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Erratum

Analytic surface germs with minimal Pythagoras number

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The proof of Proposition 3.1 in [Fe2] is not correct. It fails at page 735, line -10 where it is claimed the following:

(∗) *Indeed, suppose $\mathcal{Z}(J)$ singular. Then there exists a function f in $\mathcal{P}(\mathcal{Z}(J)) \setminus \Sigma(\mathcal{Z}(J))$* ([Sch]).

Before entering in further details, we recall briefly the involved notation. Let X be an analytic set germ (at the origin of $\mathbb{R}^n$); we denote by $\mathcal{O}(X)$ the ring of germs of analytic functions on X. If $X \subset \mathbb{R}^n$ we have $\mathcal{O}(X) = \mathbb{R}\{x_1, \ldots, x_n\}/\mathcal{J}(X)$, where $\mathcal{J}(X)$ is the ideal of all analytic function germs vanishing on X. If $I \subset \mathbb{R}\{x_1, \ldots, x_n\}$ is an ideal, $\mathcal{Z}(I)$ denotes the *zero set of* I and $\omega(I)$ stands for the minimal order of a series in I. We recall that a germ $f \in \mathcal{O}(X)$ is *positive semi-definite* or *psd* if it is ≥ 0 on X. We denote by $\mathcal{P}(X)$ the set of all psd's of X and by $\Sigma(X)$ (resp. $\Sigma_q(X)$) the set of all sums of squares (resp. q squares) of elements of $\mathcal{O}(X)$. Morever, $p[X]$ stands for the Pythagoras number of $\mathcal{O}(X)$, that is, the least integer $p \geq 1$ such that $\Sigma(X) = \Sigma_p(X)$. If such an integer does not exists we say that $p[X] = +\infty$.

Recall that J in (∗) denotes an ideal of $\mathbb{R}\{x, y, z\}$ such that $\mathcal{Z}(J)$ is a curve germ. The claim (∗) is not true, because if we take the singular curve germ $Y \subset \mathbb{R}^3$ given by the equations $xy = 0$, $z = 0$ we have, in view of [Sch, 3.9], that $\mathcal{P}(Y) = \Sigma(Y)$. Nevertheless, the statement of Proposition 3.1 is true and we include here a correct proof of such result.

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Proposition 0.1. *Let $X \subset \mathbb{R}^3$ be a mixed surface germ. Then $\mathcal{P}(X) = \Sigma(X)$ if and only if X is equivalent to the union of a plane and a transversal line. Furthermore, in this case, $p[X] = 2$.*

Proof. First, we prove that if X is the union of a plane π and a transversal line ℓ then every $f \in \mathcal{P}(X)$ is a sum of squares of analytic function germs. Indeed, after a change of coordinates the ideal of X is (zx, zy) and, every non unit f in $\mathcal{O}(X)$ can be written uniquely as $f_1(x, y) + f_2(z)$ where $f_1 \in \mathbb{R}\{x, y\}$, $f_2 \in \mathbb{R}\{z\}$. Note that $f(0, 0) = 0$, $g(0) = 0$. Now, $f = f_1(x, y) + f_2(z) \in \mathcal{P}(X)$ if and only if $f_1 \in \mathcal{P}(\pi)$ and $f_2 \in \mathcal{P}(\ell)$, or equivalently $f_1(x, y) = a(x, y)^2 + b(x, y)^2$ and $f_2(z) = c(z)^2$. Thus, $f = f_1 + f_2 \equiv (a + c)^2 + b^2$ in $\mathcal{O}(X)$.

Conversely, if $\mathcal{P}(X) = \Sigma(X)$, by [Fe1, 2.1], $\omega(\mathcal{J}(X)) = 2$. Let I (resp. J) be the ideal of the union of the components of X of dimension 2 (resp. 1). Then $\mathcal{J}(X) = I \cap J$. Moreover, since the ideal $I \subset \mathbb{R}\{x, y, z\}$ has height 1, it is principal, and we write $I = (\varphi)$ with $\varphi \in \mathbb{R}\{x, y, z\}$. One can check that $\mathcal{J}(X) = I \cdot J$; hence, $2 = \omega(\mathcal{J}(X)) = \omega(I) + \omega(J)$. Thus, $\omega(I) = \omega(J) = 1$ and we may assume that $I = (z)$ and $J = (\psi_1, \psi_2)$ where $\psi_j \in \mathbb{R}\{x, y, z\}$ and $1 = \omega(\psi_1) \leq \omega(\psi_2)$. Let us see that we can suppose that $\psi_1 = x$.

Otherwise, we may assume that the initial form of ψ_1 is equal to z and after a suitable analytic change of coordinates we may assume moreover that $\psi_1 = z + 2F(x, y)$ for certain analytic series $F(x, y)$ of order ≥ 2. We have that $\mathcal{J}(X) = (z(z + 2F(x, y)), z\phi_2)$. Note that since

$$z(z + 2F(x, y)) = (z + F(x, y))^2 - F^2(x, y) \in \mathcal{J}(X),$$

the following equality holds for X:

$$|z + F(x, y)| = |F(x, y)|.$$

On the other hand, since $\omega(F) \geq 2$, we can write $F(x, y) = \sum_{i+j=2} a_{ij}(x, y)x^i y^j$, and locally at the origin

$$|F(x, y)| \leq \sum_{i+j=2} c_{ij}|x|^i |y|^j, \qquad c_{ij} = 1 + |a_{ij}(0, 0)|.$$

Then, the function

$$\frac{\sum_{i+j=2} c_{ij}|x|^i |y|^j}{x^2 + y^2}$$

is continuous in the unit circle, thus bounded on that circle, say by c^2 for some real number $c > 0$. From the homogeneity we deduce that

$$|F(x, y)| \leq \sum_{i+j=2} c_{ij}|x|^i |y|^j \leq c^2(x^2 + y^2).$$

Thus, we get that

$$|z + F(x, y)| = |F(x, y)| \leq c^2(x^2 + y^2).$$

and the analytic function germ $h = c^2(x^2 + y^2) + F(x, y) + z$ is positive semi-definite on X. However, since h has order 1 and $\mathcal{J}(X)$ has order 2, h cannot be a sum of squares in $\mathcal{O}(X)$, a contradiction.

Thus, in what follows, we may assume that $J = (x, \phi_2)$ where $\phi_2 \in \mathbb{R}\{y, z\}$ is a series of order ≥ 1. We are to prove that after a new change of coordinates $\phi_2(y, z) = y$, hence $J = (x, y)$, which means that X is (equivalent) to the union of a plane and a transversal line. To that end, we begin by proving that $\mathcal{P}(Y) = \Sigma(Y)$ for the curve germ Y of $\mathbb{R}^2$ given by $Y: \ z\phi_2(y, z) = 0$.

Indeed, note that $\mathcal{J}(Y) = z\phi_2\mathbb{R}\{y, z\}$. Let $f(y, z) \in \mathcal{P}(Y)$ be a psd analytic function germ on $Y = \{z\phi_2 = 0\}$. Then $f(y, z)$ is psd on $X = \{z\phi_2 = 0, zx = 0\} \subset \mathbb{R} \times Y$. Since $\mathcal{P}(X) = \Sigma(X)$, there exist analytic functions $a_1, \ldots, a_p, b_1, b_2 \in \mathbb{R}\{x, y, z\}$ such that

$$\begin{aligned} f(y, z) = a_1^2(x, y, z) + \cdots + a_p^2(x, y, z) + zxb_1(x, y, z) \\ + z\phi_2(y, z)b_2(x, y, z). \end{aligned}$$

Making $x = 0$ in the previous equality we get that

$$f(y, z) = a_1^2(0, y, z) + \cdots + a_p^2(0, y, z) + z\phi_2(y, z)b_2(0, y, z).$$

Thus, $\mathcal{P}(Y) = \Sigma(Y)$.

Next, since $\omega(\phi_2) \geq 1$, by [Sch, 3.9], we have that Y is analytically equivalent to the union of two transversal lines. Hence, after a suitable change of coordinates, we may assume that $\phi_2 = y$, and we are done. $\square$

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