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Some remarks on the computation of Pythagoras numbers of real irreducible algebroid curves through Gram matrices

José F. Fernando [a,1] and Ronan Quarez [b,*]

[a] *Depto. Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain*
[b] *UFR Mathémathiques, Campus de Beaulieu, Université de Rennes 1, 35042 Rennes cedex, France*

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Abstract

We clarify a difficulty that appears in [R. Quarez, J. Algebra 238 (2001) 139] to bound the Pythagoras number of a real irreducible algebroid curve by its multiplicity.

Let A be a real irreducible algebroid curve. In [2] an algorithm is developed to estimate the length of a sum of squares in A. This algorithm involves the use of Gram matrices and gives bounds for the Pythagoras number $p(A)$ of A. In particular, the bound $p(A) \leqslant \operatorname{mult}(A)$ follows from this algorithm. Our purpose here is to clarify a difficulty that appears in the *General Case* of the proof of Theorem 3.4 in [2], in relation with the existence of some limits of rational functions whose expressions are not explicit. We will use all notations in [2] and follow closely the proof there. Now, let $F = a_1^2 + \cdots + a_s^2$ be a sum of squares in A and set $\omega = \omega(F)/2$. To estimate the minimum number of squares needed to express F as a sum of squares we use Gram matrices as follows. We may assume that $0 < \omega < c$, see [2]. We write $\Gamma = \langle n_0, \dots, n_{m-1}\rangle$ with $n_i = \inf\{k \in \Gamma\colon\ k \equiv i \mod m\}$ and ϕ_{n_i} the canonical generator of order n_i. Let $\Theta = \Theta_0 \cup \cdots \cup \Theta_{m-1}$ where $\Theta_i = \{\phi_{n_i}, \phi_{n_i}\phi_{n_0}, \dots, \phi_{n_i}\phi_{n_0}^{u_0-1}\}$ and $u_0 = E[2c/n_0]+1$. We

* Corresponding author.
E-mail addresses: josefer@mat.ucm.es (J.F. Fernando), quarez@maths.univ-rennes1.fr (R. Quarez).
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proceed similarly to the *Preparation* part of the proof of Theorem 3.4 in [2], and observe that if $\{n'_1, n'_2, \dots, n'_m\} = \{n_0, n_1, \dots, n_{m-1}\}$ we can replace the ordered system of linear generators $\Theta = \Theta_0 \cup \dots \cup \Theta_{m-1}$ by $\Theta' = \Theta'_1 \cup \dots \cup \Theta'_m$ where

$$\Theta'_i = \left\{\psi_{n'_i}, \psi_{n'_i}\phi_{n_0}, \dots, \psi_{n'_i}\phi_{n_0}^{u_0-1}\right\} \quad \text{for } i = 1, \dots, r \leqslant m, \quad \text{and}$$

$$\psi_{n'_i} = \phi_{n'_i} + \sum_{j=r+1}^{m} \lambda_{ij}\phi_{n'_j}\phi_{n_0}^{\alpha_j}, \quad \lambda_{ij} \in R,\ \alpha_j \geqslant 0, \tag{$*$}$$

$$\Theta'_i = \left\{\phi_{n'_i}, \phi_{n'_i}\phi_{n_0}, \dots, \phi_{n'_i}\phi_{n_0}^{u_0-1}\right\} \quad \text{for } i = r+1, \dots, m.$$

Let us now proceed with the *General Case*. Let $H_0 \in \mathfrak{M}_{mu_0 \times mu_0}$ be the restricted Gram matrix canonically associated to F relative to Θ, and $M \in \mathfrak{M}_{mu_0 \times s}(R)$ be a matrix such that

$$F \equiv \Theta M M^t \Theta^t = \Theta H_0 \Theta^t \mod t^{\omega + c}.$$

We can write

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_m \end{pmatrix}, \qquad H_0 = \begin{pmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mm} \end{pmatrix},$$

where $M_i = (m^i_{kl}) \in \mathfrak{M}_{u_0 \times s}(R)$ and $H_{ij} = (h^{ij}_{k\ell})_{1 \leqslant k,\, \ell \leqslant u_0} = M_i M_j^t \in \mathfrak{M}_{u_0 \times u_0}(R)$. Let $\beta_i = \inf\{k\colon h^{ii}_{kk} \neq 0\} = \inf\{k\colon v^i_k = (m^i_{k1}, \dots, m^i_{ks}) \neq 0\}$; if β_i does not exist we will take $\beta_i = 0$. Consider the matrix

$$h = \left(h^{ij}_{\beta_i, \beta_j}\right)_{1 \leqslant i,\, j \leqslant m} = \left(\langle v^i_{\beta_i}, v^j_{\beta_j}\rangle\right)_{1 \leqslant i,\, j \leqslant m},$$

where $\langle \cdot, \cdot \rangle$ designs the usual dot product. Let $\operatorname{rk}(h) = r \leqslant m$. Reordering the elements $n_0, n_1, \dots, n_{m-1}$ and renaming them as $n'_1, \dots, n'_m$, we can suppose that:

$$\text{the vectors } \left\{v^1_{\beta_1}, \dots, v^r_{\beta_r}\right\} \text{ are independent,}$$

$$v_{\beta_j} = \sum_{i=1}^{r} \lambda_{ji} v_{\beta_i} \quad \text{for all } j = r+1, \dots, m,$$

$$\beta_i \geqslant \beta_j \quad \text{for all } i = 1, \dots, r \text{ and } j = r+1, \dots, m \text{ such that } \lambda_{ji} \neq 0.$$

Indeed, it is enough to take $n'_1 = n_{i_1}$ such that $\beta_{i_1} \geqslant \beta_i$ for all $i = 1, \dots, m$; $n'_2 = n_{i_2}$ such that the vectors $\{v^{i_1}_{\beta_{i_1}}, v^{i_2}_{\beta_{i_2}}\}$ are independent and $\beta_{i_2} \geqslant \beta_i$ for all i such that the vectors $\{v^{i_1}_{\beta_{i_1}}, v_{\beta_i}\}$ are independent. Proceeding inductively we obtain r independent vectors $\{v^{i_1}_{\beta_1}, \dots, v^{i_r}_{\beta_r}\}$ with the desired conditions. Since $(h^{ij}_{\beta_i, \beta_j})_{1 \leqslant i,\, j \leqslant r}$ is the matrix of the usual dot product with respect to the linearly independent system $\{v^1_{\beta_1}, \ldots, v^r_{\beta_r}\}$, we have $\det(h^{ij}_{\beta_i,\beta_j})_{1\leqslant i,\, j\leqslant N} \neq 0$ for any $N = 1, \ldots, r$. Here we recall the following immediate but useful fact from linear algebra: given $v^1_{\beta_1}, \ldots, v^r_{\beta_r} \in \mathbb{R}^s$,

$$\mathrm{rk}\{v^1_{\beta_1}, \ldots, v^r_{\beta_r}\} = \mathrm{rk}\big(\langle v^i_{\beta_i}, v^j_{\beta_j}\rangle\big)_{1\leqslant i,\, j\leqslant m} = r.$$

Therefore, if $r = m$, using the construction of [2, 3.4] we are done. Hence, we can suppose $r < m$. Let p be an integer and consider the matrix $J_p \in \mathfrak{M}_{u_0\times u_0}(R)$, such that $J_p = 0$ if $p \leqslant 0$ and

$$J_p = \begin{pmatrix} 0 & 0 \\ I_{u_0-p+1} & 0 \end{pmatrix} \quad \text{if } p > 0.$$

Let $\lambda \in \mathfrak{M}_{(m-r)u_0\times ru_0}(R)$ be the matrix

$$\lambda = (\lambda_{ji} J_{\beta_i-\beta_j+1}) = \begin{pmatrix} \lambda_{r+1,1} J_{\beta_1-\beta_{r+1}+1} & \cdots & \lambda_{r+1,r} J_{\beta_r-\beta_{r+1}+1} \\ \vdots & \ddots & \vdots \\ \lambda_{m,1} J_{\beta_1-\beta_m+1} & \cdots & \lambda_{m,r} J_{\beta_r-\beta_m+1} \end{pmatrix}$$

and $V \in \mathfrak{M}_{mu_0\times mu_0}(R)$ the matrix

$$V = \begin{pmatrix} I_{ru_0} & 0 \\ -\lambda & I_{(m-r)u_0} \end{pmatrix}.$$

Then, one checks easily that

$$V^{-1} = \begin{pmatrix} I_{ru_0} & 0 \\ \lambda & I_{(m-r)u_0} \end{pmatrix}.$$

And therefore

$$F \equiv \Theta V^{-1}\big(VMM^tV^t\big)\big(V^{-1}\big)^t\Theta^t = \Theta' H' (\Theta')^t \mod t^{\omega+c},$$

where $H' = VMM^tV^t$ and $\Theta' = \Theta V^{-1}$. Moreover, since we always consider the restricted Gram matrix modulo $t^{\omega+c}$, we can suppose that the ordered system of linear generators $\Theta' = \Theta'_1 \cup \cdots \cup \Theta'_m$ satisfies

$$\Theta'_i = \big\{\psi_{n'_i}, \psi_{n'_i}\phi_{n_0}, \ldots, \psi_{n'_i}\phi_{n_0}^{u_0-1}\big\} \quad \text{for } i = 1, \ldots, r \leqslant m, \quad \text{and}$$

$$\psi_{n'_i} = \phi_{n'_i} + \sum_{j=r+1}^{m} \lambda_{ji}\phi_{n'_j}\phi_{n_0}^{\beta_i-\beta_j}, \quad \lambda_{ij} \in R,$$

$$\Theta'_i = \big\{\phi_{n'_i}, \phi_{n'_i}\phi_{n_0}, \ldots, \phi_{n'_i}\phi_{n_0}^{u_0-1}\big\} \quad \text{for } i = r+1, \ldots, m.$$

Since $\beta_i \geqslant \beta_j$ for all $i = 1, \ldots, r$, $j = r+1, \ldots, m$ such that $\lambda_{ji} \neq 0$, it is clear that $\psi_{n'_i} \in A$ for $i = 1, \ldots, r$, and hence Θ' is an ordered system of linear generators of type $(*)$. It is not difficult to check that if we consider for H' the integers β'_i as above and the submatrix h' we have that:

$$\begin{aligned} &\beta_i = \beta'_i \quad \text{if } i = 1, \ldots, r, \\ &\beta_i < \beta'_i \quad \text{or} \quad \beta'_i = 0 \quad \text{if } i = r+1, \ldots, m, \\ &r = \operatorname{rk}(h) \leqslant \operatorname{rk}(h'). \end{aligned}$$

Therefore, after applying this construction to H finitely many times, say q, we obtain a matrix $H^{(q)} \in \mathfrak{M}_{mu_0 \times mu_0}$ and an ordered system of linear generators $\Theta^{(q)}$ such that

$$F \equiv \Theta^{(q)} H^{(q)} \big(\Theta^{(q)}\big)^t \bmod t^{\omega+c}, \qquad H^{(q)} = \begin{pmatrix} H_1^{(q)} & 0 \\ 0 & 0 \end{pmatrix},$$

where $H_1^{(q)} \in \mathfrak{M}_{r^{(q)}u_0 \times r^{(q)}u_0}(R)$ and $\operatorname{rk}(h^{(q)}) = r^{(q)} \leqslant m$. Finally, applying the construction of [2, 3.4] to $H_1^{(q)}$ we are done. $\square$

As we said before, a corollary of this algorithm is the bound $p(A) \leqslant \operatorname{mult}(A)$. This can be proved directly by means of the well-known Pfister's diagonalization trick [1]. This trick works here because: (1) Any quadratic form Q over $R[\![s]\!]$ can always be diagonalizated with a single denominator s^{2r}, because $p(\mathbb{R}[\![s]\!]) = 1$. (2) The denominator can be cleared if Q is positive semidefinite, since s generates a real ideal [3, 4.5]. (3) A is a free module of rank $m = \operatorname{mult}(A)$ over a ring $R[\![\phi]\!]$ for a suitable series $\phi \in A$, as proved in [2, 2.1]. Note however that this argument goes no further, as it gives no algorithm to study lengths of sums of squares of A.

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