions are equivalent*:

(a) $\mathscr{P}(X_0) = \Sigma_2(X_0)$.

(b) $\mathscr{P}(X_0) = \Sigma(X_0)$.

(c) *X_0 has the $\mathscr{P}\mathscr{E}$ property.*

(d) *X_0 has the $\mathscr{P}\mathscr{E}^+$ property.*

Moreover, we recall that the *Pythagoras number* $p[X_0]$ of the ring of analytic function germs $\mathscr{O}(X_0)$ of an analytic germ X_0 is the least integer $p \geqq 1$ such that every sum of squares of the ring $\mathscr{O}(X_0)$ is a sum of p squares in such ring, or $+\infty$ if such integer does not exist. In [Fe4], 1.1, it is proved again that if $X_0 \subset \mathbb{R}^3_0$ is an unmixed surface analytic germ with $p[X_0] = 2$, then X_0 belongs to the List. Thus, if we restrict our target to unmixed analytic surface germs we deduce the following surprising result:

Corollary 4.6. *Let $X_0 \subset \mathbb{R}^3_0$ be an unmixed analytic surface germ. Then the following assertions are equivalent*:

(a) $p[X_0] = 2$.

(b) $\mathscr{P}(X_0) = \Sigma(X_0)$.

(c) *X_0 has the $\mathscr{P}\mathscr{E}$ property.*

(d) *X_0 has the $\mathscr{P}\mathscr{E}^+$ property.*

5. Non-isolated singular points

In this section we study what happens with respect to the $\mathscr{P} = \Sigma$ and $\mathscr{P}\mathscr{E}$ properties around the one dimensional component of the singular set of a global analytic set X which has $\mathscr{P} = \Sigma$ and/or the $\mathscr{PE}^+$ property at the germs of all its points. Since we are working in local embedding dimension $\leqq 3$, the non-isolated singularities having such properties must be equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$ (see the List). This is because, as we have seen in 1.5 and 2.8, to have such properties globally, all the singularities have to be coherent. Related to this, the main result of this section is the following:

Theorem 5.1. *Let S be a real analytic surface in an open set $\Omega \subset \mathbb{R}^n$. Suppose that for all $x \in C = \mathrm{Sing}(S)$ the germ S_x is equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$. Let f be a positive semidefinite analytic function on S such that $C \subset \{f = 0\}$. Then, there exists an open neighbourhood W of C in S, such that f is a sum of two squares in $\mathcal{O}(W)$.*

To prove 5.1, we will need to use in a crucial way the normalization of a real analytic set. We recall here, for the sake of the reader, its definition and some of its main properties:

Definition 5.2. Let $(X, \mathcal{O}_X)$ be a real analytic space (see [Tg] for the general theory of real analytic spaces). A *normalization of* X is a pair $\big((\hat{X}, \mathcal{O}_{\hat{X}}), \pi\big)$, where $(\hat{X}, \mathcal{O}_{\hat{X}})$ is a real normal analytic space and $\pi : \hat{X} \to X$ is a surjective analytic map such that:

(a) π is proper and has finite fibers.

(b) $\hat{X} \backslash \pi^{-1}\big(\mathrm{Sing}(X)\big)$ is dense in $\hat{X}$ and the restriction

$$\pi| : \hat{X} \backslash \pi^{-1}\big(\mathrm{Sing}(X)\big) \to X \backslash \mathrm{Sing}(X)$$

is an analytic diffeomorphism.

Recall that a real analytic space $(X, \mathcal{O}_X)$ is *normal* if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is integrally closed in its total ring of fractions. Note that we have to use real analytic spaces to define the normalization because it can happen that the normalization of a real analytic set is not again a real analytic set, although it is always a real analytic space. Nevertheless, the only problem arises from the local embedding dimension of the normalization, which can be $+\infty$. Summing up [N], §VI. Lem. 2 and Thm. 4, and [Tg], §8, we get that:

Theorem 5.3. *Let $(X, \mathcal{O}_X)$ be a real coherent reduced analytic space. Then $(X, \mathcal{O}_X)$ has a coherent normalization, which is unique up to analytic equivalence, that is, if $\big((\widehat{X}_i, \mathcal{O}_{\widehat{X}_i}), \pi_i\big)$ are normalizations of $(X, \mathcal{O}_X)$ for $i = 1, 2$, then there exists an analytic diffeomorphism $\varphi : \widehat{X_1} \to \widehat{X_2}$ such that $\pi_1 = \pi_2 \circ \varphi$.*

Lemma 5.4. *Let S be a real analytic surface in an open set $\Omega \subset \mathbb{R}^n$ and let $(\hat{S}, \pi)$ be its normalization. Suppose that the germ S_x is equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$ for all $x \in C = \mathrm{Sing}(S)$. Then $\hat{S}$ is a smooth analytic surface and for all $x \in C$ there exist $y_1, y_2 \in \hat{S}$ and respective open neighbourhoods V^{y_1}, V^{y_2} of y_1, y_2 in $\hat{S}$ analytically diffeomorphic to a plane, such that*

(i) *$\pi|_{V^{y_i}} : V^{y_i} \to \pi(V^{y_i})$ is an analytic diffeomorphism for $i = 1, 2$, and*

(ii) *$W^x = \pi(V^{y_1}) \cup \pi(V^{y_2})$ is an open neighbourhood of x in S analytically diffeomorphic to the union of two transversal planes.*

Proof. Take a point $x \in C$. Since the germ S_x is equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$, there exists a neighbourhood W^x in S analytically diffeomorphic to the union of two transversal planes. Note that $\pi|_{\pi^{-1}(W^x)} : \pi^{-1}(W^x) \to W^x$ is the normalization of W^x. On the other hand, the normalization of W^x is the disjoint union of two planes. Hence, by the unicity of the normalization (see 5.3), we have $\pi^{-1}(W^x) = V^{y_1} \cup V^{y_2}$ where $\pi(y_1) = \pi(y_2) = x$, $V^{y_1} \cap V^{y_2} = \emptyset$ and V^{y_i} is an open neighbourhood of y_i in $\tilde{S}$ analytically diffeomorphic to a plane. Moreover, $\pi|_{V^{y_i}} : V^{y_i} \to \pi(V^{y_i})$ is an analytic diffeomorphism and $W^x = \pi(V^{y_1}) \cup \pi(V^{y_2})$.

Finally, since each point of $\hat{S}$ has an open neighbourhood analytically diffeomorphic to a plane, we conclude that $\hat{S}$ is a smooth analytic surface, as wanted. $\square$

Lemma 5.5. *Let $X_0 \subset \mathbb{R}^3$ be the analytic surface germ of equation $yz = 0$ and consider the analytic germs $X_{1,0} : y = 0$ and $X_{2,0} : z = 0$. Let $f_i \in \mathcal{O}(X_{i,0})$ be analytic function germs for $i = 1, 2$, which vanish on the line germ $C_0 : y = 0$, $z = 0$. Then, the function germ $f : X_0 \to \mathbb{R}$ given by $f|_{X_{i,0}} = f_i$ for $i = 1, 2$ is analytic on X_0.*

Proof. Indeed, note that

$$\mathcal{O}(X_{1,0}) = \mathcal{O}(X_0)/(y) \cong \mathbb{R}\{x, z\} \quad \text{and} \quad \mathcal{O}(X_{2,0}) = \mathcal{O}(X_0)/(z) \cong \mathbb{R}\{x, y\}.$$

Then, we may assume that

$$f_1 \in \mathbb{R}\{x, z\} \subset \mathbb{R}\{x, y, z\} \quad \text{and} \quad f_2 \in \mathbb{R}\{x, y\} \subset \mathbb{R}\{x, y, z\}.$$

Moreover, since f_1, f_2 vanish on the line germ $C_0 : y = 0$, $z = 0$, we get $f_1 = zg_1$ and $f_2 = yg_2$ where $g_1 \in \mathbb{R}\{x, z\}$ and $g_2 \in \mathbb{R}\{x, y\}$. Then, if $f = f_1 + f_2 \in \mathbb{R}\{x, y, z\}$, we have

$$g(x, 0, z) = f_1(x, 0, z) + f_2(x, 0, z) = f_1(x, z) + 0 \cdot g_2(x, 0) = f_1(x, z),$$

$$g(x, y, 0) = f_1(x, y, 0) + f_2(x, y, 0) = 0 \cdot g_1(x, 0) + f_2(x, y) = f_2(x, z).$$

Therefore, $g|_{X_{i,0}} = f_i = f|_{X_{i,0}}$ for $i = 1, 2$, and $f = g \in \mathcal{O}(X_0)$, as wanted. $\square$

Corollary 5.6. *Let S_1, S_2 be two transversal smooth analytic surfaces in an open set $\Omega \subset \mathbb{R}^n$. Let $S = S_1 \cup S_2$ and let f_i be an analytic function on S_i such that $f_i|_{S_1 \cap S_2} = 0$ for $i = 1, 2$. Then, the function $f : S \to \mathbb{R}$ given by $f|_{S_i} = f_i$ for $i = 1, 2$ is analytic on S.*

Proof. First, since S_1, S_2 are transversal smooth analytic surfaces, $C = S_1 \cap S_2$ is a smooth analytic curve and that S_x is equivalent to $X_0 : yz = 0$ for all $x \in C$.

Next, note that we only have to check the analyticity of f in the points $x \in C$. For that, it is enough to check that the germ f_x is analytic for all $x \in C$; but, this follows straightforwardly from 5.5, and we are done. $\square$

Now, we are ready to prove 5.1:

Proof of Theorem 5.1. Indeed, let $(\hat{S}, \pi)$ be the normalization of S. By 5.4, $\hat{S}$ is a smooth analytic surface. For each compact connected component $\hat{S}_k$ of $\hat{S}$ we consider a point $x_k \in \hat{S}_k \backslash \pi^{-1}(C)$. We have that $D = \bigcup_k \{x_k\}$ is a closed subset of $\hat{S}$ which does not intersect $\pi^{-1}(C)$. Note that all the connected components of $\hat{S}\backslash D$ are non compact and that $W = S\backslash\pi(D)$ is an open neighbourhood of C in S. Moreover, $(\hat{S}\backslash D, \pi|_{\hat{S}\backslash D})$ is the normalization of W.

Consider the analytic function $f \circ \pi : \hat{S}\backslash D \to \mathbb{R}$. Since f is positive semidefinite on S, we have that $f \circ \pi$ is positive semidefinite on $\hat{S}\backslash D$.

Thus, since all the connected components of the smooth analytic surface $\hat{S}\backslash D$ are non compact, by [Jw1], there exist two analytic functions $f_1, f_2 : \hat{S}\backslash D \to \mathbb{R}$ such that $f \circ \pi = f_1^2 + f_2^2$. Note that since $f|_C = 0$, we have $f \circ \pi|_{\pi^{-1}(C)} = 0$. Therefore f_1, f_2 are identically zero on $\pi^{-1}(C)$.

Consider for $j = 1, 2$, the function

$$a_j : W \to \mathbb{R}, \quad x \mapsto a_j(x) = \begin{cases} f_j \circ \pi^{-1}(x) & \text{if } x \in W\backslash C, \\ 0 & \text{if } x \in C. \end{cases}$$

Clearly, $f = a_1^2 + a_2^2$. Thus, to finish we must check that a_1, a_2 are analytic on S. For that, it is enough to prove that the functions a_i are analytic around x, for all $x \in C$.

Indeed, fix a point $x \in C$. By 5.4, there exist two different points $y_1, y_2 \in \hat{S}\backslash D$ such that $\pi(y_1) = \pi(y_2) = x$ and respective open neighbourhoods V^{y_1}, V^{y_2} of y_1, y_2 in $\hat{S}\backslash D$, analytically diffeomorphic to a plane, such that for $i = 1, 2$ the map

$$\pi|_{V^{y_i}} : V^{y_i} \to \pi(V^{y_i})$$

is an analytic diffeomorphism and $W^x = \pi(V^{y_1}) \cup \pi(V^{y_2})$ is an open neighbourhood of x in W analytically diffeomorphic to the union of two transversal planes. Consider the analytic functions

$$b_{ij} = f_j \circ (\pi|_{V^{y_i}})^{-1} : \pi(V^{y_i}) \to \mathbb{R}$$

on $\pi(V^{y_i})$ for $1 \leqq i, j \leqq 2$. Clearly, $b_{ij} \circ \pi|_{V^{y_i}} = f_j|_{V^{y_i}}$ for $1 \leqq i, j \leqq 2$.

Since f_1, f_2 are identically zero on $\pi^{-1}(C)$, the analytic functions b_{ij} vanish on $C \cap \pi(V^{y_i})$ for $1 \leqq i, j \leqq 2$. Thus, by 5.6, there exist analytic functions $b_1, b_2 \in \mathcal{O}(W^x)$ such that $b_j|_{\pi(V^{y_i})} = b_{ij}$ for $1 \leqq i, j \leqq 2$. We have that

$$a_j \circ \pi|_{V^{y_i}} = f_j|_{V^{y_i}} = b_{ij} \circ \pi|_{V^{y_i}} = b_j \circ \pi|_{V^{y_i}}$$

and $a_j|_{C \cap W^x} = b_j|_{C \cap W^x} = 0$ for $1 \leqq i, j \leqq 2$. Thus, $a_j|_{W^x} = b_j$ is analytic on W^x for $j = 1, 2$, and we are done. □

We dedicate the second part of this section to determine the topology of a real analytic surface S around a connected curve $C \subset \operatorname{Sing}(S)$ such that S_x is equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$ for all $x \in C$. We recall that germs at any closed subset Z of a real analytic set X are defined exactly as germs at a point, through neighbourhoods of Z in X. If $Y \subset X$ is a subset of X containing Z we denote by Y_Z the germ of Y at Z.

(5.7) Topology of our surfaces around the non-isolated singular points. *Let S be a real analytic surface and let $C \subset \operatorname{Sing}(S)$ be a connected curve C such that the germ S_x is equivalent to the germ $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$ for all $x \in C$. Then, S is homeomorphic around C to one of the following four surfaces*:

(i) *Two transversal planes. This is the case if C is non compact, that is, analytically diffeomorphic to a line. In particular, the germ S_C is reducible.*

(ii) *Two transversal orientable bands, that is, the union of a cylinder and a transversal circular crown. This is the case if C is compact (that is, analytically diffeomorphic to a circumference), the germ S_C is reducible, and one of the irreducible components of S_C is orientable. In particular, the other irreducible component is also orientable.*

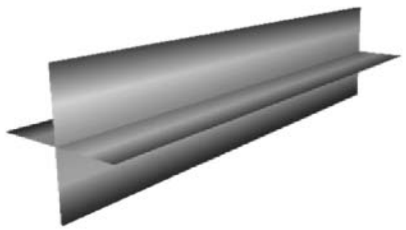
(iii) *Two transversal Moebius bands. This is the case if C is compact (that is, analytically diffeomorphic to a circumference), the germ S_C is reducible, and one of the irreducible components of S_C is non-orientable. In particular, the other irreducible component is also non-orientable.*

(iv) *Singular Moebius band. This is the case if C is compact and the germ S_C is irreducible. In this case S_C is homeomorphic to the germ at the circumference $C : x^2 + y^2 = 1$, $z = 0$ of the analytic surface S parametrized by the analytic map*:

$$\varphi : \mathbb{R} \times \left(-\frac{1}{4}, \frac{1}{4}\right) \to \mathbb{R}^3,$$

$$(\theta, \rho) \mapsto \big(\cos(2\theta), \sin(2\theta), 0\big) + \rho\left(\cos(2\theta)\cos\frac{\theta}{2}, \sin(2\theta)\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right).$$

The following table shows pictures of the previous surfaces:

(i) two transversal planes	(ii) two transversal orientable bands
	
(iii) two transversal Moebius bands	(iv) singular Moebius band
	

Remark 5.8. Consider the Moebius band M parametrized by the map

$$\psi : \mathbb{R} \times \left(-\frac{1}{4}, \frac{1}{4}\right) \to \mathbb{R}^3,$$

$$(\theta, \rho) \mapsto (\cos\theta, \sin\theta, 0) + \rho\left(\cos\theta\cos\frac{\theta}{2}, \sin\theta\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right);$$

and the map

$$\pi : M \to \mathbb{R}^3, \quad (x, y, z) \mapsto \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}, z\right).$$

Then, (M, π) is the normalization of the singular Moebius band S described above.

Proof of 5.7. First, let $(\hat{S}, \pi)$ be the normalization of S. By 5.4, we deduce that $\pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \to C$ is a 2:1 local diffeomorphism. We distinguish two cases:

Case 1. *If C is non compact, that is, analytically diffeomorphic to a straight-line, then $\pi^{-1}(C)$ is the disjoint union of two straight-lines and S is homeomorphic around C to two transversal planes.*

Indeed, since C is non compact and $\pi|_{\pi^{-1}(C)}$ is a local diffeomorphism, each connected component of $\pi^{-1}(C)$ is non compact, hence, analytically diffeomorphic to a straigh-line. Thus, if L is a connected component of $\pi^{-1}(C)$, the local diffeomorphism $\pi|_L : L \to C$ defines a global diffeomorphism between L and $\pi(L)$. Consequently, $\pi(L)$ is an open subset of C. On the other hand, since π is proper and L is closed (because it is an analytic subset of $\tilde{S}$), we conclude that $\pi(L) = C$. Thus, $\pi|_{\pi^{-1}(C)}$ being a 2:1 map, we conclude that $\pi^{-1}(C)$ is the disjoint union of two straight-lines L_1, L_2.

Next, since L_1, L_2 are closed disjoint subsets of $\hat{S}$, we can find disjoint open neighbourhoods W_i of L_i in $\hat{S}$ which are moreover analytically diffeomorphic to a plane. One can check that $W = \pi(W_1 \cup W_2)$ is an open neighbourhood of C in W homeomorphic to the union of *two transversal planes*. □

Case 2. *If C is compact, that is, analytically diffeomorphic to a circumference, then $\pi^{-1}(C)$ is either a circumference or the disjoint union of two. In the first case, S is homeomorphic around C to a singular Moebius band, and in the second one, S is homeomorphic around C either to two transversal orientable bands or to two transversal Moebius bands.*

Indeed, since C is compact and $\pi|_{\pi^{-1}(C)}$ is proper, we deduce that each connected component of $\pi^{-1}(C)$ is compact, hence, analytically diffeomorphic to a circumference. Moreover, since $\pi|_{\pi^{-1}(C)}$ is a local diffeomorphism, the image under π of each connected component of $\pi^{-1}(C)$ is the whole C. Thus, $\pi|_{\pi^{-1}(C)}$ being a 2:1 map, we conclude that $\pi^{-1}(C)$ is either a circumference or the disjoint union of two. We distinguish both situations:

(5.8.1) *If $\pi^{-1}(C)$ is the disjoint union of two circumferences C_1, C_2, then S is homeomorphic around C either to two transversal orientable bands or to two transversal Moebius bands.*

Since C_1, C_2 are disjoint closed subsets of $\hat{S}$, there exist disjoint open neighbourhoods W_i of C_i in $\hat{S}$ which are moreover homeomorphic either to a cylinder or to a Moebius band. Thus, it is enough to see that W_1 is orientable if and only if W_2 is orientable.

Indeed, assume that W_1 is orientable and fix an orientation in W_1. Take $x \in C$; we know that x has a neighbourhood W^x analytically diffeomorphic to the union of two transversal planes, say $X = \{yz = 0\} \subset \mathbb{R}^3$. Let $\psi : W^x \to X$ be such a diffeomorphism. We may assume that ψ maps $W_1 \cap W^x$ on the plane $\{z = 0\}$ and $W_2 \cap W^x$ on the plane $\{y = 0\}$. We also assume that W_1 induces (through ψ) in $\{y = 0\}$ the orientation given by the basis $\{(1,0,0),(0,1,0)\}$. After this choice, we consider in $W_2 \cap W^x$ the orientation induced by ψ and the basis $\{(1,0,0),(0,0,1)\}$ on the plane $\{z = 0\}$. One can check that this procedure determines an orientation in W_2; and, W_2 is orientable. The converse follows interchanging the roles of W_1 and W_2.

Thus, $W_1 \cup W_2$ is homeomorphic either to *two transversal oriented bands* or to *two transversal Moebius bands*. $\square$

(5.8.2) *If $\pi^{-1}(C) = C_0$ is a circumference, then S is homeomorphic around C to a singular Moebius band.*

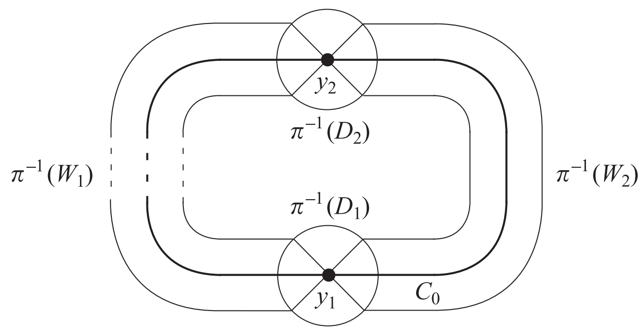
Indeed, let $x_0 \in C$ and $y_1, y_2 \in C_0$ be such that $\pi(y_1) = \pi(y_2) = x_0$. Note that $L = C \backslash \{x_0\}$ is an analytic curve in $S \backslash \{x_0\}$ analytically diffeomorphic to a straight-line. By the Case 1, L has an open neighbourhood in $S \backslash \{x_0\}$ analytically diffeomorphic to the union of two transversal planes. Let W_1, W_2 be the irreducible components of such open neighbourhood. On the other hand, since S_{x_0} is analytically diffeomorphic to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$, x_0 has an open neighbourhood in S analytically diffeomorphic to the union of two transversal planes. We denote by D_1, D_2 the irreducible components of such neighbourhood. We shrink W_1, W_2 and D_1, D_2 in such a way that:

(i) $V = \pi^{-1}(W_1 \cup W_2 \cup D_1 \cup D_2)$ defines a neighbourhood of C_0 in $\hat{S}$ homeomorphic either to a cylinder or to a Moebius band.

(ii) $\pi^{-1}(W_i) \cap \pi^{-1}(W_j) = \emptyset$ if $i \neq j$.

(iii) $\pi^{-1}(W_i) \cap \pi^{-1}(D_j)$ is a non-empty connected open set for $1 \leqq i, j \leqq 2$.

(iv) W_i, D_j are still analytically diffeomorphic to a plane.

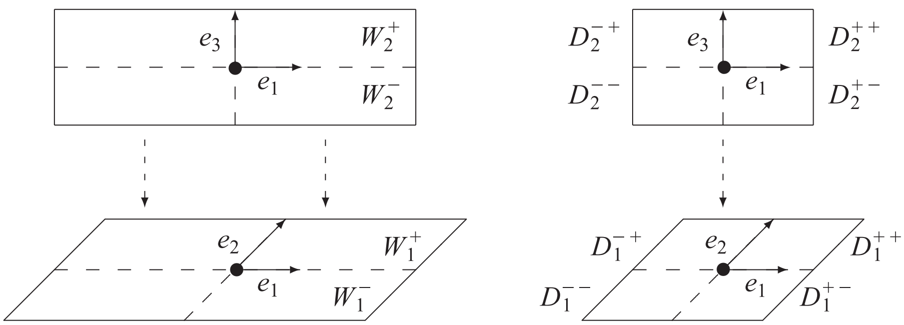


Let $\varphi : \{yz = 0\} \subset \mathbb{R}^3 \to W_1 \cup W_2$ and $\psi : \{yz = 0\} \subset \mathbb{R}^3 \to D_1 \cup D_2$ be analytic diffeomorphisms such that $\varphi(\{z = 0\}) = W_1$, $\varphi(\{y = 0\}) = W_2$, $\psi(\{z = 0\}) = D_1$ and $\psi(\{y = 0\}) = D_2$. Consider the open sets:

$$\begin{aligned}
W_1^{\varepsilon} &= \varphi(\{z = 0, \varepsilon y > 0\}), && \varepsilon = \pm,\\
W_2^{\varepsilon} &= \varphi(\{y = 0, \varepsilon z > 0\}), && \varepsilon = \pm,\\
D_1^{\delta} &= \psi(\{z = 0, \delta x > 0\}), && \delta = \pm,\\
D_2^{\delta} &= \psi(\{y = 0, \delta x > 0\}), && \delta = \pm,\\
D_1^{\delta\varepsilon} &= \psi(\{z = 0, \delta x > 0, \varepsilon y > 0\}), && \delta = \pm,\ \varepsilon = \pm,\\
D_2^{\delta\varepsilon} &= \psi(\{y = 0, \delta x > 0, \varepsilon z > 0\}), && \delta = \pm,\ \varepsilon = \pm.
\end{aligned}$$

Up to shrinking the sets W_1, W_2, if necessary, we can suppose moreover that for $\varepsilon = \pm$ and $\delta = \pm$ the set $W_i^{\varepsilon} \cap D_j^{\delta}$ is either empty or a subset of $D_j^{\delta\rho}$ for some $\rho = \pm$. We may also assume, using the symmetries with respect to the planes $y = 0$ and $z = 0$ in the domain of ψ, that $W_i^{\varepsilon} \cap D_i^{+} \subset D_i^{+\varepsilon}$ for $i = 1, 2$ and $\varepsilon = \pm$.

Consider the vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ and the diagrams:



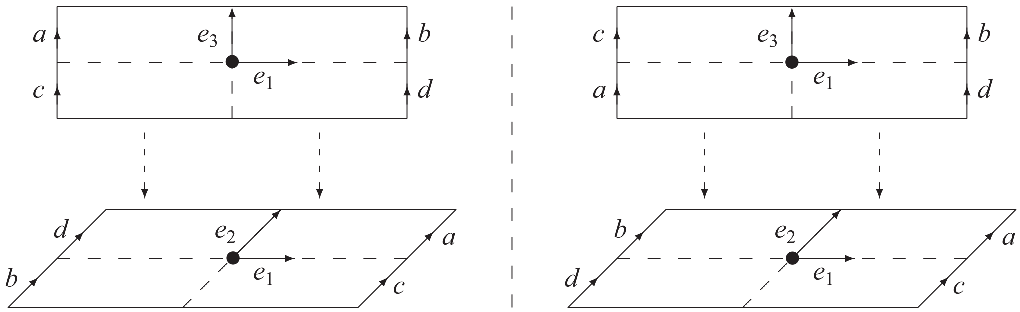
Since $W_1 \cap D_2^-$ must contain a non empty open set, $W_1^+ \cap D_2^- \subset D_2^{-\varepsilon}$ for some $\varepsilon = \pm$. If $W_1^+ \cap D_2^- \subset D_2^{-+}$, we deduce, using the diagrams above that

$$\begin{aligned}
W_1^+ \cap D_2^- &\subset D_2^{-+}, & W_1^- \cap D_2^- &\subset D_2^{--},\\
W_2^+ \cap D_1^- &\subset D_1^{--}, & W_2^- \cap D_1^- &\subset D_1^{-+}.
\end{aligned}$$

Otherwise, we have:

$$\begin{aligned}
W_1^+ \cap D_2^- &\subset D_2^{--}, & W_1^- \cap D_2^- &\subset D_2^{-+},\\
W_2^+ \cap D_1^- &\subset D_1^{-+}, & W_2^- \cap D_1^- &\subset D_1^{--}.
\end{aligned}$$

Thus, $W = \pi(V) = W_1 \cup W_2 \cup D_1 \cup D_2$ is homeomorphic to one of the two surfaces given by the identifications:



Using a suitable symmetry, one can check that both surfaces are homeomorphic, and in fact, both are homeomorphic to a *singular Moebius band*. Finally, since $W \setminus C$ is connected, we deduce that $V \setminus C_0 = \pi^{-1}(W) \setminus C_0$ is connected. Hence, V is homeomorphic to a Moebius band, as wanted. □

6. Global results

In this section we finally prove 1.6. The key result to prove it is the following:

Theorem 6.1. *Let S be a real coherent analytic surface in an open set $\Omega \subset \mathbb{R}^n$ such that $\mathscr{P}(S_x) = \Sigma_2(S_x)$ for all $x \in \mathrm{Sing}(S)$. Suppose that any non-isolated singularity S_x of S is equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$. Then, $\mathscr{P}(S) = \Sigma_6(S)$.*

Remark 6.2. Note that if an analytic function germ $g \in \mathcal{O}(S_x)$ belongs to $\Sigma_2(S_x)$, then it can be written as $g = a^2 + b^2 = (a + \sqrt{-1}b)(a - \sqrt{-1}b)$ for certain analytic function germs $a, b \in \mathcal{O}(S_x)$. Hence, g is reducible in the ring $\mathcal{O}(S_x) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{O}(\tilde{S}_x)$ of holomorphic function germs on the complexification $\tilde{S}_x$ of S_x. □

Thus, the *complexification* of a real analytic set will play a crucial role for the proof of 6.1. We recall here, for the sake of the reader, its definition and some of its main properties.

Definition 6.3. Let X be a real analytic set in an open set $\Omega \subset \mathbb{R}^n$. A complexification $\tilde{X}$ of X is a complex analytic set $\tilde{X}$ in an open neighbourhood U of Ω in $\mathbb{C}^n$ such that:

(i) X is a closed subset of $\tilde{X}$ and $X = \tilde{X} \cap \mathbb{R}^n$.

(ii) $\mathcal{O}(\tilde{X}_x) = \mathcal{O}(X_x) \otimes_{\mathbb{R}} \mathbb{C}$ for all $x \in X$.

Some of the most relevant properties of the complexification are summarized in the following result ([Tg], §3, §4).

Theorem 6.4. *Let X be a real coherent analytic set in an open set $\Omega \subset \mathbb{R}^n$. Then*:

(a) *Existence*: *There exists a complexification $\tilde{X}$ of X.*

(b) *Uniqueness*: *If $\tilde{X}_1$ and $\tilde{X}_2$ are complexifications of X, then $\tilde{X}_1 \cap \tilde{X}_2$ is a complexification of X.*

(c) *X has in $\tilde{X}$ a fundamental system of open neighbourhoods which are Stein spaces.*

Consider the usual conjugation $\sigma : \mathbb{C}^n \to \mathbb{C}^n$, $z \mapsto \bar{z} = (\overline{z_1}, \dots, \overline{z_n})$, whose fixed points are $\mathbb{R}^n$. A subset $A \subset \mathbb{C}^n$ is (σ-)*invariant* if $\sigma(A) = A$; obviously, $A \cap \sigma(A)$ is the biggest invariant subset of A. The restriction of σ to an invariant complexification $\tilde{X}$ of a real analytic set X defines an involution in $\tilde{X}$ whose fixed set is X. Let $F : \tilde{X} \to \mathbb{C}$ be a holomorphic function. We say that F is (σ-)*invariant* if $F(z) = \overline{F \circ \sigma(z)}$ for all $z \in \tilde{X}$. This implies that F restricts to a real analytic function on X.

Now, we are ready to prove 6.1.

Proof of Theorem 6.1. Let $f \in \mathscr{P}(S)$ be a positive semidefinite analytic function on S. We have to check that f is a sum of 6 squares of analytic functions on S. The proof of this fact runs in several steps:

Step 1. *Initial preparation.*

First of all, let $\{C_j\}_{j \in J}$ be the irreducible components of dimension 1 of $\mathrm{Sing}(S)$. By the hypothesis about the non-isolated singular points of S, each C_j is a connected smooth analytic curve.

Let S_1 be the union of the global irreducible components of S over which f is not identically 0 and S_2 the union of those over which f is identically 0. Note that S_1, S_2 are closed subsets of S whose union is S and whose intersection is an analytic subset of $\mathrm{Sing}(S)$ which necessarily has dimension $\leqq 1$.

Next, we write

$$f^{-1}(0) \cap S_1 = D_1 \cup \bigcup_{k \in K} Y_k \cup \bigcup_{i \in I} C_i$$

as the union of its irreducible components, where $I \subset J$, D_1 is a discrete set and each Y_k is an irreducible analytic curve not contained in $\mathrm{Sing}(S)$. □

Step 2. *Study of the behaviour of f around the regular points of the one dimensional part of its zero set.*

Write $Y = \bigcup_{k \in K} Y_k$. There exists a discrete set $D_2 \subset Y$ satisfying the following property: *For each $k \in K$, there exist an analytic function $h_k : S \to \mathbb{R}$ and a positive integer $m_k \geqq 1$ such that for all $x \in Y_k \backslash D_2$ the curve germ $Y_x = Y_{k,x}$ is regular, the surface germ S_x is regular, $\mathscr{I}_{Y,x} = h_k \mathscr{O}_{S,x}$ and $f_x \mathscr{O}_{S,x} = h_k^{2m_k} \mathscr{O}_{S,x}$.*

Indeed, fix $k \in K$ and choose any regular point $a \in Y_k$ off $\mathrm{Sing}(S)$. By Cartan's Theorem A, the ideal $\mathscr{I}_{Y_k,a}$ is generated by finitely many global analytic functions $f_1, \dots, f_r \in \mathscr{O}(S)$ which vanish on Y_k, and at least one of the germs $f_{i,a}$ does not belong to $\mathscr{I}_{Y,a}^2$. Say this is true for $i = 1$ and write $h_k = f_1$. Since a is a regular point of Y_k, we have that $h_{k,a}$ generates $\mathscr{I}_{Y_k,a}$. In fact, since Y_k is coherent, because it is a curve, $h_{k,x}$ generates $\mathscr{I}_{Y_k,x}$ for all $x \in Y_k$ close enough to a. Now, consider the coherent sheaf of ideals

$$\mathscr{I}_x = (h_{k,x}\mathcal{O}_{S,x} : \mathscr{J}_{Y_k,x}), \quad x \in S,$$

and notice that $\mathscr{I}_x = \mathcal{O}_{S,x}$ if and only if $h_{k,x}$ generates $\mathscr{J}_{Y_{k,x}}$. Hence, the support

$$\operatorname{supp}(\mathcal{O}_S/\mathscr{I}) = \{x \in Y_k : h_{k,x} \text{ does not generate } \mathscr{J}_{Y_k,x}\}$$

is a closed analytic set Y_k' that does not contain Y_k. As Y_k is an irreducible curve, $E_k = Y_k \cap Y_k'$ is a discrete subset of Y_k. Thus, $h_{k,x}$ generates $\mathscr{J}_{Y_{k,x}}$ for all $x \in Y_k \backslash E_k$. Consider the discrete subset

$$E_k' = E_k \cup \operatorname{Sing}(Y_k) \cup \big(\operatorname{Sing}(X) \cap Y_k\big) \cup \bigcup_{\alpha \neq k} (Y_\alpha \cap Y_k)$$

of Y_k. For each $x \in Y_k \backslash E_k$ we have that $Y_x = Y_{k,x}$ is a regular curve germ, $h_{k,x}$ generates $\mathscr{J}_{Y_{k,x}} = \mathscr{J}_{Y_x}$ and S_x is a regular surface germ.

Next, take a regular point $b \in Y_k \backslash E_k'$. We have that $f_b = v_b h_{k,b}^{\alpha_b}$ for some integer $\alpha_b \geqq 1$ and an analytic germ $v_b \notin \mathscr{J}_{Y_k,b}$. Thus, since Y_k has only one branch at b (because it is a regular point of Y_k), we deduce that $v(x) \neq 0$ for x close to b and v_x is a unit for such x's. Consequently, for $x \neq b$ close enough to b we have

$$h_{k,x}^{\alpha_b} | f_x \quad \text{and} \quad f_x | h_{k,x}^{\alpha_b}.$$

Now, we consider the coherent sheaf of ideals $\mathscr{H}$ of $\mathcal{O}_S$ given by

$$\mathscr{H}_x = (h_{k,x}^{\alpha_b} : f_x) \cap (f_x : h_{k,x}^{\alpha_b}), \quad x \in S.$$

The support

$$\operatorname{supp}(\mathcal{O}_S/\mathscr{H}) = \{x \in X : h_{k,x}^{\alpha_b} \nmid f_x \text{ or } f_x \nmid h_{k,x}^{\alpha_b}\}$$

is an analytic set Y_k'' that does not contain Y_k, hence $E_k'' = Y_k \cap Y_k''$ is a discrete subset of Y_k. If $x \in Y_k \backslash (E_k' \cup E_k'')$, then $f_x = h_{k,x}^{\alpha_b} v_x$ for certain unit $v_x \in \mathcal{O}_{S,x}$; since f_x is a positive semidefinite analytic function we have moreover that $\alpha_b = 2m_k$ is even.

Since $\{Y_k\}_{k \in K}$ is a locally finite family of closed subsets of S and the set $E_k' \cup E_k''$ is a discrete subset of Y_k for all $k \in K$, we deduce that $D_2 = \bigcup_{k \in K} (E_k' \cup E_k'')$ is a discrete subset of S. Moreover, note that D_2 has the desired property. $\square$

Step 3. *Construction of a coherent sheaf which represents f locally as a sum of two squares.*

Consider the analytic set $C = \bigcup_{i \in I} C_i$ and the discrete set $D = D_1 \cup (D_2 \backslash C)$. Let $\tilde{S}$, $\tilde{Y}$ and $\tilde{C}$ be respective complexifications of S, Y and C (which exist because S, Y and C are real coherent analytic sets) such that $\tilde{Y}$ and $\tilde{C}$ are closed subsets of $\tilde{S}$ and f can be extended holomorphically to a function $F : \tilde{S} \to \mathbb{C}$.

Next, we write $D = \{x_\ell\}_{\ell \in L}$. For each $\ell \in L$, let $V^{x_\ell} \subset \tilde{S}$ be an open neighbourhood of x_ℓ in $\tilde{S}$ such that $\operatorname{Cl}_{\tilde{S}}(V^{x_\ell}) \cap \tilde{C} = \emptyset$ for all $\ell \in L$, and $\{\operatorname{Cl}_{\tilde{S}}(V^{x_\ell})\}_{\ell \in L}$ is a locally finite family of disjoint closed sets in $\tilde{S}$.

Moreover, for each $\ell \in L$ we have $\mathscr{P}(S_{x_\ell}) = \Sigma_2(S_{x_\ell})$. Thus, there exist analytic function germs $a_{\ell,x_\ell}, b_{\ell,x_\ell} \in \mathcal{O}_{S,x_\ell}$ such that $f_{x_\ell} = a_{\ell,x_\ell}^2 + b_{\ell,x_\ell}^2$. Shrinking V^{x_ℓ}, if necessary, we may assume that there exist holomorphic functions $A_\ell, B_\ell : V^{x_\ell} \to \mathbb{C}$ such that the analytic functions $A_\ell|_{V^{x_\ell} \cap S}$ and $B_\ell|_{V^{x_\ell} \cap S}$ define respectively at x_ℓ the germs a_{ℓ,x_ℓ} and b_{ℓ,x_ℓ}. Note that $F_z = A_{\ell,z}^2 + B_{\ell,z}^2$ for all $z \in V^{x_\ell}$.

By 5.1, there exist an open neighbourhood $W \subset S$ of C in S and two analytic functions $g_1, g_2 : W \to \mathbb{R}$ such that $f|_W = g_1^2 + g_2^2$. Choose an open neighbourhood V_0 of C in $\tilde{S}$ on which g_1, g_2 extend to holomorphic functions G_1, G_2. We may assume, after reducing the open set V_0 if necessary, that $\mathrm{Cl}_{\tilde{S}}(V^{x_\ell}) \cap \mathrm{Cl}_{\tilde{S}}(V_0) = \emptyset$ for all $\ell \in L$.

For each $k \in K$, let V_k be a neighbourhood of Y_k in $\tilde{S}$ on which we can extend h_k to a holomorphic function $H_k : V_k \to \mathbb{C}$. For each $x \in Y_k \setminus D_2$ we consider an open neighbourhood $V^x \subset \mathrm{Reg}(\tilde{S}) \setminus (D \cup \tilde{C}) \subset V_k$ of x in $\tilde{S}$ such that:

(i) $F^{-1}(0) \cap V^x = \tilde{Y} \cap V^x$.

(ii) $\tilde{Y}_z$ is a regular complex curve germ for all $z \in V^x \cap \tilde{Y}$.

(iii) $H_{k,z}$ generates $\mathscr{J}_{\tilde{Y},z}$ for all $z \in V^x$.

(iv) $F_z \mathcal{O}_{\tilde{S},z} = H_{k,z}^{2m_k} \mathcal{O}_{\tilde{S},z}$ for all $z \in V^x$.

Next, since for each $x \in S_2 \setminus S_1$ the function f is identically 0 around x, there exists an open neighbourhood V_1 of $S_2 \setminus S_1$ in $\tilde{S}$ such that $F|_{V_1} \equiv 0$.

Consider the open neighbourhood

$$V = \big(\tilde{S} \setminus F^{-1}(0)\big) \cup V_0 \cup V_1 \cup \bigcup_{\ell \in L} V^{x_\ell} \cup \bigcup_{x \in Y \setminus D_2} V^x$$

of S in $\tilde{S}$. We consider an invariant complexification of S contained in V which is moreover a Stein space. To simplify notation we denote again by $\tilde{S}$ such complexification. Let σ be the involution of $\tilde{S}$ induced by the complex conjugation of $\mathbb{C}^n$.

We denote again by V^{x_ℓ}, V_0, V_k, V^x and V_1 the intersections of such open sets with $\tilde{S}$ and by A_ℓ, B_ℓ, H_k, G_1, G_2 the restrictions of such holomorphic functions to the respective open set V^{x_ℓ}, V_k or V_0 where they are defined. We also denote $\tilde{Y}$ and $\tilde{C}$ the intersections of such sets with $\tilde{S}$, and by F the restriction of F to $\tilde{S}$.

Next, consider the subsheaf of ideals $\mathscr{F}$ of $\mathcal{O}_{\tilde{S}}$ given by

$$\mathscr{F}_z = \begin{cases} (A_{\ell,z} + \sqrt{-1} B_{\ell,z}) \mathcal{O}_{\tilde{S},z} & \text{if } z \in V^{x_\ell} \text{ and } \ell \in L, \\ H_{k,z}^{m_k} \mathcal{O}_{\tilde{S},z} & \text{if } z \in V^x,\ x \in Y_k \setminus D_2 \text{ and } k \in K, \\ (G_{1,z} + \sqrt{-1} G_{2,z}) \mathcal{O}_{\tilde{S},z} & \text{if } z \in V_0, \\ (0) & \text{if } z \in V_1, \\ \mathcal{O}_{\tilde{S},z} & \text{if } z \in \tilde{S} \setminus F^{-1}(0). \end{cases}$$

Let us see that $\mathscr{F}$ is a well-defined coherent sheaf on the Stein space $\tilde{S}$.

Indeed, consider the open covering

$$\mathscr{V} = \{\tilde{S}\backslash F^{-1}(0), V_0, V_1\} \cup \{V^{x_\ell}\}_{\ell \in L} \cup \{V^x\}_{x \in Y\backslash D_2}$$

of $\tilde{S}$. For each open set $U \in \mathscr{V}$, there exists a holomorphic function $\Psi_U : U \to \mathbb{C}$ such that $F|_U = \Psi_U \overline{\Psi_U \circ \sigma}$ and $\mathscr{F}|_U = \Psi_U \mathscr{O}_{\tilde{S}}|_U$. Next, take $z \in \tilde{S}$. We distinguish several cases:

(a) If $F(z) \neq 0$, we have $(\mathscr{F}|_U)_z = \mathscr{O}_{\tilde{S},z}$ for all $U \in \mathscr{V}$ such that $z \in U$.

(b) If $F(z) = 0$ and $z \in V_1$, then $(\mathscr{F}|_U)_z = (0)$ for all $U \in \mathscr{V}$ such that $z \in U$.

(c) If $F(z) = 0$ and $z \notin V_1$ we proceed as follows. By the construction of the open covering $\mathscr{V}$, we just have to consider the case when $z \in V^x \cap U$ where $x \in Y_k\backslash D_2$ for some $k \in K$, and U is another open set of $\mathscr{V}$. Let us see that in such case $(\mathscr{F}|_{V^x})_z = (\mathscr{F}|_U)_z$.

We have that $(\mathscr{F}|_{V^x})_z = H_{k,z}^{m_k} \mathscr{O}_{\tilde{S},z}$ and $(\mathscr{F}|_U)_z = \Psi_{U,z}\mathscr{O}_{\tilde{S},z}$. Thus, we have to check that

$$H_{k,z}^{m_k} \mathscr{O}_{\tilde{S},z} = \Psi_z \mathscr{O}_{\tilde{S},z}.$$

Recall that by the choice of V^x we have that $\tilde{Y}_z$ is a regular complex curve germ, $\tilde{S}_z$ is a regular complex surface germ, $\mathscr{J}_{\tilde{Y},z} = H_{k,z}\mathscr{O}_{\tilde{S},z}$ and $H_{k,z}^{2m_k}\mathscr{O}_{\tilde{S},z} = F_z\mathscr{O}_{\tilde{S},z}$. In particular, there exists an integer $\ell \geqq 0$ and an analytic germ $G_z \in \mathscr{O}_{\tilde{S},z}\backslash \mathscr{J}_{\tilde{Y},z}$ such that $\Psi_z = H_{k,z}^{\ell} G_z$. Thus, $\overline{\Psi_z \circ \sigma} = H_{k,z}^{\ell}\overline{G_z \circ \sigma}$, and therefore

$$H_{k,z}^{2m_k}\mathscr{O}_{\tilde{S},z} = F_z\mathscr{O}_{\tilde{S},z} = (\Psi_z\overline{\Psi_z \circ \sigma})\mathscr{O}_{\tilde{S},z} = (H_{k,z}^{2\ell} G_z \overline{G_z \circ \sigma})\mathscr{O}_{\tilde{S},z}.$$

Hence, since $\mathscr{O}_{\tilde{S},z}$ is a unique factorization domain (because $\tilde{S}_z$ is a regular germ), $\ell = m_k$ and $G_z \in \mathscr{O}_{\tilde{S},z}$ is a unit. Therefore, we deduce that $\Psi_z\mathscr{O}_{\tilde{S},z} = H_{k,z}^{m_k}\mathscr{O}_{\tilde{S},z}$, as wanted. □

Step 4. *Representation of f as a sum of six squares in $\mathscr{O}(\tilde{S})$.*

Now, since $\tilde{S}$ is a Stein space of dimension 2 we have by [Co] that there exist global sections $F_1, F_2, F_3 : \tilde{S} \to \mathbb{C}$ which generate $\mathscr{F}$. Consider the holomorphic invariant function

$$F' = F_1\overline{F_1 \circ \sigma} + F_2\overline{F_2 \circ \sigma} + F_3\overline{F_3 \circ \sigma}.$$

We have that F' restricts to S to a real analytic function f' which is a sum of six squares in $\mathscr{O}(S)$. Moreover, we get that $F'_x\mathscr{O}_{\tilde{S},x} = F_x\mathscr{O}_{\tilde{S},x}$ for all $x \in S$.

Indeed, fix $x \in S$. By the construction of $\mathscr{F}$, there exists a holomorphic function germ $\Psi_x \in \mathscr{O}_{\tilde{S},x}$ such that $F_x = \Psi_x\overline{\Psi_x \circ \sigma}$ and $\mathscr{F}_x = \Psi_x\mathscr{O}_{\tilde{S},x}$. Thus, there exist holomorphic function germs $\Phi_{1,x}, \Phi_{2,x}, \Phi_{3,x} \in \mathscr{O}_{\tilde{S},x}$ such that $F_{\lambda,x} = \Psi_x\Phi_{\lambda,x}$ for $\lambda = 1,2,3$. Moreover, since $\mathscr{F}_x = (F_{1,x}, F_{2,x}, F_{3,x})\mathscr{O}_{\tilde{S},x}$, we may assume that $\Phi_{1,x}$ is a unit. Hence,

$$F'_x = \sum_{\ell=1}^{3} F_{\ell,x}\overline{F_{\ell,x} \circ \sigma} = (\Psi_x\overline{\Psi_x \circ \sigma})\sum_{\ell=1}^{3} \Phi_{\ell,x}\overline{\Phi_{\ell,x} \circ \sigma} = F_x u_x$$

where $u_x = \sum_{\ell=1}^{3} \Phi_{\ell,x}\overline{\Phi_{\ell,x} \circ \sigma} \in \mathscr{O}_{\tilde{S},x}$ is a unit. Thus, $F'_x\mathscr{O}_{\tilde{S},x} = F_x\mathscr{O}_{\tilde{S},x}$ for all $x \in S$.

Now, since f and f' are positive semidefinite analytic functions and $f_x \mathcal{O}_{S,x} = f'_x \mathcal{O}_{S,x}$, there exist an open neighbourhood Ω_1 of S_1 in S and a strictly positive semidefinite analytic function $u : \Omega_1 \to \mathbb{R}$ such that $f|_{\Omega_1} = f'|_{\Omega_1} u^2$. Thus, f is a sum of six squares of analytic functions on Ω_1. Let $a_1, \dots, a_6 \in \mathcal{O}(\Omega_1)$ be such that

$$f|_{\Omega_1} = a_1^2 + \cdots + a_6^2.$$

Since $f|_{S_2} \equiv 0$, each a_i is identically 0 on $S_2 \cap \Omega_1$. Hence, we can extend analytically each a_i by 0 to the whole S. Therefore f is a sum of six squares of analytic functions on S, and we are done. $\square$

Now, we are ready to prove 1.6:

Proof of Theorem 1.6. First, it is clear that (a) implies (b), and (b) implies (c). Next, let us check that (c) implies (d). Indeed, suppose (c) holds. By 2.5, for all $x \in X$ the germ X_x has the $\mathscr{P}\mathscr{E}^+$ property. Hence, by 4.1, all the germs X_x belong to the List. Moreover, by 2.8 (b), none of the germs X_x can be equivalent to Whitney's umbrella singularity because otherwise X does not have the $\mathscr{P}\mathscr{E}$ property. Thus, we deduce that each germ X_x is coherent, and (d) holds.

Finally, by 4.5, [ABFR2], 1.1, and 6.1 we have that (d) implies (a), and we are done. $\square$

We finish this section with a collection of examples of analytic surfaces X_n, of embedding dimension $n+1$, which have $\mathscr{P} = \Sigma$.

Examples 6.5. The *Veronese cone* $X_n \subset \mathbb{R}^{n+1}$, $n \geqq 2$ (cone over the rational normal curve) is the analytic surface given by the equations

$$F_{ij} = x_i x_j - x_{i-1} x_{j+1} = 0, \quad 1 \leqq i \leqq j \leqq n-1,$$

and whose complexification in $\mathbb{C}^n$ is parametrized by $\gamma(z, w) = (z^n, z^{n-1}w, \dots, zw^{n-1}, w^n)$, (see [Ha]). It can be proved that X_n is a coherent surface and that $\operatorname{Sing}(X_n) = \{0\}$. Note that the surface germ $X_{n,0}$ has embedding dimension $n+1$. We also have that $\mathscr{P}(X_{n,0}) = \Sigma_2(X_{n,0})$ for all $n \geqq 2$ (see [Fe4], 4.1). Hence, by 6.1, we deduce that $\mathscr{P}(X_n) = \Sigma(X_n)$ for all $n \geqq 2$.

7. Conjectures and open questions

For higher local embedding dimension the situation is quite more delicate and we have not achieved concluding results. The most remarkable are those referring to analytic curve germs developed in Section 3 (see 3.2, 3.9 and 3.10). In view of such results, we consider that the predictable behaviour about the $\mathscr{P} = \Sigma$ and $\mathscr{P}\mathscr{E}$ properties for analytic curves can be summarized in the following two conjectures:

Conjecture 7.1. *Let $X_0 \subset \mathbb{R}^n$ be an analytic curve germ. The following assertions are equivalent*:

(a) $\mathscr{P}(X_0) = \Sigma(X_0)$.

(b) X_0 *is equivalent to a union of* $r \leqq n$ *independent lines through the origin.*

(c) X_0 *has the* $\mathscr{PE}$ *property.*

(d) X_0 *has the* $\mathscr{PE}^+$ *property.*

Conjecture 7.2. *Let* C *be an analytic curve in an open set* $\Omega \subset \mathbb{R}^n$. *The following assertions are equivalent*:

(a) $\mathscr{P}(C) = \Sigma(C)$.

(b) $\mathscr{P}(C_x) = \Sigma(C_x)$ *for each* $x \in C$.

(c) C_x *is equivalent to a union of independent lines through the origin for each* $x \in C$.

(d) C *has the* $\mathscr{PE}$ *property.*

(e) C_x *has the* $\mathscr{PE}$ *property for each* $x \in C$.

(f) C_x *has the* $\mathscr{PE}^+$ *property for each* $x \in C$.

Notice that both conjectures are true for local embedding dimension $n \leqq 3$ (see 3.1, [Sch], 3.9, [ABFR2], 1.1, and 1.6). Moreover,

(i) Conjecture 7.1 holds, by 3.10, for analytic germs $X_0 \subset \mathbb{R}^n_0$ which have only nonsingular branches, and therefore

(ii) Conjecture 7.2 holds for analytic curves $X \subset \mathbb{R}^n$ whose irreducible components are all regular curves.

With respect to Conjecture 7.1 we have, by [Sch], 3.9, that the assertions (a) and (b) are equivalent. Also, (a) implies (c) and (c) implies (d). Thus, to prove 7.1 it would be enough to prove that (d) implies (b).

Furthermore, as it is proved in [ABFR2], 1.1, the assertions (a), (b) and (c) in Conjecture 7.2 are equivalent. Moreover, (a) implies (d), (b) implies (e), (e) implies (f) and by 2.5 also (d) implies (f). In fact, note that if Conjecture 7.1 is true, then (f) will imply (c) and Conjecture 7.2 will hold. In general, every local *situation* on which 7.1 holds, gives us a global situation on which 7.2 holds.

Remark 7.3. The suitable strategy to prove that (d) implies (b) in Conjecture 7.1 could be to improve 3.9. In view of the proof of 3.1, we need to compute enough terms of primitive parametrizations of all the irreducible components of the germ X_0, which has the $\mathscr{PE}^+$ property. For that, we just use certain function germs that must be necessarily in the ideal $\mathscr{J}(X_0)$. However, such function germs maybe do not generate $\mathscr{J}(X_0)$. As we have seen in 3.1, the discussion is already too cumbersome for embedding dimension $n = 3$, hence, it seems extremely difficult for $n \geqq 4$. $\square$

We finish here with several questions that arise naturally from the results we have presented in this work:

(7.4) Open questions. (1) Does Conjectures 7.1 and/or 7.2 hold true?

(2) Is the $\mathscr{PE}$ property hereditary? That is, if the $\mathscr{PE}$ property holds for a real analytic set X, does it also hold for the germs at all its points?

(3) Are $\mathscr{P} = \Sigma$ and $\mathscr{PE}$ equivalent properties for a global analytic set of dimension $\leqq 2$ and local embedding dimension $\geqq 4$?

(4) As we have seen in 1.5 the $\mathscr{P} = \Sigma$ property for a global analytic set X is hereditary and implies the coherence of X and that $\dim X \leqq 2$. The questions now are if the following statements hold true:

(A) *A global analytic surface S in an open set $\Omega \subset \mathbb{R}^n$ has $\mathscr{P} = \Sigma$ if and only if S is coherent and $\mathscr{P} = \Sigma$ holds for the germs at all its points.*

Referring to this, in 6.1 we have proved that a real coherent analytic surface S such that $\mathscr{P}(S_x) = \Sigma_2(S_x)$ for all $x \in \operatorname{Sing}(S)$ and any non-isolated singularity S_x of S is equivalent to $X_0 = \{yz = 0\} \subset \mathbb{R}^3_0$, has $\mathscr{P} = \Sigma$. Note that in the proof of 6.1, both hypotheses about the singular points, which are always true for local embedding dimension $\leqq 3$, play a crucial role.

However, for embedding dimension $\geqq 4$ we do not even know if $\mathscr{P}(X_0) = \Sigma_2(X_0)$ holds always true for a singularity X_0 with $\mathscr{P} = \Sigma$. Moreover, we neither know which are the non-isolated singularities with $\mathscr{P} = \Sigma$.

(B) *A global analytic set X in an open set $\Omega \subset \mathbb{R}^n$ has the $\mathscr{PE}$ property if and only if X is coherent and the $\mathscr{PE}$ property holds for the germs at all its points.*

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