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On Łojasiewicz's inequality and the Nullstellensatz for rings of semialgebraic functions ☆

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ABSTRACT

In this article we present versions of Łojasiewicz's inequality and the Nullstellensatz for the ring of bounded semialgebraic functions on an arbitrary semialgebraic set M. We also prove that the classical Łojasiewicz inequality and Nullstellensatz for the ring of semialgebraic functions on a semialgebraic set M work if and only if M is locally compact.

1. Introduction

A subset $M \subset \mathbb{R}^n$ is said to be *basic semialgebraic* if it can be written as

$$M = \{x \in \mathbb{R}^n:\ f(x) = 0,\ g_1(x) > 0,\ \ldots,\ g_m(x) > 0\}$$

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for some polynomials $f, g_1, \dots, g_m \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$. The finite unions of basic semialgebraic sets are called *semialgebraic sets*. A continuous function $f : M \to \mathbb{R}$ is said to be *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{n+1}$. Usually a semialgebraic function is a function that is not necessarily continuous and whose graph is semialgebraic. However, since most of the semialgebraic functions in this article are continuous, we omit the continuity condition for simplicity when we refer to them and write *functions whose graph is semialgebraic* for those, which are not necessarily continuous. For further readings about semialgebraic sets and functions we refer the reader to [2, §2].

The sum and product of functions, defined pointwise, endow the set $\mathcal{S}(M)$ of semialgebraic functions on M with a natural structure of a commutative ring whose unity is the function with constant value 1. In fact $\mathcal{S}(M)$ is an $\mathbb{R}$-algebra if we identify each real number r with the constant function. The simplest examples of semialgebraic functions on M are the restrictions of polynomials in n variables to M. Other relevant ones are the Euclidean distance function $\operatorname{dist}(\cdot, N)$ for a given semialgebraic set $N \subset M$, the absolute value of a semialgebraic function, the maximum and the minimum of a finite family of semialgebraic functions, the inverse and the k-root of a semialgebraic function whenever these operations are well-defined.

It is obvious that the subset $\mathcal{S}^*(M)$ of bounded semialgebraic functions on M is a real subalgebra of $\mathcal{S}(M)$. We denote either $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$ with $\mathcal{S}^\diamond(M)$ if the involved statements or arguments are valid for both rings. For each $f \in \mathcal{S}^\diamond(M)$ and each semialgebraic subset $N \subset M$ we denote $Z_N(f) = \{x \in N\colon\ f(x) = 0\}$ and $D_N(f) = M \setminus Z_N(f)$. If $N = M$, we say that $Z_M(f)$ is the *zero set* of f.

Łojasiewicz's inequality is one of the main results in Real Algebraic Geometry. Its first versions are independently due to L. Hörmander [11] and S. Łojasiewicz [12]. They invented them as the main ingredient in their solutions to the so-called "division problem" stated by L. Schwartz [15] concerning the division of a distribution by a polynomial or, more generally, by an analytic function.

Precisely, Hörmander's version states that given a polynomial $f \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$, there exist positive real numbers c, μ such that $c \operatorname{dist}(x, Z_M(f))^\mu \leqslant |f(x)|$ for every $x \in \mathbb{R}^n$ with $\|x\| \leqslant 1$. On the other hand, Łojasiewicz stated (without proof) that given a compact set $K \subset \mathbb{R}^n$, an open neighborhood $\Omega \subset \mathbb{R}^n$ of K and an analytic function $f : \Omega \to \mathbb{R}$, there exist positive real numbers c, μ such that $c \operatorname{dist}(x, Z_M(f))^\mu \leqslant |f(x)|$ for all $x \in K$.

When dealing with semialgebraic functions, a useful version of this classical result appears in [2, 2.6.6–7], which provides a Nullstellensatz for semialgebraic functions as a byproduct (see Corollary 3.3). Namely,

Theorem 1.1 *(Łojasiewicz's inequality). Let $M \subset \mathbb{R}^n$ be a locally compact semialgebraic set and $f, g \in \mathcal{S}(M)$ be such that $Z_M(f) \subset Z_M(g)$. Then*

(i) *There exist a positive integer ℓ and $h \in \mathcal{S}(M)$ such that $g^\ell = fh$.*
(ii) *If $c = \sup\{|h(x)|\colon\ x \in M\}$ exists, then $|g(x)|^\ell \leqslant c|f(x)|$ for each $x \in M$.*

Remarks 1.2. (i) The previous result, and in fact the corresponding Nullstellensatz, is no longer true if M is not locally compact, see Proposition 3.4. A very representative example of this situation is the following one proposed in [2, 2.6.5]. Consider the semialgebraic set $M := \{y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$ and the semialgebraic functions $g(x, y) = x^2 + y^2$ and $f(x, y) = y$. Their zero sets are $Z_M(f) = Z_M(g) = \{(0,0)\}$. However, for each $\ell \in \mathbb{N}$ the limit at the origin of the semialgebraic function $h_\ell := \frac{g^\ell}{f} = \frac{(x^2+y^2)^\ell}{y}$ does not exist.

(ii) Observe that Theorem 1.1(ii) says nothing if $c = +\infty$. However, if $c < +\infty$, it is equivalent to Theorem 1.1(i), even if $M \subset \mathbb{R}^n$ is an arbitrary semialgebraic set. More precisely, *let $M \subset \mathbb{R}^n$ be a semialgebraic set and $f, g \in \mathcal{S}^\diamond(M)$ be such that $Z_M(f) \subset Z_M(g)$. If there exist a constant $c > 0$ and a positive integer $\ell \geqslant 1$ such that $|g(x)|^\ell \leqslant c|f(x)|$ for each $x \in M$, then there exists $h \in \mathcal{S}^\diamond(M)$ such that $g^{2\ell+1} = fh$.*

Indeed, for each $x \in M$ we have $g^{2\ell}(x) \leqslant c^2 f^2(x)$. Thus, the function $h_0 : M \to \mathbb{R}$ given by

$$h_0(x) := \begin{cases} \frac{g^{2\ell+1}(x)}{f^2(x)} & \text{if } x \in D_M(f), \\ 0 & \text{if } x \in Z_M(f) \end{cases}$$

is continuous because $Z_M(f) \subset Z_M(g)$ and the quotient $\frac{g^{2\ell}}{f^2}$ is bounded on $D_M(f)$. Moreover, $h_0 \in \mathcal{S}(M)$, and in fact it is bounded if g is bounded. Since $h_0 f^2 = g^{2\ell+1}$, we deduce that $h = f h_0 \in \mathcal{S}^\diamond(M)$ satisfies the required condition.

In view of Remark 1.2(ii), we say in the following that Łojasiewicz's inequality does not hold for a semialgebraic set M if there exist semialgebraic functions $f, g \in \mathcal{S}(M)$ such that $Z_M(f) \subset Z_M(g)$ but $g \notin \sqrt{f\mathcal{S}(M)}$.

Of course, Theorem 1.1(i) can be understood as a Nullstellensatz for principal ideals. To approach the announced Nullstellensatz for arbitrary ideals (see Corollary 3.3), and since the common zero set Z of the semialgebraic functions of a prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$ provides almost no information about such $\mathfrak{p}$ because Z is either empty or a singleton (see Proposition 2.3), we are led to consider the *z-filter* consisting of the collection of the zero sets of all functions in $\mathfrak{p}$ (see Section 3.1). As it is well-known, this is a classical idea used to study rings of continuous functions, which has been compiled in full detail in [10]. On the other hand, the use of these kinds of filters is a usual technique in Real Algebra (see for instance [1, II.1.6] and [2, 7.1, 7.5]).

In any case, the main goal of this work is to develop a similar theory (Łojasiewicz's inequality and Nullstellensatz) to approach the case of bounded semialgebraic functions. The existence of non-units in $\mathcal{S}^*(M)$ with empty zero set requires to generalize these z-filters in order to obtain a similar Łojasiewicz's inequality, which has been revealed as a crucial tool in Real Geometry. Even more, the bounded case can be done without the local compactness assumption. Namely,

Theorem 1.3. *Let f, g be two bounded semialgebraic functions on the semialgebraic set M such that each maximal ideal of $\mathcal{S}^*(M)$ containing f contains g, too. Then $g^\ell = fh$ for a suitable positive integer ℓ and a function $h \in \mathcal{S}^*(M)$. In particular, $|g|^\ell \leqslant (\sup_M(|h|))|f|$ on M.*

Clearly, this result (translated to the language of maximal spectra of semialgebraic rings in Theorem 3.10) can be understood as the counterpart of the classical Łojasiewicz inequality, stated in Theorem 1.1(ii), for rings of bounded semialgebraic functions. Its importance lies amongst others in the fact that it provides a Nullstellensatz for the ring $\mathcal{S}^*(M)$ as a byproduct where M is an arbitrary semialgebraic set (see Corollary 3.9). In contrast, the Nullstellensatz for $\mathcal{S}(M)$ is only true if M is locally compact (see Proposition 3.4). To prove this fact, it is indispensable to analyze the set $M_{\mathrm{lc}} \subset M$ of those points in M having a compact neighborhood in M. In fact, such set is moreover semialgebraic (see Lemma 2.8).

The article is organized as follows. In Section 2 we introduce most of the used terminology and we prove that every non-locally compact semialgebraic set M contains a semialgebraic subset C that is closed in M and that is semialgebraically homeomorphic to the triangle $T := \{(x, y) \in \mathbb{R}^2\colon\ 0 < y \leqslant x \leqslant 1\} \cup \{(0, 0)\}$. This last result is the key to prove that Łojasiewicz's inequality and the corresponding Nullstellensatz are no longer true for non-locally compact semialgebraic sets. In Section 3 we develop the main results of this work concerning Łojasiewicz's inequality and the Nullstellensatz for rings of semialgebraic and bounded semialgebraic functions on semialgebraic sets. In fact, we prove that Theorem 1.1 can be also obtained as a byproduct of Theorem 1.3.

To finish this Introduction, we would like to point out that Łojasiewicz's inequalities and Nullstellensätze are crucial tools for the study of chains of prime ideals in rings of semialgebraic and bounded semialgebraic functions (see [6]) and to determine the Krull dimension of the rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set (see [7] for further details).

2. Preliminaries on semialgebraic sets and functions

In this section we present some preliminary terminology and useful results for this work.

2.1. Basics on semialgebraic sets and functions

Sometimes it will be advantageous to assume that the semialgebraic set M we are working with is bounded. Such assumption can be done without loss of generality as we see in the next remark. We denote the open and closed balls of $\mathbb{R}^n$ of center $x \in \mathbb{R}^n$ and radius ε with $\mathbb{B}_n(x, \varepsilon)$ and $\overline{\mathbb{B}}_n(x, \varepsilon)$. Their common boundary is denoted with $\mathbb{S}^{n-1}(x, \varepsilon)$.

Remark 2.1. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. The semialgebraic homeomorphism

$$\varphi : \mathbb{B}_n(0,1) \to \mathbb{R}^n, \quad x \mapsto \frac{x}{\sqrt{1 - \|x\|^2}}$$

induces a ring isomorphism $\mathcal{S}(M) \to \mathcal{S}(N)$, $f \mapsto f \circ \varphi$, where $N = \varphi^{-1}(M)$, that maps $\mathcal{S}^*(M)$ onto $\mathcal{S}^*(N)$. So if necessary, we may always assume that M is bounded.

The following result, which concerns the representation of closed semialgebraic subsets of a semialgebraic set as zero sets of semialgebraic functions, is well-known and will be used along this work.

Lemma 2.2. *Let Z be a closed semialgebraic subset of the semialgebraic set $M \subset \mathbb{R}^n$. Then there exists $h \in \mathcal{S}^*(M)$ such that $Z = Z_M(h)$.*

Proof. Take for instance $h = \min\{1, \operatorname{dist}(\cdot, Z)\}$. $\square$

In contrast to ideals of polynomial rings, the zero set of a prime ideal $\mathfrak{p}$ of $\mathcal{S}^\diamond(M)$ provides no substantial information about $\mathfrak{p}$ because it is either a point or the empty set.

Proposition 2.3. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and $\mathfrak{p}$ a prime ideal of $\mathcal{S}^\diamond(M)$. Then the set $Z := \{x \in M\colon f(x) = 0\ \forall f \in \mathfrak{p}\}$ is either empty or a singleton.*

Proof. Suppose by contradiction that Z contains two distinct points p, q. Let $r > 0$ be the Euclidean distance between p and q and B_1 and B_2 the open balls centered at p of respective radii $r_1 := r/3$ and $r_2 := 2r/3$. Consider the closed semialgebraic sets in $\mathbb{R}^n$ defined as $C_1 := \mathbb{R}^n \setminus B_1$ and $C_2 := \operatorname{Cl}_{\mathbb{R}^n}(B_2)$. By Lemma 2.2 there exist $f_1, f_2 \in \mathcal{S}^*(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(f_i) = C_i$. Clearly, the product $f_1 f_2$ vanishes identically on $\mathbb{R}^n$, hence, on M. Thus, if we write $g_i := f_i|_M$ for $i = 1, 2$, we have $g_1 g_2 \in \mathfrak{p}$ and therefore either g_1 or g_2 belongs to $\mathfrak{p}$. But g_1 does not vanish at p and g_2 does not vanish at q, which is a contradiction. $\square$

This result suggests to introduce some classical definitions.

Definitions and notations 2.4. An ideal $\mathfrak{a}$ of $\mathcal{S}^\diamond(M)$ is said to be *fixed* if all functions in $\mathfrak{a}$ vanish simultaneously at some point of M. Otherwise the ideal $\mathfrak{a}$ is *free*.

Given a point $p \in M$, we denote the fixed ideal of $\mathcal{S}(M)$ (resp. $\mathcal{S}^*(M)$) consisting of those functions vanishing at p with $\mathfrak{m}_p$ (resp. $\mathfrak{m}^*_p$). Distinct points produce distinct maximal ideals and $\{\mathfrak{m}_p\}_{p \in M}$ (resp. $\{\mathfrak{m}^*_p\}_{p \in M}$) constitutes the collection of all fixed maximal ideals of $\mathcal{S}(M)$ (resp. $\mathcal{S}^*(M)$).

We denote the collection of all maximal ideals of $\mathcal{S}^*(M)$ with $\beta^*_s M$. Given a function $f \in \mathcal{S}^*(M)$, we write

$$\mathcal{Z}_{\beta^*_s M}(f) := \{\mathfrak{m} \in \beta^*_s M\colon\ f \in \mathfrak{m}\} \quad \text{and} \quad \mathcal{D}_{\beta^*_s M}(f) := \beta^*_s M \setminus \mathcal{Z}_{\beta^*_s M}(f).$$

Notice that the map $\phi : M \to \beta_s^* M$, $p \mapsto \mathfrak{m}_p^*$ is injective; thus, we identify M with $\phi(M)$. This provides the equalities $D_M(f) = \mathcal{D}_{\beta_s^* M}(f) \cap M$ and $Z_M(f) = \mathcal{Z}_{\beta_s^* M}(f) \cap M$.

2.2. Maximal ideals associated to semialgebraic paths

Concerning free maximal ideals of $\mathcal{S}^*(M)$, which are deeply studied in [9] and [8], we are mainly interested in the simplest class of them: those associated to semialgebraic paths. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Consider a semialgebraic path $\alpha : (0,1] \to M$, that is, a continuous map whose components are semialgebraic functions. We claim:

(2.2.1) *The set* $\mathfrak{m}_\alpha^* = \{f \in \mathcal{S}^*(M):\ \lim_{t\to 0}(f \circ \alpha)(t) = 0\}$ *is a maximal ideal of* $\mathcal{S}^*(M)$.

Of course, the ideal $\mathfrak{m}_\alpha^*$ is free if and only if α cannot be extended to a (continuous) semialgebraic path $[0,1] \to M$.

Before proving (2.2.1), we need the following preliminary result. Recall that given an open semialgebraic set $U \subset \mathbb{R}^n$, a function $f \in \mathcal{S}(U)$ is said to be a *Nash* function on U if it is moreover analytic (see [2, 8.1.6–8]).

Lemma 2.5. *Let* $I := (a,b) \subset \mathbb{R}$ *be an open interval with* $-\infty \leqslant a < b \leqslant +\infty$ *and let* $f \in \mathcal{S}(I)$ *be a semialgebraic function. Then*

(i) *There exists a finite subset* $F \subset I$ *such that the restriction* $h = f|_{I \setminus F}$ *is a Nash function.*
(ii) *There exists* $c \in I$ *such that the restriction* $f|_{[c,b)}$ *is a monotone function.*
(iii) *If* f *is moreover bounded, there exist* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to b} f(x)$.

Proof. (i) As the graph of f is a 1-dimensional semialgebraic subset of $\mathbb{R}^2$, it is a finite union of singletons $\{p_1, \dots, p_n\}$ and 1-dimensional Nash manifolds (see [2, 2.9.10]). Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$ be the projection onto the first coordinate. Then the set $F = \{\pi_1(p_1), \dots, \pi_1(p_n)\}$ satisfies the statement.

(ii) If f is constant on a subinterval (c,b) of I, the result is evident. Otherwise the zero set $Z_{I \setminus F}(f')$ of the derivative f' of f is a union of singletons and intervals where none of them is of the form (c,b) because it is semialgebraic. In other words, $Z_{I \setminus F}(f') \subset (a, c_0)$ for some $c_0 < b$, and it is enough to choose $c = c_0$. Note that in this case $f|_{[c,b)}$ is either increasing or decreasing, according to the sign of f' in $[c,b)$.

(iii) It is enough to prove that the limit of f at b exists. This is obvious if f is constant on a subinterval $J = [c,b) \subset I$. Hence, we can suppose without loss of generality that f is decreasing on J. Since f is a bounded function, $f(J)$ is a bounded interval and as f is decreasing on J, there exists $\lambda \in \mathbb{R}$ such that $f(J) = (\lambda, f(c)]$. Note that $\mathrm{Cl}_{\mathbb{R}}(f(J)) \setminus f(J) = \{\lambda\}$ and so $\lim_{x\to b} f(x) = \lambda$. $\square$

Now the claim in (2.2.1) follows straightforwardly from Lemma 2.5:

Proof of Statement (2.2.1). It follows from Lemma 2.5 that $\lim_{t\to 0}(f \circ \alpha)(t) \in \mathbb{R}$ exists for each function $f \in \mathcal{S}^*(M)$. Once this is done, note that $\mathfrak{m}_\alpha^*$ is the kernel of the ring epimorphism $\mathcal{S}^*(M) \to \mathbb{R}$, $f \mapsto \lim_{t\to 0}(f \circ \alpha)(t)$. $\square$

Remark 2.6. With the notation above, suppose there exists $\lim_{t\to 0} \alpha(t) = p \in M$. This includes the case in which the path α is locally constant around 0. Then $\mathfrak{m}_\alpha^* = \mathfrak{m}_p^*$.

We next study some properties about local compactness of semialgebraic sets.

2.3. Local compactness

Locally compact Hausdorff spaces are characterized as spaces, which admit a Hausdorff compactification by a single point [14, 3.29.1]. On the other hand, locally closed semialgebraic subsets of $\mathbb{R}^n$

are those that can be embedded as closed semialgebraic subsets of some $\mathbb{R}^m$. Local closedness has revealed to be an important property in the semialgebraic setting for the validity of results, which are in the core of semialgebraic geometry, such as Łojasiewicz's inequality. But as it is well-known, locally closed subsets of $\mathbb{R}^n$ coincide with the locally compact ones (see [3, §9.7, Propositions 12–13]). In fact, if $M \subset \mathbb{R}^n$ is locally compact, then $M = U \cap \mathrm{Cl}_{\mathbb{R}^n}(M)$ where $U := \mathbb{R}^n \setminus (\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M)$ is an open subset of $\mathbb{R}^n$. Of course, if $M \subset \mathbb{R}^n$ is a semialgebraic set, both $\mathrm{Cl}_{\mathbb{R}^n}(M)$ and U are semialgebraic; hence, each locally compact semialgebraic set $M \subset \mathbb{R}^n$ is the intersection of a closed and an open semialgebraic subset of $\mathbb{R}^n$.

We will see in Section 3 that only locally compact semialgebraic sets satisfy a Łojasiewicz inequality or a Nullstellensatz for its ring of semialgebraic functions. The clue result to prove this is the following:

Lemma 2.7. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set, which is not locally compact. Then M contains a semialgebraic set C that is closed in M and semialgebraically homeomorphic to the triangle $T := \{(x, y) \in \mathbb{R}^2:\ 0 < y \leqslant x \leqslant 1\} \cup \{(0, 0)\}$.*

The proof of this lemma requires a certain analysis of the set of points of M that have a compact neighborhood in M. Its construction is the main goal of [5, 9.14–9.21].

Lemma 2.8. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Define*

$$\rho_0(M) = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M \quad \text{and} \quad \rho_1(M) = \rho_0\big(\rho_0(M)\big) = \mathrm{Cl}_{\mathbb{R}^n}\big(\rho_0(M)\big) \cap M.$$

Then $M_{\mathrm{lc}} = M \setminus \rho_1(M)$ is a locally compact semialgebraic set, which coincides with the set of points of M that have a compact neighborhood in M.

Assume we have already proved Lemma 2.8 and let us show Lemma 2.7.

Proof of Lemma 2.7. We may assume $0 \in \rho_1(M)$. By Lemma 2.8 the origin is not an isolated point of M. By [2, 9.3.6] there exist a positive real number $\varepsilon > 0$ and a semialgebraic homeomorphism $\varphi : \overline{\mathbb{B}}_n(0, \varepsilon) \to \overline{\mathbb{B}}_n(0, \varepsilon)$ such that

(i) $\|\varphi(x)\| = \|x\|$ for every $x \in \overline{\mathbb{B}}_n(0, \varepsilon)$,
(ii) $\varphi|_{\mathbb{S}^{n-1}(0,\varepsilon)}$ is the identity map,
(iii) $\varphi^{-1}(M \cap \overline{\mathbb{B}}_n(0, \varepsilon))$ is the cone with vertex 0 and basis $M \cap \mathbb{S}^{n-1}(0, \varepsilon)$.

Consider the semialgebraic homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\psi(x) := \begin{cases} x & \text{if } x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}_n(0, \varepsilon), \\ \varphi(x) & \text{if } x \in \overline{\mathbb{B}}_n(0, \varepsilon). \end{cases}$$

In the following we identify M with $\psi^{-1}(M)$. Since $0 \in \rho_1(M)$, this point has no compact neighborhood in M (see Lemma 2.8). In particular $M \cap \overline{\mathbb{B}}_n(0, \varepsilon)$, which is the cone with vertex 0 and basis $N := \mathbb{S}^{n-1}(0, \varepsilon) \cap M$, is not compact. This implies that the basis N is not compact and so it is not closed in $\mathbb{R}^n$. Choose a point $q \in \mathrm{Cl}_{\mathbb{R}^n}(N) \setminus N$. By the Curve Selection Lemma [2, 2.5.5] there exists a semialgebraic path $\alpha : [0, 1] \to \mathbb{R}^n$ such that $\alpha(0) = q$ and $\alpha((0, 1]) \subset N$. After shrinking the domain of α if necessary, we may assume that $\alpha|_{(0,1]}$ is a homeomorphism onto its image $K := \alpha((0, 1]) \subset N$. Thus, K is a closed subset of N and it is homeomorphic to the interval $(0, 1]$.

Let C be the cone with vertex 0 and basis K. A straightforward computation shows that C, which is a closed semialgebraic subset of M, is homeomorphic to T via the semialgebraic homeomorphism

$$T \to C, \quad (s, t) \mapsto \begin{cases} s\alpha(t/s) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0 \end{cases}$$

whose inverse map is defined by

$$C \to T, \quad x \mapsto \begin{cases} (\|x\|/\varepsilon)(1, \alpha^{-1}(\varepsilon x/\|x\|)) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This finishes the proof. □

We proceed to prove the remaining result Lemma 2.8.

Proof of Lemma 2.8. We check first $M \setminus \rho_1(M) = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus \mathrm{Cl}_{\mathbb{R}^n}(\rho_0(M))$. Observe $\mathrm{Cl}_{\mathbb{R}^n}(M) = M \sqcup \rho_0(M)$ and $\mathrm{Cl}_{\mathbb{R}^n}(\rho_0(M)) = \rho_0(M) \sqcup \rho_1(M)$. Thus,

$$\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus \mathrm{Cl}_{\mathbb{R}^n}\big(\rho_0(M)\big) = \big(M \sqcup \rho_0(M)\big) \setminus \big(\rho_0(M) \sqcup \rho_1(M)\big) = M \setminus \rho_1(M).$$

Consequently, $M_{\mathrm{lc}} = M \setminus \rho_1(M) = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus \mathrm{Cl}_{\mathbb{R}^n}(\rho_0(M))$ is a locally closed set and so it is by Section 2.3 locally compact. Note that

$$\rho_1(M) = \mathrm{Cl}_{\mathbb{R}^n}\big(\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M\big) \cap M$$

is a closed subset of M. If N denotes the set of points of M having a compact neighborhood in M, we deduce that $M_{\mathrm{lc}} = M \setminus \rho_1(M)$ is contained in N since M_{lc} is locally compact and open in M.

Conversely, let $x \in N$ and K be a compact neighborhood of x in M. Let W be an open subset of $\mathbb{R}^n$ such that $x \in W$ and $M \cap W \subset K$. Thus,

$$x \in \mathrm{Cl}_{\mathbb{R}^n}(M) \cap W = \mathrm{Cl}_{\mathbb{R}^n}(M \cap W) \cap W \subset K \subset M$$

or equivalently W is a neighborhood of x in $\mathbb{R}^n$ such that $W \cap (\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M) = \varnothing$. Hence, $x \notin \mathrm{Cl}_{\mathbb{R}^n}(\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M) \cap M = \rho_1(M)$, that is, $x \in M_{\mathrm{lc}} = M \setminus \rho_1(M)$, as wanted. □

3. Łojasiewicz's inequalities and Nullstellensätze

We introduce several preliminary notions and remarks, which allow us to state the Nullstellensatz for the ring of semialgebraic functions on a semialgebraic set properly. Whenever we consider an ideal of $\mathcal{S}^\diamond(M)$, we refer to a proper ideal of $\mathcal{S}^\diamond(M)$.

3.1. Filters in rings of semialgebraic functions and z-ideals

Let $\mathcal{Z}_M$ be the collection of all closed semialgebraic subsets of M, which coincides by Lemma 2.2 with the family of zero sets of semialgebraic functions on M. Let $\mathcal{P}(\mathcal{Z}_M)$ be the set of all subsets of $\mathcal{Z}_M$. Recall that a subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{Z}_M)$ is a z-filter on M if it satisfies the following properties:

(i) $\varnothing \notin \mathcal{F}$.
(ii) Given $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$.
(iii) Given $Z \in \mathcal{F}$ and $Z' \in \mathcal{Z}_M$ such that $Z \subset Z'$, then $Z' \in \mathcal{F}$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{S}(M)$. One can check straightforwardly that

(i) *The family* $\mathcal{Z}[\mathfrak{a}] := \{Z_M(f)\colon\ f \in \mathfrak{a}\}$ *is a z-filter on* M.
(ii) *If* $\mathcal{F}$ *is a z-filter, then* $\mathcal{J}(\mathcal{F}) := \{f \in \mathcal{S}(M)\colon\ Z_M(f) \in \mathcal{F}\}$ *is an ideal of* $\mathcal{S}(M)$ *satisfying* $\mathcal{Z}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$.

Definition 3.1. An ideal $\mathfrak{a}$ of $\mathcal{S}(M)$ is a *z-ideal* if $\mathcal{J}(\mathcal{Z}[\mathfrak{a}]) = \mathfrak{a}$. That is, whenever there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}(M)$ satisfying $Z_M(f) \subset Z_M(g)$, we have $g \in \mathfrak{a}$.

Remark 3.2. Notice that the equality $\mathcal{Z}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$ implies that $\mathcal{J}(\mathcal{F})$ is a z-ideal whenever $\mathcal{F}$ is a z-filter. Observe that each z-ideal is a radical ideal because $Z_M(f) = Z_M(f^k)$ for each $f \in \mathcal{S}(M)$ and each $k \geqslant 1$.

3.2. Łojasiewicz's inequality and Nullstellensatz

We are now ready to present the Nullstellensatz for the ring of semialgebraic functions on a semialgebraic set.

Corollary 3.3 *(Nullstellensatz). Let $M \subset \mathbb{R}^n$ be a locally compact semialgebraic set and $\mathfrak{a}$ an ideal of $\mathcal{S}(M)$. Then $\mathcal{J}(\mathcal{Z}[\mathfrak{a}]) = \sqrt{\mathfrak{a}}$ and $\mathfrak{a}$ is a z-ideal if and only if $\mathfrak{a}$ is a radical ideal. In particular, each prime ideal of $\mathcal{S}(M)$ is a z-ideal.*

Proof. Let $g \in \mathcal{S}(M)$ be such that $Z_M(g) \in \mathcal{Z}[\mathfrak{a}]$. Then there exists $f \in \mathfrak{a}$ such that $Z_M(f) = Z_M(g)$ and by Theorem 1.1 there exist $\ell \geqslant 1$ and $h \in \mathcal{S}(M)$ such that $g^\ell = fh \in \mathfrak{a}$, that is, $g \in \sqrt{\mathfrak{a}}$. The rest of the statement follows from Remark 3.2 and the fact that all prime ideals are radical ideals. $\square$

Let us see that if M is not locally compact, Łojasiewicz's inequality, stated in Theorem 1.1, does not hold for M and in addition there exist prime ideals in $\mathcal{S}(M)$, which are not z-ideals. More precisely,

Proposition 3.4. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set, which is not locally compact. Then*

(i) *Łojasiewicz's inequality does not hold for M.*
(ii) *The ring $\mathcal{S}(M)$ has fixed prime ideals that are not z-ideals.*

We need some preliminary results for the proof. Namely,

Lemma 3.5. *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets. Write $Y = M \setminus N$ and take $b \in \mathcal{S}^*(N)$. Let $h \in \mathcal{S}^\diamond(M)$ be such that $Y \subset Z_M(h)$. Then the product $(h|_N)b$ can be extended continuously by 0 to a semialgebraic function that belongs to $\mathcal{S}^\diamond(M)$.*

Proof. Since b is bounded on N and h vanishes identically on Y, we deduce

$$\lim_{x \to p} (h|_N b)(x) = 0$$

for all $p \in Y \cap \mathrm{Cl}_M(N)$. Thus, $(h|_N)b$ can be extended continuously by 0 to entire M. The graph of such an extension, which is the union $\mathrm{graph}(h|_N b) \cup (Y \times \{0\})$, is a semialgebraic set. So such an extension is an element of $\mathcal{S}^\diamond(M)$. $\square$

Lemma 3.6. *Let $N \subset M \subset \mathbb{R}^n$ be semialgebraic sets such that N is closed in M and let $\mathfrak{a}$ be a radical ideal of $\mathcal{S}(N)$, which is not a z-ideal. Let $\jmath : N \hookrightarrow M$ be the inclusion map and $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$, $f \mapsto f|_N = f \circ \jmath$ the induced homomorphism. Then $\mathfrak{b} := \phi^{-1}(\mathfrak{a})$ is a radical ideal but not a z-ideal.*

Proof. It is immediate to check that $\mathfrak{b}$ is radical, so let us prove that it is not a z-ideal. Since N is closed in M, the homomorphism ϕ is surjective by the semialgebraic version of the Tietze–Urysohn Lemma [4]. Suppose now by contradiction that $\mathfrak{b}$ is a z-ideal. Since $\mathfrak{a}$ is not a z-ideal, there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}(N) \setminus \mathfrak{a}$ such that $Z_N(f) \subset Z_N(g)$. Let $F, G \in \mathcal{S}(M)$ be such that $\phi(F) = f$ and $\phi(G) = g$. By Lemma 2.2 there exists $H \in \mathcal{S}(M)$ such that $Z_M(H) = N$. Consider the semialgebraic functions $F_1 := F^2 + H^2$ and $G_1 := G^2 + H^2$. Then

$$F_1|_N = f^2, \qquad G_1|_N = g^2 \quad \text{and} \quad Z_M(F_1) = Z_N(f) \subset Z_N(g) = Z_M(G_1).$$

Moreover, $F_1 \in \mathfrak{b}$ because $\phi(F_1) = f^2 \in \mathfrak{a}$. Thus, $G_1 \in \mathfrak{b}$ and therefore $g^2 = \phi(G_1) \in \mathfrak{a}$. Since $\mathfrak{a}$ is radical, we conclude $g \in \mathfrak{a}$, which is a contradiction. $\square$

Now we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. By Lemma 2.7 there exists a semialgebraic subset $C \subset M$, which is closed in M, and a semialgebraic homeomorphism

$$\psi : C \to T := \{(x, y) \in \mathbb{R}^2 \colon 0 < y \leqslant x \leqslant 1\} \cup \{p = (0, 0)\}.$$

By Lemma 2.2 there exists $c \in \mathcal{S}^*(M)$ such that $Z_M(c) = C$.

(i) Consider the semialgebraic functions $g(x, y) = y$ and $h(x, y) = x^2 + y^2$ on T. Let $g_1 := g \circ \psi$, $h_1 := h \circ \psi \in \mathcal{S}(C)$. Let $G_1, H_1 \in \mathcal{S}(M)$ be semialgebraic functions, which extend[1] g_1, h_1 respectively. The semialgebraic functions $G := G_1^2 + c^2$ and $H := H_1^2 + c^2$ satisfy $Z_M(G) = Z_M(H) = \{\psi^{-1}(p)\}$. Suppose by contradiction that there exist $\ell \geqslant 2$ and $F \in \mathcal{S}(M)$ such that $H^\ell = GF$ and so $(H|_C)^\ell = (G|_C)(F|_C)$. After composition with ψ^{-1}, we deduce the existence of $f \in \mathcal{S}(T)$ such that $h^{2\ell} = g^2 f$, that is, the quotient

$$f = \frac{h^{2\ell}}{g^2} = \frac{(x^2 + y^2)^{2\ell}}{y^2}$$

is continuous on T, which is a contradiction. Therefore Łojasiewicz's inequality does not hold for M.

(ii) Since C is closed, it is by Lemma 3.6 enough to find a fixed prime ideal in $\mathcal{S}(C)$, which is not a z-ideal. Even more, the semialgebraic homeomorphism $\psi : C \to T$ induces a ring isomorphism $\psi^* : \mathcal{S}(T) \to \mathcal{S}(C)$, $f \mapsto f \circ \psi$ and $Z_T(f) = \psi(Z_C(\psi^*(f)))$ for every $f \in \mathcal{S}(T)$. Thus, we only have to prove the existence of a fixed prime ideal in $\mathcal{S}(T)$, which is not a z-ideal.

(3.4.1) We claim:

$$\mathfrak{p} := \{f \in \mathcal{S}(T) \colon \exists \varepsilon > 0 \mid f \textit{ is continuously extended by } 0 \textit{ to } T \cup ((0, \varepsilon] \times \{0\})\}$$

is a fixed prime ideal of $\mathcal{S}(T)$, which is not a z-ideal.

Indeed, it is clear that $\mathfrak{p}$ is closed under addition. Let $f \in \mathfrak{p}$ and $g \in \mathcal{S}(T)$. Since the origin p belongs to T, there exists a neighborhood W of p in T on which g is bounded. Thus, by Lemma 3.5 there exists $\varepsilon > 0$ such that fg can be extended continuously by 0 to $T \cup ([0, \varepsilon] \times \{0\})$, that is, $fg \in \mathfrak{p}$ and so $\mathfrak{p} \subset \mathfrak{m}_p$ is a fixed ideal of $\mathcal{S}(T)$. Moreover, $\mathfrak{p}$ is not a z-ideal because the semialgebraic functions $g_1 := x^2 + y^2$ and $g_2 := y$ satisfy $Z_T(g_1) = Z_T(g_2) = \{p\}$ and $g_2 \in \mathfrak{p}$ while $g_1 \notin \mathfrak{p}$.

We check now that $\mathfrak{p}$ is prime. Let $h_1, h_2 \in \mathcal{S}(T)$ be such that $h_1 h_2 \in \mathfrak{p}$. Since $1/(1 + |h_1|)$ and $1/(1+|h_2|)$ are units in $\mathcal{S}(T)$, it is enough to check that either $f_1 := h_1/(1+|h_1|)$ or $f_2 := h_2/(1+|h_2|)$ lies in $\mathfrak{p}$. Note that both f_1 and f_2 are bounded functions.

Let $X_1 := \mathrm{Cl}_{\mathbb{R}^3}(\mathrm{graph}(f_1))$ and $X_2 := \mathrm{Cl}_{\mathbb{R}^3}(\mathrm{graph}(f_2))$, which are compact bidimensional semialgebraic sets. By [2, 2.8.13] each $C_i := X_i \setminus \mathrm{graph}(f_i)$ is a semialgebraic curve whose projection onto the plane $\{z = 0\}$ is the segment $(0, 1] \times \{0\}$.

By [2, 2.9.10] each curve $C_i \subset \mathbb{R} \times \{0\} \times \mathbb{R}$ is the disjoint union of finitely many points $p_{i\ell}$ and a finite number of Nash curves M_{ik} and each of them is Nash diffeomorphic to an open interval $(0, 1)$. Note that each curve M_{ik} is either contained in a vertical line $\{(a, 0)\} \times \mathbb{R}$ or it has only finitely many points with vertical tangent. Thus, there exist only finitely many values $a \in (0, 1]$ such that the line $\{(a, 0)\} \times \mathbb{R}$ either passes through one of the points $p_{i\ell}$ or it contains some curve M_{ik} or it

[1] Recall the already mentioned semialgebraic version of the Tietze–Urysohn Lemma [4].

is the tangent line to some M_{ik} at one of its points. Denote the set of such values with J and let $b \in (0,1] \setminus J$. Let us see that both functions f_1, f_2 can be extended continuously to the point $(b,0)$. Fix $i = 1,2$ and observe that the line $\{(b,0)\} \times \mathbb{R}$ meets the curve C_i in finitely many points. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the first two coordinates.

Let $\delta > 0$ be such that the closure $\overline{B}$ of the open ball B of center $(b,0)$ and radius δ has the following properties

(1) $\overline{B}_1 = \overline{B} \cap \{y \geqslant 0\} \subset \mathrm{Cl}_{\mathbb{R}^2}(T) \setminus \{p\}$.
(2) There exists an index k such that the closed interval $[b-\delta, b+\delta]$ is Nash diffeomorphic via the projection onto the first coordinate to a closed subset of the Nash curve M_{ik}.

One can check that the restriction

$$\varphi := \pi|_Z : Z := \mathrm{Cl}_{\mathbb{R}^3}\big(\pi^{-1}(B \cap T)\big) \to \pi(Z) = \overline{B}_1$$

is a semialgebraic bijection and φ is a semialgebraic homeomorphism as Z is compact. Let $q := (b,0,s) = \varphi^{-1}(b,0)$. It is clear that f_i can be extended continuously to the point $(b,0)$ by setting $f_i(b,0) = s$.

Therefore there exists a finite set $J \subset (0,1]$ such that both f_1 and f_2 can be extended continuously to $T \cup ((0,1] \setminus J) \times \{0\}$. Thus, they can be extended continuously to $T \cup I_1$ for some interval $I_1 = (0,\varepsilon_1] \times \{0\}$ with $\varepsilon_1 > 0$. Since $f_1 f_2 \in \mathfrak{p}$, we may assume that $f_1 f_2$ can be extended continuously by 0 to $T \cup I_1$. By the semialgebraicity of f_1 and f_2 we can assume the existence of $\varepsilon_2 \in (0,\varepsilon_1)$ such that the continuous extension of f_1 to $T \cup ((0,\varepsilon_2] \times \{0\})$ vanishes identically on $(0,\varepsilon_2] \times \{0\}$, that is, $f_1 \in \mathfrak{p}$. Consequently, $\mathfrak{p}$ is a fixed prime ideal of $\mathcal{S}(T)$, which is not a z-ideal. $\square$

Our next aim is to develop a similar theory to approach the case of bounded semialgebraic functions. The existence of non-units in $\mathcal{S}^*(M)$ with empty zero set requires a generalization of the z-filters used above to obtain a similar Łojasiewicz's inequality. It is worthwhile to mention that in contrast to the ring $\mathcal{S}(M)$, this can be done without the local compactness assumption on M.

3.3. Filters in rings of bounded semialgebraic functions and z^*-ideals

Recall that a function $f \in \mathcal{S}(M)$ is a unit if and only if $Z_M(f) = \varnothing$. However, this is no longer true in the bounded case: Given a bounded semialgebraic function with empty zero set, its inverse in $\mathcal{S}(M)$ could be unbounded. Recall that in a general commutative ring with unity an element is a unit if and only if it does not belong to any maximal ideal. This leads us to handle all maximal ideals in $\mathcal{S}^*(M)$ and not only the ones corresponding to points in M. Observe that with the notations in Section 2.4 a function $f \in \mathcal{S}^*(M)$ is a unit if and only if $\mathcal{Z}_{\beta_s^* M}(f) = \varnothing$. The family of all sets $\mathcal{Z}_{\beta_s^* M}(f)$ for $f \in \mathcal{S}^*(M)$ is denoted with $\mathcal{Z}_{\beta_s^* M}$. Recall that a subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{Z}_{\beta_s^* M})$ is a z^*-filter on M if it satisfies the following properties:

(i) $\varnothing \notin \mathcal{F}$.
(ii) Given $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$.
(iii) Given $Z \in \mathcal{F}$ and $Z' \in \mathcal{Z}_{\beta_s^* M}$ such that $Z \subset Z'$, then $Z' \in \mathcal{F}$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{S}^*(M)$. One can check almost straightforwardly that

(i) *The family* $\mathcal{Z}_{\beta_s^* M}[\mathfrak{a}] := \{\mathcal{Z}_{\beta_s^* M}(f)\colon\ f \in \mathfrak{a}\}$ *is a* z^**-filter on* M.
(ii) *If* $\mathcal{F}$ *is a* z^**-filter, then* $\mathcal{J}(\mathcal{F}) := \{f \in \mathcal{S}^*(M)\colon\ \mathcal{Z}_{\beta_s^* M}(f) \in \mathcal{F}\}$ *is an ideal of* $\mathcal{S}^*(M)$ *such that* $\mathcal{Z}_{\beta_s^* M}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$.

Definition 3.7. An ideal $\mathfrak{a}$ of the ring $\mathcal{S}^*(M)$ is a z^**-ideal* if $\mathcal{J}(\mathcal{Z}_{\beta_s^* M}[\mathfrak{a}]) = \mathfrak{a}$, that is, whenever there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}^*(M)$ satisfying $\mathcal{Z}_{\beta_s^* M}(f) \subset \mathcal{Z}_{\beta_s^* M}(g)$, we have $g \in \mathfrak{a}$.

Remark 3.8. Notice that the equality $\mathcal{Z}_{\beta_s^* M}[\mathcal{J}(\mathfrak{F})] = \mathfrak{F}$ implies that $\mathcal{J}(\mathfrak{F})$ is a z^*-ideal whenever $\mathfrak{F}$ is a z^*-filter. Note also that each z^*-ideal is a radical ideal because $\mathcal{Z}_{\beta_s^* M}(f) = \mathcal{Z}_{\beta_s^* M}(f^k)$ for all $f \in \mathcal{S}^*(M)$ and all $k \geqslant 1$.

3.4. Łojasiewicz's inequality for bounded semialgebraic functions

The analogous result to Corollary 3.3 for bounded semialgebraic functions is the following Nullstellensatz whose proof requires some preliminary results.

Corollary 3.9 *(Nullstellensatz). Let $M \subset \mathbb{R}^n$ be a semialgebraic set and $\mathfrak{a}$ an ideal of $\mathcal{S}^*(M)$. Then $\mathcal{J}(\mathcal{Z}_{\beta_s^* M}[\mathfrak{a}]) = \sqrt{\mathfrak{a}}$ and $\mathfrak{a}$ is a z^*-ideal if and only if $\mathfrak{a}$ is a radical ideal. In particular, each prime ideal of $\mathcal{S}^*(M)$ is a z^*-ideal.*

The crucial tool to prove the Nullstellensatz is again a Łojasiewicz inequality that takes the following formulation (equivalent to the one already stated in Theorem 1.3).

Theorem 3.10 *(Łojasiewicz's inequality). Let $M \subset \mathbb{R}^n$ be a semialgebraic set and $f, g \in \mathcal{S}^*(M)$ be such that $\mathcal{Z}_{\beta_s^* M}(f) \subset \mathcal{Z}_{\beta_s^* M}(g)$. Then there exist $h \in \mathcal{S}^*(M)$ and a positive integer ℓ such that $g^\ell = fh$. In particular, $|g|^\ell \leqslant (\sup_M(|h|))|f|$ on M.*

Remarks 3.11. (i) As we have already observed in Remark 1.2(ii), the existence of an integer $\ell \geqslant 1$ and a constant $c > 0$ such that $|g|^\ell \leqslant cf$ on M guarantees in our context the existence of $h \in \mathcal{S}^*(M)$ such that $g^{2\ell+1} = hf$.

(ii) The previous result plays an important role in the study of non-refinable chains of prime ideals in rings of bounded semialgebraic functions (see [6] for further details). In fact, Theorem 3.10 is crucial to prove a useful criterion of primality of ideals of $\mathcal{S}(M)$ (see [6, 5.4]) that is strongly inspired by the corresponding result in [10, 2.9] concerning rings of continuous functions.

On the other hand, it follows from [2, 7.1.23] that given a free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$, the family of prime ideals of $\mathcal{S}^*(M)$ containing the prime ideal $\mathfrak{m} \cap \mathcal{S}^*(M)$ constitutes a chain. Theorem 3.10 is an essential tool to describe the *immediate successor* of $\mathfrak{m} \cap \mathcal{S}^*(M)$, that is, the smallest prime ideal of $\mathcal{S}^*(M)$ that contains $\mathfrak{m} \cap \mathcal{S}^*(M)$ properly. For further details see [6, §6]. It is strongly inspired by the corresponding result for rings of continuous functions developed in [13, 6] and [10, 14.25–27].

Assume for a while that Theorem 3.10 is proved and let us use it to prove the Nullstellensatz, stated in Corollary 3.9, as its straightforward consequence.

Proof of Corollary 3.9. Let $g \in \mathcal{S}^*(M)$ be such that $\mathcal{Z}_{\beta_s^* M}(g) \in \mathcal{Z}_{\beta_s^* M}[\mathfrak{a}]$. Then there exists $f \in \mathfrak{a}$ such that $\mathcal{Z}_{\beta_s^* M}(f) = \mathcal{Z}_{\beta_s^* M}(g)$. By Theorem 3.10 there exist a positive integer ℓ and $h \in \mathcal{S}^*(M)$ such that $g^\ell = fh \in \mathfrak{a}$, that is, $g \in \sqrt{\mathfrak{a}}$. The rest of the statement follows from Remark 3.8 and the fact that all prime ideals are radical ideals. □

We are led to prove Theorem 3.10. Our proof is inspired by [2, 2.6.4].

Proof of Theorem 3.10. As observed in Remark 2.1, we may assume $M \subset \mathbb{B}_n(0,1)$. For each $u \in \mathbb{R}$ we define the semialgebraic subset $M_u := \{y \in M\colon\ u|g(y)| = 1\}$. Let us see

(3.10.1) If $M_u \neq \varnothing$, then $\sup\{1/|f(y)|\colon\ y \in M_u\} < +\infty$.

Otherwise there exists a sequence $\{y_m\}_{m\geqslant 1} \subset M_u$ such that $\lim_{m\to+\infty} f(y_m) = 0$. Consider the graph Γ of the restriction $h := f|_{M_u} : M_u \to \mathbb{R}$. Since $M_u \subset \mathbb{B}_n(0,1)$ is a bounded subset of $\mathbb{R}^n$, its closure $\mathrm{Cl}_{\mathbb{R}^n}(M_u)$ is compact. So we may assume after substituting $\{y_m\}_{m\geqslant 1}$ by one of its subsequences if necessary that $\lim_{m\to+\infty} y_m = y \in \mathrm{Cl}_{\mathbb{R}^n}(M_u)$ exists. Note that the point $(y, 0) \in \mathrm{Cl}_{\mathbb{R}^n}(\Gamma)$.

Hence, by the Curve Selection Lemma [2, 2.5.5] there exists a semialgebraic path $\gamma : [0,1] \to \mathbb{R}^{n+1}$ such that $\gamma(0) = (y, 0)$ and $\gamma((0,1]) \subset \Gamma$. For each $t \in [0,1]$ we write $\gamma(t) := (\alpha(t), \nu(t)) \in \mathbb{R}^n \times \mathbb{R}$. Then $\alpha : [0,1] \to \mathbb{R}^n$ is a semialgebraic path such that $\alpha(0) = y$, $\alpha((0,1]) \subset M_u$ and $\nu(t) = (f \circ \alpha)(t)$ for all $t \in (0,1]$. Hence, $\lim_{t\to 0}(f \circ \alpha)(t) = 0$, that is, $f \in \mathfrak{m}^*_\alpha$. This implies that also $g \in \mathfrak{m}^*_\alpha$ or equivalently $\lim_{t\to 0}(g \circ \alpha)(t) = 0$ since $\mathcal{Z}_{\beta^*_s M}(f) \subset \mathcal{Z}_{\beta^*_s M}(g)$. But this is impossible because $|g|_{M_u}| \equiv \frac{1}{u} \in \mathbb{R}$. This proves (3.10.1).

Consider now the non-necessarily continuous function

$$v : \mathbb{R} \to [0, +\infty), \quad u \mapsto v(u) = \begin{cases} 0 & \text{if } M_u = \varnothing, \\ \sup\{1/|f(y)|: \ y \in M_u\} & \text{otherwise} \end{cases}$$

whose graph is semialgebraic. Note that the function v is identically 0 on $(-\infty, 0]$. We claim:

(3.10.2) *The restriction* $v_r = v|_{[0,r]}$ *is bounded for every* $r > 0$.

Indeed, assume by contradiction the existence of $r > 0$ and of a sequence $\{u_m\}_{m\geqslant 1} \subset [0,r]$ such that $v(u_m) > m$ for all $m \geqslant 1$. Thus, by definition of the function v there exists a sequence $\{y_m\}_{m\geqslant 1}$ such that $\frac{1}{|f(y_m)|} > m$ and $y_m \in M_{u_m}$ for all $m \geqslant 1$. Since $\mathrm{Cl}_{\mathbb{R}^n}(M)$ is compact, we may assume that the sequence $\{y_m\}_{m\geqslant 1}$ converges to a point $z \in \mathrm{Cl}_{\mathbb{R}^n}(M)$ and so the sequence $\{(y_m, f(y_m))\}_{m\geqslant 1}$ converges to the point $(z, 0)$. On the other hand, as $[0,r]$ is compact, we may assume that $\{u_m\}_{m\geqslant 1}$ converges to a point $a \in [0,r]$ and $|g(y_m)|u_m = 1$ because $y_m \in M_{u_m}$. Therefore

$$\lim_{m\to+\infty} |g(y_m)| = \lim_{m\to+\infty} \frac{1}{u_m} = \frac{1}{a}.$$

Since g is bounded, a cannot be 0 and so the previous limit is well-defined. Consider next the semialgebraic set

$$T := \{(u, y, f(y)) \in [0,r] \times M \times \mathbb{R}: \ u|g(y)| = 1\}.$$

The points $(u_m, y_m, f(y_m)) \in T$ and so $(a, z, 0) \in \mathrm{Cl}_{\mathbb{R}^n}(T)$. By the Curve Selection Lemma [2, 2.5.5] there exists a semialgebraic path $\varphi := (\rho, \eta, \mu) : [0,1] \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ such that $\varphi(0) = (\rho(0), \eta(0), \mu(0)) = (a, z, 0)$ and

$$\varphi|_{(0,1]} = \left(\frac{1}{|(g \circ \eta)|_{(0,1]}|}, \eta|_{(0,1]}, (f \circ \eta)|_{(0,1]}\right).$$

Therefore $\lim_{t\to 0}(f \circ \eta)(t) = 0$, that is, $f \in \mathfrak{m}^*_\eta$ and so $g \in \mathfrak{m}^*_\eta$ because $\mathcal{Z}_{\beta^*_s M}(f) \subset \mathcal{Z}_{\beta^*_s M}(g)$. This means $\lim_{t\to 0}(g \circ \eta)(t) = 0$, which is impossible, because

$$\lim_{t\to 0} \frac{1}{|(g \circ \eta)(t)|_{(0,1]}|} = a \in \mathbb{R}.$$

This proves (3.10.2).

(3.10.3) On the other hand, by [2, 2.6.1] there exist $c, s \in \mathbb{R}$ and a positive integer $p \geqslant 1$ such that $v(u) \leqslant cu^p$ for every u such that $|u| \geqslant s$. Additionally, as we have just seen, there exists $L > 0$ such that $0 \leqslant v|_{[-s,s]} \leqslant L$. Now let us prove that the function

$$h_1 : M \to \mathbb{R}, \quad y \mapsto h_1(y) := \begin{cases} g^p(y)/f(y) & \text{if } y \in D_M(f), \\ 0 & \text{if } y \in Z_M(f) \end{cases}$$

is bounded. Of course, it is enough to check that h_1 is bounded on $D_M(f)$. Let $y_0 \in D_M(f)$. If $g(y_0) = 0$, then $h_1(y_0) = 0$. Thus, we may assume $g(y_0) \neq 0$ and denote $u_0 := \frac{1}{|g(y_0)|}$. Suppose first that $|g(y_0)| \leqslant \frac{1}{s}$ or equivalently $u_0 \geqslant s$. Then

$$\left|\frac{g^p(y_0)}{f(y_0)}\right| \leqslant \frac{1}{u_0^p} \sup\{1/|f(y)|\colon\ y \in M_{u_0}\} = \frac{v(u_0)}{u_0^p} \leqslant c.$$

Suppose now $|g(y_0)| > \frac{1}{s}$, that is, $u_0 < s$. Then

$$\left|\frac{g^p(y_0)}{f(y_0)}\right| \leqslant |g(y_0)|^p \sup\left\{\frac{1}{|f(y)|}\colon\ y \in M_{u_0}\right\} = |g(y_0)|^p v(u_0) \leqslant \sup\{|g(y)|^p\colon\ y \in M\} \cdot L.$$

Since g is bounded, we conclude that also h_1 is bounded. Therefore we obtain by Lemma 3.5 that $h = gh_1 \in \mathcal{S}^*(M)$ because $Z_M(f) \subset Z_M(g)$ as $\mathcal{Z}_{\beta_s^* M}(f) \subset \mathcal{Z}_{\beta_s^* M}(g)$. If $\ell := p + 1$, we get $g^\ell = fgh_1 = fh$. $\square$

Remark 3.12. The proof above shows that in order to obtain an equality of the form $g^\ell = fh$, it is sufficient to require $g \in \mathfrak{m}_\alpha^*$ for each semialgebraic path $\alpha : (0, 1] \to M$ such that $f \in \mathfrak{m}_\alpha^*$.

We finish this work with an alternative proof of Theorem 1.1 obtained as an almost straightforward consequence of Theorem 3.10 and Remark 3.12. Namely,

Alternative proof of Theorem 1.1. Since M is locally compact, it is locally closed (see Section 2.3) and by [2, 2.2.9] M can be embedded in some $\mathbb{R}^m$ as a closed semialgebraic subset. Thus, we assume in the following that $M \subset \mathbb{R}^n$ is a closed semialgebraic subset of $\mathbb{R}^n$. Let $f, g \in \mathcal{S}(M)$ be such that $Z_M(f) \subset Z_M(g)$ and consider the bounded semialgebraic functions on M

$$f_1 := \frac{f}{(1 + \|x\|)(1 + |f|)} \in \mathcal{S}^*(M) \quad \text{and} \quad g_1 := \frac{g}{(1 + \|x\|)(1 + |g|)} \in \mathcal{S}^*(M).$$

Taking Remark 3.12 into account, it is enough to check that $g_1 \in \mathfrak{m}_\alpha^*$ for each semialgebraic path $\alpha : (0, 1] \to M$ such that $f_1 \in \mathfrak{m}_\alpha^*$ in order to apply Theorem 3.10 to f_1 and g_1. Indeed, let $\alpha : (0, 1] \to M$ be a semialgebraic path such that $f_1 \in \mathfrak{m}_\alpha^*$. If $\mathfrak{m}_\alpha^*$ is a fixed maximal ideal of $\mathcal{S}^*(M)$, there exists a point $p \in M$ such that $\mathfrak{m}_\alpha^* = \mathfrak{m}_p^*$ and

$$0 = f_1(p) = \frac{f(p)}{(1 + \|p\|)(1 + |f(p)|)}$$

because $f_1 \in \mathfrak{m}_p^*$. Hence, $f(p) = 0$ and so $g(p) = 0$ because $Z_M(f) \subset Z_M(g)$. Thus,

$$g_1(p) = \frac{g(p)}{(1 + \|p\|)(1 + |g(p)|)} = 0,$$

that is, $g_1 \in \mathfrak{m}_p^* = \mathfrak{m}_\alpha^*$.

If $\mathfrak{m}_\alpha^*$ is a free ideal, the semialgebraic path $\alpha : (0, 1] \mapsto M$ cannot be extended to a continuous semialgebraic path $[0, 1] \mapsto M$ (see (2.2.1)). Since M is closed in $\mathbb{R}^n$, this implies that α cannot be extended to a semialgebraic path $[0, 1] \mapsto \mathbb{R}^n$. Thus, by Lemma 2.5 the semialgebraic function $\|\alpha\| : (0, 1] \to \mathbb{R}$ is unbounded. On the other hand, as the semialgebraic function $\frac{1}{1+\|\alpha\|} : (0, 1] \to \mathbb{R}$ is bounded, the limit $\lim_{t \to 0} \frac{1}{1+\|\alpha\|} = c \in \mathbb{R}$ exists by Lemma 2.5. In fact, $c = 0$ because $\|\alpha\| : (0, 1] \to \mathbb{R}$ is unbounded. Thus, using Lemma 2.5 once more,

$$\lim_{t\to 0}(g_1\circ\alpha)(t)=\lim_{t\to 0}\left(\frac{1}{1+\|\alpha(t)\|}\right)\left(\frac{g(\alpha(t))}{1+|g(\alpha(t))|}\right)=0.$$

Therefore, also $g_1 \in \mathfrak{m}^*_\alpha$ and by Theorem 3.10 and Remark 3.12 there exist $h_1 \in \mathcal{S}^*(M)$ and a positive integer $\ell \geqslant 1$ such that $g_1^\ell = f_1 h_1$. Hence, $g^\ell = fh$ where

$$h := h_1 \frac{(1+\|x\|)^{\ell-1}(1+|g|)^\ell}{1+|f|} \in \mathcal{S}(M)$$

and we are done. □

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