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On the Pythagoras Numbers of Real Analytic Rings[1]

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We show that the Pythagoras number of a real analytic ring of dimension 2 is finite, bounded by a function of the multiplicity and the codimension.

1. INTRODUCTION AND MAIN RESULTS

The Pythagoras number of a ring A is the smallest integer $p(A) = p \geq 1$ such that any sum of squares of A is a sum of p squares, and $p(A) = +\infty$ if such an integer does not exist. This is a very delicate invariant whose study has deserved a lot of attention from specialists in number theory, quadratic forms, real algebra, and real geometry. Well-known examples are the following: $p(\mathbb{Z}) = 4$ (Lagrange's famous theorem), $n + 2 \leq p(\mathbb{R}(x_1, \ldots, x_n)) \leq 2^n$ [Pf, CEP], $p(\mathbb{R}[x_1, \ldots, x_n]) = +\infty$ for $n \geq 2$ [ChDLR]. We refer the reader to [ChDLR, BCR] for further details. A special important case is that of local rings. The most general result here is that $p(A) = +\infty$ for any local regular ring A of dimension ≥ 3 [ChDLR]; there are also several finiteness results for local regular rings of dimension 2 in the so-called geometric cases [Sch]. However, there is a serious lack of information without the regularity assumption. In this paper we deal with this matter for real local analytic rings, that is, for the local rings of real analytic spaces.

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To summarize what is already known, we fix the following notation. Let X be an analytic set germ (at the origin of $\mathbb{R}^n$); we denote by $\mathcal{O}(X)$ the ring of germs of analytic functions on X and by $\mathcal{M}(X)$ its total ring of fractions. As $X \subset \mathbb{R}^n$ we have $\mathcal{O}(X) = \mathbb{R}\{x\}/I$, where $I = I(X)$ is the ideal of all analytic function germs vanishing on X. For irreducible X, the ideal $I(X)$ is prime, $\mathcal{O}(X)$ is a domain, and $\mathcal{M}(X)$ is a field; for arbitrary X, we have $I(X) = \cap \mathfrak{p}_i$, where the $\mathfrak{p}_i$'s are the ideals of the irreducible components X_i of X, and $\mathcal{M}(X) = \prod \mathcal{M}(X_i)$. For instance, $\mathcal{O}(\mathbb{R}^n)$ is the ring $\mathbb{R}\{x\}$ of convergent power series in $x = (x_1, \dots, x_n)$, and $\mathcal{M}(\mathbb{R}^n)$ is the field of fractions $\mathbb{R}(\{x\})$ of $\mathbb{R}\{x\}$.

From now on, $p[X]$ stands for the Pythagoras number of $\mathcal{O}(X)$ and $p(X)$ for that of $\mathcal{M}(X)$.

These Pythagoras numbers may be very different. If X is a curve germ, $p(X) = 1$ and $p[X]$ is finite [Rz1]. However, $p[X]$ can be arbitrarily large (see [Or], where there is an algorithmic approach to the estimation of $p[X]$ for X with fixed value semigroup); we will see in Section 5 that one can even find curve germs $X \subset \mathbb{R}^3$ with $p[X] \to +\infty$. On the other hand, if X is irreducible $p[X]$ is bounded by the multiplicity [Qz]. We also know that $p[X] = 1$ exactly for the so-called Arf germs [CaRz].

In higher dimensions, let us look first at $p(X)$. We have the following result (that could be considered folklore):

PROPOSITION 1.1. *Let $X \subset \mathbb{R}^n$ be an analytic surface germ. Then*

$$2 \leq p(X) \leq 2m,$$

where m is the maximum multiplicity of a two dimensional irreducible component of X.

Proof. We can suppose that X is irreducible. After a linear change, $\mathcal{O}(X)$ is a finitely generated module over $\mathbb{R}\{x_1, x_2\}$, and $\mathcal{M}(X)$ is a linear space of finite dimension m over the field $K = \mathbb{R}(\{x_1, x_2\})$ (see for instance [Rz2, II.2.3]). Suppose first that $p(X) = 1$. Then by the Diller–Dress descent theorem [DiDr] we would have $p(K) = 1$, against the known fact that $p(K) = 2$. Now we prove the upper bound using an argument attributed to Pfister.

Let $z_1, \dots, z_m$ be a basis of $\mathcal{M}(X)$ over K. Now, given a sum of squares $f = f_1^2 + \cdots + f_r^2$, we can write $f_i = f_{i1}z_1 + \cdots + f_{im}z_m \in \mathcal{M}(X)$ with $f_{ij} \in K$. Thus we consider the quadratic form over K

$$Q(\mathsf{z}_1, \dots, \mathsf{z}_m) = \sum_{i=1}^{r} L_i(\mathsf{z}_1, \dots, \mathsf{z}_m)^2, \qquad L_i(\mathsf{z}_1, \dots, \mathsf{z}_m) = f_{i1}\mathsf{z}_1 + \cdots + f_{im}\mathsf{z}_m,$$

so that $f = Q(z_1, \dots, z_m)$. Clearly we only need to see that $Q(\mathtt{z})$ can be written as a sum of $2m$ squares of linear forms over K.

To show that, we diagonalize $Q(\mathtt{z})$: after a linear change $\mathtt{t}_i = H_i(\mathtt{z})$ we find $Q(\mathtt{z}) = d_1\mathtt{t}_1^2 + \cdots + d_m\mathtt{t}_m^2$, $d_i \in K$. Hence

$$Q(\mathtt{z}) = d_1H_1(\mathtt{z})^2 + \cdots + d_mH_m(\mathtt{z})^2,$$

for some d_i's, which by the definition of $Q(\mathtt{z})$ are sums of squares. As $p(K) = 2$, we have $d_i = a_i^2 + b_i^2$ and

$$Q = \sum_{i=1}^{m}\left(a_1^2 + b_i^2\right)H_i^2 = \sum_{i=1}^{m} a_i^2H_i^2 + \sum_{i=1}^{m} b_i^2H_i^2.$$

We are done. ∎

The same argument would bound $p(X)$ in any dimension d, had we a bound for $p(\mathbb{R}^d)$. However, only the bound $p(\mathbb{R}^3) \le 8$ is available at the moment [Jw].

We next turn to $p[X]$. From the result quoted above about local regular rings we see that $p[\mathbb{R}^n] = +\infty$ for $n \ge 3$. Actually, a bit more is true: $p[X] = +\infty$ if X has dimension ≥ 4 [Rz3]; of course, one expects the same for dimension 3. Thus, we are left with surface germs.

For these, one has $p[X] \ge p(X) \ge 2$, and the oldest example computed was $p[\mathbb{R}^2] = p(\mathbb{R}^2) = 2$ [BoRi]. The same was proved later for Brieskorn's singularity, the two planes, Whitney's umbrella [Rz3], the cone [FeRz], and a few more: the singularities $z^2 = x^3 + y^4$, $x^2y - y^3$, $x^3 - xy^3$ and deformations $z^2 = x^2 + (-1)^ky^k (k \ge 3)$ of the two planes and $z^2 = x^2y + (-1)^ky^k (k \ge 4)$ of Whitney's umbrella [Fe1]. There are also some 2-dimensional cones X with arbitrary embedding dimension for which again $p[X] = p(X) = 2$ [Fe2]. However, there was no general result concerning finiteness. Our main goal here is to settle this. First, concerning lower bounds, we have:

THEOREM 1.2. *Let $X \subset \mathbb{R}^n$ be an analytic germ, $n \ge 3$. Then*

$$p[X] \ge P\big(n, 2E\big[\tfrac{1}{2}(\omega(I(X)) - 1)\big]\big),$$

where $E[\cdot]$ represents the integral part of a real number, $\omega(I(X))$ is the minimum order of a series in $I(X)$, and $P(n,m)$ is the Pythagoras number of homogeneous forms over $\mathbb{R}$ of degree m in n variables [ChLR].

Proof. We consider an homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree $2E[\frac{1}{2}(\omega(I(X)) - 1)]$ which is the sum of $p = P(n, 2E[\frac{1}{2}(\omega(I(X)) - 1)])$ squares, but not less than p. Since f is an SOS in $\mathbb{R}^n$, then it is also an SOS in X. If $p[X] < p$, then

$$f = h_1^2 + \cdots + h_{p-1}^2 + h,$$

where $h \in I(X)$. Since f has degree $2E[\frac{1}{2}(\omega(I(X)) - 1)] < \omega(I(X))$, comparing initial forms we obtain that

$$f = g_1^2 + \cdots + g_{p-1}^2,$$

where $g_1, \ldots, g_{p-1} \in \mathbb{R}[x_1, \ldots, x_n]$ are homogeneous polynomials, and so f is a sum of $p-1$ squares, a contradiction. Thus we have

$$p[X] \geq P\big(n, 2E\big[\tfrac{1}{2}(\omega(I(X)) - 1)\big]\big).$$

∎

Remarks 1.3. (a) In view of [ChDLR, 4.16] we deduce that for $n \geq 3$

$$P(n, m) \geq E[\log_2(m+2)].$$

This already gives the bound

$$p[X] \geq E[\log_2(\omega(I(X)) + 1)].$$

Furthermore, by [ChDLR, 4.6′] we obtain

if $m \geq 2^{i+2} + 2^i - 4i - 6$ then $P(n, m) \geq 2i$
if $m \geq 2^{i+2} + 2^{i+1} + 2^i - 4i - 8$ then $P(n, m) \geq 2i + 1$.

This bound is better than the previous one.

(b) From [ChLR, 6.4] we see that for fixed $m \geq 2$ there exists a constant $\gamma_1(m)$ such that

$$P(n, m) \geq \gamma_1(m) n^{m/2}.$$

In particular if $m = 2$ then $P(n, 2) = n$ and therefore, if $\omega(I(X)) \geq 3$, we have

$$p[X] \geq P(n, 2) \geq n.$$

(c) For embedding dimension $n = 3$ there exist curve and surface germs with the Pythagoras number arbitrarily large: in Section 5, we will obtain irreducible curve germs in $\mathbb{R}^3$ whose ideals have orders arbitrarily large.

Concerning upper bounds, in Section 4 we will show:

THEOREM 1.4. *Let $X \subset \mathbb{R}^n$, $n \geq 3$, an analytic surface germ. Then*

$$p[X] \leq 2\, \mathrm{mult}_{\mathrm{T}}(X)^{n-2},$$

where $\mathrm{mult}_{\mathrm{T}}(X)$ *denotes the* total multiplicity *of X, that is, the sum of the multiplicities of* all *the irreducible components and not only the maximum dimension ones.*

The proof of the upper bound involves, in fact, the so-called *Strong Question* [ChDLR, Sect. 2(Q_2)], namely: Let A_0 be a commutative ring with $p(A_0) < +\infty$, and let A be an A_0-algebra that can be generated by m elements as an A_0-module. Is it true that $p(A) \leq p(A_0) \cdot m$?

The argument of 1.1 above actually settles this strong question for any *field* A_0. Then a suitable generalization of this idea works for $A_0 = R[t]$, where R is a real closed field [ChDLR]. Here we are able to solve the cases $A_0 = \mathbb{R}(\{x\})[y], \mathbb{R}\{x\}[y]$, and $\mathbb{R}\{x, y\}$. This requires diagonalization of quadratic forms over $\mathbb{C}\{x, y\}$, which we obtain in Section 3 following some ideas of Djoković, after some discussion in Section 2 of positive elements in two variables.

2. DEFINITENESS

The purpose of this section is to characterize the positive semidefinite elements in two variables and to show that they are sums of *two* squares.

In what follows $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. In this section, $\mathbb{K}\{x\}$ is the ring of convergent series in one single variable x, and $\mathbb{K}(\{x\})$ its quotient field. Furthermore, we denote by $\Omega_{\mathbb{K}}$ the ring of convergent Puiseux series with coefficients in $\mathbb{K}$, and by $\Phi_{\mathbb{K}}$ its quotient field. If $\alpha \in \Omega_{\mathbb{K}}$ we put $\mathrm{q}(\alpha) = \min\{n \in \mathbb{N} : \alpha \in \mathbb{K}\{x^{1/n}\}\}$ and $\omega(\alpha) = r/n$ if $\alpha = a_r x^{r/n} + a_{r+1} x^{(r+1)/n} + \cdots$; then

$$\mathrm{q}(\alpha/\beta) = \min\{n \in \mathbb{N} : \alpha/\beta \in \mathbb{K}(\{x^{1/n}\}\}) = \mathrm{l.c.m}\{\mathrm{q}(\alpha), \mathrm{q}(\beta)\mathrm{p}\}$$

and $\omega(\alpha/\beta) = \omega(\alpha) - \omega(\beta)$

The field $\mathbb{R}(\{x\})$ has only two orderings, determined by the sign of x:

(+) $(\mathbb{R}(\{x\}), <)$. In this ordering x is positive and its real closure is given by the inclusion $\mathbb{R}(\{x\}) \subset \Phi_{\mathbb{R}}$. A series $f = a_r x^r + a_{r+1} x^{r+1} + \cdots$ is >0 when $a_r > 0$, and $f/g > 0$ if and only if $fg > 0$.

(−) $(\mathbb{R}(\{x\}), \prec)$. In this ordering x is negative and its real closure is given by the embedding $\sigma : \mathbb{R}(\{x\}) \to \Phi_{\mathbb{R}} : f(x) \mapsto f(-x)$. A series $f = a_r x^r + a_{r+1} x^{r+1} +$ is $\succ 0$ when $(-1)^r a_r > 0$, and $f/g \succ 0$ if and only if $fg \succ 0$.

The general notion of semidefiniteness is the following:

DEFINITION 2.1. Let A be a commutative ring and $f \in A$. We say that f is *positive semidefinite* or *psd in* A if $f(\alpha) \geq 0$ for every prime cone $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha) \in \mathrm{Spec}_r(A)$; that is, the class of f is ≥ 0 in the field $(qf(A/\mathfrak{p}_\alpha), \leq_\alpha)$.

In our case we have:

PROPOSITION 2.2. *Let* $f \in \mathbb{R}\{x\}[y]$, $f \neq 0$. *The following assertions are equivalent*:

(a) *f is psd in the ring* $\mathbb{R}\{x\}[y]$.

(b) *f is psd in the field* $qf(\mathbb{R}\{x\}[y])$.

(c) *For every* $\xi \in \Phi_{\mathbb{R}}$ *the Puiseux series* $f(x, \xi)$, $f(-x, \xi)$ *are positive in* $\Phi_{\mathbb{R}}$.

If any, hence all, of these assertions holds true we write $f \geq 0$.

Proof. Clearly, (a) $\Rightarrow$ (b) and the converse is also true, since $\mathbb{R}\{x\}[y]$ being regular, every prime cone has a generalization which is a total ordering. Thus, we are reduced to prove (a) $\Rightarrow$ (c) $\Rightarrow$ (b).

(a) $\Rightarrow$ (c). This follows readily since every $\xi \in \Phi_{\mathbb{R}}$ and $\varepsilon = \pm 1$ define a prime cone α of $\mathbb{R}\{x\}[y]$ by $h \mapsto h(\varepsilon x, \xi)$.

(c) $\Rightarrow$ (b). Suppose now that there exists an ordering in $\mathbb{R}(\{x\})[y]$ such that $f < 0$. Then by the Artin–Lang Theorem (see [La, XI]), there exists an $\mathbb{R}(\{x\})$-homomorphism $\varphi : \mathbb{R}(\{x\})[y] \to \Phi_{\mathbb{R}}$ such that $\varphi(f) < 0$. Now, by the description of the two unique orderings of $\mathbb{R}(\{x\})$ and their respective real closures, if $\varphi(y) = \xi$, then $\varphi(f) = f(\varepsilon x, \xi) < 0$ with $\varepsilon = \pm 1$, against (c). ∎

THEOREM 2.3. *Every positive semidefinite element of* $\mathbb{R}\{x\}[y]$ *is a sum of* 2 *squares in* $\mathbb{R}\{x\}[y]$.

Proof. Let $H \in \mathbb{R}\{x\}[y]$ be psd. Since H is positive in all total orderings of $\mathbb{R}\{x\}[y]$, then H is an SOS in $\mathbb{R}(\{x\})(y)$. This field has Pythagoras number 2 [ChDLR], hence H is a sum of 2 squares, and this implies that it is a sum of two squares in $\mathbb{R}\{x\}[y]$ (see [ChDLR]). ∎

Now we introduce some definiteness notions for matrices over $\mathbb{K}(\{x\})[y]$. Given a matrix $a = (a_{ij})_{1 \leq i, j \leq n}$ with coefficients in $\mathbb{C}(\{x\})[y]$, we consider its *transpose conjugated* $a^* = \overline{a}^t = (\overline{a_{ji}})_{1 \leq i, j \leq n}$.

DEFINITIONS 2.4. Let a be a matrix with coefficients in $\mathbb{K}(\{x\})[y]$. Then:

- If $a = a^*$, $z^*az \in \mathbb{R}(\{x\})[y]$ for every $z \in \mathbb{K}(\{x\})^n$. We say that $a \geq 0$ if and only if $z^*az \geq 0$ for every $z \in \mathbb{K}(\{x\})^n$.
- We say that a is *anisotropic* if $z^*az \neq 0$ for every $z \in \mathbb{K}(\{x\})^n$, $z \neq 0$.

Remarks 2.5. (a) Let $a, b, c \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ be such that $\det(b) \neq 0$ and $a = b^*cb$. Then $a \geq 0$ if and only if $c \geq 0$.

First, $a \geq 0$ means that in particular $a = a^*$ and so, since $\det(b) \neq 0$, we have $c = c^*$. Let $z_0 \in \mathbb{C}(\{x\})^n$ and consider $y_0 = \mathrm{Adj}(b^t)z_0$. We have

$$0 \leq y_0^* a y_0 = z_0^* \mathrm{Adj}(b^t)^* b^* c b \mathrm{Adj}(b^t) z_0 = \overline{\det(b)}\det(b) z_0^* c z_0$$
$$= |\det(b)|^2 z_0^* c z_0.$$

Now since $|\det(b)|^2 \neq 0$ we conclude that $z_0^* c z_0 \geq 0$ and so $c \geq 0$. The converse is trivial.

(b) If $a \geq 0$, then every $z \in \mathbb{C}(\{x\})[y]^n$ satisfies $z^* a z \geq 0$.

Indeed, if $z \in \mathbb{C}(\{x\})[y]^n$, $z \neq 0$, we take a matrix $b \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ with determinant not equal to zero whose first column is z_0. By the previous remark $c = b^* a b \geq 0$ and therefore $z^* a z = c_{11} \geq 0$.

(c) Let $a = a^* \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ be such that $\det(a) \neq 0$ and $a \geq 0$. Then a is anisotropic.

Let $z_0 \in \mathbb{C}(\{x\})^n$, $z_0 \neq 0$, and let $b \in \mathfrak{M}_n(\mathbb{C}(\{x\}))$ be an invertible matrix, whose first column is z_0. Since $a \geq 0$ and b is invertible, $c = b^* a b \geq 0$, and $\det(c) \neq 0$. Furthermore, $z_0^* a z_0 = u_1^* c u_1 = c_{11} \geq 0$ (where u_i is the vector whose ith coordinate is 1 and the others are all zero). If $c_{11} = 0$ we show that $c_{1j} = 0$ for $2 \leq j \leq n$ and then $\det(c) = 0$, which is impossible. If there exists $2 \leq j \leq n$ such that $c_{1j} \neq 0$, say $j = 2$, we take $z_0^* = (-\overline{c_{22}} - \frac{1}{2}, c_{12}, 0, \ldots, 0) \neq 0$ and, since $c \geq 0$, we obtain $z_0^* c z_0 = -(c_{22} + 1)|c_{12}|^2 \geq 0$ and then, since $c_{12} \neq 0$, we would have $1 + c_{22} \leq 0$ which is false, because $c_{22} = u_2^* c u_2 \geq 0$.

3. DIAGONALIZATION IN TWO VARIABLES

In this section, we study the diagonalization of positive semidefinite quadratic forms over $\mathbb{R}(\{x\})[y]$, $\mathbb{R}\{x\}[y]$, and $\mathbb{R}\{x, y\}$.

Our approach is based on some ideas developed by Djoković in [Dj] where he proves that any psd matrix $a \in \mathfrak{M}_n(\mathbb{R}[x])$ can be expressed as $a = b_1 b_1^t + b_2 b_2^t$, where $b_1, b_2 \in \mathfrak{M}_n(\mathbb{R}[x])$. Thereafter, a diagonal matrix a will be denoted by $a = \langle a_1, \ldots, a_n \rangle$, where $a_1, \ldots, a_n$ are the elements of the main diagonal of a. We will use the following basic result on principal ideal domains [Hu, VII.2]:

THEOREM 3.1. *Let D be a principal ideal domain and $a \in \mathfrak{M}_n(D)$ a matrix of rank r. Then there exist two invertible matrices u, v a diagonal matrix*

$$e = \langle e_1, \ldots, e_r, 0, \ldots, 0 \rangle,$$

such that $e_1 | e_2 | \cdots | e_r$, and $a = uev$. Furthermore, the ideals $(e_1), (e_2), \ldots, (e_r)$ are unique, and the elements $e_1, e_2, \ldots, e_r$ of the main diagonal e' are called the invariant factors of a. *The diagonal matrix $e = \langle e_1, \ldots, e_r, 0, \ldots, 0 \rangle$ is a* matrix of invariant factors of a.

We begin with the following lemma.

LEMMA 3.2. *Let a be a matrix with coefficients in $\mathbb{C}(\{x\})[y]$ of rank r such that $a = a^*$. Then there exist $a_1 \in \mathfrak{M}_r(\mathbb{C}(\{x\})[y])$ and $u \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ invertible, such that*

$$a = u^* \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} u$$

and $\det(a_1) \neq 0$.

Proof. If $a = uev$ as in the previous theorem, then $a = uev(u^*)^{-1}u^* = uepu^*$ with $p = v(u^*)^{-1}$ invertible. Since $a = a^*$ then $ep = p^*e^*$ and since $e = \langle e_1, \ldots, e_r, 0, \ldots, 0\rangle$ is diagonal, we have

$$ep = \begin{pmatrix} e_1 p_{11} & e_1 p_{12} & \cdots & e_1 p_{1n} \\ \vdots & \vdots & & \vdots \\ e_r p_{r1} & e_r p_{r2} & \cdots & e_r p_{rn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{p_{11}e_1} & \cdots & \overline{p_{r1}e_r} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \\ \overline{p_{n1}e_1} & \cdots & \overline{p_{rn}e_r} & 0 & \cdots & 0 \end{pmatrix} = p^*e^*$$

and therefore

$$p = \begin{pmatrix} p_1 & 0 \\ * & p_2 \end{pmatrix},$$

where p_1 and p_2 are invertible matrices. Furthermore $\det(p) = \det(p_1) \times \det(p_2) \neq 0$, because p is invertible. If $e' = \langle e_i, \ldots, e_r\rangle$ then

$$ep = \begin{pmatrix} e'p_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\det(e'p_1) = \det(e')\det(p_1) \neq 0$. Moreover, e' is a matrix of invariant factors of $a_1 = I_r e' p_1$. ∎

Remark 3.3. Let a be a matrix with coefficients in $\mathbb{C}(\{x\})[y]$ such that $a = a^*$; then we can suppose that the invariant factors of a are elements of $\mathbb{R}(\{x\})[y]$.

Indeed, by 3.1 there exist two invertible matrices u, v and a diagonal matrix of invariant factors $e = \langle e_1, \dots, e_r, 0, \dots, 0\rangle$, such that $a = uev$. Since $a = a^* = v^* e^* u^*$ then $e^* = \langle \overline{e_1}, \dots, \overline{e_r}, 0, \dots, 0\rangle$ is also a matrix of invariant factors of a, hence $(e_i) = (\overline{e_i})$ for $i = 1, \dots, r$. Therefore, for every i there exists $u_i \in \mathbb{C}\{x\}$, $u_i \neq 0$ such that $\overline{e_i} = u_i e_i$. It is easy to see that $u_i \overline{u_i} = 1$, and so u_i is a unit of $\mathbb{C}\{x\}$. Thus, there exists $w_i \in \mathbb{C}\{x\}$ such that $w_i^2 = u_i$, and in particular $w_i \overline{w_i} = 1$. Now,

$$\overline{w_i e_i} = \overline{w_i e_i} = \overline{w_i} u_i e_i = (\overline{w_i} w_i) w_i e_i = w_i e_i,$$

and we consider the diagonal matrix $\hat{e} = \langle w_1 e_1, \dots, w_r e_r, 0, \dots, 0\rangle$ and the invertible matrix $q = \langle \overline{w_1}, \dots, \overline{w_r}, 1, \dots, 1\rangle$. Then $a = u\hat{e}(qv)$ and $\hat{e}$ is a matrix of invariant factors of a with coefficients in $\mathbb{R}(\{x\})[y]$.

Lemma 3.4. *Let $\xi \in \Phi_{\mathbb{R}} \subset \Phi_{\mathbb{C}}$ be a Puiseux series. Then the irreducible polynomial of ξ over $\mathbb{C}(\{x\})$ belongs to $\mathbb{R}(\{x\})[y]$.*

Proof. Indeed, if $\xi \in \Phi_{\mathbb{R}}$ there exists $g \in \mathbb{R}(\{x\})$ such that $\xi = g(x^{1/q})$ where $q = q(\xi)$, and the irreducible polynomial of ξ over $\mathbb{C}(\{x\})$ is $Q(x, y) = \prod_{k=0}^{q-1}(y - g(e^{2k\pi i/q}x^{1/q}))$. Furthermore, we have

$$\begin{aligned}
\overline{Q(x, y)} &= \prod_{k=0}^{q-1} \overline{(y - g(e^{2k\pi i/q}x^{1/q}))} = \prod_{k=0}^{q-1} (y - \overline{g(e^{2k\pi i/q}x^{1/q})}) \\
&= \prod_{k=0}^{q-1} (y - g(\overline{e^{2k\pi i/q}}x^{1/q})) = \prod_{k=0}^{q-1} (y - g(e^{-2k\pi i/q}x^{1/q})) \\
&= \prod_{k=0}^{q-1} (y - g(e^{2(q-k)\pi i/q}x^{1/q})) = Q(x, y)
\end{aligned}$$

and so $Q \in \mathbb{R}(\{x\})[y]$. ∎

Proposition 3.5. *Let $a \geq 0$ with* $\det(a) \neq 0$ *and suppose that $a = ep$ where $e \in \mathfrak{M}_n(\mathbb{R}(\{x\})[y])$ is a matrix of invariant factors of a and $p \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ is invertible. Then for every $1 \leq i \leq n$ there exists $\varepsilon_i = \pm 1$ such that $\varepsilon_i e_i > 0$. Furthermore, replacing e_i by $\varepsilon_i e_i$ we can suppose that $e_i \geq 0$ for every i.*

Proof. Suppose that there exists some index i such that $\pm e_i$ is not positive semidefinite, and let $i = \ell$ be the first. Since e_ℓ is not definite, then it has a root $\xi \in \Phi_{\mathbb{R}}$ of odd multiplicity $\lambda(\ell)$; the irreducible polynomial Q_ℓ of ξ over $\mathbb{C}(\{x\})[y]$ divides e_ℓ.

We will denote by $\lambda(k)$ the multiplicity (possibly 0) of ξ as a root of e_k, that is, $\lambda(k) = \max\{r \in \mathbb{N} : Q_\ell^r | e_k\}$. Since $e_1, \dots, e_{\ell-1} \geq 0$ and $e_1 | \cdots | e_n$, we have $\lambda(1) \leq \cdots \leq \lambda(\ell - 1) < \lambda(\ell) \leq \lambda(\ell + 1) \leq \cdots \leq \lambda(n)$. Let us see that $Q_\ell | p_{ij}$ if $1 \leq i \leq \ell \leq j \leq n$.

First, since $a \geq 0$, then $a_{\ell\ell} = e_\ell p_{\ell\ell} \geq 0$ and so, since $\lambda(\ell)$ is odd, we have $Q_\ell | p_{\ell\ell}$. On the other hand, since

$$ep = \begin{pmatrix} e_1 p_{11} & e_1 p_{12} & \cdots & e_1 p_{1\ell} & \cdots & e_1 p_{1n} \\ e_2 p_{21} & e_2 p_{22} & \cdots & e_2 p_{2\ell} & \cdots & e_2 p_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ e_\ell p_{\ell 1} & e_\ell p_{\ell 2} & \cdots & e_\ell p_{\ell\ell} & \cdots & e_\ell p_{\ell n} \\ \vdots & \vdots & & \vdots & & \vdots \\ e_n p_{n1} & e_n p_{n2} & \cdots & e_n p_{n\ell} & \cdots & e_n p_{nn} \end{pmatrix}$$

and $ep = p^* e^*$, then $e_i p_{ij} = \overline{e_j p_{ji}} = e_j \overline{p_{ji}}$. Hence we obtain $Q_\ell | p_{ij}$ if $i < \ell \leq j$, because $\lambda(i) < \lambda(\ell) \leq \lambda(j)$.

Now we see that if $i = \ell < j$ then $Q_\ell | p_{\ell j}$. Indeed we consider

$$v_j(\rho, \mu) = (0, \ldots, 0, \overset{(\ell)}{\rho}, 0, \ldots, 0, \overset{(j)}{\mu}, 0, \ldots, 0),$$

where $\rho, \mu \in \mathbb{C}(\{x\})[y]$, and since $ep \geq 0$ we get

$$\begin{aligned} 0 \leq v_j(\rho, \mu)^* ep v_j(\rho, \mu) &= (\bar{\rho}, \bar{\mu}) \begin{pmatrix} e_\ell p_{\ell\ell} & e_\ell p_{\ell j} \\ e_j p_{j\ell} & e_j p_{jj} \end{pmatrix} \begin{pmatrix} \rho \\ \mu \end{pmatrix} \\ &= (\bar{\rho}, \bar{\mu}) \begin{pmatrix} e_\ell p_{\ell\ell} & e_\ell p_{\ell j} \\ e_\ell \overline{p_{\ell j}} & e_j p_{jj} \end{pmatrix} \begin{pmatrix} \rho \\ \mu \end{pmatrix}. \end{aligned}$$

By 2.5 (b), for $\rho = -e_\ell p_{\ell j}$, $\mu = e_\ell p_{\ell\ell}$ we have

$$e_\ell \overline{p_{\ell\ell}} (e_\ell p_{\ell\ell} e_j p_{jj} - e_\ell^2 p_{\ell j} \bar{p}_{\ell j}) \geq 0.$$

Furthermore, since $a = ep \geq 0$ and $\det(a) \neq 0$, then by 2.5 (c), a is anisotropic. Therefore, we have $e_\ell p_{\ell\ell} \neq 0$ and $e_\ell p_{\ell\ell} \geq 0$ hence $e_\ell p_{\ell\ell} e_j p_{jj} - e_\ell^2 p_{\ell j} \bar{p}_{\ell j} \geq 0$.

Thus, since $e_\ell | e_j$ we obtain $e_j = e_\ell d_j$ and so $d_j p_{\ell\ell} p_{jj} - |p_{\ell j}|^2 \geq 0$. Now, since ξ is a root of $p_{\ell\ell}$ then $-|p_{\ell j}(\xi)|^2 \geq 0$, whence $|p_{\ell j}(\xi)|^2 = 0$, which implies $Q_\ell | p_{\ell j}$ or $Q_\ell | \overline{p}_{\ell j}$ (because Q_ℓ is irreducible). But, since $\xi \in \Phi_{\mathbb{R}}$ then (by 3.4) $Q_\ell \in \mathbb{R}(\{x\})[y]$ and therefore $Q_\ell | p_{\ell j}$.

Finally, since

$$p(\xi)=\begin{pmatrix} p_{11}(\xi) & \cdots & p_{1\ell-1}(\xi) & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & & \vdots\\ p_{\ell-11}(\xi) & \cdots & p_{\ell-1\ell-1}(\xi) & 0 & \cdots & 0\\ p_{\ell 1}(\xi) & \cdots & p_{\ell\ell-1}(\xi) & 0 & \cdots & 0\\ p_{\ell+11}(\xi) & \cdots & p_{\ell+1\ell-1}(\xi) & p_{\ell+1\ell}(\xi) & \cdots & p_{\ell+1n}(\xi)\\ \vdots & & \vdots & \vdots & & \vdots\\ p_{n1}(\xi) & \cdots & p_{n\ell-1}(\xi) & p_{n1}(\xi) & \cdots & p_{nn}(\xi) \end{pmatrix}$$

we deduce $\det(p)(\xi)=\det(p(\xi))=0$, because the first ℓ rows are linearly dependent. However, since p is invertible we conclude $\det(p)=g\in\mathbb{C}(\{x\})\backslash\{0\}$, hence $0=\det(p)(\xi)=g$, a contradiction. ∎

LEMMA 3.6. *Let $e_1,\ldots,e_n\in\mathbb{R}(\{x\})[y]$ such that $e_1|e_2|\cdots|e_n$ and $e_i\geq 0$ for $1\leq i\leq n$. Then there exists $l_1,\ldots,l_n\in\mathbb{C}(\{x\})[y]$ such that $l_1|\cdots|l_n$ and $e_i=l_i\bar{l}_i$ for $1\leq i\leq n$.*

Proof. We argue by induction. For $n=1$, since $e_1\geq 0$ then $e_1=d_1/t^{2r_1}$ where $r_1\geq 0$, $d_1\geq 0$. By 2.3, there exist $\xi_1,\eta_1\in\mathbb{R}\{x\}[y]$ such that $d_1=\xi_1^2+\eta_1^2$, and so, $e_1=\xi_1^2/t^{r_1}+\eta_1^2/t^{r_1}$. Therefore, it suffices to take $l_1=\xi_1/t^{r_1}+i(\eta_1/t^{r_1})$.

Suppose the claim is true for $n-1$: *there exist $l_1,\ldots,l_{n-1}\in\mathbb{C}(\{x\})[y]$ such that $l_1|\cdots|l_{n-1}$ and $e_i=l_i\bar{l}_i$ for $1\leq i\leq n-1$.* Then, since $e_{n-1}|e_n$ we have $e_n=e_{n-1}v_n$ and so, since $e_{n-1},e_n\geq 0$, we obtain also $v_n\geq 0$. Hence, $v_n=q_n\overline{q_n}$ where $q_n\in\mathbb{C}(\{x\})[y]$. Thus, $e_n=l_{n-1}\overline{l_{n-1}}q_n\overline{q_n}=(l_{n-1}q_n)(\overline{l_{n-1}q_n})$ and it suffices to take $l_n=l_{n-1}q_n$. ∎

THEOREM 3.7. *Let $a\in\mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ be a matrix such that $a\geq 0$ and* $\det(a)\neq 0$. *Then there exist $b,c\in\mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ such that* $\det(b)\neq 0$, *c is invertible, $c\geq 0$, and $a=b^*cb$.*

Proof. In view of 3.2, 3.3, and 3.5 there exist invertible matrices $u,p\in\mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ and $e\in\mathfrak{M}_n(\mathbb{R}(\{x\})[y])$ a matrix of invariant factors of a such that $a=u^*epu$ and the elements of the main diagonal of e are psd. Thus, by 3.6 there exist $l_1,\ldots,l_n\in\mathbb{C}(\{x\})[y]$ such that $l_1|\cdots|l_n$ and $e_i=l_i\bar{l}_i$ for $1\leq i\leq n$. Hence we construct a matrix $l=\langle l_1,\ldots,l_n\rangle$ such that

$e = l^*l$ and $l_1|l_2|\cdots|l_n$. Since $ep = p^*e^*$ then

$$ep = l^*lp = \begin{pmatrix} \bar{l}_1 l_1 p_{11} & \bar{l}_1 l_1 p_{12} & \cdots & \bar{l}_1 l_1 p_{1n} \\ \bar{l}_2 l_2 p_{21} & \bar{l}_2 l_2 p_{22} & \cdots & \bar{l}_2 l_2 p_{2n} \\ \vdots & \vdots & & \vdots \\ \bar{l}_n l_n p_{n1} & \bar{l}_n l_n p_{n2} & \cdots & \bar{l}_n l_n p_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{l}_1 l_1 \overline{p_{11}} & \bar{l}_2 l_2 \overline{p_{21}} & \cdots & \bar{l}_n l_n \overline{p_{n1}} \\ \bar{l}_1 l_1 \overline{p_{12}} & \bar{l}_2 l_2 \overline{p_{22}} & \cdots & \bar{l}_n l_n \overline{p_{n2}} \\ \vdots & \vdots & & \vdots \\ \bar{l}_1 l_1 \overline{p_{1n}} & \bar{l}_2 l_2 \overline{p_{2n}} & \cdots & \bar{l}_n l_n \overline{p_{nn}} \end{pmatrix} = p^*l^*l = p^*e^*.$$

We seek a matrix $q \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ such that

$$l^*lp = l^*ql = \begin{pmatrix} \bar{l}_1 q_{11} l_1 & \bar{l}_1 q_{12} l_2 & \ldots & \bar{l}_1 q_{1n} l_n \\ \bar{l}_2 q_{21} l_1 & \bar{l}_2 q_{22} l_2 & \ldots & \bar{l}_2 q_{2n} l_n \\ \vdots & \vdots & & \vdots \\ \bar{l}_n q_{n1} l_1 & \bar{l}_n q_{n2} l_n & \ldots & \bar{l}_n q_{nn} l_n \end{pmatrix}.$$

Comparing the two expressions above, and taking into account that $l_1|l_2|\cdots|l_n$, we deduce

$$q_{ij} = \begin{cases} \overline{p_{ji}\frac{\bar{l}_j}{\bar{l}_i}} & \text{if } i < j \\ p_{ii} & \text{if } i = j \\ p_{ij}\frac{l_i}{l_j} & \text{if } i > j. \end{cases}$$

Finally, since

$$\begin{aligned} \det(e)\det(p) = \det(ep) = \det(l^*ql) \\ = \det(l^*)\det(q)\det(l) = \det(e)\det(q) \end{aligned}$$

and $\det(e) \neq 0$, then $\det(p) = \det(q)$, and we conclude that q is invertible.
We conclude by taking $b = lu$ and $c = q$. ∎

After all the preceding preparation, we can finally prove:

THEOREM 3.8. *Let $a \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ be a matrix such that $a \geq 0$. Then there exists $b \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ such that $a = b^*b$.*

Proof. By 3.2 there exist $u \in \mathfrak{M}_n(\mathbb{C}(\{x\})[y])$ invertible and $a' \in \mathfrak{M}_r(\mathbb{C}(\{x\})[y])$ ($r = rk(a)$) such that $a = u^*\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}u$, $a' \geq 0$, and $\det(a') \neq 0$. Furthermore, in view of 3.7 there exist $b_1, c \in \mathfrak{M}_r(\mathbb{C}(\{x\})[y])$ such that $\det(b_1) \neq 0, c$ is invertible, $c \geq 0$, and $a' = b_1^* c b_1$. Now, since c is ≥ 0 and invertible, it is also anisotropic by 2.5 (c), and so by [Dj, Sect. 5, Proposition 4] (for $F = \mathbb{C}(\{x\})$) there exist matrices $v \in \mathfrak{M}_r(\mathbb{C}(\{x\})[y])$ invertible and $d \in \mathfrak{M}_r(\mathbb{R}(\{x\}))$ diagonal and ≥ 0 (by 2.5 (a)), such that $c = v^* d v$. Since every $d_i \in \mathbb{R}(\{x\})$ is psd, there exists $g_i \in \mathbb{R}(\{x\})$ such that $d_i = g_i^2$. Therefore, if $g = \langle g_1, \ldots, g_r \rangle$ we have $a' = b_1^* v^* g g v b_1 = (g v b_1)^* (g v b_1)$. Whence, we take $b = \begin{pmatrix} gvb_1 & 0 \\ 0 & 0 \end{pmatrix}u$ to conclude the proof. ∎

COROLLARY 3.9. *Let $L_1, \ldots, L_r$ be linear forms in m variables over $A_0 = \mathbb{R}(\{x\})[y]$, $\mathbb{R}\{x\}[y]$, or $\mathbb{R}\{x, y\}$, and $\varphi = L_1^2 + \cdots + L_r^2$. There exist linear forms $Q_1, \ldots, Q_{2m}$ over A_0 such that $\varphi = Q_1^2 + \cdots + Q_{2m}^2$.*

Proof. Suppose first that $A_0 = \mathbb{R}(\{x\})[y]$. Let $a \in \mathfrak{M}_m(\mathbb{R}(\{x\})[y])$ be the matrix associated to the quadratic form φ. Since $a \geq 0$ then for every $z \in \mathbb{R}(\{x\})^m$ we have $z^t a z \geq 0$. This is also true for $z \in \mathbb{C}(\{x\})^m$. Indeed, if we take $y = u + iv \in \mathbb{C}(\{x\})^m$ we have $(u+iv)^* a(u+iv) = (u^t - iv^t)a(u + iv) = (u^t a u + v^t a v) + i(u^t a v - v^t a u) = u^t a u + v^t a v \geq 0$.

Thus, by 3.8 (taking $n = m$) there exists $b \in \mathfrak{M}_m(\mathbb{C}(\{x\})[y])$ such that $a = b^* b$. Since $b = b_1 + i b_2$ with $b_1, b_2 \in \mathfrak{M}_m(\mathbb{R}(\{x\})[y])$ then $a = (b_1^t - i b_2^t)(b_1 + i b_2) = b_1^t b_1 + b_2^t b_2 + i(b_1^t b_2 - b_2^t b_1)$ and therefore

$$a = b_1 b_1^t + b_2 b_2^t$$

$$b_1^t b_2 = b_2^t b_1.$$

Thus, $\varphi = \mathrm{z} a \mathrm{z}^t = \mathrm{z} b_1 b_1^t \mathrm{z}^t + \mathrm{z} b_2 b_2^t \mathrm{z}^t$ (where $\mathrm{z} = (\mathrm{z}_1, \ldots, \mathrm{z}_m)$) and so; there exist linear forms $Q_1, \ldots, Q_{2m}$ over $\mathbb{R}(\{x\})[y]$ such that $\varphi = Q_1^2 + \cdots + Q_{2m}^2$.

Now we consider the case $A_0 = \mathbb{R}\{x\}[y]$. Let $a \in \mathfrak{M}_m(\mathbb{R}\{x\}[y]) \subset \mathfrak{M}_m(\mathbb{R}(\{x\})[y])$ the matrix associated to the quadratic form φ. As we have seen there exist two matrices $b_1, b_2 \in \mathfrak{M}_m(\mathbb{R}(\{x\})[y])$ such that $a = b_1 b_1^t + b_2 b_2^t$.

Let, now, $c_1, c_2 \in \mathfrak{M}_m(\mathbb{R}\{x\}[y])$, $r \geq 0$ such that $b_k = c_k / x^r$. We deduce

$$a_{ii} = \frac{\left(c_{i1}^{(1)}\right)^2 + \cdots + \left(c_{im}^{(1)}\right)^2 + \left(c_{i1}^{(2)}\right)^2 + \cdots + \left(c_{im}^{(2)}\right)^2}{x^{2r}} \in \mathbb{R}\{x\}[y]$$

hence $x^r | c_{ij}^{(k)}$ for every i, j, k, and consequently $b_1, b_2 \in \mathfrak{M}_m(\mathbb{R}\{x\}[y])$. Thus, we can finish as before but now we know that the forms Q_i are defined over $\mathbb{R}\{x\}[y]$.

Finally we suppose $A_0 = \mathbb{R}\{x, y\}$. Fix $k \geq 1$, let A_{ik} be the jet of degree k of L_i ($1 \leq i \leq r$), and consider the quadratic form $\varphi_k = A_{1k}^2 + \cdots + A_{rk}^2$

over $\mathbb{R}\{x\}[y]$, which is positive semidefinite. By the previous case, there exist linear forms $Q_{1k}, \ldots, Q_{2m,k}$ over $\mathbb{R}\{x\}[y]$ such that $\varphi_k = Q_{1k}^2 + \cdots + Q_{2m,k}^2 = \varphi \bmod (x, y)^{k+1}$.

Since we can do this for every $k \geq 1$, by Artin's Approximation Lemma [Ku *et al.*, Ar], there exist linear forms $Q_1, \ldots, Q_{2m}$ over $\mathbb{R}\{x, y\}$ such that $\varphi = Q_1^2 + \cdots + Q_{2m}^2$. ∎

THEOREM 3.10. *Let A be a ring which is a finite generated module, say by m generators, over $A_0 = \mathbb{R}(\{x\})[y]$, $\mathbb{R}\{x\}[y]$, or $\mathbb{R}\{x, y\}$. Then $p(A) \leq 2m$.*

Proof. Let $z_1, \ldots, z_m$ the generators of A and

$$f = (a_{11}z_1 + \cdots + a_{1m}z_m)^2 + \cdots + (a_{r1}z_1 + \cdots + a_{rm}z_m)^2.$$

We consider the quadratic form

$$\varphi = (a_{11}\mathrm{z}_1 + \cdots + a_{1m}\mathrm{z}_m)^2 + \cdots + (a_{r1}\mathrm{z}_1 + \cdots + a_{rm}\mathrm{z}_m)^2,$$

which we write $\varphi = \mathrm{z}b\mathrm{z}^t$, $\mathrm{z} = (\mathrm{z}_1, \ldots, \mathrm{z}_m)$, and $b \in \mathfrak{M}_m(A_0)$; φ is clearly positive semidefinite. Then by 3.9, there exist linear forms $Q_1, \ldots, Q_{2m}$ such that $\varphi = Q_1^2 + \cdots + Q_{2m}^2$. Finally,

$$f = \varphi(z_1, \ldots, z_m) = Q_1(z_1, \ldots, z_m)^2 + \cdots + Q_{2m}(z_1, \ldots, z_m)^2$$

which is a sum of $2m$ squares of A. ∎

4. THE UPPER BOUND

Here we get the announced upper bound for the Pythagoras number of a surface germ using some of their numerical invariants.

First, we recall that the *multiplicity* $\mathrm{mult}(X)$ of an analytic germ X, which is usually defined through the Hilbert polynomial of the local ring $\mathscr{O}(X)$, can be more geometrically characterized using *Noether normalizations* as follows: if X is irreducible and $\mathscr{O}_d \subset \mathscr{O}(X)$ is a general Noether normalization, then

$$\mathrm{mult}(X) = \mathrm{mult}(\mathscr{O}(X)) = \mathrm{mult}(\mathbb{R}\{x\}/I(X)) = [qf(\mathscr{O}(X)) : qf(\mathscr{O}_d)].$$

[JP, 4.2.23]. If X is reducible of dimension d and $X_1, \ldots, X_r$ are the irreducible components of X of dimension d then $\mathrm{mult}(X) = \sum_{i=1}^r \mathrm{mult}(X_i)$ [JP, 4.2.33]. In addition, we will need the *total multiplicity* $\mathrm{mult}_{\mathrm{T}}(X)$ of X, which is the sum of the multiplicities of *all* the irreducible components and not only those of dimension d.

Now we can prove:

THEOREM 4.1. *Let $X \subset \mathbb{R}^n$ be a surface germ. Then*

$$p[X] \leq 2\ \mathrm{mult}_{\mathrm{T}}(X)^{\mathrm{codim}(X)}.$$

Proof. We denote by m the minimum number of generators of $\mathcal{O}(X)$ as an $\mathcal{O}_d$-module. In view of 3.10 we only need to show

$$m \leq \mathrm{mult}_{\mathrm{T}}(X)^{\mathrm{codim}(X)}.$$

To simplify notations, we suppose that X is an analytic germ of dimension d. Let $I(X) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ be the reduced primary descomposition of $I(X)$ where $\mathrm{ht}(\mathfrak{p}_i) = r_i = n - d_i$. Using Rückert's Local Parametrization [Rz2, II.2.3, 3], it is easy to check that there exists a linear change in $\mathcal{O}_n$ such that the inclusions $\mathcal{O}_{d_i} \subset \mathcal{O}_n/p_i$ are all general Noether normalizations. Then

$$[qf(\mathcal{O}_n/\mathfrak{p}_i) : qf(\mathcal{O}_{d_i})] = \mathrm{mult}(\mathcal{O}_n/\mathfrak{p}_i), \qquad 1 \leq j \leq s.$$

Let P_{ij} be the irreducible polynomial of the element $\theta_{ij} = x_j + \mathfrak{p}_i \in \mathcal{O}_n/\mathfrak{p}_i$ over $qf(\mathcal{O}_{d_i})$ for $1 \leq i \leq s$, $d+1 \leq j \leq n$. By [Rz2, II.3], these P_{ij}'s are Weierstrass polynomials of $\mathcal{O}_{d_i}[T]$ of degrees $\leq [qf(\mathcal{O}_n/\mathfrak{p}_i) : qf(\mathcal{O}_{d_i})] = \mathrm{mult}(\mathcal{O}_n/\mathfrak{p}_i)$.

We next consider the polynomials

$$P_j\big(x^{(i)}, x_j\big) = \prod_{i=1}^{s} P_{ij}\big(x^{(i)}, x_j\big),$$

where $x^{(i)} = (x_1, \ldots, x_{d_i})$. It is clear that they are in $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_S = I(X)$ and their degrees (with respect to x_j) are $\leq \sum_{i=1}^{s} \mathrm{mult}(\mathcal{O}_n/\mathfrak{p}_i) = \mathrm{mult}_{\mathrm{T}}(X)$. Dividing successively by them, we see that the monomials

$$x_{d+1}^{\nu_{d+1}} \cdots x_n^{\nu_n} : 0 \leq \nu_{d+1}, \ldots, \nu_n < \mathrm{mult}_{\mathrm{T}}(X)$$

generate $\mathcal{O}(X)$ as an $\mathcal{O}_d$-module, and so, $m \leq \mathrm{mult}_{\mathrm{T}}(X)^{n-d}$. ∎

5. EXAMPLES

Here we show that we cannot obtain an upper bound for the Pythagoras number of a germ surface depending only on its multiplicity, which explains the use of the total multiplicity in the previous section. To begin with we prove:

PROPOSITION 5.1. *For every $q \in \mathbb{N}$ there exist analytic curve germs $Y \subset \mathbb{R}^3$ with Pythagoras number $\geq q$.*

Proof. By 1.2 it suffices to show that for every $k \geq 1$ there exists an irreducible curve germ Y_k, such that $\omega(\mathcal{I}(Y_k)) > k$; in fact, by 1.3 (a) we deduce that the curve germ $Y = Y_{2^q}$ has Pythagoras number $\geq q$.

To prove the previous assertion, we consider three relatively prime integers $a_k < b_k < c_k$, and we claim these integers can be chosen for the parametrized curve germ $Y_k : x = t^{a_k},\ y = t^{b_k},\ z = t^{c_k}$ to verify the required condition.

We simplify the notation dropping some indices k. The ideal $\mathfrak{p} = I(Y_k$ is the kernel of the homomorphism $\mathbb{R}\{x, y, z\} \to \mathbb{R}\{t\} : x, y, z \mapsto t^a, t^b, t^c$, and we find a system of generators of $\mathfrak{p}$ as follows. Let

$$F = \sum_{\nu} a_\nu x^{\nu_1} y^{\nu_2} z^{\nu_3} \in \mathbb{R}\{x, y, z\}$$

be such that

$$F(t^a, t^b, t^c) = \sum_{d=1}^{\infty} \left(\sum_{a\nu_1 + b\nu_2 + c\nu_3 = d} a_\nu t^d \right) = 0.$$

We consider the polynomials

$$F_d = \sum_{a\nu_1 + b\nu_2 + c\nu_3 = d} a_\nu x^{\nu_1} y^{\nu_2} z^{\nu_3} \in I(Y_k),$$

so that $F = \sum_{d=1}^{\infty} F_d$. It is known that there exist three binomials

$$P_\alpha = x^{\alpha_1} - y^{\alpha_2} z^{\alpha_3}, \qquad P_\beta = y^{\beta_2} - x^{\beta_1} z^{\beta_3}, \qquad P_\gamma = z^{\gamma_3} = x^{\gamma_1} y^{\gamma_2}$$

(where $a\alpha_1 = b\alpha_2 + c\alpha_3$, $b\beta_2 = a\beta_1 + c\beta_3$, $c\gamma_3 = a\gamma_1 + b\gamma_2$) which generate the kernel of the homomorphism $\mathbb{R}[x, y, z] \to \mathbb{R}[t] : x, y, z \mapsto t^a, t^b, t^c$ (see (Kz, V, Sect. 3]). Thus, for every $d \geq 1$ there exist polynomials $A_d, B_d, C_d \in \mathbb{R}[x, y, z]$ such that $F_d = A_d P_\alpha + B_d P_\beta + C_d P_\gamma$, and

$$F = \left(\sum_d A_d \right) P_\alpha + \left(\sum_d B_d \right) P_\beta + \left(\sum_d C_d \right) P_\gamma,$$

hence $I(Y_k) = (P_\alpha, P_\beta, P_\gamma)$.

Once we have these generators of $I(Y_k)$, we see that the orders $\omega_\alpha = \omega(P_\alpha)$, $\omega_\beta = \omega(P_\beta)$, $\omega_\gamma = \omega(P_\gamma)$ are $\geq k$ for the following choice: $a = p$, a prime number $\geq k^2 + 2$, $b = p(p-1) + k$, and $c = p^2 + 1$.

(i) $\omega_\alpha \geq k$. Since $a\alpha_1 = b\alpha_2 + c\alpha_3 \geq b(\alpha_2 + \alpha_3)$ and $a < b$, then $\alpha_1 \geq \alpha_2 + \alpha_3$, and therefore $\omega_\alpha = \alpha_2 + \alpha_3$. On the other hand, for the chosen a, b, and c, we obtain

$$p\alpha_1 = \big((p-1)\alpha_2 + p\alpha_3\big)p + k\alpha_2 + \alpha_3$$

hence, $p \leq k\alpha_2 + \alpha_3 \leq k(\alpha_2 + \alpha_3) = k\omega_\alpha$, and we conclude $\omega_\alpha \geq p/k \geq k$.

(ii) $\omega_\beta \geq k$. Since $b\beta_2 = a\beta_1 + c\beta_3 \geq c\beta_3$ and $b < c$, then $\beta_2 > \beta_3$. Furthermore,

$$(p\beta_3 + \beta_1 - (p-1)\beta_2)p = k\beta_2 - \beta_3 > 0$$

hence $k\beta_2 \geq p$ and $\beta_2 \geq p/k \geq k$. If $\omega_\beta = \beta_2 \leq \beta_1 + \beta_3$ we are done. Therefore, we suppose $\omega_\beta = \beta_1 + \beta_3$ and $\beta_1 \leq k$ (otherwise there is nothing to prove). Then $p\beta_3 + \beta_1 > (p-1)\beta_2 \geq (p-1)p/k$ and so $\omega_\beta \geq \beta_3 \geq (p-1)/k - \beta_1/p \geq (p-1)/k - k/p \geq (p-2)/k \geq k$.

(iii) $\omega_\gamma \geq k$. Since $c\gamma_3 = a\gamma_1 + b\gamma_2 \leq b(\gamma_1 + \gamma_2)$ and $c > b$, then $\gamma_1 + \gamma_2 \geq \gamma_3 = \omega_\gamma$. On the other hand

$$(p\gamma_3 - (p-1)\gamma_2 - \gamma_1)p = k\gamma_2 - \gamma_3,$$

and we have three subcases:

(a) $k\gamma_2 < \gamma_3$. Then $\omega_\gamma = \gamma_3 \geq p \geq k$.

(b) $k\gamma_2 = \gamma_3$. Then $\gamma_2 \geq 1$ and $\omega_\gamma = \gamma_3 \geq k$.

(c) $k\gamma_2 > \gamma_3$. Then $k\gamma_2 \geq p$ and $p\gamma_3 > (p-1)\gamma_2 \geq (p-1)p/k$, and we conclude $\omega_\gamma = \gamma_3 \geq (p-1)/k \geq k$. ∎

COROLLARY 5.2. *For every $q \geq 1$ there exist analytic surface germs $X \subset \mathbb{R}^3$ of multiplicity* 1 *and Pythagoras number $\geq q$.*

Proof. Choose a curve germ $Y \subset \mathbb{R}^3$ with Pythagoras number $\geq q$. Let $X = Y \cup \{z = 0\}$ which is an analytic surface germ, such that $\mathrm{mult}(X) = \mathrm{mult}(\{z = 0\}) = 1$ (although its total multiplicity is $\mathrm{mult}(Y) + 1$, hence very big) and $p[X] \geq p[Y] \geq q$, because $\mathcal{O}(Y)$ is a quotient of $\mathcal{O}(X)$. ∎

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