International Journal of Mathematics
Vol. 25, No. 7 (2014) 1450071 (18 pages)

DOI: 10.1142/S0129167X14500712

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On the complements of 3-dimensional convex polyhedra as polynomial images of $\mathbb{R}^3$

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Received 7 March 2014
Accepted 20 June 2014
Published 14 July 2014

Dedicated to Julio Castellanos on the occasion of his 60th birthday

Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron of dimension n. Denote $\mathcal{S} := \mathbb{R}^n \setminus \mathcal{K}$ and let $\overline{\mathcal{S}}$ be its closure. We prove that for $n = 3$ the semialgebraic sets $\mathcal{S}$ and $\overline{\mathcal{S}}$ are polynomial images of $\mathbb{R}^3$. The former techniques cannot be extended in general to represent the semialgebraic sets $\mathcal{S}$ and $\overline{\mathcal{S}}$ as polynomial images of $\mathbb{R}^n$ if $n \geq 4$.

Keywords: Polynomial maps and images; complement of a convex polyhedra; first and second trimming positions; dimension 3.

Mathematics Subject Classification 2010: 14P10, 52B10, 52B55, 90C26

1. Introduction

A map $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a *polynomial map* if its components $f_k \in \mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ are polynomials. Analogously, f is a *regular map* if its components can be represented as quotients $f_k := \frac{g_k}{h_k}$ of two polynomials $g_k, h_k \in \mathbb{R}[\mathtt{x}]$ such that h_k never vanishes on $\mathbb{R}^n$. During the last decade we have approached the following problem:

To determine which subsets $\mathcal{S} \subset \mathbb{R}^m$ are polynomial or regular images of $\mathbb{R}^n$.

We refer the reader to [9] for the first proposal of studying this problem and some particular related ones as the "quadrant problem" [4]. By Tarski–Seidenberg's Principle 1.4 of [2], the image of an either polynomial or regular map is a semialgebraic set. A subset $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* if it has a description as a finite

Boolean combination of polynomial equations and inequalities that we will call a *semialgebraic* description.

The effective representation of a subset $\mathcal{S} \subset \mathbb{R}^m$ as a polynomial or regular image of $\mathbb{R}^n$ reduces the study of certain classical problems in Real Geometry to its study in $\mathbb{R}^n$ with the advantage that this avoids contour conditions. Examples of these problems are Optimization or Positivstellensätze [5, 7]. The latter can be generalized to not necessarily closed basic semialgebraic sets (as the exterior of a convex polyhedron) using this type of representations. Polynomial representations are advantageous in comparison to regular representations because they avoid denominators and in this work we show that the quite natural polynomial constructions devised in [5, 12] still work for dimension 3 in the unbounded (remaining) case. For higher dimension, our results in [5] concerning regular images are still the best for the unbounded case.

We are far away from solving the representation problems stated above in its full generality but we have developed significant progresses in two ways:

General conditions. We obtain general conditions that a semialgebraic subset $\mathcal{S} \subset \mathbb{R}^m$ that is either a polynomial or a regular image of $\mathbb{R}^n$ must satisfy [5, 8, 11]. The most remarkable one states that the set of points at infinity of a polynomial image of $\mathbb{R}^n$ is connected.

Ample families. We show constructively that ample families of significant semialgebraic sets are either polynomial or regular images of $\mathbb{R}^n$ (see [3, 4, 6, 7, 12]).

A distinguished family of semialgebraic sets is the one constituted by those whose boundary is piecewise linear, that is, semialgebraic sets that admit a semialgebraic description involving only linear equations. Many of them cannot be polynomial or regular images of $\mathbb{R}^n$, but it seems natural to wonder what happens with: convex polyhedra, their interiors (as topological manifolds with boundary), their complements and the complements of their interiors. As the 1-dimensional case is completely determined in [3], we assume dimension ≥ 2 in the following.

We proved in [6] that all n-dimensional convex polyhedra and their interiors are regular images of $\mathbb{R}^n$. As many convex polyhedra are bounded and the images of non-constant polynomial maps are unbounded, the suitable approach there was to consider only regular maps. Concerning the representation of unbounded polygons as polynomial images see [13]. In [7], we prove that the complement $\mathbb{R}^n \setminus \mathcal{K}$ of a convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ that does not disconnect $\mathbb{R}^n$ and the complement $\mathbb{R}^n \setminus \operatorname{Int} \mathcal{K}$ of its interior are regular images of $\mathbb{R}^n$. If $\mathcal{K}$ is in addition bounded or has dimension $d < n$, then $\mathbb{R}^n \setminus \mathcal{K}$ and $\mathbb{R}^n \setminus \operatorname{Int} \mathcal{K}$ are polynomial images of $\mathbb{R}^n$.

If $\mathcal{K} \subset \mathbb{R}^3$ is a 3-dimensional unbounded convex polyhedron, the techniques we developed in [7] can be squeezed out to represent the semialgebraic sets $\mathbb{R}^3 \setminus \mathcal{K}$ and $\mathbb{R}^3 \setminus \operatorname{Int} \mathcal{K}$ as polynomial images of $\mathbb{R}^3$. Analogous results appear in [12] for dimension 2.

Theorem 1.1. *Let* $\mathcal{K} \subset \mathbb{R}^3$ *be a* 3*-dimensional unbounded convex polyhedron that does not disconnect* $\mathbb{R}^3$*. Then* $\mathbb{R}^3 \setminus \mathcal{K}$ *and* $\mathbb{R}^3 \setminus \operatorname{Int} \mathcal{K}$ *are polynomial images of* $\mathbb{R}^3$*.*

A convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ disconnects $\mathbb{R}^n$ if and only if it is a *layer*, that is, it is affinely equivalent to $[-a, a] \times \mathbb{R}^{n-1}$ for some $a \in \mathbb{R}$, which reduces to a hyperplane if $a = 0$. These layers are particular cases of *degenerate* convex polyhedra that do not have vertices. We will refer to convex polyhedra that have at least one vertex as *non-degenerate.*

Trimming positions with respect to a facet

The proof of Theorem 1.1 is based on a "placing problem" that involves the following definitions introduced in [7, 4.2 and 6.1]. We simplify them here to ease the discussion.

Consider the fibration of $\mathbb{R}^n$ induced by the projection

$$\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1} \times \{0\}, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, 0).$$

The fiber $\pi_n^{-1}(a, 0)$ is a parallel line to the vector $\vec{e}_n := (0, \ldots, 0, 1)$ for each $a \in \mathbb{R}^{n-1}$. Given an n-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$, the intersection $\mathcal{I}_a := \pi_n^{-1}(a, 0) \cap \mathcal{K}$ can be either empty or a closed interval (either bounded or unbounded). Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and $\mathcal{F}$ one of its facets (faces of dimension $n - 1$). Consider the set

$$\mathfrak{A}_{\mathcal{K}} := \{a \in \mathbb{R}^{n-1} : \mathcal{I}_a \neq \varnothing, (a, 0) \notin \mathcal{I}_a\}.$$

We say:

(1) $\mathcal{K}$ is in *first trimming position with respect to the facet* $\mathcal{F}$ if:

  (i) $\mathcal{F} \subset \{x_{n-1} = 0\}$ and $\mathcal{K} \subset \{x_{n-1} \leq 0\}$.
  (ii) For all $a \in \mathbb{R}^{n-1}$ the interval $\mathcal{I}_a$ is bounded.
  (iii) The set $\mathfrak{A}_{\mathcal{K}}$ is bounded.

(2) $\mathcal{K}$ is in *second trimming position with respect to the facet* $\mathcal{F}$ if:

  (i) $\mathcal{F} \subset \{x_n = 0\}$ and $\mathcal{K} \subset \{x_n \leq 0\}$.
  (ii) The set $\mathfrak{A}_{\mathcal{K}}$ is bounded.

A strategy to place an unbounded convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ in a trimming position with respect to a facet requires to determine under which conditions the set $\mathfrak{A}_{\mathcal{K}}$ is bounded. This characterization is approached in Lemma 3.1. Using this result, we obtain the following placing results for the 3-dimensional case.

Proposition 1.2 (3-dimensional polyhedra and first trimming position). *Let* $\mathcal{K} \subset \mathbb{R}^3$ *be a non-degenerate* 3*-dimensional unbounded convex polyhedron. Then:*

(i) *If* $\mathcal{K}$ *has facets with non-parallel unbounded edges, then it can be placed in first trimming position with respect to any of them.*

(ii) *If all unbounded edges of $\mathcal{K}$ are parallel, then $\mathcal{K}$ has at least one bounded facet and it can be placed in first trimming position with respect to any of its bounded facets.*

Proposition 1.3 (3-dimensional polyhedra and second trimming position). *Let $\mathcal{K} \subset \mathbb{R}^3$ be a non-degenerate, 3-dimensional unbounded convex polyhedron. We obtain*:

(i) *If $\mathcal{K}$ has facets with non-parallel unbounded edges, then it can be placed in second trimming position with respect to any of them.*
(ii) *If all unbounded edges of $\mathcal{K}$ are parallel, then $\mathcal{K}$ can be placed in second trimming position with respect to any of its bounded facets.*

1.1. *Sketch of the proof of Theorem* 1.1

Proof of Theorem 1.1. The proof of Theorem 1.1 is conducted similarly to the proofs of [7, Theorems 5.1(ii) and 7.1(ii)] where we proved analogous results involving regular maps instead of polynomial ones. The clue is to use [7, Remarks 5.3 and 7.3] in addition that explain the circumstances under which one can switch from regular maps to polynomials ones in the statements of [7, Theorems 5.1(ii) and 7.1(ii)]. To that end, we need to guarantee first that the following fact holds.

($*$) *Any non-degenerate 3-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^3$ has a* facet $\mathcal{F}$ *such that $\mathcal{K}$ can be placed in first and second trimming positions with respect to $\mathcal{F}$.*

The fact ($*$) follows from Propositions 1.2 and 1.3. Without entering into full detail because the proof is very similar to those of [7, Theorems 5.1(ii) and 7.1(ii)], we explain the strategy of the proof adapted to our 3-dimensional case next.

It consists of an induction process on the number of facets of the convex polyhedron $\mathcal{K}$ and is supported mainly by the following two facts:

(1) A degenerated 3-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^3$ can be placed as the product $\mathcal{P} \times \mathbb{R}$ where $\mathcal{P} \subset \mathbb{R}^2$ is a 2-dimensional convex polygon. In [12], it is proved that $\mathbb{R}^2 \setminus \mathcal{P}$ and $\mathbb{R}^2 \setminus \operatorname{Int} \mathcal{P}$ are polynomial images of $\mathbb{R}^2$, say by polynomial maps $f_0, g_0 : \mathbb{R}^2 \to \mathbb{R}^2$, respectively. Then $\mathbb{R}^3 \setminus \mathcal{K}$ and $\mathbb{R}^3 \setminus \operatorname{Int} \mathcal{K}$ are the images of the polynomial maps $(f_0, \mathrm{id}_{\mathbb{R}}) : \mathbb{R}^3 \to \mathbb{R}^3$ and $(g_0, \mathrm{id}_{\mathbb{R}}) : \mathbb{R}^3 \to \mathbb{R}^3$, respectively.
(2) Let $\mathcal{K} := \{h_1 \geq 0, \ldots, h_m \geq 0\}$ be a non-degenerate 3-dimensional unbounded convex polyhedron with m facets and suppose that each h_i is a linear equation. Assume by ($*$) that $\mathcal{K} \subset \mathbb{R}^3$ can be placed in first and second trimming positions with respect to its facet $\mathcal{F}$ that is contained in the hyperplane $\{h_1 = 0\}$. Let $\mathcal{K}_\times := \{h_2 \geq 0, \ldots, h_m \geq 0\}$ be the 3-dimensional unbounded convex polyhedron with $m - 1$ facets obtained "after eliminating the facet $\mathcal{F}$ from $\mathcal{K}$":
   - *As $\mathcal{K}$ can be placed in first trimming position with respect to $\mathcal{F}$ by ($*$), there exists a polynomial map $f_1 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f_1(\mathbb{R}^3 \setminus \mathcal{K}_\times) = \mathbb{R}^3 \setminus \mathcal{K}$* by [7, Lemma 4.8].

- *As* $\mathcal{K}$ *can be placed in both first and second trimming positions with respect to* $\mathcal{F}$ *by* $(*)$, *there exists a polynomial map* $g_1 : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $g_1(\mathbb{R}^3 \setminus \operatorname{Int} \mathcal{K}_\times) = \mathbb{R}^3 \setminus \operatorname{Int} \mathcal{K}$ by [7, Lemmas 4.8 and 6.8].

Now, by induction hypothesis (and taking into account (1) if $\mathcal{K}_\times$ is degenerated) there exist polynomial maps $f_2, g_2 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f_2(\mathbb{R}^3) = \mathbb{R}^3 \setminus \mathcal{K}_\times$ and $g_2(\mathbb{R}^3) = \mathbb{R}^3 \setminus \operatorname{Int} \mathcal{K}_\times$, so the compositions $f := f_2 \circ f_1$ and $g := g_2 \circ g_1$ satisfy the required conditions.

We refer the reader to [7, Theorems 5.1(ii) and 7.1(ii) and Remarks 5.3 and 7.3] for a more detailed presentation. □

The limit to extend Theorem 1.1 to dimensions ≥ 4 by using our former techniques relies on the fact that the following property is exclusive for convex 2-dimensional polyhedra.

Property 1.4. *For any convex polygon* $\mathcal{K} \subset \mathbb{R}^2$ *there exist a vectorial line* $\vec{\ell}$ *and a hyperplane* H *such that the projection* $\pi : \mathbb{R}^2 \to H$ *with direction* $\vec{\ell}$ *satisfies the identity* $\pi(\mathcal{K}) = \pi(\mathcal{K} \cap H)$.

In Appendix A, we exhibit the existence of n-dimensional unbounded convex polyhedra that can be placed neither in first trimming position with respect to any of its facets nor in second trimming position with respect to any unbounded facet.

1.2. *Structure of the paper*

The paper is organized as follows. All basic notions and (standard) notation appear in Sec. 2. The reading can be started directly in Sec. 3 and referred to the preliminaries only when needed. In Sec. 3, we analyze the boundedness of the set $\mathfrak{A}_{\mathcal{K}}$ and provide tools to approach the proofs of Propositions 1.2 and 1.3, which are developed in Sec. 4.

2. Preliminaries on Convex Polyhedra

We begin by introducing some preliminary terminology and notations concerning convex polyhedra. For a detailed study of the main properties of convex sets we refer the reader to [1, 10, 14]. An affine hyperplane of $\mathbb{R}^n$ will be denoted as $H := \{x \in \mathbb{R}^n : \ h(x) = 0\} \equiv \{h = 0\}$ for a linear equation h. It determines two *closed half-spaces*

$$H^+ := \{x \in \mathbb{R}^n : \ h(x) \geq 0\} \equiv \{h \geq 0\}$$
$$\text{and} \quad H^- := \{x \in \mathbb{R}^n : \ h(x) \leq 0\} \equiv \{h \leq 0\}.$$

We use an overlying arrow $\vec{\cdot}$ when referring to vectorial staff.

2.1. *Generalities on convex polyhedra*

A subset $\mathcal{K} \subset \mathbb{R}^n$ is a *convex polyhedron* if it can be described as the finite intersection $\mathcal{K} := \bigcap_{i=1}^r H_i^+$ of closed half-spaces H_i^+. To describe $\mathcal{K} = \mathbb{R}^n$ as a convex polyhedron we allow this family of half-spaces to be empty. The dimension $\dim(\mathcal{K})$ of $\mathcal{K}$ is its dimension as a topological manifold with boundary. By [1, 12.1.5], there exists a unique minimal family $\mathfrak{H} := \{H_1, \ldots, H_m\}$ of affine hyperplanes of $\mathbb{R}^n$, which is empty only if $\mathcal{K} = \mathbb{R}^n$, such that $\mathcal{K} = \bigcap_{i=1}^m H_i^+$. We refer to this family as the *minimal presentation* of $\mathcal{K}$. We assume that we choose the linear equation h_i of each H_i such that $\mathcal{K} \subset H_i^+$.

2.1.1. *Faces of the boundary of a convex polyhedron*

The *facets* or $(n-1)$-*faces* of $\mathcal{K}$ are the intersections $\mathcal{F}_i := H_i \cap \mathcal{K}$ for $1 \leq i \leq m$. Only the convex polyhedron $\mathbb{R}^n$ has no facets. Each facet $\mathcal{F}_i := H_i^- \cap \bigcap_{j=1}^m H_j^+$ is a convex polyhedron contained in H_i. The convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a topological manifold with boundary whose interior is $\operatorname{Int}\mathcal{K} = \bigcap_{i=1}^m (H_i^+ \setminus H_i)$ and its boundary is $\partial\mathcal{K} = \bigcup_{i=1}^m \mathcal{F}_i$. For $0 \leq j \leq n-2$ we define inductively the j-*faces* of $\mathcal{K}$ as the facets of the $(j+1)$-faces of $\mathcal{K}$, which are again convex polyhedra. The 0-faces are the *vertices* of $\mathcal{K}$ and the 1-faces are the *edges* of $\mathcal{K}$. Obviously if $\mathcal{K}$ has a vertex, then $m \geq n$. A convex polyhedron of $\mathbb{R}^n$ is *non-degenerate* if it has at least one vertex. Otherwise, we say that the convex polyhedron is *degenerate*.

2.1.2. *Supporting hyperplanes of a convex polyhedron*

A *supporting hyperplane* of a convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a hyperplane H of $\mathbb{R}^n$ that intersects $\mathcal{K}$ and satisfies $\mathcal{K} \subset H^+$ or $\mathcal{K} \subset H^-$. This is equivalent to $\varnothing \neq \mathcal{K} \cap H \subset \partial\mathcal{K}$. The intersection of $\mathcal{K}$ with a supporting hyperplane H is a face of $\mathcal{K}$ and conversely each face of $\mathcal{K}$ is the intersection of $\mathcal{K}$ with some supporting hyperplane. In particular, the vertices of a convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ are those points $p \in \mathcal{K}$ for which there exists a (supporting) hyperplane $H \subset \mathbb{R}^n$ such that $\mathcal{K} \cap H = \{p\}$.

2.1.3. *Cones*

The *cone of vertex* p *and whose basis is the bounded convex polyhedron* $\mathcal{P} \subset \mathbb{R}^n$ is

$$\mathcal{C} := \{\lambda p + (1-\lambda) q : q \in \mathcal{P}, 0 \leq \lambda \leq 1\}.$$

Given $\vec{v}_1, \ldots, \vec{v}_r \in \mathbb{R}^n$, we define the *cone generated by the vectors* $\{\vec{v}_1, \ldots, \vec{v}_r\}$ as the set $\vec{\mathcal{C}} := \{\sum_{i=1}^r \lambda_i \vec{v}_i : \lambda_i \geq 0\}$ and denote $\mathcal{C}_p := p + \vec{\mathcal{C}}$ for $p \in \mathbb{R}^n$. Given two points $p, q \in \mathbb{R}^n$, we denote the *segment connecting* p *and* q with $\overline{pq} := \{\lambda p + (1-\lambda) q : 0 \leq \lambda \leq 1\}$ and given a vector $\vec{v} \in \mathbb{R}^n$, we denote the *half-line of extreme* p *and direction* $\vec{v}$ with $p\vec{v} := \{p + \lambda \vec{v} : \lambda \geq 0\}$.

2.2. *Recession cone of a convex polyhedron*

We associate to each convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ its *recession cone*, see [4, Chap. 1]. Fix a point $p \in \mathcal{K}$ and denote $\vec{\mathfrak{C}}(\mathcal{K}) := \{\vec{v} \in \mathbb{R}^n : p\vec{v} \subset \mathcal{K}\}$. *Then* $\vec{\mathfrak{C}}(\mathcal{K})$ *is a convex cone and does not depend on the choice of* p. The set $\vec{\mathfrak{C}}(\mathcal{K})$ is called the *recession* *cone* of $\mathcal{K}$. If $\mathcal{K} := \bigcap_{i=1}^r H_i^+$, then

$$\vec{\mathfrak{C}}(\mathcal{K}) = \bigcap_{i=1}^r \vec{\mathfrak{C}}(H_i^+) = \bigcap_{i=1}^r \vec{H_i}^+.$$

Clearly, $\vec{\mathfrak{C}}(\mathcal{K}) = \{\mathbf{0}\}$ if and only if $\mathcal{K}$ is bounded. In addition, if $\mathcal{P} \subset \mathbb{R}^n$ is a non-degenerate convex polyhedron and $k \geq 1$, then $\vec{\mathfrak{C}}(\mathbb{R}^k \times \mathcal{P}) = \mathbb{R}^k \times \vec{\mathfrak{C}}(\mathcal{P})$. Recall that each degenerate convex polyhedron can be written as the product of a non-degenerate convex polyhedron times an Euclidean space.

2.2.1. *Description of a convex polyhedron in terms of its recession cone*

An important property of a bounded convex polyhedron is that it coincides with the *convex hull* of the set of its vertices. A general non-degenerate convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ can be described as follows, see [14, Chap. 1]. *Let* $\mathfrak{V} := \{p_1, \ldots, p_r\}$ *be the set of vertices of* $\mathcal{K}$ *and* $\mathfrak{A} := \{\mathcal{A}_1, \ldots, \mathcal{A}_s\}$ *the set of unbounded edges of* $\mathcal{K}$. *Write* $\mathcal{A}_j := q_j\vec{v}_j$ *for* $j = 1, \ldots, s$. *Then*:

(i) $\vec{\mathfrak{C}}(\mathcal{K}) = \{\sum_{j=1}^s \lambda_j\vec{v}_j : \lambda_1, \ldots, \lambda_s \geq 0\}$ *or* $\{\mathbf{0}\}$ *if* $\mathfrak{A} = \varnothing$.

(ii) $\mathcal{K} = \mathcal{K}_0 + \vec{\mathfrak{C}}(\mathcal{K})$ *where* $\mathcal{K}_0$ *is the bounded convex polyhedron of vertices* $p_1, \ldots, p_r$.

2.3. *Facing upwards positions for convex polyhedra*

The proof of Theorem 1.1 has been reduced to show that non-degenerate unbounded convex polyhedra of $\mathbb{R}^3$ can be placed in a specific form. As an initial step we introduce the concept of *facing upwards positions*.

Definition 2.1. An unbounded non-degenerate convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is in *facing upwards position with respect to the hyperplane* $\vec{\Pi}$ of $\mathbb{R}^n$ (shortly, FU-position w.r.t. $\vec{\Pi}$) if there exists a hyperplane Π parallel to $\vec{\Pi}$ that intersects all unbounded edges of $\mathcal{K}$ and such that all vertices of $\mathcal{K}$ belong to the open half-space $\operatorname{Int}\Pi^-$. The hyperplane Π is called a *sawing hyperplane* for $\mathcal{K}$. Any hyperplane $\Pi' \subset \Pi^+$ (parallel to $\vec{\Pi}$) is also a sawing hyperplane for $\mathcal{K}$.

Let $\vec{h}$ be a linear equation of $\vec{\Pi}$. We say that the FU-position of $\mathcal{K}$ with respect to $\vec{\Pi}$ is *optimal* if the minimum of $\vec{h}|_{\mathcal{K}}$ is attained exactly in one point (which must be a vertex of $\mathcal{K}$).

2.3.1. *Connection with bounded convex polyhedra*

Each unbounded non-degenerate convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ can be placed in optimal FU-position w.r.t. the hyperplane $\vec{\Pi} := \{x_n = 0\}$ in such a way that it does not intersect the hyperplane $\Pi_0 := \{x_n = 0\}$. Under this hypothesis there exists a natural bridge between non-degenerate unbounded convex polyhedra and bounded ones. Denote the hyperplane at infinity of the projective space $\mathbb{RP}^n$ with

$\mathsf{H}_\infty(\mathbb{R})$. Write $\widehat{\mathcal{K}} := \mathrm{Cl}_{\mathbb{RP}^n}(\mathcal{K}) = \mathcal{K} \sqcup \mathcal{K}_\infty$ where $\mathcal{K}_\infty := \mathrm{Cl}_{\mathbb{RP}^n}(\mathcal{K}) \cap \mathsf{H}_\infty(\mathbb{R})$ and consider the involution

$$\phi : \mathbb{RP}^n \to \mathbb{RP}^n, \quad (x_0 : x_1 : \cdots : x_{n-1} : x_n) \mapsto (x_n : x_1 : \cdots : x_{n-1} : x_0)$$

induced by the birational map

$$f := \phi|_{\mathbb{R}^n} : \mathbb{R}^n \dashrightarrow \mathbb{R}^n, \quad (x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n) := \left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n}\right).$$

Then $\mathcal{K}' := \mathrm{Cl}(f(\mathcal{K})) = \phi(\widehat{\mathcal{K}}) \subset \mathbb{R}^n \equiv \{y_0 \neq 0\}$ *is a bounded convex polyhedron and one of its faces is* $\mathcal{E}' := \phi(\mathcal{K}_\infty)$. *Moreover,* $\phi(p + \vec{\mathfrak{C}}(\mathcal{K})) \cup \phi(\mathcal{K}_\infty)$ *is the closed cone* $\mathcal{C}_{\phi(p)}$ *of base* $\phi(\mathcal{K}_\infty)$ *and vertex* $\phi(p)$ *for each* $p \in \mathcal{K}$.

Proof. We only check $\phi(\widehat{\mathcal{K}}) \subset \mathbb{R}^n$. As $\mathcal{K}$ is in optimal FU-position w.r.t. the hyperplane $\vec{\Pi} := \{x_n = 0\}$, no half-line contained in $\mathcal{K}$ can be parallel to Π_0. As in addition $\mathcal{K} \cap \Pi_0 = \varnothing$, we conclude $\widehat{\mathcal{K}} \cap \mathrm{Cl}_{\mathbb{RP}^n}(\Pi_0) = \varnothing$. If we take $\mathrm{Cl}_{\mathbb{RP}^n}(\Pi_0)$ as the hyperplane at infinity of $\mathbb{RP}^n$, it holds that $\phi(\widehat{\mathcal{K}})$ is compact and so $\phi(\widehat{\mathcal{K}}) \subset \mathbb{R}^n$. □

2.3.2. *Construction of supporting hyperplanes*

There is an easy procedure to find supporting hyperplanes for a non-degenerate unbounded convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ in FU-position w.r.t. a hyperplane $\vec{\Pi}$.

Lemma 2.2. *Let* Π *be a sawing hyperplane for* $\mathcal{K}$ *and* W *a supporting hyperplane in* Π *of the convex polyhedron* $\mathcal{P} := \mathcal{K} \cap \Pi$. *Let* $p \in W \cap \mathcal{P}$ *be a vertex of* $\mathcal{P}$ *and* $\mathcal{A}$ *the unbounded edge of* $\mathcal{K}$ *such that* $\{p\} = \mathcal{A} \cap \Pi$. *Then the affine subspace* H *of* $\mathbb{R}^n$ *generated by* W *and* $\mathcal{A}$ *is a supporting hyperplane of* $\mathcal{K}$.

2.3.3. *Recession cone with maximal dimension*

We finish with a technical result concerning the recession cone $\vec{\mathfrak{C}}(\mathcal{K})$ of an n-dimensional non-degenerate unbounded convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ in FU-position w.r.t. $\vec{\Pi}$ when $\dim(\vec{\mathfrak{C}}(\mathcal{K})) = n$.

Lemma 2.3. *Assume* $\dim(\vec{\mathfrak{C}}(\mathcal{K})) = n$, *let* $\vec{v} \in \mathrm{Int}\,\vec{\mathfrak{C}}(\mathcal{K})$ *and consider a finite set* $\mathfrak{G} \subset \mathbb{R}^n$. *Then there exists a hyperplane* Π' *parallel to* $\vec{\Pi}$ *such that* $p\vec{v} \cap (\mathcal{K} \cap \Pi')$ *is a singleton for each* $p \in \mathfrak{G}$.

Proof. We use freely the straightforward fact:

For each $p \in \mathbb{R}^n$ *the intersection* $p\vec{v} \cap \mathcal{K}$ *is a half-line* $p_1\vec{v} \subset \mathcal{K}$.

Write $\vec{\Pi} := \{\vec{h} = 0\}$ and let $\Pi_0 := \{h_0 := a_0 + \vec{h} = 0\}$ be a sawing hyperplane for $\mathcal{K}$. Write $\mathcal{A}_i := q_i\vec{v}_i$ where q_i is a vertex of $\mathcal{K}$ and the vector $\vec{v}_i \in \vec{\mathfrak{C}}(\mathcal{K})$ for $i = 1, \ldots, s$. Let b_i be the intersection point between $\mathcal{A}_i$ and Π_0. Let $\mu_i > 0$ be such that $b_i = q_i + \mu_i \vec{v}_i$. As $0 = h_0(b_i) = h_0(q_i) + \mu_i \vec{h}(\vec{v}_i)$ and $h_0(q_i) < 0$, we deduce $\vec{h}(\vec{v}_i) = -\frac{1}{\mu_i} h_0(q_i) > 0$. By Sec. 2.2.1, there exist $\lambda_i \geq 0$ not all zero such that $\vec{v} = \sum_{i=1}^{s} \lambda_i \vec{v}_i$. Thus,

$$\vec{h}(\vec{v}) = \vec{h}\left(\sum_{i=1}^{s} \lambda_i \vec{v}_i\right) = \sum_{i=1}^{s} \lambda_i \vec{h}(\vec{v}_i) > 0.$$

For each $p \in \mathfrak{G}$ choose $p' \in \mathcal{K}$ such that $p\vec{v} \cap \mathcal{K} = p'\vec{v}$. We take $\Pi' := \{h = 0\}$ parallel to $\vec{\Pi}$ such that $p' \in \operatorname{Int} \Pi'^{-}$ for each $p \in \mathfrak{G}$. We claim: $p\vec{v} \cap (\mathcal{K} \cap \Pi') \neq \varnothing$ *for each* $p \in \mathfrak{G}$.

Indeed, fix $p \in \mathfrak{G}$. As $\vec{h}(\vec{v}\,) > 0$ and $h(p') < 0$, there exists $t > 0$ such that $h(p' + t\vec{v}\,) = 0$, so $p' + t\vec{v} \in p'\vec{v} \cap \Pi' = p\vec{v} \cap (\mathcal{K} \cap \Pi')$. Obviously, as $p'\vec{v}$ is a half-line and $p' \notin \Pi'$, the intersection $p\vec{v} \cap (\mathcal{K} \cap \Pi')$ is a singleton, as required. □

3. Characterization of the Boundedness of the Set $\mathfrak{A}_{\mathcal{K}}$

In this section, we characterize when the set $\mathfrak{A}_{\mathcal{K}}$ of a non-degenerate unbounded convex polyhedron $\mathcal{K}$ is bounded. Recall

$$\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1} \times \{0\}, \quad x := (x_1, \ldots, x_n) \mapsto (x', 0) := (x_1, \ldots, x_{n-1}, 0)$$

and denote the vectorial line generated by the vector $\vec{e}_n := (0, \ldots, 0, 1)$ with $\vec{\ell}_n$.

Lemma 3.1 (Boundedness of $\mathfrak{A}_{\mathcal{K}}$). *Let* $\mathcal{K} \subset \mathbb{R}^n$ *be a non-degenerate unbounded convex polyhedron. Then the set*

$$\mathfrak{A}_{\mathcal{K}} := \{a \in \mathbb{R}^{n-1} : \mathcal{I}_a := \pi_n^{-1}(a, 0) \cap \mathcal{K} \neq \varnothing, (a, 0) \notin \mathcal{I}_a\}$$

is bounded if and only if the following conditions hold:

(i) $\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \{x_n = 0\}$.
(ii) *There exists a hyperplane* $\Pi \subset \mathbb{R}^n$ *parallel to* $\vec{\ell}_n$ *such that*: *it meets all unbounded edges of* $\mathcal{K}$ *that are non-parallel to* $\vec{\ell}_n$; *all vertices of* $\mathcal{K}$ *and all unbounded edges of* $\mathcal{K}$ *parallel to* $\vec{\ell}_n$ *are contained in* $\operatorname{Int} \Pi^{-}$ *and* $\pi_n(\mathcal{K} \cap \Pi) = (\mathcal{K} \cap \Pi) \cap \{x_n = 0\}$.

3.1. *Hyperplane sections of a convex polyhedron*

In order to prove Lemma 3.1, we need to understand the generic sections of an n-dimensional non-degenerate unbounded convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ parallel to a hyperplane $\vec{\Pi} := \{\vec{h} = 0\}$.

Lemma 3.2. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_s$ *be the unbounded edges of* $\mathcal{K}$. *Assume that the first* k *are not parallel to* $\vec{\Pi}$ *and the remaining* $s - k$ *are parallel to* $\vec{\Pi}$. *For each* $i = 1, \ldots, s$ *write* $\mathcal{A}_i := q_i \vec{v}_i$ *where* q_i *is a vertex of* $\mathcal{K}$ *and* $\vec{v}_i \in \vec{\mathfrak{C}}(\mathcal{K})$. *Let* Π_0 *be a hyperplane parallel to* $\vec{\Pi}$ *that meets* $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *and such that all vertices of* $\mathcal{K}$ *and the unbounded edges* $\mathcal{A}_{k+1}, \ldots, \mathcal{A}_s$ *are contained in* $\operatorname{Int} \Pi_0^{-}$. *For each hyperplane* $\Pi \subset \Pi_0^{+}$ *we* *have*:

(i) $\mathcal{P} := \mathcal{K} \cap \Pi$ *is the convex polyhedron whose vertices are the intersections of* Π *with* $\mathcal{A}_1, \dots, \mathcal{A}_k$ *and its recession cone is* $\vec{\mathfrak{C}}(\mathcal{P}) = \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-) = \{\sum_{j=k+1}^s \lambda_j \vec{v}_j : \lambda_j \geq 0\}$.

(ii) $\mathcal{K} \cap \Pi^+ = \mathcal{P} + \vec{\mathfrak{C}}(\mathcal{K}) = \mathcal{P} + \{\sum_{j=1}^k \lambda_j \vec{v}_j : \lambda_j \geq 0\}$.

Remark 3.3. If $k = s$, then $\mathcal{K}$ is in FU-position w.r.t. $\vec{\Pi}$ and it holds:

(i) $\mathcal{P} := \mathcal{K} \cap \Pi \neq \varnothing$ is bounded and its vertices are the intersections of Π with $\mathcal{A}_1, \dots, \mathcal{A}_s$.

(ii) $\mathcal{K} \cap \Pi^-$ is bounded and $\mathcal{K} \cap \Pi^+ = \mathcal{P} + \vec{\mathfrak{C}}(\mathcal{K})$.

Proof of Lemma 3.2. (i) Let p be a vertex of $\mathcal{P}$. As $p \in \partial\mathcal{K}$, we choose a face $\mathcal{E}$ of $\mathcal{K}$ of the smallest dimension between those containing p. Clearly, $p \in \operatorname{Int}\mathcal{E}$. As the vertices of $\mathcal{K}$ are contained in $\operatorname{Int}\Pi^-$ and $\Pi \subset \Pi^+$, the dimension of $\mathcal{E}$ is ≥ 1. Let W be the affine subspace generated by $\mathcal{E}$ and notice $W \cap \Pi = \{p\}$ (otherwise we would have a face of $\mathcal{P}$ crossing the vertex p). As Π is a hyperplane, $\dim W = 1$. Thus, $\mathcal{E} := \mathcal{A}_j$ is an (unbounded) edge of $\mathcal{K}$ non-parallel to $\vec{\Pi}$. On the other hand, it is clear that the intersections of Π with $\mathcal{A}_1, \dots, \mathcal{A}_k$ are vertices of $\mathcal{P}$.

Next we prove $\vec{\mathfrak{C}}(\mathcal{P}) = \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-) = \vec{\mathcal{C}} := \{\sum_{j=k+1}^s \lambda_j \vec{v}_j : \lambda_j \geq 0\}$.

Indeed, let $h := a + \vec{h}$ be a linear equation of Π such that $\Pi^+ = \{h \geq 0\}$. It holds

$$\vec{h}(\vec{v}_i) \begin{cases} > 0 & \text{for } i = 1, \dots, k, \\ = 0 & \text{for } i = k+1, \dots, s. \end{cases} \tag{3.1}$$

If $\vec{v} \in \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-)$, then $\vec{v} \in \vec{\mathfrak{C}}(\mathcal{K})$ and $\vec{h}(\vec{v}) \leq 0$. As $\vec{\mathfrak{C}}(\mathcal{K}) = \{\sum_{i=1}^s \lambda_i \vec{v}_i : \lambda_i \geq 0\}$, there exist $\zeta_i \geq 0$ such that $\vec{v} = \sum_{j=1}^s \zeta_j \vec{v}_j$. By Eq. (3.1),

$$0 \geq \vec{h}(\vec{v}) = \vec{h}\left(\sum_{i=1}^s \zeta_i \vec{v}_i\right) = \sum_{i=1}^s \zeta_i \vec{h}(\vec{v}_i) = \sum_{i=1}^k \zeta_i \vec{h}(\vec{v}_i) \geq 0,$$

so $\sum_{j=1}^k \zeta_j \vec{h}(\vec{v}_j) = 0$. As $\zeta_j \geq 0$ and $\vec{h}(\vec{v}_j) > 0$ for $j = 1, \dots, k$, we deduce that each $\zeta_j = 0$, so $\vec{v} = \sum_{i=k+1}^s \zeta_i \vec{v}_i \in \vec{\mathcal{C}}$. Consequently,

$$\vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-) \subset \vec{\mathcal{C}} \subset \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{\Pi} = \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi) = \vec{\mathfrak{C}}(\mathcal{P}) \subset \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-).$$

(ii) The vertices of $\mathcal{K} \cap \Pi^+$ are the intersections of Π with the unbounded edges $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $\mathcal{K}$. Thus, the convex hull of the set consisting of those vertices is contained in $\mathcal{P}$ and so $\mathcal{K} \cap \Pi^+ = \mathcal{P} + \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^+)$. As $\vec{\mathfrak{C}}(\mathcal{K}) = \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^+)$ and $\vec{\mathfrak{C}}(\mathcal{P}) = \{\sum_{j=k+1}^s \lambda_j \vec{v}_j : \lambda_j \geq 0\}$, we deduce

$$\begin{aligned} \mathcal{K} \cap \Pi^+ &= \mathcal{P} + \vec{\mathfrak{C}}(\mathcal{K}) \\ &= \mathcal{P} + \vec{\mathfrak{C}}(\mathcal{P}) + \left\{\sum_{j=1}^k \lambda_j \vec{v}_j : \lambda_j \geq 0\right\} = \mathcal{P} + \left\{\sum_{j=1}^k \lambda_j \vec{v}_j : \lambda_j \geq 0\right\}, \end{aligned}$$

as required. $\square$

3.2. *Proof of Lemma* 3.1

Proof. Denote the vertices of $\mathcal{K}$ with $p_1, \ldots, p_r$ and the unbounded edges of $\mathcal{K}$ with $\mathcal{A}_1, \ldots, \mathcal{A}_s$. Assume that $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are ordered in such a way that the first k are not parallel to $\vec{\ell}_n$ and the remaining $s-k$ are parallel to $\vec{\ell}_n$. For each $i = 1, \ldots, s$ we write $\mathcal{A}_i := q_i\vec{v}_i$ where q_i is a vertex of $\mathcal{K}$ and $\vec{v}_i \in \vec{\mathfrak{C}}(\mathcal{K})$ for $i = 1, \ldots, s$.

Assume that $\mathfrak{A}_{\mathcal{K}}$ is bounded. We prove first (i) $\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \{x_n = 0\}$. Write $H := \{x_n = 0\}$ and let us see that if $\vec{v} := (v_1, \ldots, v_n) \in \vec{\mathfrak{C}}(\mathcal{K})$, then

$$\vec{w} := \vec{\pi}_n(\vec{v}\,) = (v_1, \ldots, v_{n-1}, 0) \in \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{H}.$$

Indeed, if $\vec{v} \in \vec{\ell}_n$, then $\vec{\pi}_n(\vec{v}\,) = \mathbf{0} \in \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{H}$, so assume $\vec{v} \notin \vec{\ell}_n$. Let $p \in \mathcal{K}$. As $\vec{v} \in \vec{\mathfrak{C}}(\mathcal{K})$, the half-line $p\vec{v} \subset \mathcal{K}$. In addition $\vec{v} \notin \vec{\ell}_n$, so $\vec{w} \in \vec{H}\backslash\{\mathbf{0}\}$. As the set $\mathfrak{A}_{\mathcal{K}}$ is bounded and $\pi_n(p\vec{v}\,) = \pi_n(p)\vec{w}$ is a half-line, there exists a point $q \in p\vec{v}$ such that the half-line $\pi_n(q\vec{v}\,) = \pi_n(q)\vec{w}$ does not meet $\mathfrak{A}_{\mathcal{K}} \times \{0\} \subset H$. For each $t > 0$ the point $q + t\vec{v} \in \mathcal{K} \cap \pi_n^{-1}(\pi_n(q) + t\vec{w})$ while $\pi_n(q) + t\vec{w} \notin \mathfrak{A}_{\mathcal{K}} \times \{0\}$. As $\mathcal{K} \cap \pi_n^{-1}(\pi_n(q) + t\vec{w}) \neq \varnothing$ and $\pi_n(q) + t\vec{w} \notin \mathfrak{A}_{\mathcal{K}} \times \{0\}$, we conclude taking into account the definition of $\mathfrak{A}_{\mathcal{K}}$ that $\pi_n(q) + t\vec{w} \in \mathcal{K}$. Thus, the half-line $\pi_n(q)\vec{w} \subset \mathcal{K} \cap H$, so $\vec{w} \in \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{H}$. Therefore,

$$\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K})) \subset \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{H}$$

and consequently $\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \{x_n = 0\}$ because the other inclusion is trivial.

To show (ii) we distinguish two cases:

Case 1. If $k = 0$, the edges $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are parallel to $\vec{\ell}_n$. Then $\mathcal{K}$ is in FU-position w.r.t. $\{x_n = 0\}$ and by Remark 3.3 the projection $\pi_n(\mathcal{K})$ is bounded. We choose a hyperplane Π parallel to the line $\vec{\ell}_n$ such that $\mathcal{K} \subset \operatorname{Int}\Pi^-$. Clearly, Π enjoys the required conditions because all edges of $\mathcal{K}$ are parallel to Π, $\mathcal{K} \subset \operatorname{Int}\Pi^-$ and $\mathcal{K} \cap \Pi = \varnothing$.

Case 2. If $k > 0$, we know $\vec{\mathfrak{C}}(\mathcal{K}) = \{\sum_{i=1}^s \lambda_i\vec{v}_i : \lambda_i \geq 0\}$ and if $\vec{w}_i := \vec{\pi}_n(\vec{v}_i)$, we deduce that

$$\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K})) = \left\{\sum_{i=1}^{s} \lambda_i\vec{w}_i : \lambda_i \geq 0\right\} = \left\{\sum_{i=1}^{k} \lambda_i\vec{w}_i : \lambda_i \geq 0\right\} \neq \{\mathbf{0}\}$$

is an unbounded cone whose unique vertex is $\mathbf{0}$ (otherwise $\mathcal{K}$ would be degenerate). Observe that $\vec{w}_i \neq 0$ for each $i = 1, \ldots, k$ because the edges $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are not parallel to $\vec{\ell}_n$.

Let W be a supporting hyperplane of $\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K}))$ in $\{x_n = 0\}$ such that

$$\vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K})) \cap W = \{\mathbf{0}\}$$

and consider the hyperplane $\vec{\Pi} := \vec{W} + \vec{\ell}_n = \{\vec{h} = 0\}$. We may assume $\vec{h}(\vec{w}_i) > 0$ for $i = 1, \ldots, k$. As $\vec{v}_i - \vec{w}_i$ is parallel to $\vec{\ell}_n$, we have $\vec{h}(\vec{v}_i) = \vec{h}(\vec{w}_i) > 0$.

Let $\Pi := \{h = 0\}$ be a hyperplane parallel to $\vec{\Pi}$ such that all vertices p_i of $\mathcal{K}$ and the set $\mathfrak{A}_{\mathcal{K}} \times \{0\}$ are contained in $\operatorname{Int}\Pi^-$. As $\vec{h}(\vec{v_i}) = \vec{h}_0(\vec{v_i}) > 0$, it holds that Π meets the edges $\mathcal{A}_i = p_i\vec{v}_i$ for $i = 1, \ldots, k$. Since $\vec{\ell}_n \subset \vec{\Pi}$,

$$\pi_n^{-1}(\mathfrak{A}_{\mathcal{K}} \times \{0\}) \subset \pi_n^{-1}(\pi_n(\{h < 0\})) = \{h < 0\} = \operatorname{Int}\Pi^-$$

and so $\pi_n(x) \in \mathcal{K} \cap \Pi \cap \{x_n = 0\}$ for each $x \in \mathcal{K} \cap \Pi$. Thus,

$$\pi_n(\mathcal{K} \cap \Pi) = (\mathcal{K} \cap \Pi) \cap \{x_n = 0\}.$$

Assume now that (i) and (ii) hold. To prove that $\mathfrak{A}_{\mathcal{K}}$ is bounded we distinguish two cases:

Case 1. $\mathcal{K} \cap \Pi = \varnothing$, or equivalently, all unbounded edges of $\mathcal{K}$ are parallel to the line $\vec{\ell}_n$. Then $\mathcal{K}$ is in FU-position w.r.t. $\{x_n = 0\}$ and by Remark 3.3 the projection $\pi_n(\mathcal{K})$ is a bounded convex polyhedron. Thus, $\mathfrak{A}_{\mathcal{K}} \times \{0\} \subset \pi_n(\mathcal{K})$ is a bounded set.

Case 2. $\mathcal{K} \cap \Pi \neq \varnothing$. By hypothesis (ii) and Lemma 3.2,

$$\mathcal{K} \cap \Pi^+ = (\mathcal{K} \cap \Pi) + \vec{\mathfrak{C}}(\mathcal{K}) \quad \text{and} \quad \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-) = \left\{ \sum_{j=k+1}^{s} \lambda_j \vec{v}_j : \lambda_j \geq 0 \right\}. \tag{3.2}$$

The vertices of $\mathcal{K} \cap \Pi^-$ are the vertices of $\mathcal{K}$ and the intersections $\{y_j\} := \mathcal{A}_j \cap \Pi$ for $j = 1, \ldots, k$. By Sec. 2.2.1, $\mathcal{K} \cap \Pi^- = \mathcal{K}_0 + \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-)$ where $\mathcal{K}_0$ is the convex hull of $\{p_1, \ldots, p_r, y_1, \ldots, y_k\}$. As $\vec{\pi}_n(\vec{v}_j) = \mathbf{0}$ for $j = k+1, \ldots, s$,

$$\pi_n(\mathcal{K} \cap \Pi^-) = \pi_n(\mathcal{K}_0 + \vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-)) = \pi_n(\mathcal{K}_0) + \vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K} \cap \Pi^-)) = \pi_n(\mathcal{K}_0),$$

which is a bounded set. Thus, if we prove $\pi_n(\mathcal{K} \cap \Pi^+) = \mathcal{K} \cap \Pi^+ \cap \{x_n = 0\}$, the set $\mathfrak{A}_{\mathcal{K}} \times \{0\} \subset \pi_n(\mathcal{K}_0)$ is bounded.

Indeed, by hypotheses (i), (ii) and Eq. (3.2) it holds

$$\begin{aligned}\pi_n(\mathcal{K} \cap \Pi^+) &= \pi_n(\mathcal{K} \cap \Pi) + \vec{\pi}_n(\vec{\mathfrak{C}}(\mathcal{K}))\\ &= ((\mathcal{K} \cap \Pi) + \vec{\mathfrak{C}}(\mathcal{K})) \cap \{x_n = 0\} = \mathcal{K} \cap \Pi^+ \cap \{x_n = 0\},\end{aligned}$$

as required. □

4. Placing into Trimming Positions

4.1. *First trimming position*

The purpose of the first part of this section is to prove Proposition 1.2. For the sake of clearness, we divide the proof into two parts.

Proof of Proposition 1.2 for unbounded facets. Assume $\mathcal{K}$ is in FU-position w.r.t. $\vec{\Pi} = \{x_n = 0\}$. The proof is conducted in several steps. While reading the proof it may be useful to look at Figs. 1 and 2.

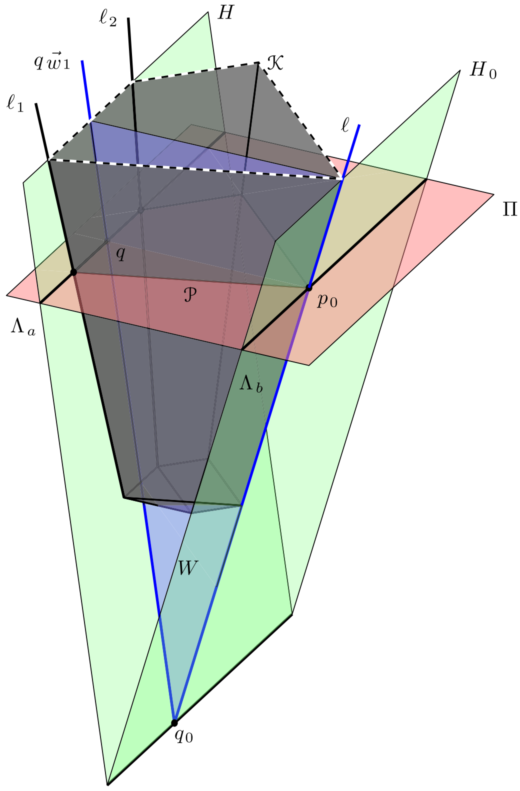

Fig. 1. Unbounded convex polyhedron.

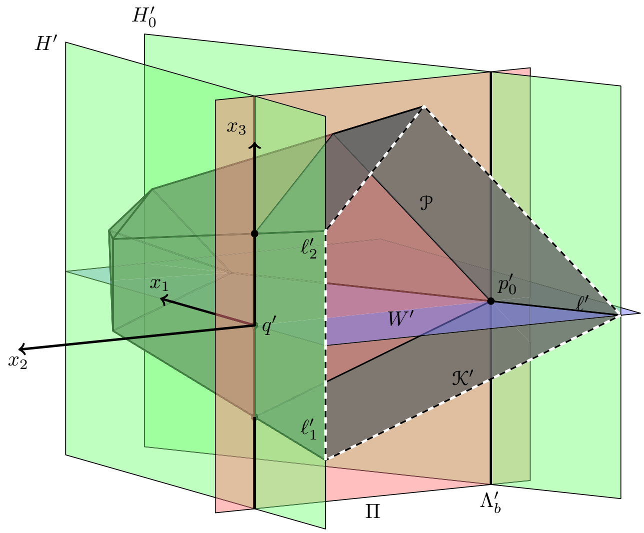

Fig. 2. First trimming position for the unbounded convex polyhedron.

Step 1: *Restrictions for the facet* $\mathcal{F}$. Denote $\widehat{\mathcal{K}} := \mathrm{Cl}_{\mathbb{RP}^3}(\mathcal{K})$ and $\mathcal{K}_\infty := \mathrm{Cl}_{\mathbb{RP}^3}(\mathcal{K}) \cap \mathsf{H}_\infty(\mathbb{R})$. Consider the involutive homography

$$\phi : \mathbb{RP}^3 \to \mathbb{RP}^3, \quad (x_0 : x_1 : x_2 : x_3) \mapsto (x_3 : x_1 : x_2 : x_0) = (y_0 : y_1 : y_2 : y_3).$$

By Sec. 2.3.1, the convex polyhedron $\mathcal{K}' := \phi(\widehat{\mathcal{K}}) \subset \mathbb{R}^3 \equiv \{y_0 \neq 0\}$ is bounded and one of its faces is $\mathcal{E}' := \phi(\mathcal{K}_\infty)$, which needs not to be a facet. Now we distinguish:

Case 1. If $\mathcal{E}'$ has dimension 0, choose any unbounded facet $\mathcal{F}$ of $\mathcal{K}$.

Case 2. If $\mathcal{E}'$ has dimension 1 ($\mathcal{E}'$ is an edge of $\phi(\widehat{\mathcal{K}})$) or 2 ($\mathcal{E}'$ is a facet of $\phi(\widehat{\mathcal{K}})$), choose any unbounded facet $\mathcal{F}$ with no parallel unbounded edges. Note $\dim(\mathrm{Cl}(\phi(\mathcal{F})) \cap \mathcal{E}') = 1$.

Denote the plane generated by the facet $\mathcal{F}$ with H.

Step 2: *Choice of a suitable sawing plane* Π *and an auxiliary vector* $\vec{w}$. Let Π be a sawing plane for $\mathcal{K}$ (parallel to $\vec{\Pi}$). If $\dim(\mathcal{E}') = 2$, the dimension of $\mathcal{K}_\infty$ is 2, the dimension of $\mathrm{Cl}_{\mathbb{RP}^3}(\mathcal{F}) \cap \mathcal{K}_\infty$ is 1 and by Sec. 2.3.1 the dimension of $\vec{\mathfrak{C}}(\mathcal{F})$ is 2. Since $\mathcal{K}$ is in FU-position w.r.t. $\vec{\Pi}$, $\mathcal{F}$ is in FU-position w.r.t. $\vec{\Pi} \cap \vec{H}$ inside H. Let $\mathfrak{G}$ be the (finite) set constituted by the intersections of H with the unbounded edges of $\mathcal{K}$ that are not parallel to H and fix $\vec{w} \in \mathrm{Int}\,\vec{\mathfrak{C}}(\mathcal{F})$. By Lemma 2.3, we may assume in addition that the intersection $p\vec{w} \cap (\mathcal{F} \cap \Pi)$ is a singleton for each $p \in \mathfrak{G}$.

Step 3: *Construction of an auxiliary supporting hyperplane* H_0 *of* $\mathcal{K}$. Denote $\mathcal{P} := \mathcal{K} \cap \Pi$ and note that $\mathcal{F} \cap \Pi = H \cap \mathcal{P}$ is one of its edges. The map

$$\rho : \Pi \to \Pi/(\vec{H} \cap \vec{\Pi}), \quad x \mapsto x + (\vec{H} \cap \vec{\Pi})$$

is continuous (with respect to the quotient topology of $\Pi/(\vec{H} \cap \vec{\Pi})$). As $\mathcal{P} \subset \Pi$ is compact and $\Pi/(\vec{H} \cap \vec{\Pi})$ is homeomorphic to $\mathbb{R}$ (with its usual topology), $\rho(\mathcal{P}) \equiv [a, b]$ is a compact interval (non-trivial because $\mathcal{P}$ has dimension 2). Clearly, $\rho(H \cap \Pi)$ is one of the extremes of this interval and we assume $\Lambda_a := \rho^{-1}(a) = H \cap \Pi$. Note that $\Lambda_b := \rho^{-1}(b)$ is a supporting line of $\mathcal{P}$ in Π, so $\Lambda_b \cap \mathcal{P}$ is either an edge or a vertex of $\mathcal{P}$. We pick a vertex $p_0 \in \Lambda_b \cap \mathcal{P}$, which is the intersection of Π with an unbounded edge $\mathcal{A}$ of $\mathcal{K}$. Denote the line generated by $\mathcal{A}$ with ℓ.

Let H_0 be the supporting plane of $\mathcal{K}$ generated by Λ_b and ℓ (Lemma 2.2). We can assume $\mathcal{K} \subset H^+ \cap H_0^+$.

Step 4: *Construction of a plane* W *that contains the line* ℓ *and such that* $W \cap \mathcal{F}$ *is a half-line*. In order to achieve this, we analyze two possible situations:

Case 1. If ℓ is parallel to H, choose a plane W that contains ℓ and meets $\mathrm{Int}\,\mathcal{F} \cap \Pi$ in a point q.

Case 2. If ℓ is not parallel to H, then it meets H in a singleton, say $\ell \cap H = \{q_0\}$. Besides,

$$\mathrm{Cl}_{\mathbb{RP}^3}(\ell) \cap \mathrm{Cl}_{\mathbb{RP}^3}(H) \cap \mathsf{H}_\infty(\mathbb{R}) = \varnothing \quad \text{implies } \mathrm{Cl}_{\mathbb{RP}^3}(\mathcal{A}) \cap \mathrm{Cl}_{\mathbb{RP}^3}(\mathcal{F}) \cap \mathcal{K}_\infty = \varnothing,$$

so the dimension of $\mathcal{K}_\infty$ is 2. Recall now the auxiliary vector $\vec{w}$ fixed in Step 2. By our choice of Π (see Step 2), the intersection $q_0\vec{w} \cap (\mathcal{F} \cap \Pi)$ is a point q. Denote the plane of $\mathbb{R}^3$ that contains the coplanar lines ℓ and the line through q_0 that is parallel to $\vec{w}$ with W.

Step 5: Let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto W in the direction of $\vec{H} \cap \vec{\Pi}$. Then: $\pi(\mathcal{K} \cap \Pi) = (\mathcal{K} \cap \Pi) \cap W$, and $\vec{\pi}(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{W}$.

Indeed, as $p_0 \in \Lambda_b \cap \Pi$, $q \in \Lambda_a \cap \Pi$ and these lines are parallel to $\vec{H} \cap \vec{\Pi}$, the line $\Pi \cap W$ (through the points p_0 and q) satisfies $\pi(\mathcal{K} \cap \Pi) = (\mathcal{K} \cap \Pi) \cap W$. Next we check: $\vec{\pi}(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{W}$.

Notice first that $\vec{H}^+ \cap \vec{H}_0^+ \cap \vec{\Pi}^+ \cap \vec{W}$ is the cone generated by two vectors $\vec{w}_1, \vec{w}_2 \in \vec{\mathfrak{C}}(\mathcal{K}) \setminus \{0\}$ (not necessarily linearly independent) such that $\vec{w}_1$ generates the line $W \cap H$ and $\vec{w}_2$ the line $H_0 \cap W$. Thus,

$$\vec{H}^+ \cap \vec{H}_0^+ \cap \vec{\Pi}^+ \cap \vec{W} \subset \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{W} \subset \vec{\pi}(\vec{\mathfrak{C}}(\mathcal{K})). \tag{4.1}$$

As $\mathcal{K}$ is in FU-position w.r.t. $\vec{\Pi}$ and $\mathcal{K} \subset H^+ \cap H_0^+$, we deduce

$$\vec{\mathfrak{C}}(\mathcal{K}) \subset \vec{\mathfrak{C}}(H^+ \cap H_0^+) \cap \vec{\Pi}^+ = \vec{H}^+ \cap \vec{H}_0^+ \cap \vec{\Pi}^+.$$

As $\vec{H} \cap \vec{\Pi} \subset \vec{H}_0$ and $\mathrm{Im}(\vec{\pi}) = \vec{W}$, we have

$$\vec{\pi}(\vec{\mathfrak{C}}(\mathcal{K})) \subset \vec{H}^+ \cap \vec{H}_0^+ \cap \vec{\Pi}^+ \cap \vec{W}. \tag{4.2}$$

Combining Eqs. (4.1) and (4.2), we conclude $\vec{\pi}(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \vec{W}$.

Step 6: After a change of coordinates that transforms H^+ into $\{x_2 \leq 0\}$, W into $\{x_3 = 0\}$ and the line $\vec{H} \cap \vec{\Pi}$ into the line $\{x_1 = x_2 = 0\}$, we conclude by Lemma 3.1 and Remark 3.3 that the convex polyhedron $\mathcal{K}$ is in first trimming position with respect to the facet $\mathcal{F}$ (see Fig. 2), as required. □

Proof of Proposition 1.2(ii) for bounded facets. We check first that $\mathcal{K}$ has a bounded facet. Assume that the unbounded edges of $\mathcal{K}$ are parallel to the vector $\vec{e}_3$. Let $\mathcal{A} := p\vec{e}_3$ be an unbounded edge of $\mathcal{K}$. Let $\mathcal{F}$ be a facet of $\mathcal{K}$ not parallel to $\vec{e}_3$ such that $p \in \mathcal{F}$. It holds $\vec{\mathfrak{C}}(\mathcal{F}) \subset \vec{\mathfrak{C}}(\mathcal{K}) = \{\lambda\vec{e}_3 : \lambda \geq 0\}$. As $\mathcal{F}$ is not parallel to $\vec{e}_3$, we obtain $\vec{\mathfrak{C}}(\mathcal{F}) = \{\mathbf{0}\}$, so $\mathcal{F}$ is bounded.

Fix a bounded facet $\mathcal{F}_0$ of $\mathcal{K}$ and let $H \subset \mathbb{R}^3$ be the plane generated by $\mathcal{F}_0$. As the unbounded edges of $\mathcal{K}$ are parallel, $\vec{\mathfrak{C}}(\mathcal{K}) = \{\lambda\vec{v} : \lambda \geq 0\}$. Let $\vec{\ell}$ be the line generated by $\vec{v}$. As $\mathcal{F}_0 = \mathcal{K} \cap H$ is a bounded facet, $\vec{H} \cap \vec{\ell} = \{\mathbf{0}\}$. Let now Π be a plane parallel to H that meets all unbounded edges of $\mathcal{K}$ and such that all bounded faces of $\mathcal{K}$ are contained in $\operatorname{Int}\Pi^-$. By Remark 3.3, $\mathcal{P} := \mathcal{K} \cap \Pi$ is a bounded convex polygon, $\mathcal{K} \cap \Pi^-$ is a bounded convex polyhedron and

$$\mathcal{K} \cap \Pi^+ = \mathcal{P} + \vec{\mathfrak{C}}(\mathcal{K}) = \mathcal{P} + \{\lambda\vec{v} : \lambda \geq 0\}.$$

Thus, $\mathcal{K} \cap \Pi^+$ is affinely equivalent to $\mathcal{P} \times [0, +\infty[$. After a change of coordinates we assume:

- H is the plane $\{x_2 = 0\}$ and $\mathcal{K} \subset \{x_2 \leq 0\}$,
- Π is the plane $\{x_2 = -1\}$ and $\pi_3(\mathcal{P}) = \mathcal{P} \cap \{x_3 = 0\}$ (recall here the projection Property 1.4 for convex polygons described in Sec. 1),

- $\vec{v} = -\vec{e}_2$, so $\vec{\mathfrak{C}}(\mathcal{K}) = \{-\lambda\vec{e}_2 : \lambda \geq 0\} \subset \{x_3 = 0\}$ and $\pi_3(\vec{\mathfrak{C}}(\mathcal{K})) = \vec{\mathfrak{C}}(\mathcal{K}) \cap \{x_3 = 0\}$.

By Lemma 3.1 and Remark 3.3, $\mathcal{K}$ is in first trimming position with respect to the facet $\mathcal{F}_0$. $\square$

4.2. *Second trimming position*

Our purpose now is to prove Proposition 1.3.

Proof of Proposition 1.3. (i) Assume that $\mathcal{K} \cap \{x_3 = 0\} = \mathcal{F}$ is one of its unbounded facets with non-parallel unbounded edges $\mathcal{A}_1$ and $\mathcal{A}_2$. Let H_i be the plane of $\mathbb{R}^3$ generated by the facet $\mathcal{F}_i$ such that $\mathcal{A}_i = \mathcal{F} \cap \mathcal{F}_i$. As $H_1 \cap \{x_3 = 0\}$ and $H_2 \cap \{x_3 = 0\}$ are non-parallel lines, we may assume keeping the plane $\{x_3 = 0\}$ invariant that $H_i := \{x_i = 0\}$. Changing the signs of the variables if necessary, we assume $\mathcal{K} \subset \{x_1 \geq 0, x_2 \geq 0, x_3 \leq 0\}$. It remains to show that $\mathfrak{A}_{\mathcal{K}}$ is bounded. Consider now the projection $\pi_3 : \mathbb{R}^3 \to \mathbb{R}^2, (x_1, x_2, x_3) \mapsto (x_1, x_2, 0)$.

It is clear that $\mathfrak{A}_{\mathcal{K}} \subset \pi_3(\mathcal{K}) \subset \{x_1 \geq 0, x_2 \geq 0\}$. Take the extremes $p_1 = (0, b, 0)$ and $p_2 = (a, 0, 0)$ of the unbounded edges $\mathcal{A}_1$ and $\mathcal{A}_2$. Note that $a, b \geq 0$. The bounded convex polygon

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, \frac{x_1}{a+1} + \frac{x_2}{b+1} - 1 \leq 0\right\}$$

contains $\mathfrak{A}_{\mathcal{K}}$, so it is bounded. Thus, $\mathcal{K}$ is in second trimming position with respect to $\mathcal{F}$.

(ii) If $\mathcal{F}$ is a bounded facet of $\mathcal{K}$, then $\mathcal{F}$ is not parallel to the unbounded edges of $\mathcal{K}$. We may assume that $\mathcal{K} \subset \{x_3 \leq 0\}$, $\mathcal{K} \cap \{x_3 = 0\} = \mathcal{F}$ is a facet of $\mathcal{K}$ and the unbounded edges of $\mathcal{K}$ are parallel to the vector $\vec{e}_3$. Thus, $\mathcal{K}$ is in FU-position w.r.t. $\vec{\Pi} := \{-x_3 = 0\}$. By Sec. 2.2.1 and as $\vec{\pi}_3(\vec{e}_3) = \mathbf{0}$, we deduce that $\pi_3(\mathcal{K})$ is bounded, so $\mathfrak{A} \subset \pi_3(\mathcal{K})$ is also bounded. Thus, $\mathcal{K}$ is in second trimming position with respect to $\mathcal{F}$, as required. $\square$

Acknowledgments

First author is supported by Spanish GR MTM2011-22435 while the second author is a external collaborator of this project.

The authors are very grateful to S. Schramm for a careful reading of the final version and for the suggestions to refine its redaction.

Appendix A. Limitations of Trimming Positions

In this Appendix, we construct unbounded convex polyhedra that cannot be placed in first or second trimming position with respect to their facets. The clue is that Property 1.4 fails for $n > 2$. The key result is the following lemma. It can be proved by perturbing appropriately the vertices of a *regular cross-polytope* of $\mathbb{R}^n$, which is the convex hull of the set of points consisting of all permutations of $(\pm 1, 0, \ldots, 0) \in \mathbb{R}^n$.

Lemma A.1 (Convex polyhedra without sectional projections). *For each* $n \geq 3$ *there exists an* n*-dimensional bounded convex polyhedron* $\mathcal{K} \subset \mathbb{R}^n$ *such that for each line* $\vec{\ell}$ *and each hyperplane* H *that is not parallel to* $\vec{\ell}$ *it holds* $\pi(\mathcal{K}) \neq \mathcal{K} \cap H$ *where* $\pi : \mathbb{R}^n \to H$ *is the linear projection onto* H *in the direction of* $\vec{\ell}$.

Corollary A.2 (Counterexamples to the trimming positions). *Let* $n \geq 4$ *and* $\mathcal{P} \subset \mathbb{R}^{n-1}$ *be a bounded convex polyhedron without sectional projections. Define* $\mathcal{K} := \mathcal{P} \times [0, +\infty[\subset \mathbb{R}^n$. *Then* $\mathcal{K}$ *can be placed neither in first trimming position with respect to any of its facets nor in second trimming position with respect to any of its unbounded facets.*

Proof. Denote $\mathcal{F}_0 := \mathcal{K} \cap \{x_n = 0\} = \mathcal{P} \times \{0\}$, which is a facet of $\mathcal{K}$, and let $\vec{\ell}$ be the direction of the unbounded edges of $\mathcal{K}$. Suppose that $\mathcal{K}$ is placed in first trimming position with respect to one of its facets. By Lemma 3.1, there exists a hyperplane $\Pi \subset \mathbb{R}^n$ such that: its direction $\vec{\Pi}$ contains the line $\vec{\ell}_n$ generated by $\vec{e}_n$, Π meets each unbounded edge of $\mathcal{K}$ in a singleton, the vertices of $\mathcal{K}$ are contained in $\operatorname{Int} \Pi^-$ and $\pi_n(\mathcal{K} \cap \Pi) = \mathcal{K} \cap \Pi \cap \{x_n = 0\}$.

Consider the projection $\rho : \mathbb{R}^n \to \mathbb{R}^n$ in the direction of the line $\vec{\ell}$ onto the hyperplane H generated by $\mathcal{F}_0$. As Π meets the unbounded edges of $\mathcal{K}$ in a singleton and all of them are parallel to $\vec{\ell}$, we deduce that Π is not parallel to $\vec{\ell}$, so $g := \rho|_\Pi : \Pi \to H$ is an affine bijection. By Remark 3.3, $\mathcal{P}_1 := \mathcal{K} \cap \Pi$ is the bounded convex polyhedron whose vertices are the intersections of the unbounded edges of $\mathcal{K}$ (all of them parallel to $\vec{\ell}$) with the hyperplane Π. As $\vec{\rho}(\vec{\ell}) = \{\mathbf{0}\}$, we deduce $\rho(\mathcal{P}_1) = \mathcal{F}_0$. If we define $\vec{r} := \vec{g}(\vec{\ell}_n)$ and $W := g(\Pi \cap \{x_n = 0\}) \subset H$, the projection $\pi := (g \circ \pi_n \circ g^{-1}) : H \to H$ in the direction of $\vec{r}$ onto the hyperplane W satisfies $\pi(\mathcal{F}_0) = \mathcal{F}_0 \cap W$, which is against the fact that $\mathcal{F}_0 := \mathcal{P} \times \{0\}$ has no sectional projections. Thus, $\mathcal{K}$ cannot be placed in first trimming position with respect to any of its facets.

Suppose next that $\mathcal{K}$ is placed in second trimming position with respect to one of its unbounded facets $\mathcal{F}_0$, which is contained in the hyperplane $\{x_n = 0\}$. The unbounded edges of $\mathcal{K}$ are parallel to a line $\vec{\ell}$ contained in the hyperplane $\{x_n = 0\}$, so they are not parallel to the line $\vec{\ell}_n$ generated by $\vec{e}_n$. By Lemma 3.1, there exists a hyperplane $\Pi \subset \mathbb{R}^n$ such that: its direction $\vec{\Pi}$ contains the line $\vec{\ell}_n$, the hyperplane Π meets the unbounded edges of $\mathcal{K}$ (which are all not parallel to $\vec{\ell}_n$), the bounded edges of $\mathcal{K}$ are contained in the open half-space $\operatorname{Int} \Pi^-$ and $\pi_n(\mathcal{K} \cap \Pi) = \mathcal{K} \cap \Pi \cap \{x_n = 0\}$. Proceeding analogously to the previous paragraph, we achieve a contradiction. Thus, $\mathcal{K}$ cannot be placed in second trimming position with respect to any of its unbounded facets. $\square$

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