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José F. Fernando · J. M. Gamboa · Jesús M. Ruiz

Finiteness problems on Nash manifolds and Nash sets

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Abstract. We study here several *finiteness* problems concerning *affine* Nash manifolds M and Nash subsets X. Three main results are: (i) A Nash function on a semialgebraic subset Z of M has a Nash extension to an open *semialgebraic* neighborhood of Z in M. (ii) A Nash set X that has only normal crossings in M can be covered by *finitely many* open semialgebraic sets U equipped with Nash diffeomorphisms $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that $U \cap X = \{u_1 \cdots u_r = 0\}$. (iii) Every affine Nash manifold N with corners is a closed subset of an affine Nash manifold M where the Nash closure of the boundary ∂N of N has only normal crossings and N can be covered with *finitely many* open semialgebraic sets U such that each intersection $N \cap U$ is of the form $\{u_1 \geq 0, \dots, u_r \geq 0\}$ for a Nash diffeomorphism $(u_1, \dots, u_m) : U \to \mathbb{R}^m$.

Keywords. Finiteness, Nash functions and Nash sets, semialgebraic sets, Nash manifolds with corners, extension, normal crossings at a point, normal crossings divisor

1. Introduction and statements of the main results

This work is devoted to several questions concerning semialgebraic and Nash sets, Nash manifolds and Nash functions. In a way those questions always refer to the comparison of the Euclidean and the *semialgebraic topology* ([DK2], [K]). The latter is not a true topology, and as we will explain soon the word that always appears in this connection is *finiteness*. Here we present the framework within which the problems arise, formulate them rigorously and state our main results. The proofs will be developed in the following sections.

A subset $Z \subset \mathbb{R}^n$ is *semialgebraic* when it has a description by a finite boolean combination of polynomial equations and inequalities, which we will call a *semialgebraic* description. A (not necessarily continuous) function $f : S \to T$ is *semialgebraic* if its graph is a semialgebraic set (in particular S and T are semialgebraic). Among semialgebraic objects, we will focus on affine Nash manifolds and Nash functions. We present

J. F. Fernando, J. M. Gamboa: Departamento de Álgebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain;
e-mail: josefer@mat.ucm.es, jmgamboa@mat.ucm.es

J. M. Ruiz: Departamento de Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain; e-mail: jesusr@mat.ucm.es

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a careful and detailed study of these types of objects and their main properties in Section 2; however, we anticipate here some definitions to make clearer the statements of our main results.

Definition 1.1. An *affine Nash manifold* is a pure dimensional semialgebraic subset M of some affine space $\mathbb{R}^n$ that is a smooth submanifold of an open subset of $\mathbb{R}^n$. A *Nash function* on an open semialgebraic set $U \subset M$ is a semialgebraic smooth function on U. A *Nash subset* of U is the zero set of a Nash function on U.

It is relevant to remark here the following. A smooth submanifold M of $\mathbb{R}^n$ is a locally compact set, hence M is open in its closure $\overline{M}$. Thus, when M is semialgebraic, the set $C = \overline{M} \setminus M$ is a closed semialgebraic subset of $\mathbb{R}^n$. Consequently, M is a *closed submanifold of the open semialgebraic set* $\mathbb{R}^n \setminus C$.

We can determine which semialgebraic sets are contained in affine Nash manifolds of the same dimension. First, recall that a point x of a semialgebraic set Z of dimension m is *regular* if the germ Z_x is the germ of an affine Nash manifold of dimension m. We start Section 3 with the proof of the following fact:

Proposition 1.2. *Let $Z \subset \mathbb{R}^n$ be a locally compact semialgebraic set such that for each point $x \in Z$ the analytic closure $\overline{Z}_x^{\mathrm{an}}$ of the germ Z_x is regular of constant dimension m. Then Z is a closed subset of an affine Nash manifold $M \subset \mathbb{R}^n$ of dimension m.*

Let M be an affine Nash manifold. Since open semialgebraic sets form a basis of the Euclidean topology, we have the *sheaf of germs of Nash functions $\mathcal{N}_M$ on M*. The ring $\mathcal{N}(M)$ of global Nash functions on M is then the ring of global cross-sections of $\mathcal{N}_M$.

Note that we actually have Nash functions on open *not necessarily semialgebraic* subsets of M, and most problems require understanding which data associated to $\mathcal{N}_M$ are in fact semialgebraic. In particular, if Z is a semialgebraic subset of M, then a Nash function on Z is a cross-section of $\mathcal{N}_M$ over Z, which according to general sheaf theory, is given by a Nash function $f : U \to \mathbb{R}$ defined on some open *not necessarily semialgebraic* neighborhood of Z. In Section 3 we will prove that this U can be chosen semialgebraic:

Theorem 1.3. *Let M be an affine Nash manifold and let Z be a semialgebraic subset of M. Let $f : Z \to \mathbb{R}$ be a Nash function. Then f has a Nash extension defined on an open semialgebraic neighborhood U of Z.*

In this vein, we see how the study of Nash functions depends heavily on the so-called *semialgebraic topology,* defined by choosing as open sets the open semialgebraic sets. Although it is a fake topology, because unions of open semialgebraic sets may not be semialgebraic, it is the source of a deeper and better understanding of many questions. As mentioned before, the problem here is *finiteness,* that is, the possibility of *finite* descriptions for properties of local nature in the Euclidean topology. We will come back to this in several places.

In this paper we also pay special attention to Nash normal crossings. This has two different aspects. First, the local notion:

Definition 1.4. Let X be a Nash subset of an affine Nash manifold M. We say that X *has only normal crossings at a point* $x \in X$ if there are analytic local coordinates $u = (u_1, \dots, u_m)$ of M at x such that $X_x = \{u_1 \cdots u_r = 0\}$ for some r. The number r is the *multiplicity* of X at x, denoted $\operatorname{mult}(X, x)$. We say that X *has only normal crossings in* $U \subset M$ if it has only normal crossings at all $x \in U$.

The first result of Section 4 will be that the points at which a Nash set has only normal crossings form a semialgebraic set:

Proposition 1.5. *Let X be a Nash subset of an affine Nash manifold M. Then*

$$U = \{x \in M : \textit{either } x \notin X \textit{ or } X \textit{ has only normal crossings at } x\}$$

is an open semialgebraic subset of M.

This is a natural statement, but a delicate matter that involves deep results like *M. Artin's approximation* [Ar] and *semialgebraic triviality*. Next, also in Section 4, we prove a finiteness result for this local normal crossings notion:

Theorem 1.6. *Let X be a Nash subset of an affine Nash manifold M. Suppose that X has only normal crossings in M. Then X can be covered by finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that $U \cap X = \{u_1 \cdots u_r = 0\}$.*

In fact we will prove a version of this for a larger class of semialgebraic sets. We need that version to deduce:

Theorem 1.7. *Let $Z \subset M$ be a locally compact semialgebraic subset of an affine Nash manifold M. Suppose that for every point $x \in Z$ there is an integer r and a coordinate system $(u_1, \dots, u_m)$ of M at x such that $\overline{Z}^{\mathrm{an}}_y = \{u_1 \cdots u_r = 0\}_y$ for every $y \in Z$ close enough to x. Then there is an open semialgebraic neighborhood Ω of Z in M and a Nash subset X of Ω which has only normal crossings in Ω such that $X_x = \overline{Z}^{\mathrm{an}}_x$ for every $x \in Z$. In particular, the Nash closure X of Z in Ω satisfies the desired conditions.*

Proposition 1.2 for $m = n - 1$ is a particular case of this result, as non-singular hypersurfaces can be seen as the simplest non-singular normal crossings. Next, we have the global concept of normal crossings:

Definition 1.8. A *Nash normal crossings divisor of M* is a Nash subset $X \subset M$ whose irreducible components are non-singular hypersurfaces $X_1, \dots, X_p$ of M in general position. This means that at every point $x \in X_{i_1}, \dots, X_{i_r}$ with $x \notin X_i$ for $i \neq i_k$, the tangent hyperplanes $T_x X_{i_1}, \dots, T_x X_{i_r}$ are linearly independent in the tangent space $T_x M$.

Examples 1.9. It is important to compare this global notion and the preceding local one. The following examples might be clarifying.

(1) The algebraic set $X = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2(1 + x)\} \subset M = \mathbb{R}^2$ is an irreducible Nash set. Since it is singular at the origin, it is not a normal crossings divisor of $\mathbb{R}^2$. However, X has only normal crossings at all points of $\mathbb{R}^2$.

(2) Let $a = (-1, 0)$ and $Y = X \setminus \{a\} \subset M_1 = \{x > -1\}$. The Nash irreducible components of Y in M_1 are the non-singular hypersurfaces

$$Y_1 = \{(x, y) \in M_1 : y = x\sqrt{1+x}\} \quad \text{and} \quad Y_2 = \{(x, y) \in M_2 : y = -x\sqrt{1+x}\},$$

which meet transversally at the origin; hence, Y is a Nash normal crossings divisor of the Nash manifold M_1.

Of course, it is clear that a normal crossing divisor has only normal crossings, but it is also clear that there is something more to it. In Section 5 we will prove a result concerning regular systems of parameters in the ring of global Nash functions, and then deduce which Nash sets with only normal crossings are normal crossings divisors.

The last section, Section 6, is devoted to affine Nash manifolds with corners, which are closely related to normal crossings. The basic definitions run as in the case of affine Nash manifolds. Our general reference for the smooth theory is [MO], where a comprehensive presentation is offered.

Definition 1.10. An *affine Nash manifold with corners* is a pure dimensional semialgebraic subset N of some affine space $\mathbb{R}^n$ that is a smooth submanifold with corners of an open subset of $\mathbb{R}^n$. A *Nash function* on an open semialgebraic set $U \subset N$ is a semialgebraic smooth function on U. The *boundary* ∂N of N is the set of points $x \in N$ at which the germ N_x is not regular.

As remarked after Definition 1.1, an affine Nash manifold with corners is also a *closed submanifold with corners of an open semialgebraic subset of* $\mathbb{R}^n$.

Taking into acount that for each point x of an affine Nash manifold N with corners the germ $\overline{N}_x^{\text{an}}$ is regular of dimension m, we can apply Proposition 1.2 and embed N in an affine Nash manifold. However, we are able to prove more, including finiteness:

Theorem 1.11. *Let $N \subset \mathbb{R}^n$ be an affine Nash manifold with corners. Then N is a closed subset of an affine Nash manifold $M \subset \mathbb{R}^n$ of the same dimension, say m, in such a way that:*

(i) *The Nash closure X of ∂N in M has only normal crossings in M and $N \cap X = \partial N$.*
(ii) *For every $x \in \partial N$ the analytic closure of the germ ∂N_x is X_x.*
(iii) *M can be covered with finitely many open semialgebraic subsets U equipped with Nash diffeomorphisms $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that*

$$\begin{cases} U \subset N \text{ or } U \cap N = \emptyset & \text{if } U \text{ does not meet } \partial N, \\ U \cap N = \{u_1 \geq 0, \dots, u_r \geq 0\} \text{ for a suitable } r \geq 1 & \text{if } U \text{ meets } \partial N. \end{cases}$$

Then, we will characterize affine Nash manifolds with corners for which there is an embedding as above where X is a Nash normal crossings divisor. For this we need another definition: a *face* of an affine Nash manifold N with corners is the closure in N of a connected component of $\operatorname{Reg}(\partial N)$, $\dim(N) = m$. Since ∂N is semialgebraic, the faces are semialgebraic and finitely many. We also remark that the Nash closure of each face is irreducible, hence the Nash closures of the faces are the irreducible components of the Nash closure of the boundary. Note also the corresponding local fact. Since every pair of germs $N_x \subset \overline{N}_x^{\mathrm{an}}$ is like a pair $\{x_1 \geq 0, \ldots, x_r \geq 0\}_0 \subset \mathbb{R}^m_0$, the irreducible components of $\overline{\partial N}_x^{\mathrm{an}}$ are the analytic closures of the connected components of $\operatorname{Reg}(\partial N)_x$, which are all different. After this preparation, the result we will prove is the following:

Theorem 1.12. *Let $N \subset \mathbb{R}^n$ be an affine Nash manifold with corners,* $\dim(N) = m$*. The following assertions are equivalent:*

(i) *N is contained in an affine Nash manifold $M \subset \mathbb{R}^n$ where the Nash closure of ∂N is a Nash normal crossings divisor.*
(ii) *Every face D of N is contained in an affine Nash manifold $X \subset \mathbb{R}^n$ of dimension $m - 1$.*
(iii) *The number of faces of N that contain every given point $x \in \partial N$ coincides with the number of connected components of the germ* $\operatorname{Reg}(\partial N)_x$.
(iv) *Every face of N is an affine Nash manifold with corners.*

If that is the case, the manifold M in (i) *can be chosen such that* the Nash closure in M of every face D meets N exactly along D.

The proofs of Theorems 1.11 and 1.12 depend heavily on all our previous results. We also prove in Section 6 that if the faces of an affine Nash manifold with corners are again affine Nash manifolds with corners, then so are the faces of the faces, the faces of the faces of the faces, and so on.

2. Preliminaries

The purpose of this section is to provide a careful analysis of the objects that we study along this work. Our main reference for all that follows is [BCR]. We also introduce set germs into our discussion. For analytic and semianalytic sets germs we refer to [N] and [Ł]. And to complete now general references, let us mention [Mt] for commutative algebra.

(2.1) Generalities on semialgebraic sets and affine Nash manifolds. By elimination of quantifiers, a subset $Z \subset \mathbb{R}^n$ is semialgebraic if it has a description by a first order formula possibly with quantifiers. Such freedom gives easy semialgebraic descriptions for topological operations: the interiors, closures and borders of semialgebraic sets are again semialgebraic. Recall that semialgebraicity is preserved by linear maps and even by semialgebraic maps; in fact, elimination of quantifiers is equivalent to the fact that linear projections of semialgebraic sets are again semialgebraic. Concerning topological conditions, we recall that $Z \subset \mathbb{R}^n$ is *locally compact* if and only if it is open in its closure, or equivalently, when it is the intersection of a closed set F and an open set U; if Z is semialgebraic, F and U can be chosen semialgebraic, namely $F = \overline{Z}$ and $U = \mathbb{R}^n \setminus (\overline{Z} \setminus Z)$. In what follows, $\overline{Z}$ stands for the closure of Z and $\operatorname{Int}(Z)$ for the interior of Z (in the space we are working in). In case we must clarify the space where closures or interiors are taken, we will use a subscript to specify that space.

(2.1.1) As is well known, by means of the semialgebraic version of the Tietze–Urysohn extension lemma (see [DK1]), two disjoint closed semialgebraic subsets C_1, C_2 of a semialgebraic set $Z \subset \mathbb{R}^n$ can be separated by disjoint open semialgebraic subsets U_1, U_2 of Z. This fact will be used freely along this work and it is useful, for instance, to separate the connected components of a semialgebraic set or to prove the following statement concerning refinements:

(2.1.2) *Let $Z \subset \mathbb{R}^n$ be a semialgebraic set and let $Z = \bigcup_{i=1}^p Z_i$ be an open semialgebraic covering of Z. Then for each $\ell = 1, \dots, p$ there is an open semialgebraic subset $Z_i' \subset Z_i$ of Z such that $\overline{Z_i'} \subset Z_i$ and $Z = \bigcup_{i=1}^p Z_i'$.*

Indeed, suppose we have already constructed open semialgebraic subsets $Z_1', \dots, Z_{r-1}'$ of Z (maybe none) such that $\overline{Z_i'} \subset Z_i$ and $\bigcup_{j=1}^{r-1} Z_j' \cup \bigcup_{k=r}^p Z_k = Z$. Consider the closed semialgebraic subset $C_r = Z \setminus (\bigcup_{j=1}^{r-1} Z_j' \cup \bigcup_{k=r+1}^p Z_k)$ of Z which is contained in the open semialgebraic subset Z_r of Z. By (2.1.1), there is an open semialgebraic subset Z_r' of Z such that $C_r \subset Z_r' \subset \overline{Z_r'} \subset Z_r$. Proceeding in this way up to $r = p$, we are done.

(2.1.3) The *dimension* $\dim(Z)$ of a semialgebraic set Z is the dimension of its *algebraic Zariski closure;* set $d = \dim(Z)$. The *local dimension* $\dim(Z_x)$ *of Z at a point $x \in Z$* is the dimension $\dim(U)$ of a small enough open semialgebraic neighborhood U of x in Z. The dimension d of Z coincides with the maximum of those local dimensions. For any fixed k, the set of points $x \in Z$ such that $\dim(Z_x) = k$ is semialgebraic.

(2.1.4) Next, recall that a semialgebraic subset $M \subset \mathbb{R}^n$ is an affine Nash manifold (of dimension m) if and only if every point $x \in M$ has an open neighborhood U in $\mathbb{R}^n$ equipped with a Nash diffeomorphism $(u_1, \dots, u_n) : U \to \mathbb{R}^n$ that maps x to the origin and such that $V = U \cap M = \{u_{m+1} = 0, \dots, u_n = 0\}$. The restriction map $(u_1, \dots, u_m)|_V : V \to \mathbb{R}^m$ is a Nash diffeomorphism, and we say that $(u_1, \dots, u_m)$ are *coordinates of M at x*. In fact, as is well known, M can be covered with finitely many semialgebraic domains $V \subset M$ of such Nash diffeomorphisms $(u_1, \dots, u_n) : V \to \mathbb{R}^m$; this is a typical *finiteness* result that will be used freely along this work and that we include in Lemma 2.2 to illustrate the type of "tricks" and techniques we use in this paper. As is well known, semialgebraic smooth functions are in fact analytic, so that affine Nash manifolds are analytic manifolds. Again we refer here to [BCR, §8].

(2.1.5) As we have already pointed out, a point x of a d-dimensional semialgebraic set $Z \subset \mathbb{R}^n$ is *regular* if the germ Z_x is the germ of an affine Nash manifold of dimension d. The set $\mathrm{Reg}(Z)$ of regular poins of Z is a non-empty open semialgebraic subset of Z, whose complement $Z \setminus \mathrm{Reg}(Z)$ is a closed semialgebraic subset of Z of dimension $\le d-1$. If the local dimension of Z at x is d, then x is the limit of a sequence of regular points of Z.

(2.1.6) Moreover, a semialgebraic set $N \subset \mathbb{R}^n$ is an affine Nash manifold with corners (of dimension m) if and only if every point $x \in N$ has an open neighborhood U in $\mathbb{R}^n$ equipped with a Nash diffeomorphism $(u_1, \dots, u_n) : U \to \mathbb{R}^n$ that maps x to the origin such that $U \cap N = \{u_1 \ge 0, \dots, u_r \ge 0, u_{m+1} = 0, \dots, u_n = 0\}$ for some $0 \le r \le m$.

Since $(u_1, \dots, u_n)$ are coordinates at $x \in N$, the germ $\overline{N}^{\text{an}}_x = \{u_{m+1} = 0, \dots, u_n = 0\}_x$ is regular of dimension m. The *boundary* ∂N of N is the set of points with $r \geq 1$, that is, the set of those points of N for which the germ N_x is not regular. Properly speaking, *corners* are the points with $r > 1$; if there are no corners, N is an *affine Nash manifold with boundary*. Clearly, $N \setminus \partial N = \operatorname{Reg}(N)$, which is semialgebraic, and so the boundary is semialgebraic too. Of course, affine Nash manifolds with corners are analytic manifolds with corners.

Lemma 2.2. *Let $M \subset \mathbb{R}^n$ be an affine Nash manifold of dimension m. Then there is a finite open* (*semialgebraic*) *covering $M = \bigcup_i M_i$ by affine Nash manifolds $M_i \subset M$, each Nash diffeomorphic to $\mathbb{R}^m$.*

Proof. First we may assume, by [BCR, 9.3.10], that M is an open subset of $\mathbb{R}^m$. Since, M is, by [BCR, 2.9.10], a finite disjoint union of Nash submanifolds, each Nash diffeomorphic to some Euclidean space, it is enough to prove the following statement and to proceed recursively afterwards.

(2.2.1) *Let $S \subset M$ be a semialgebraic set. Then there exist finitely many open semialgebraic sets $U_j \subset M$, each Nash diffeomorphic to $\mathbb{R}^m$, such that* $\dim(S \setminus \bigcup_j U_j) < \dim S = d$.

Indeed, since $\dim(S \setminus \operatorname{Reg}(S)) < \dim S$, we may assume that S is an affine Nash manifold. Moreover, by [BCR, 9.3.10] once more, we may assume that there are Nash functions $h_{d+1}, \dots, h_m : V \to \mathbb{R}$ defined on an open semialgebraic neighborhood V of S in M such that $S = \{h_{d+1} = 0, \dots, h_m = 0\}$ and $\operatorname{rk}(h_{d+1}, \dots, h_m) = m - d$ everywhere. In fact, again by [BCR, 2.9.10], we may assume that S is Nash diffeomorphic to $\mathbb{R}^d$ via a Nash diffeomorphism $\psi : S \to \mathbb{R}^d$. Next, by [BCR, 8.9.2–4], there exist an open (tubular) semialgebraic neighborhood $U \subset V$ of S and a strictly positive semialgebraic function $\alpha : S \to \mathbb{R}$, which we may assume morever Nash after approximation, such that the map

$$\Phi : W = \{(x, z) \in S \times \mathbb{R}^{m-d} : \|z\| < \alpha(x)\} \to U, \qquad (x, z) \mapsto x + \sum_{j=d+1}^{m} z_j \operatorname{grad}(h_j)(x),$$

is a Nash diffeomorphism. Finally W is Nash diffeomorphic to $\mathbb{R}^m$ via the Nash diffeomorphism

$$\Psi : W \to \mathbb{R}^m, \qquad (x, z) \mapsto \left(\psi(x), \frac{z}{\sqrt{\alpha(x)^2 - \|z\|^2}}\right),$$

and we are done. □

A basic topological fact is that semialgebraic sets can always be triangulated. It enters in an essential way in the proofs of Proposition 1.2 and Theorem 1.3. We summarize here the notation. Once again, we refer to [BCR, §9]. One fundamental consequence of the existence of triangulations is the so-called *triviality of semialgebraic functions,* which we will use in Section 4. As usual, we refer to [BCR, 9.3.1].

(2.3) Triangulations of semialgebraic sets. Let $Z \subset \mathbb{R}^n$ be a semialgebraic set and let $T_1, \ldots, T_q$ be semialgebraic subsets of Z.

Suppose first that Z is compact. Then there is a finite simplicial complex K and a semialgebraic homeomorphism $\Phi : K \to Z$ such that each preimage $\Phi^{-1}(T_i)$ is the union of some open simplices of K; furthermore, we may assume that the restriction of Φ to every open simplex of K is a Nash embedding [BCR, 9.2.3]. We say that (K, Φ) is a *triangulation of Z compatible with $T_1, \ldots, T_q$*.

If Z is not compact, we can consider any semialgebraic compactification Z^* of Z, that is, Z^* is a compact semialgebraic set and Z is a dense subset of Z^*. Then choose a triangulation of Z^* compatible with $T_1, \ldots, T_q$ and with the remainder $Z^* \setminus Z$; then we obtain a "triangulation" of Z compatible with $T_1, \ldots, T_q$ by dropping the simplices contained in the remainder. In this case, if Z is locally compact, we can consider its one-point compactification, so that the remainder reduces to the "infinite point", which is a vertex of the triangulation, and the only thing we drop when returning to Z. For this reason, whenever we stress Z to be locally compact, it is to recall triangulations are made via the one-point compactification.

The closed simplices of K are denoted as usual by σ, and the corresponding open simplices by σ^0. Recall that if τ, σ are two simplices of K, then the corresponding open simplices τ^0, σ^0 are either disjoint or they coincide. Since the restrictions $\Phi|_{\sigma^0}$ are Nash embeddings, their images $\Gamma = \Phi(\sigma^0)$ are affine Nash manifolds, Nash diffeomorphic to $\mathbb{R}^d$ with $d = \dim(\sigma)$. Actually, we will mainly deal with these Γ's, which we call *strata*. These strata form a partition $\mathcal{G}$ of Z called a *stratification compatible with $T_1, \ldots, T_q$*. In particular, every semialgebraic set splits into a finite pairwise disjoint union of affine Nash manifolds.

(2.3.1) Moreover, since the closure in Z of a stratum of $\mathcal{G}$ is a finite union of strata of $\mathcal{G}$, one deduces that if Σ, Γ are strata of $\mathcal{G}$, then either $\Sigma \subset \overline{\Gamma}$ or $\Sigma \cap \overline{\Gamma} = \emptyset$. Therefore, if $\mathcal{G}_\Sigma = \{\Gamma \in \mathcal{G} : \Sigma \subset \overline{\Gamma}\}$, then the semialgebraic set $\bigcup_{\Gamma \in \mathcal{G}_\Sigma} \Gamma$ is an open semialgebraic neighborhood of Σ in Z; this fact will be used throughout.

(2.3.2) We write down for further reference a few connectedness properties.

(1) Every stratum Γ is *connected at every point* $x \in \overline{\Gamma}$, that is, x has a basis of neighborhoods V in Z with connected intersection $V \cap \Gamma$. This, together with the fact that Γ is an affine Nash manifold, implies that the analytic closure $\overline{\Gamma}_x^{\text{an}}$ of the germ Γ_x is an irreducible analytic germ of dimension $\dim(\Gamma)$.

(2) Suppose the stratum Γ is *adherent* to another stratum Σ, that is, $\Sigma \subset \overline{\Gamma}$. Then Σ has a basis of neighborhoods V in Z with connected intersection $V \cap \Gamma$. This is clear if we think that topologically Σ is included in Γ as an open face of an open simplex. Following the terminology in (1), we say that Γ is *connected at Σ*.

(3) Suppose that Z is locally compact and it is triangulated via the one-point compactification. Then, for any two strata Σ, Γ the intersection $\overline{\Sigma} \cap \overline{\Gamma}$ is connected. Indeed, this just says that in a triangulation a non-empty intersection of two simplices is a face of both, hence connected. Even if we must drop the remainder, this is just a vertex, and it causes no trouble. □

(2.4) Piecewise Nash functions. Semialgebraic functions, even if not necessarily continuous, are not far from being Nash functions; in fact, they are Nash after subdivision. We make this idea rigorous through two statements that will be needed later.

(2.4.1) *Let $f : M \to \mathbb{R}$ be a semialgebraic function defined on an affine Nash manifold M. Then the set U of points $x \in M$ at which f is Nash is open semialgebraic. Furthermore, $M \setminus U$ has codimension ≥ 1.*

That U is open is clear. Now, let $G \subset M \times \mathbb{R}$ be the graph of f, which is a semialgebraic set, and consider $R = \operatorname{Reg}(G)$, which is an affine Nash manifold of the same dimension as M. Let $T \subset M$ denote the set of critical values of the Nash map $\pi|_R : R \to M$, which is a semialgebraic set by the semialgebraic Sard theorem [BCR, 9.6.2]. The function f is Nash at the point $x \in M$ if and only if $(x, f(x)) \in R$ and x is a regular value of the Nash map $\pi|_R : R \to M$. Consequently, $U = \pi(R) \setminus T$ is a semialgebraic set. We also get the assertion on the codimension of the non-Nash points. Indeed, since $\pi|_G : G \to M$ is bijective, $(\pi|_G)^{-1}(U) = R \setminus (\pi|_G)^{-1}(T)$ and $M \setminus U = M \setminus (\pi(R) \setminus T) = \pi(G \setminus R) \cup T$; now

$$\begin{cases} \dim(\pi(G \setminus R)) = \dim(G \setminus R) < \dim(G) = \dim(M) \quad \text{(by (2.1.3))}, \\ \dim(T) < \dim(M) \quad \text{(by the Sard theorem)}, \end{cases}$$

and so $\dim(M \setminus U) < \dim(M)$.

From the preceding result and the fact that every semialgebraic set is a finite union of affine Nash manifolds, we readily deduce:

(2.4.2) *Let $f : S \to \mathbb{R}$ be a semialgebraic function. Then there is a finite partition $S = \bigcup_i S_i$ into affine Nash manifolds S_i such that all restrictions $f|_{S_i} : S_i \to \mathbb{R}$ are Nash functions.* □

One main feature of Nash functions is how close they are to polynomials.

(2.5) Algebraicity of Nash functions. First of all, Nash functions are algebraic over the polynomials. An analytic function $f : U \to \mathbb{R}$ defined on an open connected subset U of a connected affine Nash manifold $M \subset \mathbb{R}^n$ is Nash if and only if there is a polynomial $P(\mathtt{x}, \mathtt{t}) \in \mathbb{R}[\mathtt{x}, \mathtt{t}] = \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{t}]$ which is not identically zero on M such that $P(x, f(x)) = 0$ for all $x \in U$. This describes $f(x)$ as one of the real roots of the polynomial $P(x, \mathtt{t})$ for x off the set $\Pi \subset M$ where all coefficients of P vanish simultaneously. This set Π is semialgebraic of dimension $< \dim(M)$.

But on the other hand, we have the so-called *Artin–Mazur description* [BCR, 8.4.4], which we formulate as follows: *Given an affine Nash manifold M and finitely many Nash functions $f_1, \dots, f_r : M \to \mathbb{R}$, there are a non-singular real algebraic set $V \subset \mathbb{R}^p$ and a Nash diffeomorphism $\varphi : M' \to M$ from an open and closed semialgebraic subset M' of V such that $f_1 \circ \varphi, \dots, f_r \circ \varphi$ are the restrictions to M' of polynomial functions on V.*

Let us next turn to Nash sets (see [BCR, §8]).

(2.6) Nash sets. Let M be an affine Nash manifold, and $\mathcal{N}(M)$ its ring of global Nash functions. This ring is noetherian, and there is a satisfactory theory for its ideals and the associated zero sets. The *zero set* $\mathcal{Z}(I)$ of an ideal $I \subset \mathcal{N}(M)$ is the set $X \subset M$ of points at which all functions in I vanish; we say that X is a *Nash subset* of M. As I is finitely generated, X is the zero set of finitely many global Nash functions $f_1, \dots, f_r$ and in fact X is the zero set of one Nash function $f = f_1^2 + \cdots + f_r^2$ in M (see Definition 1.1). Of course, every Nash set X is a semialgebraic set. The first examples of Nash subsets are (closed) Nash submanifolds of M [Sh, II.5.4].

The *Nash ideal* of a set $Z \subset M$ is the ideal $\mathcal{J}_{\mathcal{N}}(Z)$ of all Nash functions vanishing on Z; the Nash set $X = \mathcal{Z}(\mathcal{J}_{\mathcal{N}}(Z))$ is the *Nash closure* of Z in M, and is the smallest Nash subset of M containing Z. If Z is a semialgebraic set, both X and Z have the same dimension, which coincides with the Krull dimension of the ring $\mathcal{N}(M)/\mathcal{J}_{\mathcal{N}}(Z) = \mathcal{N}(M)/\mathcal{J}_{\mathcal{N}}(X)$. The ideal of a point $x \in M$ is a maximal ideal of $\mathcal{N}(M)$, usually denoted by $\mathfrak{m}_x$.

Of course, Nash subsets of M have finite decompositions into *Nash irreducible components,* which are the zero sets of the associated primes of their ideals in the ring $\mathcal{N}(M)$.

The theory of Nash sets relies on the properties of the so-called *finite Nash sheaves,* which is related to the finiteness previously mentioned. These sheaves have a good *coherence* behavior [CRSh2].

There is an algebraic notion of regularity for Nash sets that be considered. Namely, a point x of a Nash set X is *regular in the algebraic sense* if the localization of the quotient $\mathcal{N}(M)/\mathcal{J}_{\mathcal{N}}(X)$ at the maximal ideal of x is a regular ring. We remark for the record that this notion only depends on X. The set of algebraically regular points is a non-empty open semialgebraic subset of X, dense in the set of points of maximum dimension of X (the Artin–Mazur description reduces this to the corresponding well known fact for real algebraic varieties). An algebraically regular point is regular (of dimension $\dim(X)$) as defined in (2.1.5). The converse is not true, and it is a difficult matter (see [G]). To illustrate this fact, consider for instance the Nash set $X = \{xz(x^2 + y^2) - y^4 = 0\}$ and pick a point $x = (0, 0, a)$ with $a \neq 0$. Then the ring $(\mathcal{N}(M)/\mathcal{J}_{\mathcal{N}}(X))_{\mathfrak{m}_x}$ is not regular, while the germ X_x is regular. To check this last fact, consider the parametrizations

$$\varphi_\varepsilon : \{t > 0\} \to X \cap \{\varepsilon z > 0\}, \qquad (s, t) \mapsto \varepsilon((s^2 + t^2)s^2, (s^2 + t^2)st, t^4),$$

for $\varepsilon = \pm 1$, whose images cover $X \setminus \{0\}$.

Let us mention here this: *if X is an affine Nash manifold, then all points of X are algebraically regular* [Sh, II.5.6]. We will prove a more general version of this, for X coherent (see Remark 5.3).

Now we turn to the analytic underlying structure of affine Nash manifolds.

(2.7) Analytic structure. Let M be an affine Nash manifold. Then M is also an analytic manifold, and we consider the sheaf $\mathcal{O}_M$ of germs of analytic functions on M, and the ring $\mathcal{O}(M)$ of global analytic functions on M. Of course, $\mathcal{N}_M \subset \mathcal{O}_M$ and $\mathcal{N}(M) \subset \mathcal{O}(M)$.

We recall that $\mathcal{O}(M)$ need not be noetherian (unless M is compact). A global analytic set $X \subset M$ is the zero set of finitely many global analytic functions, and its analytic ideal $\mathcal{J}_{\mathcal{O}}(X)$ is the ideal of all global analytic functions vanishing on X. More generally, we have the analytic ideal $\mathcal{J}_{\mathcal{O}}(Z)$ of any given set $Z \subset M$; the global analytic subset $X = \mathcal{Z}(\mathcal{J}_{\mathcal{O}}(Z))$ of M is the *global analytic closure* of Z, and it is the smallest global analytic set containing Z. There are also decompositions into global analytic irreducible components, but they are not finite in general; we do not enter into any details here. The first reference for this is [WB].

These global analytic sets must be distinguished from local analytic sets, that is, sets that are locally described by finitely many analytic equations. Local analytic sets are the classical concern of analytic geometry, but *in the real case they need not be global*. A local analytic set X is global when it is the real part of a complex analytic set. The classical condition that implies this is coherence: X is *coherent* if the global analytic functions in $\mathcal{J}_{\mathcal{O}}(X)$ generate all ideals $\mathcal{J}_{\mathcal{O}}(X_x)$ of germs of analytic functions vanishing on the germ X_x ($x \in M$). For instance, locally finite unions of analytic manifolds are coherent. The best reference is the classical article [Ca].

The interplay between Nash and analytic features is a recurrent theme, and often analytic means make the understanding of Nash matters easier and better. From the local viewpoint, that is, for germs, the comparison is perfect, due to M. Artin's approximation results [BCR, §8.3, §8.6]. The global compact case is more difficult but similar in the use of approximation tools [CRSh1]. The global non-compact case has to be studied in a different way, but there is a clear picture of the situation [CRSh2], [CSh]. We will need the following facts, which were open problems in recent references like [BCR, 8.6.10], then something surely known to the specialists, but as far as we know not in print till [FG, 2.10, 3.2]:

Proposition 2.8. *Let M be an affine Nash manifold and let Z be a semialgebraic subset of M. Then:*

(i) *The ideal $\mathcal{J}_{\mathcal{N}}(Z)$ of Nash functions vanishing on Z generates the ideal $\mathcal{J}_{\mathcal{O}}(Z)$ of global analytic functions vanishing on Z. In particular, the Nash and the analytic closures of Z coincide.*

(ii) *If Z is global analytic, then it is Nash, and its Nash irreducible components are also its global analytic irreducible components.*

Proof. Let $d = \dim(Z)$. Denote by X the Nash closure of Z and let $X = X_1 \cup \cdots \cup X_r$ be the decomposition of X into Nash irreducible components; set $\mathfrak{p}_i = \mathcal{J}_{\mathcal{N}}(X_i)$. We claim that for each $i = 1, \ldots, r$,

(2.8.1) $\mathfrak{p}_i \mathcal{O}(M) = \mathcal{J}_{\mathcal{O}}(X_i)$ *and X_i is an irreducible global analytic set.*

Indeed, by [CSh, Corollary 2], $\mathfrak{q}_i = \mathfrak{p}_i \mathcal{O}(M)$ is a prime ideal of $\mathcal{O}(M)$. Now suppose $f \in \mathcal{O}(M)$ vanishes on X_i, that is, $f \in \mathcal{J}_{\mathcal{O}}(X_i)$. Then, pick a regular point x of X_i, in the algebraic sense, and notice that $f_x \in \mathcal{J}_{\mathcal{O}}(X_{ix}) = \mathfrak{q}_i \mathcal{O}_{M,x}$, and the stalk at x of the coherent sheaf $\mathcal{F} = (\mathfrak{q}_i : f)$ is $\mathcal{O}_x$ (we borrow this sheaf from [AL, proof of 3.1]). By Cartan's Theorem A, the coherent sheaf $\mathcal{F}$ is generated by its global sections, and so there exist $g_1, \ldots, g_m \in (\mathfrak{q}_i : f) \subset \mathcal{O}(M)$ such that $1 = \sum_{k=1}^m a_k g_{k,x}$ for certain function germs $a_k \in \mathcal{O}_{M,x}$. This implies that $g_\ell(x) \neq 0$ for some index ℓ, hence $g_\ell \notin \mathfrak{q}_i$. Since $g_\ell f \in \mathfrak{q}_i$ and $\mathfrak{q}_i$ is a prime ideal, it follows that $f \in \mathfrak{q}_i$. Thus $\mathfrak{q}_i = \mathcal{J}_{\mathcal{O}}(X_i)$, as desired. It follows that X_i is an irreducible global analytic set, because as remarked above, $\mathfrak{q}_i$ is a prime ideal of $\mathcal{O}(M)$. The claim is proved.

Now, let us prove that the global analytic closure Y of Z coincides with its Nash closure X. Since $Z \subset Y \subset X$ and $d = \dim(Z) = \dim(X)$, the dimension of Y is also d. By (2.8.1), all X_i's are irreducible, so that every global analytic irreducible component of dimension d of Y must coincide with one X_i of dimension d. Denote by X' the union of those X_i's, which is the union of the global analytic irreducible components of Y of dimension d, and by Y'' the union of the global analytic irreducible components of Y of dimension $< d$. In this situation Y'' is the global analytic closure of the semialgebraic set $Z' = Z \setminus X' \subset Y''$, which has dimension $< d$, and by induction on d we conclude that Y'' is also the Nash closure of Z', and in particular a Nash set. Thus $Y = X' \cup Y''$ is a Nash set, and a fortiori the Nash closure X of Z.

Summing up, we have proved that the Nash closure and the analytic closure of Z coincide, that their Nash and global analytic irreducible components are the same, namely the X_i's, and the claim (2.8.1). Consequently, from the properties of coherent sheaves we get

$$\begin{aligned}\mathcal{J}_{\mathcal{O}}(Z) = \mathcal{J}_{\mathcal{O}}(X) = \bigcap_i \mathcal{J}_{\mathcal{O}}(X_i) = \bigcap_i \mathfrak{p}_i \mathcal{O}(M) = \Big(\bigcap_i \mathfrak{p}_i\Big)\mathcal{O}(M)\\ = \mathcal{J}_{\mathcal{N}}(X)\mathcal{O}(M) = \mathcal{J}_{\mathcal{N}}(Z)\mathcal{O}(M),\end{aligned}$$

which completes the proof of (i) and provides half of (ii).

Finally, if Z is global analytic, then it is its own analytic closure, hence its own Nash closure, as already seen. This means that Z is a Nash set. Thus we have the missing half of (ii). □

In fact, we will use (ii) above in Section 4 in the proof of Theorem 1.7, and in Section 5 to compare semialgebraic and algebraic regularity for coherent Nash sets, improving what is commented in (2.6).

3. Extensions of Nash functions and embeddings of semialgebraic sets

We start by proving that if all the germs of a locally compact semialgebraic set have non-singular analytic closure of the same dimension, then the set is contained in an affine Nash manifold of that dimension. The idea is that all those non-singular analytic closures glue together into an analytic manifold, and the main difficulty is to guarantee the glueing is semialgebraic. Roughly speaking, one considers a suitable stratification of the Zariski closure of the given set and then drops enough strata to get rid of all spurious singularities. This is the argument in full:

(3.1) *Proof of Proposition 1.2.* Let $U \subset \mathbb{R}^n$ be an open semialgebraic neighborhood of Z in which Z is closed. Let $\overline{Z}^{\text{zar}}$ stand for the Zariski closure of Z in $\mathbb{R}^n$ and set $X = U \cap \overline{Z}^{\text{zar}}$. By the hypothesis, Z has pure dimension m and $\dim(X) = m$. Let $\mathcal{G}$ be a stratification of X compatible with Z as in (2.3); note that X is locally compact, hence we use the one-point compactification. Henceforth all topological operations are in X, except otherwise specified by a subscript. The set Z is a union of strata, from which we select those of dimension m, say $\Gamma_1, \dots, \Gamma_r$. Since Z is closed in X and has pure dimension m, we have $Z = \overline{\Gamma_1} \cup \cdots \cup \overline{\Gamma_r}$, and $\overline{Z}_x^{\mathrm{an}} = \overline{\Gamma}_{1x}^{\mathrm{an}} \cup \cdots \cup \overline{\Gamma}_{rx}^{\mathrm{an}}$ for every $x \in Z$. Now, given $j = 1, \ldots, r$ and $x \in \overline{\Gamma_j}$, we deduce that $\overline{\Gamma}_{jx}^{\mathrm{an}} = \overline{Z}_x^{\mathrm{an}}$ because $\Gamma_j \subset Z$ has dimension m and $\overline{Z}_x^{\mathrm{an}}$ is regular of dimension m.

(3.1.1) *Fix $j = 1, \ldots, r$. There is an affine Nash manifold $M_j \subset U$ containing $\overline{\Gamma_j}$ such that $\overline{Z}_x^{\mathrm{an}} = M_{jx}$ for every point $x \in \overline{\Gamma_j}$.*

The key property to show this is the following remark.

(3.1.2) *Consider a stratum $\Sigma \in \mathcal{G}$. If $\Sigma_x \subset \overline{Z}_x^{\mathrm{an}}$ for some $x \in \overline{\Sigma} \cap \overline{\Gamma_j}$, then $\Sigma_y \subset \overline{Z}_y^{\mathrm{an}}$ for all $y \in \overline{\Sigma} \cap \overline{\Gamma_j}$.*

Indeed, consider the intersection $Q = \overline{\Sigma} \cap \overline{\Gamma_j}$, which is connected by the properties of stratifications ((2.3.2)(3)), and the subset $B = \{y \in Q : \Sigma_y \subset \overline{Z}_y^{\mathrm{an}}\}$. This set is non-empty, because it contains x. We will conclude that $B = Q$ by proving that B is an open and closed subset of Q.

Let $y \in Q$ and let $Y \subset X$ be an analytic manifold such that $\overline{Z}_y^{\mathrm{an}} = Y_y$; in particular, there is a neighborhood V of y such that $Z \cap V \subset Y$. Then for every $z \in Q \cap V$, $Z_z \subset Y_z$, and since Y_z is a non-singular analytic germ of dimension m, we have $\overline{Z}_z^{\mathrm{an}} = Y_z$.

From this we deduce that B is open in Q. Suppose $y \in B$. Then $\Sigma_y \subset \overline{Z}_y^{\mathrm{an}}$ and so shrinking V we have $\Sigma \cap V \subset Y$; hence, for every $z \in Q \cap V$ we conclude that $\Sigma_z \subset Y_z = \overline{Z}_z^{\mathrm{an}}$.

Now, to see that B is closed in Q, let $y \in Q$ be such that $\Sigma_z \subset \overline{Z}_z^{\mathrm{an}}$ for $z \in Q \cap V$ close enough to y. The above preparation gives the inequality

$$\dim(\Sigma_y \cap \overline{Z}_y^{\mathrm{an}}) = \dim(\Sigma_y \cap Y_y) \geq \dim(\Sigma_z \cap Y_z) = \dim(\Sigma_z \cap \overline{Z}_z^{\mathrm{an}}),$$

so that

$$\dim(\Sigma_y) \geq \dim(\Sigma_y \cap \overline{Z}_y^{\mathrm{an}}) \geq \dim(\Sigma_z \cap \overline{Z}_z^{\mathrm{an}}) = \dim(\Sigma_z) = \dim(\Sigma_y),$$

and $\dim(\Sigma_y \cap \overline{Z}_y^{\mathrm{an}}) = \dim(\Sigma_y)$. Since Σ is an affine Nash manifold connected at y, the analytic germ $\overline{\Sigma}_y^{\mathrm{an}}$ is irreducible ((2.3.2)(1)). Hence, by the dimension condition just shown, $\overline{\Sigma}_y^{\mathrm{an}}$ is the analytic closure of $\Sigma_y \cap \overline{Z}_y^{\mathrm{an}}$. Since the analytic germ $\overline{Z}_y^{\mathrm{an}}$ contains $\Sigma_y \cap \overline{Z}_y^{\mathrm{an}}$, it also contains its analytic closure $\overline{\Sigma}_y^{\mathrm{an}}$. We conclude that $\Sigma_y \subset \overline{\Sigma}_y^{\mathrm{an}} \subset \overline{Z}_y^{\mathrm{an}}$, and so $y \in B$. The proof of (3.1.2) is finished.

Let us now prove (3.1.1). Consider the family of strata

$$\mathcal{F} = \{\Sigma \in \mathcal{G} : \Sigma_x \subset \overline{Z}_x^{\mathrm{an}} \text{ for some } x \in \overline{\Sigma} \cap \overline{\Gamma_j}\}.$$

We know that $\overline{\Gamma}_{jx}^{\mathrm{an}} = \overline{Z}_x^{\mathrm{an}}$ for all $x \in \overline{\Gamma_j}$, hence $\Gamma_j \in \mathcal{F}$. Next consider the closed semialgebraic subsets of X:

$$T_j = \bigcup_{\Sigma \in \mathcal{F}} \overline{\Sigma} \quad \text{and} \quad S_j = \bigcup_{\Sigma \notin \mathcal{F}} \overline{\Sigma}.$$

Note that $X = T_j \cup S_j$. Let us check that $M_j = \operatorname{Reg}(T_j) \subset T_j \subset X \subset U$ is the affine Nash manifold (3.1.1) asks for. It suffices to see that

(3.1.3) $T_{jx} = \overline{Z}_x^{\mathrm{an}}$ *for every point* $x \in \overline{\Gamma_j}$.

Fix $x \in \overline{\Gamma_j}$. First we see that $T_{jx} \subset \overline{Z}_x^{\mathrm{an}}$. It is enough to show that $\Sigma_x \subset \overline{Z}_x^{\mathrm{an}}$ for each $\Sigma \in \mathcal{F}$. If $x \notin \overline{\Sigma}$ there is nothing to prove; otherwise, $x \in \overline{\Sigma} \cap \overline{\Gamma_j}$. Then, by definition of $\mathcal{F}$ and (3.1.2), $\Sigma_x \subset \overline{Z}_x^{\mathrm{an}}$.

For the converse inclusion, observe first that $\dim(\overline{Z}_x^{\mathrm{an}} \cap \Sigma_x) < m$ for $\Sigma \notin \mathcal{F}$. Indeed, since $\overline{Z}_x^{\mathrm{an}}$ is regular of dimension m, if m were the dimension of $\overline{Z}_x^{\mathrm{an}} \cap \Sigma_x$, then $\overline{Z}_x^{\mathrm{an}}$ would be the analytic closure of $\overline{Z}_x^{\mathrm{an}} \cap \Sigma_x$, and so $\overline{Z}_x^{\mathrm{an}}$ would be contained in $\overline{\Sigma}_x^{\mathrm{an}}$. But then $\overline{\Sigma}_x^{\mathrm{an}}$ would be irreducible of dimension m ((2.3)(1) again), and in fact $\overline{Z}_x^{\mathrm{an}} = \overline{\Sigma}_x^{\mathrm{an}}$. Consequently, $\Sigma_x \subset \overline{Z}_x^{\mathrm{an}}$, and $\Sigma \in \mathcal{F}$, a contradiction. Once we have proved this, we deduce that $\dim(\overline{Z}_x^{\mathrm{an}} \cap S_{jx}) < m$, and so the germ $\overline{Z}_x^{\mathrm{an}} \setminus S_{jx}$ is dense in the germ $\overline{Z}_x^{\mathrm{an}}$. Since $\overline{Z}_x^{\mathrm{an}} \setminus S_{jx} \subset \overline{Z}_x^{\mathrm{an}} \cap T_{jx}$, the latter germ is also dense in $\overline{Z}_x^{\mathrm{an}}$. But T_j is closed, so that $\overline{Z}_x^{\mathrm{an}} \cap T_{jx} = \overline{Z}_x^{\mathrm{an}}$. This finishes the proof of (3.1.3), and as remarked before, that of (3.1.1).

Next we modify the affine Nash manifold M_j a little. Notice that, since $\overline{\Gamma_j} \subset M_j \cap Z$ (see (3.1.3) and use the regularity of the m-dimensional analytic germs $\overline{Z}_x^{\mathrm{an}}$), we have

$$\overline{\Gamma}_{jx} \subset M_{jx} \cap Z_x = \overline{Z}_x^{\mathrm{an}} \cap Z_x = Z_x$$

for each point $x \in \overline{\Gamma_j}$. Hence, $M_j \cap Z$ is a semialgebraic neighborhood of $\overline{\Gamma_j}$ in Z, that is, $\operatorname{Int}_Z(M_j \cap Z)$ is an open semialgebraic subset of Z containing $\overline{\Gamma_j}$. Therefore, there exists an open semialgebraic subset U_j of X such that $\overline{\Gamma_j} \subset U_j \cap Z = \operatorname{Int}_Z(M_j \cap Z)$. Replacing M_j by $M_j \cap U_j$, we may assume that $M_j \subset U_j$ and $M_j \cap Z = U_j \cap Z$.

Moreover, by (2.1.1), there exists an open semialgebraic subset V_j of X such that $\overline{\Gamma_j} \subset V_j \subset \overline{V_j} \subset U_j$. Now we define $M'_j = M_j \cap V_j$, which is an affine Nash manifold of dimension m and $M = \operatorname{Reg}(M'_1 \cup \cdots \cup M'_r) \subset U$, which is also an affine Nash manifold of dimension m. Hence, all reduces to checking that $Z \subset M$, and this will follow from:

(3.1.4) $M'_{1x} \cup \cdots \cup M'_{rx} = \overline{Z}_x^{\mathrm{an}}$ *for every point* $x \in Z$.

Let us show this. Given a point $x \in Z = \overline{\Gamma_1} \cup \cdots \cup \overline{\Gamma_r}$ there exists an index ℓ such that $x \in \overline{\Gamma_\ell} \subset V_\ell$; hence, $M'_{\ell x} = M_{\ell x} = \overline{Z}_x^{\mathrm{an}}$. On the other hand, for those indices j with $x \notin \overline{V_j}$ we have $M'_{jx} = \emptyset$. Therefore, all reduces to checking that $M'_{jx} \subset \overline{Z}_x^{\mathrm{an}}$ for those indices j such that $x \in \overline{V_j} \subset U_j$. Observe that if $x \in U_j \cap Z = M_j \cap Z \subset M_j$, we get

$$Z_x = U_{jx} \cap U_{\ell x} \cap Z_x = M_{jx} \cap M_{\ell x} \cap Z_x \subset M_{jx} \cap M_{\ell x},$$

and this implies $\overline{Z}_x^{\mathrm{an}} \subset M_{jx} \cap M_{\ell x} \subset M_{\ell x} = \overline{Z}_x^{\mathrm{an}}$. Hence, $M_{\ell x} \cap M_{jx} = M_{\ell x}$, that is, $M_{\ell x} \subset M_{jx}$. Since both are irreducible analytic germs of the same dimension m, it follows that $M_{jx} = M_{\ell x} = \overline{Z}_x^{\mathrm{an}}$ and, in particular, $M'_{jx} \subset M_{jx} = \overline{Z}_x^{\mathrm{an}}$, as desired. □

Remark 3.2. Let $Z \subset \mathbb{R}^n$ be as in the preceding statement.

(1) One can rephrase the hypothesis on Z as follows: for each point $x \in Z$ there is an affine Nash manifold M of dimension m such that $M_y = \overline{Z}_y^{\mathrm{an}}$ for any $y \in Z$ close enough to x. This means that $\{\overline{Z}_y^{\mathrm{an}}\}$ defines a *continuous* section on Z of the sheaf of germs of analytic subsets of $\mathbb{R}^n$. By general sheaf theory, that section extends to a neighborhood of Z, which gives an analytic manifold M of dimension m that contains Z. But this does not give a semialgebraic neighborhood or a Nash manifold, as Proposition 1.2 does.

(2) The manifold M we find is contained in the Nash closure of Z (in X). Indeed, since Z and M have the same dimension, every Nash function that vanishes on Z, vanishes on M by the Identity Principle.

Now we are ready to prove that a Nash function on a semialgebraic set can always be extended to an open semialgebraic neighborhood.

(3.3) *Proof of Theorem 1.3.* Our manifold M is embedded in some affine space, and using a tubular neighborhood of M there, we can suppose M is an open semialgebraic subset of $\mathbb{R}^n$ or simply that $M = \mathbb{R}^n$. Thus we have a semialgebraic subset Z of $\mathbb{R}^n$ and a Nash function f in an open neighborhood A of Z in $\mathbb{R}^n$, and we want to replace A by an open semialgebraic neighborhood of Z in $\mathbb{R}^n$.

First of all, let us construct a stratification $\mathcal{G}_0$ of $\mathbb{R}^n$ compatible with Z satisfying the following property:

(3.3.1) *For every stratum $\Gamma \in \mathcal{G}_0$ whose closure meets Z there is a polynomial $P(\mathtt{x}, \mathtt{t}) \in \mathbb{R}[\mathtt{x}, \mathtt{t}] = \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{t}]$ which is identically zero at no $x \in \Gamma$, and every stratum $\Sigma \subset \overline{\Gamma} \cap Z$ has a (not necessarily semialgebraic) open neighborhood $V_\Gamma \subset A$ (depending on Σ) with connected intersection $V_\Gamma \cap \Gamma$ on which $P(x, f(x)) \equiv 0$.*

To start with, we pick a stratification $\mathcal{G}$ of $\mathbb{R}^n$ compatible with Z (see (2.3)). Let $\Gamma \in \mathcal{G}$ be a stratum of dimension n such that $\overline{\Gamma} \cap Z \neq \emptyset$. Since $\mathcal{G}$ is compatible with Z we deduce that $\overline{\Gamma} \cap Z = \Sigma_1 \cup \dots \cup \Sigma_r$ for certain strata $\Sigma_\ell \in \mathcal{G}$. Choose an open neighborhood $V_\ell \subset A$ of Σ_ℓ such that $V_\ell \cap \Gamma$ is connected. This is possible because the stratum Σ_ℓ is adherent to the stratum Γ (see (2.3.2)(2)). Then by (2.5) there is a polynomial $P_\ell(\mathtt{x}, \mathtt{t}) \in \mathbb{R}[\mathtt{x}, \mathtt{t}]$ such that $P_\ell(x, f(x)) = 0$ for all $x \in V_\ell \cap \Gamma$, and there is a semialgebraic set $\Pi_\ell \subset \Gamma$ of dimension $< n$ off which the polynomial P_ℓ does not vanish identically. Set $P = \prod_\ell P_\ell$ and $\Pi = \bigcup_\ell \Pi_\ell$. Then $P(x, f(x)) = 0$ for $x \in V_\ell \cap \Gamma$ and $\ell = 1, \dots, r$, and $P(x, \mathtt{t})$ does not vanish identically if $x \notin \Pi$, which is a semialgebraic set of dimension $< n$. Of course, all these data depend on Γ, but we omit this in the notation to make the proof readable.

Next, we pick a new stratification $\mathcal{G}'$ of $\mathbb{R}^n$ compatible with Z, with all strata in $\mathcal{G}$, and with the semialgebraic sets Π associated as above to the strata $\Gamma \in \mathcal{G}$ of dimension n whose closures meet Z. Let $\Gamma' \in \mathcal{G}'$ be a stratum of dimension n whose closure meets Z. Then $\overline{\Gamma'} \cap Z = \Sigma'_1 \cup \dots \cup \Sigma'_s$ for certain strata $\Sigma'_k \in \mathcal{G}'$, and by the compatibility assumptions on $\mathcal{G}'$, there are a stratum $\Gamma \in \mathcal{G}$ of dimension n and for each k a stratum $\Sigma_\ell \in \mathcal{G}$ contained in $\overline{\Gamma} \cap Z$ with $\Gamma' \subset \Gamma$ and $\Sigma'_k \subset \Sigma_\ell$. Also, for each k we find an open neighborhood $V'_k \subset V_\ell \subset A$ of Σ'_k such that $V'_k \cap \Gamma'$ is connected. In this situation, for the polynomial P associated to Γ we have $P(x, f(x)) = 0$ for $x \in V'_k \cap \Gamma'$ and $k = 1, \dots, s$, and what is the real goal, $P(x, \mathtt{t})$ is identically zero at no $x \in \Gamma'$. Indeed, since $\mathcal{G}'$ is compatible with the semialgebraic set Π where P vanishes identically, and this set has dimension $< n$, Γ' cannot meet Π.

This shows that $\mathcal{G}'$ satisfies (3.3.1) for strata of dimension n. Now, it is quite clear how to proceed. We turn to strata of dimension $n-1$ and can obtain (3.3.1) for them, *off some semialgebraic sets of dimension* $< n-1$. Then we stratify again $\mathbb{R}^n$, this time compatibly with Z, with all strata of the previous stratification, and with all semialgebraic sets of dimension $< n-1$ just found. By restricting to this new stratification the previous polynomials, we keep (3.3.1) for strata of dimension n and gain it for strata of dimension $n-1$. And we continue inductively, till all dimensions have been taken into account.

Once we have the stratification $\mathcal{G}_0$ of (3.3.1), for every stratum $\Gamma \in \mathcal{G}_0$ whose closure meets Z we prove the following:

(3.3.2) *There is a partition of Γ into finitely many affine Nash manifolds $T_1, \dots, T_m$ equipped with finitely many Nash functions $\alpha_{ij} : T_i \to \mathbb{R}$, $1 \le j \le r_i$, such that $\alpha_{i1}(x) < \cdots < \alpha_{ir_i}(x)$ are the real roots of $P_i(x, \mathtt{t}) \in \mathbb{R}[\mathtt{t}]$ for $x \in T_i$. Here we denote by $P_i(\mathtt{x}, \mathtt{t})$ the polynomial assigned to Γ to stress we specialize it to $x \in T_i$; since $P_i(x, \mathtt{t})$ is not identically zero and $P_i(x, f(x)) = 0$, $P_i(x, \mathtt{t})$ is not constant, so that $r_i \ge 1$.*

To prove this, first apply [BCR, 2.3.1] to each polynomial asigned to Γ to obtain a partition of Γ into finitely many semialgebraic sets T'_i and semialgebraic roots $\alpha_{ij} : T'_i \to \mathbb{R}$. Then use (2.4.2) to split the T'_i's into affine Nash manifolds T_j on which the roots are in addition Nash functions.

Having (3.3.2) we complete our preparation by stratifying once again: this time let $\mathcal{G}$ be a stratification of $\mathbb{R}^n$ compatible with Z, with all strata of $\mathcal{G}_0$ and all semialgebraic pieces $T_1, \dots, T_m$ associated to those strata. For the balance of the proof we only refer to strata of this last stratification $\mathcal{G}$.

The set Z is a union of strata of $\mathcal{G}$, and for the time being, we fix one of them, say Σ. Let $\mathcal{G}_\Sigma$ be the collection of all strata of $\mathcal{G}$ adherent to Σ, and consider the open semialgebraic neighborhood $W = \bigcup_{\Gamma \in \mathcal{G}_\Sigma} \Gamma$ of Σ (see (2.3.1)). The key property of the strata in $\mathcal{G}_\Sigma$ is the following:

(3.3.3) *Every $\Gamma \in \mathcal{G}_\Sigma$ is contained in a (unique) T_i, and there is an open neighborhood V_Γ of Σ in A such that $V_\Gamma \cap \Gamma$ is connected and f coincides with a root α_{ij} of P_i on $V_\Gamma \cap \Gamma$. We denote that root by h_Γ.*

The first part follows from the compatibility of $\mathcal{G}$ with the T_i's. For the second part, choose an open neighborhood V_Γ of Σ in A such that $V_\Gamma \cap \Gamma$ is connected and where $P_i(x, f(x)) = 0$. Now, the connected set $V_\Gamma \cap \Gamma$ is the disjoint union of the closed subsets defined by $\{f = \alpha_{i1}\}, \dots, \{f = \alpha_{ir_i}\}$, and consequently coincides with one of them. In other words, $f = \alpha_{ij}$ on $V_\Gamma \cap \Gamma$ for a unique index j. We are done.

Next, the semialgebraic functions $h_\Gamma : \Gamma \to \mathbb{R}$ provided by (3.3.3) glue into a semialgebraic function $h : W \to \mathbb{R}$ that coincides with f on $A_\Sigma = \bigcup_{\Gamma \in \mathcal{G}_\Sigma} \Gamma \cap V_\Gamma$, which is a neighborhood of Σ in A. Now we have to compare these functions $h : W \to \mathbb{R}$ for different strata Σ contained in Z. To that end we enumerate the strata involved as

$\Sigma_1, \dots, \Sigma_p$ and write $Z = \sum_{\ell=1}^{p} \Sigma_\ell$. For each ℓ we have an open semialgebraic neighborhood W_ℓ of Σ_ℓ and a semialgebraic function $h_\ell : W_\ell \to \mathbb{R}$ that coincides with f on a neighborhood $A_\ell \subset W_\ell \cap A$ of Σ_ℓ. An important fact is this:

(3.3.4) *Fix k and let ℓ be such that $\Sigma_k \cap W_\ell \neq \emptyset$. Then $W_k \subset W_\ell$ and $h_k = h_\ell|_{W_k}$.*

Indeed, since Σ_k meets W_ℓ, it meets some stratum adherent to Σ_ℓ; but then Σ_k coincides with that stratum and, in particular, Σ_k is adherent to Σ_ℓ. From this we see that any stratum adherent to Σ_k is adherent to Σ_ℓ too, which gives the inclusion $W_k \subset W_\ell$. Next, let us prove that h_ℓ restricts to h_k on W_k. To do that we pick a stratum Γ adherent to Σ_k, hence to Σ_ℓ, and show that h_k and h_ℓ are defined on Γ by the same root of $P_i(\mathtt{x}, \mathtt{t})$. Denote by V_k and V_ℓ the neighborhoods of Σ_k, Σ_ℓ in A with connected intersections with Γ, where $f = h_k$ and $f = h_\ell$ respectively. As $\Sigma_\ell \subset \overline{\Sigma_k}$ and $\Sigma_k \subset \overline{\Gamma}$, the intersection $V_{k\ell} = V_k \cap V_\ell \cap \Gamma$ is not empty. Notice that on $V_{k\ell}$ the three Nash functions f, h_k, h_ℓ on Γ coincide. Thus, the Nash functions $h_k|_\Gamma$ and $h_\ell|_\Gamma$ coincide on the nonempty open subset $V_{k\ell}$ of the stratum Γ, hence on the whole of it. We are done.

Once we know (3.3.4), consider the open semialgebraic neighborhood $W' = \bigcup_{\ell=1}^{p} W_\ell$ of Z, and for each $\ell = 1, \dots, p$ let $W'_\ell \subset W_\ell$ be an open semialgebraic set such that $W' \cap \overline{W'_\ell} \subset W_\ell$ and $W' = \bigcup_\ell W'_\ell$ (see (2.1.2)). Then on the semialgebraic set

$$S = \{x \in W' : \text{if } x \in \overline{W'_{\ell_1}} \cap \overline{W'_{\ell_2}}, \text{ then } h_{\ell_1}(x) = h_{\ell_2}(x)\} \subset W' = \bigcup_\ell W'_\ell$$

the semialgebraic function

$$h : S \to \mathbb{R}, \qquad x \mapsto h_\ell(x) \quad \text{if } x \in W'_\ell,$$

is well defined. We claim that

(3.3.5) *S contains an open neighborhood of Z on which h coincides with f.*

To prove this, fix $x \in Z$. All we need is a neighborhood of x contained in S on which h and f coincide. To find it, we enumerate the W'_ℓ's adherent to x, say

$$x \in \overline{W'_1}, \dots, \overline{W'_q} \quad \text{and} \quad x \notin \overline{W'_{q+1}}, \dots, \overline{W'_p}.$$

Since Z is the union of the Σ_ℓ's, one of them contains x, say $x \in \Sigma_k$. Then, by (3.3.4),

$$W_k \subset W_1 \cap \dots \cap W_q \quad \text{and} \quad h_1|_{W_k} = h_k, \dots, h_q|_{W_k} = h_k.$$

Then the neighborhood of x we seek is

$$U^x = A_k \setminus (\overline{W'_{q+1}} \cup \dots \cup \overline{W'_p}) \subset W_k \cap A.$$

First we see that $U^x \subset S$. Let $y \in U^x$. If $y \in \overline{W'_{\ell_1}} \cap \overline{W'_{\ell_2}}$, then $1 \le \ell_1, \ell_2 \le q$, and so $x \in \Sigma_k \cap W_{\ell_1} \cap W_{\ell_2}$. Thus, by (3.3.4), $y \in W_k \subset W_{\ell_1} \cap W_{\ell_2}$ and $h_{\ell_1}(y) = h_k(y) = h_{\ell_2}(y)$. Consequently, $y \in S$. Second, we see that h coincides on U^x with f. Indeed, as we have just shown, h coincides there with h_k, which on A_k coincides with f. Thus (3.3.5) is proved.

Finally, let U be the set of points of S at which h is Nash. This open set contains the neighborhood of Z on which h and f coincide, and by (2.4.1), U is a semialgebraic set. To conclude the proof of Theorem 1.3, just restrict h to U. □

Remarks 3.4. (1) Let again Z be a semialgebraic subset of an affine Nash manifold M. Suppose that every point in Z is adherent to $\mathrm{Int}_M(Z)$. This is the proper situation to define Nash functions through local extensions in the elementary way. Namely, *$f : Z \to \mathbb{R}$ is a Nash function if every point $x \in Z$ has a neighborhood U in M on which there is a Nash extension $F : U \to \mathbb{R}$ of $f|_{U\cap Z}$.*

The key fact is that any two such extensions $F_i : U_i \to \mathbb{R}$ have the same germ at x. Indeed, since x is adherent to $\mathrm{Int}_M(Z)$, any neighborhood U of x meets $\mathrm{Int}_M(Z)$; pick $U \subset U_1 \cap U_2$ connected. Then f_1 and f_2 coincide on the non-empty open set $U \cap \mathrm{Int}_M(Z)$, and by the Identity Principle, they coincide on U. As is well known from sheaf theory, this guarantees that there is a Nash extension F of f to an open neighborhood of Z in M. The extra information provided by Theorem 1.3 is that such a neighborhood can be taken *semialgebraic*.

(2) This local definition of Nash functions need not work in general. Consider the semialgebraic set $Z \subset E = \mathbb{R}^2$ consisting of the x-axis $y = 0$ and the closed disc $x^2 + y^2 \le 1/4$. Then the function $f : Z \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ y/(x^2 + y^2 - 1) & \text{if } x^2 + y^2 \le 1/4, \end{cases}$$

has local but not global analytic extensions (see also [FG]).

4. Local Nash normal crossings in a Nash manifold

Let M be an affine Nash manifold of dimension m and $\mathcal{N}(M)$ its ring of Nash functions. We recall (see Definition 1.4) that a Nash subset X of M has only normal crossings at $x \in X$ if there are coordinates $(u_1, \dots, u_m)$ of M at x such that $X_x = \{u_1 \cdots u_r = 0\}$ for some r. This number r, which depends only on the point x, is the multiplicity $\mathrm{mult}(X, x)$ of X at x. In a formal way, we write $\mathrm{mult}(X, x) = 0$ to mean $x \notin X$. It is clear that X has only normal crossings at x of multiplicity $r \ge 1$ if and only if the germ X_x is the union of r non-singular hypersurface germs *in general position* (this means that their tangent hyperplanes are linearly independent). Note also that the function $\mathrm{mult}(X, \cdot)$ is upper semicontinuous.

Lemma 4.1. *Let $X_1, \dots, X_r$ be the irreducible components of a Nash subset X of M which has only normal crossings in M. Then each finite union of irreducible components X_i of X has only normal crossings in M.*

Proof. Notice first that all reduces to checking that each X_i has only normal crossings in M. Since X has only normal crossings in M, X is a coherent analytic space and, using the normalization of X (see [T, §8]), one deduces that each X_i is pure dimensional of dimension $m-1$. Now, since $X_x = X_{1x} \cup \cdots \cup X_{rx}$ for all $x \in M$, it is straightforward, using again that X has only normal crossings in M, that the same happens for each X_i.

□

Next, to prove Proposition 1.5, that is, to prove that the set

$$U = \{x \in M : X \text{ has only normal crossings at } x\}$$

is semialgebraic, we need the following fact, interesting in its own right:

Proposition 4.2. *Let Z be a semialgebraic subset of an affine Nash manifold $M \subset \mathbb{R}^n$. The set of points $x \in \overline{Z}$ such that the germ Z_x has a fixed number s of connected components is semialgebraic.*

Proof. The proof exemplifies the use of *semialgebraic triviality* [BCR, 9.3.1]. Since the germs Z_x, $x \in Z$, are connected, we only care for points $x \in X = \overline{Z} \setminus Z$. For such an x, denote by $B(x,\varepsilon) \subset \mathbb{R}^n$ the open ball of center x and radius ε. By definition, Z_x has s connected components if and only if for all $\varepsilon > 0$ small enough the intersection $Z \cap B(x,\varepsilon)$ has s connected components adherent to x. Since "for all $\varepsilon > 0$ small enough" is a first order sentence (clearly it can be rewritten as "$\exists t > 0$ such that $\forall \varepsilon > 0$ with $\varepsilon < t$"), it is enough to see that the (x,ε)'s such that $Z \cap B(x,\varepsilon)$ *has s connected components adherent to x* form a semialgebraic set.

Consider the semialgebraic sets

$$S = \{(x,\varepsilon,z) \in X \times (0,\infty) \times M : z \in (Z \cap B(x,\varepsilon)) \cup \{x\}\},$$
$$P = \{(x,\varepsilon,z) \in X \times (0,\infty) \times M : z = x\}$$

and the projection $\pi : X \times (0,\infty) \times M \to X \times (0,\infty)$. Semialgebraic triviality [BCR, 9.3.2] says that there is a finite semialgebraic partition $X \times (0,\infty) = \bigcup_\ell T_\ell$ and for each ℓ two semialgebraic sets $F_\ell \supset G_\ell$, a point $a_\ell \in F_\ell$ and a semialgebraic homeomorphism $\theta_\ell : T_\ell \times F_\ell \to \pi^{-1}(T_\ell)$ such that $\pi \circ \theta_\ell$ is the projection $T_\ell \times F_\ell \to T_\ell$ and

$$\theta_\ell(T_\ell \times G_\ell) = S \cap \pi^{-1}(T_\ell) \quad \text{and} \quad \theta_\ell(T_\ell \times \{a_\ell\}) = P \cap \pi^{-1}(T_\ell);$$

notice that θ_ℓ is injective and $P \subset S$. In fact, we claim that $a_\ell \in G_\ell$.

Indeed, let $(x,\varepsilon) \in T_\ell$ and observe that

$$\theta_\ell(x,\varepsilon,a_\ell) = (x,\varepsilon,x) \in P \cap \pi^{-1}(T_\ell) \subset S \cap \pi^{-1}(T_\ell) = \theta_\ell(T_\ell \times G_\ell).$$

Now, since θ_ℓ is injective we deduce that $a_\ell \in G_\ell$.

Thus, for each $(x,\varepsilon) \in T_\ell$, θ_ℓ induces a homeomorphism $G_\ell \to (Z \cap B(x,\varepsilon)) \cup \{x\}$ that maps a_ℓ to x and $G_\ell \setminus \{a_\ell\}$ onto $Z \cap B(x,\varepsilon)$. Consequently, the number of connected components of $Z \cap B(x,\varepsilon)$ adherent to x is the number of connected components of $G_\ell \setminus \{a_\ell\}$ adherent to a_ℓ, which depends solely on T_ℓ. We conclude that the set of couples (x,ε) we are interested in, which are those (x,ε) such that $Z \cap B(x,\varepsilon)$ has s connected components adherent to x, equals the union of certain T_ℓ's (maybe none), so that they indeed form a semialgebraic set. □

Once we know this, we can proceed with:

(4.3) *Proof of Proposition 1.5.* The set U of points of M that either do not belong to X or at which X has only normal crossings is clearly open (see Definition 1.4) and it is the union of the sets $U_0 = M \setminus X$ and

$$U_r = \{x \in X : X \text{ has only normal crossings at } x, \text{ of multiplicity } r\}, \quad 1 \le r \le m.$$

We will prove the semialgebraicity of U_r for $1 \le r \le m$. Consider the Nash ideal $I = \mathcal{J}_{\mathcal{N}}(X)$ of X, which is finitely generated, by say $f_1, \dots, f_p \in \mathcal{N}(M)$. Clearly, $x \in U_r$ if and only if

(4.3.1) *There is a regular system of parameters $u_1, \dots, u_m$ of the local regular ring $\mathcal{N}_{M,x}$ such that $X_x = \{f_1 = 0, \dots, f_p = 0\}_x = \{u_1 \cdots u_r = 0\}_x$.*

We must show that this condition is semialgebraic. Before proceeding, we apply the Artin–Mazur Theorem (see 2.5) to assume that M is an open subset of a non-singular algebraic set $V \subset \mathbb{R}^n$ and $f_1, \dots, f_p$ are the restrictions to M of some polynomial functions that we denote by the same letters. Let J be the ideal of V in the polynomial ring $\mathbb{R}[\mathtt{x}] = \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$, and let $b_1, \dots, b_q$ be generators of J. Then the stalk $\mathcal{N}_{M,x}$ at a point $x \in M$ is the henselization of the localization of $\mathbb{R}[\mathtt{x}]/J$ at the ideal $(\mathtt{x} - x) = (\mathtt{x}_1 - x_1, \dots, \mathtt{x}_n - x_n)$. We denote by $\mathbb{R}[[\mathtt{x} - x]]_{\text{alg}}$ the henselization of the local ring $\mathbb{R}[\mathtt{x}]_{(\mathtt{x}-x)}$, and so $\mathcal{N}_{M,x} = \mathbb{R}[[\mathtt{x} - x]]_{\text{alg}}/(b_1, \dots, b_q)$. On the other hand, $J_x = J\mathcal{N}_{M,x}$ is generated by the polynomials b_k, and the parameters u_i are the classes modulo J_x of some $h_i \in \mathbb{R}[[\mathtt{x} - x]]_{\text{alg}}$; let B_{kx}, H_{ix} stand for the derivatives at x of the b_k, h_i.

Suppose that condition (4.3.1) holds true for a point $x \in M$. We deduce that all f_j's belong to the ideal of $\mathcal{N}_{M,x}$ generated by $u_1 \cdots u_r$; hence

(1) There are Nash function germs $g_j, a_{jk} \in \mathbb{R}[[\mathtt{x} - x]]_{\text{alg}}$ such that

$$f_j = h_1 \cdots h_r g_j + \sum\nolimits_k a_{jk} b_k. \qquad (\star)$$

On the other hand, that the u_i's form a regular system of parameters of $\mathcal{N}_{M,x}$ just means that

(2) The h_i's vanish at x, and the linear forms H_{ix} are linearly independent over $\mathbb{R}$ modulo the linear forms B_{kx}.

Let us now see how these new equivalent conditions are semialgebraic. We look at (1) as a system of polynomial equations in the unknowns $\mathtt{h}_i, \mathtt{g}_j, \mathtt{a}_{jk}$. Then we recall M. Artin's approximation theorem with bounds [Ar, 6.1]; it says that

(4.3.2) *For any integer α there exists another integer β, which only depends on n, α, the degrees of the f_j's, the degrees of the b_k's and the number of variables $\mathtt{h}_i, \mathtt{g}_j, \mathtt{a}_{ij}$, such that the polynomial equations*

$$f_j = \mathtt{h}_1 \cdots \mathtt{h}_r \mathtt{g}_j + \sum_k \mathtt{a}_{jk} b_k$$

have an exact solution in the local ring $\mathbb{R}[[\mathrm{x}-x]]_{\mathrm{alg}}$ *if they have an approximate solution modulo* $(\mathrm{x}-x)^{\beta}$*; furthermore that exact solution coincides with the approximate solution to order* α.

Now, fix $\alpha = 2$, so that the exact solution coincides with the approximate one to order 2, and define S as the set of points $x \in M$ such that:

(1*) There are polynomials $\mathrm{h}_i, \mathrm{g}_j, \mathrm{a}_{jk} \in \mathbb{R}[\mathrm{x}]$ of degree $\leq \beta$ such that

$$f_j \equiv \mathrm{h}_1 \cdots \mathrm{h}_r \mathrm{g}_j + \sum_k \mathrm{a}_{jk} b_k \bmod (\mathrm{x}-x)^{\beta}. \qquad (*)$$

(2*) The polynomials h_i vanish at x and the derivatives $\mathrm{H}_{i,x}$ at x of the polynomials h_i are linearly independent linear forms over $\mathbb{R}$ modulo the linear forms B_{kx}.

Then, if that approximate solution h_i satisfies (2*), the exact one h_i also satisfies (2). Thus, if the equation $(*)$ in (1*) has an *approximate* solution $\mathrm{h}_i, \mathrm{g}_j, \mathrm{a}_{jk} \in \mathbb{R}[\mathrm{x}]$ of degree $\leq \beta$ modulo $(\mathrm{x}-x)^{\beta}$ satisfying (2*), then the equation $(\star)$ in (1) has an *exact* solution $h_i, g_j, a_{jk} \in \mathbb{R}[[\mathrm{x}-x]]_{\mathrm{alg}}$ satisfying (2). Since the converse implication is trivial, both assertions are equivalent. Now, the existence of approximate solutions of fixed order β (described by conditions (1*) and (2*) above) is clearly a first order sentence, and we conclude that the set S of points $x \in M$ for which conditions (1) and (2) hold true (or equivalently conditions (1*) and (2*) hold true) is a semialgebraic set.

Next we analyze the exact meaning of (1) and (2); let $x \in S$. From (1) we get

$$X_x = \{f_1 = 0, \ldots, f_p = 0\}_x \cap M_x = \{u_1 \cdots u_r = 0\} \cup (\{g_1 = 0, \ldots, g_p = 0\} \cap M_x);$$

hence the irreducible decomposition of the germ X_x is

$$X_x = \{u_1 \cdots u_r = 0\} \cup (Y_1 \cup \cdots \cup Y_s), \qquad (\bullet)$$

where the Y_ℓ's are irreducible Nash germs on which no u_i vanishes identically. Thus we must get rid of those Y_ℓ's. To that end we use the topology of the germ X_x. The two properties of interest here are that if X has only normal crossings at x of multiplicity r then:

(a) X_x has pure dimension $m-1$, and
(b) $\mathrm{Reg}(X)_x$ has 2^r connected components.

Now, by (2.1.3) and Proposition 4.2, the points in X satisfying (a) and (b) form a semialgebraic set T.

Next, we claim that $U_r = S \cap T$. Indeed, it only remains to prove that for $x \in T$ the above decomposition $(\bullet)$ of X_x into irreducible components has no Y_ℓ. Indeed, suppose by way of contradiction that X_x has one irreducible component Y_k. Now, condition (a) tells us that the Y_ℓ's all have pure dimension $m-1$, and we deduce

$$\mathrm{Reg}(X)_x = \Big(\mathrm{Reg}(\{u_1 \cdots u_r = 0\}) \setminus \bigcup_\ell Y_\ell\Big) \cup \Big(\mathrm{Reg}\Big(\bigcup_\ell Y_\ell\Big) \setminus \{u_1 \cdots u_r = 0\}\Big).$$

On the right-hand side we see at least 2^r+1 connected components: 2^r coming from $\operatorname{Reg}(\{u_1\cdots u_r=0\})$ and another from $\bigcup_\ell Y_\ell$. But this contradicts (b), and we conclude that the Y_ℓ's cannot really be there. □

Next, we discuss finiteness of local normal crossings (Theorem 1.6), that is, whether we can formulate the definition of local normal crossings with finitely many open semialgebraic coordinate systems. First, we prove:

Proposition 4.4. *Let Z be a locally compact semialgebraic subset of an affine Nash manifold M. Suppose that at every point $x\in Z$ there is a coordinate system $(v_1,\dots,v_m)$ of M such that $\overline{Z}^{\text{an}}_x=\{v_1\cdots v_r=0\}_x$ for some $r\geq 1$; we will call this r the multiplicity of Z at x, and write $r=\operatorname{mult}(Z,x)$. Then:*

(i) *Every set $Z^{(r)}=\{x\in Z:\operatorname{mult}(Z,x)=r\}$ is semialgebraic.*
(ii) *Every set $Z^{(r)}$ can be covered by finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $(u_1,\dots,u_m):U\to\mathbb{R}^m$ such that $Z\cap U\subset\{u_1\cdots u_r=0\}$.*

Proof. Since Z is closed in some open semialgebraic subset of M, replacing M by that subset we can suppose that Z is closed in M. On the other hand, we can cover M with finitely many open semialgebraic subsets Nash diffeomorphic to $\mathbb{R}^m$ (see Lemma 2.2), and so we may assume that $M=\mathbb{R}^m$.

To start with, we choose a finite semialgebraic stratification $\mathcal{G}$ of $\mathbb{R}^m$ compatible with Z. Note that every stratum $\Gamma\in\mathcal{G}$ is an affine Nash manifold. We will need the following:

(4.4.1) *Let $\Gamma\in\mathcal{G}$ be a stratum of dimension $m-1$ contained in Z. Then for each point $x\in\overline{\Gamma}\subset Z$ the germ $\overline{\Gamma}^{\text{an}}_x$ is non-singular of dimension $m-1$.*

Indeed, given $x\in\overline{\Gamma}$ and since Z has only normal crossings at x, there exist non-singular hypersurface germs $X_1,\dots,X_s$, $s=\operatorname{mult}(Z,x)$, such that

$$\overline{\Gamma}^{\text{an}}_x\subset\overline{Z}^{\text{an}}_x=X_1\cup\cdots\cup X_s.$$

Since all germs $X_1,\dots,X_s$, $\overline{\Gamma}^{\text{an}}_x$ are irreducible of the same dimension, there exists an index $j=1,\dots,s$ such that $\overline{\Gamma}^{\text{an}}_x=X_j$, which implies (4.4.1).

Next, we prove (i). We fix a stratum $\Sigma\in\mathcal{G}$ contained in Z and a point $a\in\Sigma$ with $r=\operatorname{mult}(Z,a)\geq\operatorname{mult}(Z,x)$ for every $x\in\Sigma$, and set $\mathcal{G}_\Sigma=\{\Gamma\in\mathcal{G}:\Sigma\subset\overline{\Gamma}\subset Z\}$. We claim that

(4.4.2) *There exist strata $\Gamma_1,\dots,\Gamma_r\in\mathcal{G}_\Sigma$ such that $\overline{Z}^{\text{an}}_x=\overline{\Gamma}^{\text{an}}_{1x}\cup\cdots\cup\overline{\Gamma}^{\text{an}}_{rx}$ is the decomposition of the analytic germ $\overline{Z}^{\text{an}}_x$ into irreducible components for every $x\in\Sigma$. In particular,* $\operatorname{mult}(Z,\cdot)\equiv r$ *is constant on Σ.*

Indeed, the union $\Delta=\bigcup_{\Gamma\in\mathcal{G}_\Sigma}\Gamma$ is an open neighborhood of Σ in Z (see (2.3.1)), so that $\Delta_x=Z_x$ for every $x\in\Sigma$. Now, observe that

$$\bigcup_{\Gamma\in\mathcal{G}_\Sigma}\overline{\Gamma}^{\text{an}}_a=\overline{\Delta}^{\text{an}}_a=\overline{Z}^{\text{an}}_a=A_1\cup\cdots\cup A_r,$$

where $A_1, \dots, A_r$ are distinct, non-singular hypersurface germs; by (4.4.1) for each $i = 1, \dots, r$ we have $A_i = \overline{\Gamma}^{\mathrm{an}}_{ia}$ for some $\Gamma_i \in \mathcal{G}_\Sigma$ of dimension $m-1$. So far, the strata $\Gamma_1, \dots, \Gamma_r$ already obtained depend on the chosen point $a \in \Sigma$, but we see readily that they do not.

Indeed, for any other point $x \in \Sigma$ we have, by the choice of $a \in \Sigma$,

$$\overline{\Gamma}^{\mathrm{an}}_{ix} \subset \overline{Z}^{\mathrm{an}}_x = X_1 \cup \dots \cup X_s,$$

where $s \leq r$ and $X_1, \dots, X_s$ are distinct, non-singular hypersuface germs at x. Thus, by (4.4.1) every $\overline{\Gamma}^{\mathrm{an}}_{ix}$ coincides with one of the X_k's, and what we must see is that $\overline{\Gamma}^{\mathrm{an}}_{ix} \neq \overline{\Gamma}^{\mathrm{an}}_{jx}$ for $i \neq j$. Consider for $i \neq j$ the set

$$Q_{ij} = \{x \in \Sigma : \overline{\Gamma}^{\mathrm{an}}_{ix} \neq \overline{\Gamma}^{\mathrm{an}}_{jx}\} = \{x \in \Sigma : \dim(\overline{\Gamma}^{\mathrm{an}}_{ix} \cap \overline{\Gamma}^{\mathrm{an}}_{jx}) < m-1\}.$$

The first description shows, by using (4.4.1), that Q_{ij} is closed in Σ (just observe that its complement in Σ is open), while the second shows that Q_{ij} is open in Σ. Since $a \in Q_{ij}$ and Σ is connected we conclude that $Q_{ij} = \Sigma$, as desired. The claim (4.4.2) is proved.

Thus, for each $\Sigma \in \mathcal{G}$ contained in Z, the restriction $\operatorname{mult}(Z, \cdot)|_\Sigma$ is constant and so Σ is contained in some $Z^{(r)}$. Since $\mathcal{G}$ is compatible with Z, we deduce that each $Z^{(r)}$ is a finite union of some (maybe none) $\Sigma \in \mathcal{G}$; hence, in particular, each $Z^{(r)}$ is semialgebraic and so statement (i) is proved.

Now, we show (ii). Of course, it is enough to prove that each $\Sigma \subset Z^{(r)}$ can be covered by finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that $Z \cap U \subset \{u_1 \cdots u_r = 0\}$. To that end, consider (with the notation of (4.4.2)) $T_i = \overline{\Gamma_i}$ for $i = 1, \dots, r$. For each $x \in T_i$ we have $\overline{T}^{\mathrm{an}}_{ix} = \overline{\Gamma}^{\mathrm{an}}_{ix}$, which is a non-singular hypersurface germ by (4.4.1). Hence, by Proposition 1.2, there exists a Nash hypersurface $S_i \subset \mathbb{R}^m$ containing T_i, with $S_{ix} = \overline{T}^{\mathrm{an}}_{ix}$ for $x \in T_i$. In particular, by (4.4.2),

$$Z_x \subset \overline{Z}^{\mathrm{an}}_x = \overline{\Gamma}^{\mathrm{an}}_{1x} \cup \dots \cup \overline{\Gamma}^{\mathrm{an}}_{rx} = S_{1x} \cup \dots \cup S_{rx} \tag{$\bullet$}$$

for all $x \in \Sigma$. By means of [BCR, 9.3.10], we find a finite covering of S_i by semialgebraic open subsets U_{ij} of $\mathbb{R}^m$ and Nash functions $h_{ij} : U_{ij} \to \mathbb{R}$ such that $S_i \cap U_{ij} = \{h_{ij} = 0\}$ and $\operatorname{rk}(h_{ij}) = 1$ everywhere. In fact, since any affine Nash manifold of dimension $m-1$ can be covered by finitely many open semialgebraic subsets Nash diffeomorphic to $\mathbb{R}^{m-1}$, we may assume that the sets $V_{ij} = S_i \cap U_{ij}$ are Nash diffeomorphic to $\mathbb{R}^{m-1}$.

Back to Σ, note that it is contained in $T_1 \cap \dots \cap T_r \subset S_1 \cap \dots \cap S_r$, so that:

(4.4.3) *The intersections $U_{1j_1} \cap \dots \cap U_{rj_r}$ form an open semialgebraic covering of Σ. Therefore we work on every such intersection separately: On each $U = U_{1j_1} \cap \dots \cap U_{rj_r}$ we have r Nash functions $h_i = h_{ij_i}|_U : U \to \mathbb{R}$ of constant rank one such that $S_i \cap U = \{h_i = 0\}$.*

In particular, $Z_x \subset S_{1x} \cup \dots \cup S_{rx}$ for all $x \in U \cap \Sigma$ (see ($\bullet$) above), and so there exists an open, not necessarily semialgebraic, neighborhood $U' \subset U$ of $\Sigma \cap U$ such that $Z \cap U' \subset S_1 \cup \dots \cup S_r$. Consequently,

$$\Sigma \cap U \subset U' \subset U \setminus (Z \cap U \setminus (S_1 \cup \dots \cup S_r)),$$

that is,

$$W = \operatorname{Int}_M(U \setminus (Z \cap U \setminus (S_1 \cup \cdots \cup S_r)) \subset U$$

is an open semialgebraic neighborhood of $\Sigma \cap U$ such that

$$\Sigma \cap U = \Sigma \cap W \subset Z \cap W \subset S_1 \cup \cdots \cup S_r.$$

Notice that:

(4.4.4) *For each $x \in Z \cap W$ with $\operatorname{mult}(Z, x) = r$, the S_{ix}'s are the irreducible components of $\overline{Z_x}^{\mathrm{an}}$. Moreover, since this last germ has a representative which has only normal crossings at x, the Nash map $h = (h_1, \dots, h_r) : W \to \mathbb{R}^r$ has rank r at x.*

Now, for every choice of indices $\nu = (\nu_1, \dots, \nu_{m-r}) \in \{1, \dots, m\}^{m-r}$ with $\nu_i \neq \nu_j$ if $i \neq j$, consider the Nash map

$$\psi_\nu : W \to \mathbb{R}^m, \qquad x = (x_1, \dots, x_m) \mapsto (h(x), x_{\nu_1}, \dots, x_{\nu_{m-r}}),$$

and the (possibly empty) open semialgebraic subset

$$W_\nu = \{x \in W : \operatorname{rk}(\psi_\nu)(x) = m\} \subset \mathbb{R}^m.$$

Since, by (4.4.2) and (4.4.4), $\operatorname{rk}(h) \equiv r$ on $\Sigma \cap W$, we have $\Sigma \cap W \subset \bigcup_\nu W_\nu \cap W$. Consequently, we continue the argument on each W_ν separately; we denote for the sake of simplicity $A = W_\nu$ and $\psi = \psi_\nu$ for those ν such that $W_\nu \neq \emptyset$.

First, since the Nash map $\psi|_A : A \to \mathbb{R}^m$ is a local diffeomorphism, there is a finite open semialgebraic covering $A_1, \dots, A_p$ of A such that each restriction $\psi|_{A_\ell} : A_\ell \to \psi(A_\ell)$ is a homeomorphism, hence a Nash diffeomorphism, onto its image, which is an open semialgebraic subset of $\mathbb{R}^m$ (see [BCR, 9.3.9]). Since $\psi|_{A_\ell}$ is a diffeomorphism, $(y_1, \dots, y_m) = \psi(x)$ is a system of coordinates in A_ℓ; moreover, $\psi(S_i \cap A_\ell) \subset \{y_i = 0\}$ for $i = 1, \dots, r$. Thus,

$$\psi(\Sigma \cap A_\ell) \subset \psi(Z \cap A_\ell) \subset \psi((S_1 \cup \cdots \cup S_r) \cap A_\ell) \subset \{y_1 \cdots y_r = 0\}$$

and

$$\overline{\psi(Z \cap A_\ell)_y}^{\mathrm{an}} = \{y_1 \cdots y_r = 0\}$$

for all $y = \psi(x) \in \psi(Z \cap A_\ell)$ with $\operatorname{mult}(Z, x) = r$. In addition, we obtain $\psi(\Sigma \cap A_\ell) \subset \{y_1 = 0, \dots, y_r = 0\}$, because $\Sigma \subset S_1 \cap \cdots \cap S_r$.

This proves (ii) of Proposition 4.4 except for the fact that $\Omega_\ell = \psi(A_\ell)$ need not be $\mathbb{R}^m$. But:

(4.4.5) *Ω_ℓ contains a smaller neighborhood of $\psi(\Sigma \cap A_\ell)$, which is a finite union of open semialgebraic sets Nash diffeomorphic to $\mathbb{R}^m$, by diffeomorphisms that preserve the coordinate hyperplanes $y_1 = 0, \dots, y_r = 0$.*

Indeed, cover

$$\Delta_\ell = \{y_1 = 0, \dots, y_r = 0\} \cap \Omega_\ell \subset \{0\} \times \mathbb{R}^{m-r} \equiv \mathbb{R}^{m-r}$$

with finitely many semialgebraic sets Δ_ℓ^k Nash diffeomorphic to $\mathbb{R}^{m-r}$, say via Nash diffeomorphisms $v^k : \Delta_\ell^k \to \mathbb{R}^{m-r}$. Then choose a tubular neighborhood of Δ_ℓ^k in Ω_ℓ as follows. Approximate the distance to the closed semialgebraic set $\mathbb{R}^m \setminus \Omega_\ell$ by a strictly positive Nash function $\alpha : \Delta_\ell \to \mathbb{R}$ so that

$$\{(y', y'') \in \mathbb{R}^r \times \mathbb{R}^{m-r} : y'' \in \Delta_\ell,\ \|y'\| < \alpha(y'')\} \subset \Omega_\ell.$$

Then consider

$$\Omega_\ell^k = \{(y', y'') \in \mathbb{R}^r \times \mathbb{R}^{m-r} : y'' \in \Delta_\ell^k,\ \|y'\| < \alpha(y'')\}$$

and the Nash diffeomorphism

$$u : \Omega_\ell^k \to \mathbb{R}^r \times \mathbb{R}^{m-r}, \qquad (y', y'') \mapsto \left(\frac{1}{\sqrt{\alpha(y'')^2 - \|y'\|^2}} y', v^k(y'')\right),$$

which preserves the hyperplanes $y_1 = 0, \dots, y_r = 0$. We are done. □

Remark 4.5. Before we progress further, we notice a crucial difference between the deceivingly similar conditions

$$X_x = \{u_1 \cdots u_r = 0\} \quad \text{in Definition 1.4, and}$$
$$\overline{Z}_x^{\mathrm{an}} = \{v_1 \cdots v_r = 0\} \quad \text{in Proposition 4.4.}$$

In the first case X is a Nash set, which implies $X_y = \{u_1 \cdots u_r = 0\}_y$ for y close enough to x, while in the second we may have $\overline{Z}_y^{\mathrm{an}} \neq \{v_1 \cdots v_r = 0\}_y$ for y arbitrarily close to x. For instance, look at the semialgebraic set $Z = \{z = 0\} \cup \{x = 0,\ y^2 + (z-1)^2 \le 1\} \subset \mathbb{R}^3$ close enough to the origin. This "lack of continuity" of the correspondence $y \mapsto \overline{Z}_y^{\mathrm{an}}$ must be taken into account.

With this in mind, we obtain the following finiteness result concerning local normal crossings, which includes Theorem 1.6 as a particular case.

Proposition 4.6. *Let $Z \subset M$ be a locally compact semialgebraic subset. Suppose that at every point $x \in Z$ there is an integer $r \geq 1$ and a coordinate system $(v_1, \dots, v_m)$ of M such that $\overline{Z}_y^{\mathrm{an}} = \{v_1 \cdots v_r = 0\}_y$ for every $y \in Z$ close enough to x. Then Z can be covered by finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that $\overline{Z}_x^{\mathrm{an}} = \{u_1 \cdots u_r = 0\}_x$ for all $x \in Z \cap U$.*

Proof. By Proposition 4.4, we can cover Z by finitely many open semialgebraic sets U, for each of which there is a Nash diffeomorphism $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that

$$Z \cap U \subset \{u_1 \cdots u_r = 0\}$$

(where r depends on U). Furthermore, every point $x \in Z$ belongs to some U with $r = \mathrm{mult}(Z, x)$. We will modify these U's to get the condition closing the statement.

First of all:

(4.6.1) *Let $x \in Z$ have* $\mathrm{mult}(Z, x) = r$. *Then $\overline{Z}_y^{\mathrm{an}} = \{u_1 \cdots u_r = 0\}_y$ for all $y \in Z$ close enough to x.*

Indeed, clearly $\overline{Z}^{\text{an}}_x = \{u_1 \cdots u_r = 0\}_x$ and by the continuity hypothesis, there is a coordinate system $(v_1, \dots, v_m)$ of M at x such that $\overline{Z}^{\text{an}}_y = \{v_1 \cdots v_r = 0\}_y$ for $y \in Z$ close enough to x. Of course the v_i's need not coincide with the u_i's, but it follows from the equality

$$\overline{Z}^{\text{an}}_x = \{u_1 \cdots u_r = 0\}_x = \{v_1 \cdots v_r = 0\}_x$$

that, up to reordering, $\{u_i = 0\}$ and $\{v_i = 0\}$ have the same germ at x, hence they have it at all y close enough, so that

$$\overline{Z}^{\text{an}}_y = \{u_1 \cdots u_r = 0\}_y = \{v_1 \cdots v_r = 0\}_y$$

for y close enough to x. This is (4.6.1).

From (4.6.1) we deduce that the set $S = \{x \in U : \overline{Z}^{\text{an}}_x = \{u_1 \cdots u_r = 0\}_x big\}$ is a neighborhood in U of $T^{(r)} = \{x \in Z \cap U : \text{mult}(Z, x) = r\}$ and we claim that

(4.6.2) *The set S is semialgebraic.*

For, $Z \cap U \subset \{u_1 \cdots u_r = 0\}$, and so $\overline{Z}^{\text{an}}_x \subset \{u_1 \cdots u_r = 0\}_x$ for all $x \in Z \cap U$. Consequently, $x \in S$ if and only if $\text{mult}(Z, x)$ coincides with the number $\mathrm{s}(x)$ of the hypersurfaces $\{u_1 = 0\}, \dots, \{u_r = 0\}$ that contain x. Now note that $\text{mult}(Z, x)$ is a semialgebraic function by Proposition 4.4(i) and $\mathrm{s}(x)$ is a semialgebraic function too because it is defined through the semialgebraic functions u_i. We conclude that the set S, defined through the equality $\text{mult}(Z, x) = \mathrm{s}(x)$, is a semialgebraic set, as desired.

Once this is shown, the interior U' of S in U is an open semialgebraic neighborhood of $T^{(r)}$, and replacing U by U' we can suppose $\overline{Z}^{\text{an}}_x = \{u_1 \cdots u_r = 0\}_x$ for all $x \in Z \cap U$. However in doing this the diffeomorphism $u = (u_1, \dots, u_m) : U \to \mathbb{R}^m$ may not be onto anymore. Thus we have to split it again to complete the proof. This is done as in (4.4.5); we do not repeat the details. □

Finally, we show how (under some conditions) a semialgebraic set can be embedded into a Nash set which has only normal crossings.

(4.7) *Proof of Theorem 1.7.* Since Z is locally compact, we can replace M by an open semialgebraic neighborhood of Z so that Z is closed in M. Then, by Proposition 4.6, M can be covered with finitely many open semialgebraic subsets U_ℓ $(1 \le \ell \le p)$ equipped with Nash diffeomorphisms $u^\ell = (u^\ell_1, \dots, u^\ell_m) : U_\ell \to \mathbb{R}^m$ such that either $Z \cap U_\ell = \emptyset$ or

$$\overline{Z}^{\text{an}}_x = \{u^\ell_1 \cdots u^\ell_r = 0\}_x$$

for all $x \in Z \cap U_\ell$ and some $1 \le r \le m$. We denote by Y_ℓ the Nash subset of U_ℓ defined by the equation $u^\ell_1 \cdots u^\ell_r = 0$.

Now we shrink $\{U_\ell\}$ to a covering $M = \bigcup_\ell V_\ell$ with open semialgebraic sets V_ℓ such that $\overline{V_\ell} \subset U_\ell$ (see (2.1.2)). Also, we pick a stratification $\mathcal{G}$ of M compatible with Z, the sets $Z^{(r)}$, the U_ℓ's, the V_ℓ's and their closures $\overline{V_\ell}$ and the hypersurfaces $\{u^\ell_i = 0\}$.

In this situation, let $\mathcal{F}$ be the collection of all strata $\Gamma \in \mathcal{G}$ whose closures meet Z. Then the set $\Omega = \bigcup_{\Gamma \in \mathcal{F}} \Gamma$ is an open semialgebraic neighborhood of Z in M (see (2.3.1)). Consider the sets

$$\Omega_\ell = \Omega \cap V_\ell \quad \text{and} \quad X_\ell = Y_\ell \cap \Omega_\ell \quad \text{for } 1 \le \ell \le p,$$

and the union $X = \bigcup_\ell X_\ell$. We claim that the pair $X \subset \Omega$ solves our problem.

First of all we see that:

(4.7.1) $X_\ell \cap \Omega_k = X_k \cap \Omega_\ell$ *for any two indices* ℓ, k.

Indeed, $X_\ell \cap \Omega_k$ is a union of some strata Γ in $\mathcal{F}$, that is, those with $\overline{\Gamma} \cap Z \neq \emptyset$. Fix such a Γ and pick a point $x \in \overline{\Gamma} \cap Z$. Since $\overline{\Gamma} \subset \overline{V_\ell} \cap \overline{V}_k \subset U_\ell \cap U_k$, we have

$$\Gamma_x \subset X_{\ell x} \subset Y_{\ell x} = \overline{Z}_x^{\text{an}} = Y_{kx}.$$

This implies $\Gamma \subset Y_k$, because $\Gamma \subset U_k$ is a connected affine Nash manifold, and Y_k a Nash subset of U_k.

We have thus proved the inclusion $X_\ell \cap \Omega_k \subset Y_k \cap \Omega_k \subset X_k$, which gives one inclusion in (4.7.1). The other follows by symmetry.

From (4.7.1) we deduce that $X_x = Y_{\ell x} = \{u_1^\ell \cdots u_r^\ell = 0\}$ for $x \in \Omega_\ell$ $(1 \leq \ell \leq p)$. Hence X is a coherent local analytic subset of Ω, and this implies that X is a global analytic subset of Ω. As X is semialgebraic, from Proposition 2.8(ii) we conclude that X is a Nash subset of Ω.

Finally, since the Nash closure X' of Z in Ω is a union of irreducible components of X, we deduce that X' has only normal crossings in Ω (see Lemma 4.1) and we are done. □

5. Nash normal crossings divisors of a Nash manifold

Here we complete the results of the previous section with a quick review of the global notion of normal crossings. As usual, let M be an affine Nash manifold of dimension m and let $\mathcal{N}(M)$ denote its ring of Nash functions. Recall that a *Nash normal crossings divisor of* M is a Nash subset $X \subset M$ whose irreducible components are non-singular hypersurfaces $X_1, \ldots, X_p$ of M in general position (see Definition 1.8). We understand locally the notion of Nash normal crossings divisor as follows:

Proposition 5.1. *Let* $X \subset M$ *be a Nash subset and let* $J = \mathcal{J}_{\mathcal{N}}(X)$ *be its ideal in* $\mathcal{N}(M)$. *Let* $x \in X$ *and suppose that all irreducible components* $X_1, \ldots, X_r$ *of* X *through the point* x *are hypersurfaces of* M. *Denote by* $\mathfrak{m} = \mathfrak{m}_x$ *the maximal ideal of* x *in* $\mathcal{N}(M)$. *The following assertions are equivalent:*

(i) *The point* x *is regular in the algebraic sense in all* X_i, $i = 1, \ldots, r$, *and there exist local coordinates* $\{v_1, \ldots, v_m\}$ *of* M *at* x *such that* $X_x = \{v_1 \cdots v_r = 0\}_x$.

(ii) *There exists a regular system of parameters* $\{u_1, \ldots, u_m\}$ *of the local regular ring* $\mathcal{N}(M)_\mathfrak{m}$ *such that* $J\mathcal{N}(M)_\mathfrak{m}$ *is generated by the product* $u_1 \cdots u_r$.

Proof. Set $J_i = \mathcal{J}_{\mathcal{N}}(X_i) \subset \mathfrak{m}$, $i = 1, \ldots, r$, so that $J = J_1 \cap \cdots \cap J_r$; each J_i is a height one prime ideal of $\mathcal{N}(M)$. Therefore, each extension $J_i\mathcal{N}(M)_\mathfrak{m}$ is a height one prime ideal, and since $\mathcal{N}(M)_\mathfrak{m}$ is regular, $J_i\mathcal{N}(M)_\mathfrak{m}$ is principal. Hence, $J_i\mathcal{N}(M)_\mathfrak{m} = f_i\mathcal{N}(M)_\mathfrak{m}$ for some $f_i \in \mathfrak{m}$, and $J\mathcal{N}(M)_\mathfrak{m}$ is generated by the product $f_1 \cdots f_r$. This last fact holds because f_i, f_j with $i \neq j$ are relatively prime in the unique factorization domain $\mathcal{N}(M)_\mathfrak{m}$.

Now, suppose (i) holds true. Then, the germs $\{v_i = 0\}_x$ are the irreducible components of X_x, and consequently, up to reordering, $X_{ix} = \{v_i = 0\}_x$. Moreover, each irreducible component $X_{i,x}$ is a non-singular hypersurface germ, and the ideal $\mathcal{J}_{\mathcal{N}}(X_{ix}) \subset \mathcal{N}_{M,x}$ is generated by v_i. Since the ideal $\mathcal{J}(X_{ix})$ contains J_i, v_i divides f_i. On the other hand, the regularity condition says that each ring $\mathcal{N}(M)_{\mathfrak{m}}/J_i\,\mathcal{N}(M)_{\mathfrak{m}}$ is regular of dimension $m-1$, and so we can add to each $u_i = f_i$ another $m-1$ parameters to get a regular system of $\mathcal{N}(M)_{\mathfrak{m}}$. Then for a fixed i that regular system is a regular system of $\mathcal{N}_{M,x}$ too because the completions of both rings coincide; hence, u_i is irreducible in $\mathcal{N}_{M,x}$. We conclude that $u_i = w_i v_i$ for some unit $w_i \in \mathcal{N}_{M,x}$.

Since $v_1, \dots, v_m$ is a regular system of $\mathcal{N}_{M,x}$, we deduce that $u_1, \dots, u_r, v_{r+1}, \dots, v_m$ is a regular system of parameters of $\mathcal{N}_{M,x}$ too. This implies (using Jacobian criteria, for instance) that there are $u_{r+1}, \dots, u_m \in \mathfrak{m}$ such that $u_1, \dots, u_r, u_{r+1}, \dots, u_m$ is a regular system of parameters of $\mathcal{N}(M)_{\mathfrak{m}}$. We have obtained (ii).

Conversely, suppose we have (ii). Then $\{u_1, \dots, u_m\}$ are local analytic coordinates and $X_x = \{u_1 \cdots u_r = 0\}_x$. To prove the regularity of x as a point of X_i in the algebraic sense, notice that since $J\mathcal{N}(M)_{\mathfrak{m}}$ is generated by the element $u_1 \cdots u_r$, the associated primes $J_i\,\mathcal{N}(M)_{\mathfrak{m}}$ are generated each by one u_i; and we may assume that $X_{i,x} = \{u_i = 0\}_x$ for $i = 1, \dots, r$. Thus, the local ring $\mathcal{N}(M)_{\mathfrak{m}}/J_i\mathcal{N}(M)_{\mathfrak{m}} = \mathcal{N}(M)_{\mathfrak{m}}/u_i\,\mathcal{N}(M)_{\mathfrak{m}}$ is regular. □

From the previous result we immediately deduce:

Corollary 5.2. *A Nash subset X which has only normal crossings in M is a Nash normal crossings divisor of M if and only if its Nash irreducible components are all non-singular hypersurfaces of M.*

Proof. The only thing to remark here is that in a closed Nash submanifold Y of M, all points are regular in the algebraic sense (see 2.6). □

Remark 5.3. The last proof points to the importance of algebraic regularity. This is stronger than regularity as defined in (2.1.5) for arbitrary semalgebraic sets. In Section 1 we quoted the comparison result 2.6, and advanced the following improvement:

Let X be a Nash subset of an affine Nash manifold M which is a coherent analytic set. Then a point $x \in X$ is regular in the algebraic sense if and only if it is regular in the sense of (2.1.5).

Indeed, if a point $x \in X$ is regular in the sense of (2.1.5), then analytic germ X_x is non-singular, and the ring $\mathcal{O}_{M,x}/\mathcal{J}_{\mathcal{O}}(X_x)$ is a local regular ring. But X is coherent, that is, $\mathcal{J}_{\mathcal{O}}(X)$ generates $\mathcal{J}_{\mathcal{O}}(X_x)$. From this and Proposition 2.8 we deduce that also $\mathcal{J}_{\mathcal{N}}(X)$ generates $\mathcal{J}_{\mathcal{O}}(X_x)$. Thus

$$\mathcal{O}_{M,x}/\mathcal{J}_{\mathcal{N}}(X)\mathcal{O}_{M,x} = \mathcal{O}_{M,x}/\mathcal{J}_{\mathcal{N}}(X_x).$$

This local regular ring has the same adic completion as $\mathcal{N}(M)_{\mathfrak{m}}/\mathcal{J}_{\mathcal{N}}(X)\mathcal{N}(M)_{\mathfrak{m}}$. Consequently, this last local ring is regular too, and we conclude that x is regular in the algebraic sense.

As said before, the simplest example of coherent analytic sets are analytic manifolds, and in particular Nash manifolds, a case mentioned in (2.6).

6. Affine Nash manifolds with corners

In this last section we apply the previous results to affine Nash manifolds with corners. Let $N \subset \mathbb{R}^n$ be an affine Nash manifold with corners, ∂N its boundary and $m = \dim(N)$. First we see how N embeds into an affine Nash manifold where the Nash closure of ∂N has only normal crossings.

(6.1) *Proof of Theorem 1.11*. Recall that for each point $x \in N$ there is a Nash diffeomorphism $(v_1, \dots, v_n) : V^x \to \mathbb{R}^n$ defined in an open neighborhood $V^x \subset \mathbb{R}^n$ of x that maps x to the origin and such that

$$V^x \cap N = \{v_1 \geq 0, \dots, v_r \geq 0, v_{m+1} = 0, \dots, v_n = 0\}$$

(r depends on x). As mentioned in the Introduction, the analytic set germ $\overline{N}^{\mathrm{an}}_x = \{v_{m+1} = 0, \dots, v_n = 0\}_x$ is regular of dimension m and by Proposition 1.2, N is a closed subset of an affine Nash manifold $M \subset \mathbb{R}^n$ of dimension m; notice that $N \setminus \partial N$ is open in M. On the other hand, $(v_1, \dots, v_m) : V^x \cap M \to \mathbb{R}^m$ are coordinates of M at x, and

$$\overline{\partial N}^{\mathrm{an}}_y = \{v_1 \cdots v_r = 0, v_{m+1} = 0, \dots, v_n = 0\}_y$$

for all $y \in V^x \cap N$. Consequently, Theorem 1.7 applies with $Z = \partial N$, and replacing M by an open semialgebraic neighborhood of ∂N we can suppose that the Nash closure Y of ∂N in M has only normal crossings in M and that $Y_x = \overline{\partial N}^{\mathrm{an}}_x$ for all $x \in \partial N$.

Now, by Theorem 1.6, we can cover Y with finitely many open semialgebraic subsets U of M equipped with Nash diffeomorphisms $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ such that $U \cap Y = \{u_1 \cdots u_r = 0\}$. In particular, $\overline{\partial N}^{\mathrm{an}}_x = \{u_1 \cdots u_r = 0\}_x$ for all $x \in U \cap \partial N$. We keep only those U's that meet ∂N, and denote by Ω their union. Then $\Omega \cup (N \setminus \partial N)$ is an open semialgebraic neighborhood of N in M, and we can merely suppose $M = \Omega \cup (N \setminus \partial N)$. Since $N \setminus \partial N$ is an affine Nash manifold of dimension m, it can be covered with open semialgebraic sets Nash diffeomorphic to $\mathbb{R}^m$. Set $X' = Y \cap \Omega$. We claim that, substituting where needed $-u_i$ for u_i,

(6.1.1) $U \cap N = \{u_1 \geq 0, \dots, u_m \geq 0\}$ *and* $U \cap N \cap X' = U \cap \partial N$ *for all U in the semialgebraic covering of ∂N constructed above.*

To prove this, let us look at the sets $U \cap N \setminus X' \subset U \cap N \setminus \partial N$. Since the larger one is open and closed in $U \setminus \partial N$ and $\partial N \subset X'$, the smaller one is open and closed in $U \setminus X' = \{u_1 \cdots u_r \neq 0\}$. Hence $U \cap N$ contains a union of connected components of $\{u_1 \cdots u_r \neq 0\}$. Let $x \in U$ be the point with $u_1(x) = 0, \dots, u_m(x) = 0$. This point is adherent to all connected components of $\{u_1 \cdots u_r \neq 0\}$; hence $x \in N$. If $x \notin \partial N$, then N is a neighborhood of x in U and touches all connected components of $\{u_1 \cdots u_r \neq 0\}$, which means that N contains all of them. But N is closed in M, so that $U \subset N$. From this we see that $U \cap \partial N = \emptyset$, a contradiction. So $x \in \partial N$ and we have $\overline{\partial N}^{\mathrm{an}}_x = X'_x = \{u_1 \cdots u_r = 0\}_x$. Consequently, $N_x = \{u_1 \geq 0, \dots, u_r \geq 0\}_x$ (substituting where needed $-u_i$ for u_i) and $U \cap N$ cannot contain but the component of $\{u_1 > 0, \dots, u_r > 0\}$. This implies (6.1.1).

We deduce readily from (6.1.1) that $N \cap X' = \partial N$, and what remains is to prove that X' is a Nash subset of $M = \Omega \cup (N \setminus \partial N)$. To show this, we notice that X' is closed in M, because $X' = Y \cap \Omega$ is closed in Ω and X' does not meet the open set $N \setminus \partial N$. Hence, since X' is also Nash in Ω, [Sh, II.5.3] tells us that X' is a Nash subset of M. Finally, since the Nash closure X of ∂N in M is a union of some (maybe all) irreducible components of X', we deduce that X satifies all the conditions in the statement (see Lemma 4.1). □

Next, we characterize when an affine Nash manifold with corners N is contained in an affine Nash manifold M where the Nash closure of ∂N is a Nash normal crossings divisor. We separate our argument into several implications. First we recall the notation: the faces D_i of N are the closures in N of the connected components C_i of $\operatorname{Reg}(\partial N)$. We have

$$\partial N = \bigcup\nolimits_i D_i, \qquad D_i \cap C_j = \emptyset \text{ for } j \neq i, \qquad \text{and} \qquad D_i \setminus C_i \subset \bigcup_{j \neq i} D_j .$$

(6.2) *Proof of Theorem 1.12*, (i)⇒(ii)⇒(iii)⇒(iv). Let N be embedded in M as in (i). The Nash closure X_i of a face D_i is that of the connected affine Nash manifold C_i, and so it is irreducible. Since $\partial N = \bigcup_i D_i$, the X_i's are the irreducible components of the Nash closure X of ∂N, which by hypothesis are non-singular. This is condition (ii), from which we now deduce (iii).

Fix a point $x \in \partial N$. The number r of connected components of the germ $\operatorname{Reg}(\partial N)_x$ is at least the number of components C_i adherent to x, which is the number s of faces $D_i = \overline{C_i}$ through x. Thus, $r \geq s$ and let us see now that $s \geq r$.

By hypothesis, each face D_i is contained in an affine Nash manifold X_i. Since every X_i is non-singular, every germ X_{ix} is irreducible. Assume $D_1, \dots, D_s$ are the faces through x. Then

$$\partial N_x = D_{1x} \cup \cdots \cup D_{sx} \quad \text{and} \quad \overline{\partial N_x}^{\text{an}} \subset X_{1x} \cup \cdots \cup X_{sx}.$$

Thus $\overline{\partial N_x}^{\text{an}}$ has no more than s irreducible components, that is, $s \geq r$ and (iii) is proved.

Finally, let us deduce (iv) from (iii). Fix $x \in \partial N = \bigcup_i D_i$ and say $D_1, \dots, D_k$ are the faces through x. There is an open neighborhood U of x in $\mathbb{R}^n$ and a Nash diffeomorphism $(u_1, \dots, u_n) : U \to \mathbb{R}^n$ that maps x to the origin such that

$$U \cap N = \{u_1 \geq 0, \dots, u_r \geq 0,\ u_{m+1} = 0, \dots, u_n = 0\}.$$

Consequently, the connected components of $\operatorname{Reg}(\partial N)_x$ are these:

$$C'_{jx} = \{u_1 > 0, \dots, u_j = 0, \dots, u_r > 0,\ u_{m+1} = 0, \dots, u_n = 0\}, \qquad 1 \leq j \leq r.$$

Using now (iii) we deduce that the germs C_{ix} for $i = 1, \dots, r$ are also the connected components of $\operatorname{Reg}(\partial N)_x$. Therefore $k = r$ and, up to the order, $C_{ix} = C'_{ix}$. Thus

$$D_{ix} = \overline{C}_{ix} = \{u_1 \geq 0, \dots, u_i = 0, \dots, u_r \geq 0, u_{m+1} = 0, \dots, u_n = 0\}.$$

This shows that close enough to x the face D_i is an affine Nash manifold with corners. Thus, since the D_i's are semialgebraic, we conclude that they are affine Nash manifolds with corners, as wanted. □

(6.3) *Proof of Theorem 1.12*, (iv)⇒(i). By Theorem 1.11 there exists an affine Nash manifold $M \subset \mathbb{R}^n$ containing N as a closed subset, with $\dim(M) = \dim(N) = m$, such that the Nash closure X of ∂N in M has only normal crossings in M and $N \cap X = \partial N$. We must find an open semialgebraic neighborhood U of N in M where the Nash closure of ∂N is a Nash normal crossings divisor. We proceed as follows.

By (iv) each face D_i is an affine Nash manifold with corners, with $\dim(D_i) = m-1$. By Theorem 1.11, D_i is a closed subset of an affine Nash manifold Y_i of dimension $m-1$. Since $D_i \subset M$ is connected, we can choose $Y_i \subset M$ also connected. Then Y_i is contained in X because so is D_i; consequently, $N \cap Y_i \subset \partial N$.

Now, Y_i is locally compact, so that $\overline{Y_i} \setminus Y_i$ is closed in M. Since D_i is also closed in M and does not meet $\overline{Y_i} \setminus Y_i$, there is an open semialgebraic neighborhood U_i of D_i in M whose closure $\overline{U_i}$ does not meet $\overline{Y_i} \setminus Y_i$. We will need a further shrinking of U_i.

First, if $D_j \cap D_i = \emptyset$, we take U_i smaller to have $\overline{U_i} \cap D_j = \emptyset$, but if $D_j \cap D_i \neq \emptyset$ with $i \neq j$, we need a more delicate discussion. First, we claim that

(6.3.1) *The open semialgebraic set* $V_{ij} = \{x \in M : Y_{ix} \cap D_{jx} \subset D_{ix}\}$ *contains* D_i.

Indeed, if $x \in D_i$, then Y_{ix} is an irreducible component of $\overline{\partial N_x}^{\mathrm{an}}$, and $Y_{ix} \cap \partial N_x = D_{ix}$. Since $D_{jx} \subset \partial N_x$ we have

$$Y_{ix} \cap D_{jx} \subset Y_{ix} \cap \partial N_x = D_{ix}.$$

Thus $x \in V_{ij}$ as desired.

Now, (6.3.1) says that D_i does not meet the closed semialgebraic set $D_j \setminus V_{ij}$, and so we can reduce U_i so that $\overline{U_i} \cap D_j \subset V_{ij}$. By the definition of V_{ij} we deduce that

(6.3.2) $\overline{U_i} \cap Y_i \cap D_j \subset D_i$.

After all these shrinkings, write $Y_i' = U_i \cap Y_i$. We have

(6.3.3) $(\overline{Y_i'} \setminus Y_i') \cap N = \emptyset$.

Indeed, since $\overline{U_i} \cap (\overline{Y_i} \setminus Y_i) = \emptyset$, we have $\overline{Y_i'} \subset Y_i \subset X$, and so

$$\overline{Y_i'} \cap N \subset \overline{Y_i'} \cap N \cap X = \overline{Y_i'} \cap \partial N = \bigcup_j \overline{Y_i'} \cap D_j.$$

On the other hand, $\overline{Y_i'} \cap D_j \subset \overline{U_i} \cap Y_i \cap D_j$ and (6.3.2) gives

(6.3.4) $\overline{Y_i'} \cap D_j \subset D_i \subset Y_i \cap U_i = Y_i'$.

From this and the previous inclusion we deduce $\overline{Y_i'} \cap N \subset Y_i'$, and consequently (6.3.3).

Now, by (6.3.3), the open semialgebraic set $U = M \setminus \bigcup_i (\overline{Y_i'} \setminus Y_i')$ contains N. Moreover, by the definition itself, all the intersections $Y_i' \cap U$ are closed in U. Let Y_i'' denote the connected component of $Y_i' \cap U$ that contains D_i, which is again a closed submanifold of U, and consequently a Nash subset of U. Since the Y_i'''s are connected, they are the Nash closures in U of the D_i's, and they are irreducible; hence $Y = \bigcup_i Y_i''$ is the Nash closure in U of ∂N. Now, for every $x \in U$ the germ Y_x is a union of irreducible components of X_x, and since X has only normal crossings at x, also Y has only normal crossings at x (see Lemma 4.1). Finally, the irreducible components Y_i'' of Y are all non-singular hypersurfaces of U. Altogether, we conclude that $Y \subset U$ is a Nash normal crossings divisor. Thus, condition (i) follows. □

(6.4) *Proof of the additional assertion of Theorem 1.12.* Let $N \subset \mathbb{R}^n$ be an affine Nash manifold with corners, and let $M \subset \mathbb{R}^n$ be an affine Nash manifold containing N of the same dimension, say m. We write the boundary of N as the union of its faces, $\partial N = \bigcup_i D_i$, and denote by X_i the Nash closure of D_i in M. Clearly $X = \bigcup_i X_i$ is the Nash closure of ∂N in M, and each X_i is an irreducible Nash set (being the Nash closure of some connected component C_i of the affine Nash manifold $\operatorname{Reg}(\partial N)$). Consequently, $X = \bigcup_i X_i$ is a decomposition into irreducible components, except for the fact that there may be redundancies. We are to see that if the conditions of Theorem 1.12 hold true, then M can be chosen so that all X_i's are distinct and $X_i \cap N = D_i$ for each i.

First, by Theorem 1.12(i) we choose M such that X is a normal crossings divisor. Second, by Theorem 1.11 we can assume that $X \cap N = \partial N$ and $X_x = \overline{\partial N}_x^{\mathrm{an}}$ for all $x \in \partial N$. Then, suppose $X_i \cap N \neq D_i$ and pick $x \in X_i \cap N \setminus D_i \subset \partial N \setminus D_i$. It follows that X_{ix} is an irreducible component of $\overline{\partial N}_x^{\mathrm{an}}$. Since the irreducible components of $\overline{\partial N}_x^{\mathrm{an}}$ are the analytic closures of the germs at x of the C_j's adherent to x, and x is not adherent to C_i, we conclude that $X_{ix} = \overline{C}_{jx}^{\mathrm{an}} \subset X_{jx}$ for some $j \neq i$; since X_j and X_i are non-singular hypersurfaces, they coincide. Thus we see that the redundancies $X_i = X_j$ are the only obstruction we must take care of. We show now how to do it.

Suppose that X_1 is the Nash closure of the faces $D_1, \dots, D_p$ and does not meet the others: $X_1 \cap N = D_1 \cup \cdots \cup D_p$. We claim that those D_i's are disjoint. Indeed, if there were $y \in D_i \cap D_j$, then $X_{1y} \supset \overline{C}_{iy}^{\mathrm{an}} \cap \overline{C}_{jy}^{\mathrm{an}}$, and, since the two last germs are distinct of dimension $m-1$, X_{1y} would be reducible, a contradiction.

Now, since the D_i's are connected, closed and disjoint, they have connected open semialgebraic neighborhoods Y_i in X_1 with disjoint closures, and we consider the affine Nash manifold $M' = M \setminus \bigcup_i (\overline{Y}_i \setminus Y_i)$. Observe that

$$N \cap (\overline{Y}_i \setminus Y_i) = N \cap X_1 \cap (\overline{Y}_i \setminus Y_i) = \bigcup_{j=1}^{p} D_j \cap (\overline{Y}_i \setminus Y_i) = D_i \setminus Y_i = \emptyset,$$

and so $N \subset M'$; now, a little computation shows that each Y_i is an open and closed subset of $X_1 \cap M'$. Thus, the connected component Y_i' of Y_i that contains D_i is the Nash closure of D_i in M' and now $Y_i' \cap Y_j' = \emptyset$ if $i \neq j$. Thus we have replaced the common Nash closure X_1 of the faces D_i by different Nash closures for every one of them. □

From Theorem 1.12 we get:

Corollary 6.5. *Let N be an affine Nash manifold with corners whose faces are affine Nash manifolds with corners. Then the faces of the faces of N are again affine Nash manifolds with corners.*

Proof. Let C be a connected component of $\operatorname{Reg}(\partial N)$ and let $D = \overline{C}$ be the corresponding face of N. By Theorem 1.12 there exists an affine Nash manifold M containing N as a closed subset where the Nash closure X of ∂N in M is a Nash normal crossings divisor whose irreducible components X_i are the Nash closures of the faces D_i and $N \cap X_i = D_i$; say $D = D_1$. Then

$$\partial D = D \setminus C = D \cap \bigcup_{i \neq 1} D_i \subset D \cap \bigcup_{i \neq 1} X_i \subset X_1 \cap \bigcup_{i \neq 1} X_i.$$

We see that the Nash closure Y of ∂D in X_1 is contained in $X_1 \cap \bigcup_{i \neq 1} X_i$, which is a Nash normal crossings divisor of X_1. By Corollary 5.2, Y is a Nash normal crossings divisor of the affine Nash manifold X_1 and, from Theorem 1.12, it follows that every face of D is an affine Nash manifold with corners. □

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References

[AL] Adkins, W. A., Leahy, J. V.: A global real analytic Nullstellensatz. Duke Math. J. **43**, 81–86 (1976) Zbl 0353.32009 MR 0396991

[Ar] Artin, M.: Algebraic approximation of structures over complete local rings, Publ. Math. I.H.E.S. **36**, 23–58 (1969) Zbl 0181.48802 MR 0268188

[BCR] Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Ergeb. Math. Grenzgeb. 36, Springer, Berlin (1998) Zbl 0912.14023 MR 1659509

[Ca] Cartan, H.: Variétés analytiques réelles et variétés analytiques complexes. Bull. Soc. Math. France **85**, 77–99 (1957) Zbl 0083.30502 MR 0094830

[CRSh1] Coste, M., Ruiz, J. M., Shiota, M.: Approximation in compact Nash manifolds. Amer. J. Math. **117**, 905–927 (1995) Zbl 0873.32007 MR 1342835

[CRSh2] Coste, M., Ruiz, J. M., Shiota, M.: Separation, factorization and finite sheaves on Nash manifolds. Compos. Math. **103**, 31–62 (1996) Zbl 0885.14029 MR 1404997

[CSh] Coste, M., Shiota, M.: Nash functions on noncompact Nash manifolds. Ann. Sci. École Norm. Sup. (4) **33**, 139–149 (2000) Zbl 0981.14027 MR 1743722

[DK1] Delfs, H., Knebusch, M.: Separation, retractions and homotopy extension in semialgebraic spaces. Pacific J. Math. **114**, 47–71 (1984) Zbl 0548.14008 MR 0755482

[DK2] Delfs, H., Knebusch, M.: Locally Semialgebraic Spaces. Lecture Notes in Math. 1173, Springer, Berlin (1985) Zbl 0582.14006 MR 0819737

[FG] Fernando, J. F., Gamboa, J. M.: On the irreducible components of a semialgebraic set. Int. J. Math. **23**, no. 4, 1250031, 40 pp. (2012) Zbl 1271.14087 MR 2903188

[G] Galbiati, M.: Stratifications et ensemble de non-cohérence d'un espace analytique réel. Invent. Math. **34**, 113–128 (1976) Zbl 0314.32006 MR 0417443

[K] Knebusch, M.: Semialgebraic topology in the last ten years. In: Real Algebraic Geometry (Rennes, 1991), Lecture Notes in Math. 1524, Springer, 1–36 (1992) Zbl 0779.14023 MR 1226239

[Ł] Łojasiewicz, S.: Ensembles semi-analytiques. I.H.E.S., Bures-sur-Yvette (1964); http://perso. univ-rennes1.fr/michel.coste/Lojasiewicz.pdf

[MO] Margalef, J., Outerelo, E.: Differential Topology. North-Holland Math. Stud. 173, North-Holland, Amsterdam (1998) Zbl 0760.57001

[Mt] Matsumura, H.: Commutative Ring Theory. Cambridge Stud. Adv. Math. 8, Cambridge Univ. Press, Cambridge (1989) Zbl 0666.13002 MR 1011461

[N] Narasimhan, R.: Introduction to the Theory of Analytic Spaces. Lecture Notes in Math. 25, Springer, Berlin (1966) Zbl 0168.06003 MR 0217337

[Sh] Shiota, M.: Nash Manifolds. Lecture Notes in Math. 1269, Springer, Berlin (1987) Zbl 0629.58002 MR 0904479

[T] Tognoli, A.: Proprietà globali degli spazi analitici reali. Ann. Mat. Pura Appl. (4) **75**, 143–218 (1967) Zbl 0158.33001 MR 0236418

[WB] Whitney, H., Bruhat, F.: Quelques propriétés fondamentales des ensembles analytiques réels. Comment. Math. Helv. **33**, 132–160 (1959) Zbl 0100.08101 MR 0102094