ELSEVIER

Journal of Pure and Applied Algebra 179 (2003) 241–254

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Polynomial images of R^n

José F. Fernando[1], J.M. Gamboa*[,1]

Depto de Algebra, F. Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

Received 20 May 2001; received in revised form 9 January 2002
Communicated by M.-F. Roy

Abstract

Let R be a real closed field and $n \geqslant 2$. We prove that: (1) for every finite subset F of R^n, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n; and (2) for any independent linear forms $l_1, \dots, l_r$ of R^n, the semialgebraic set $\{l_1 > 0, \dots, l_r > 0\} \subset R^n$ is a polynomial image of R^n.

MSC: 14P10; 14Q99

1. Introduction

Let R be a real closed field and m, n be positive integers. A map $f = (f_1, \dots, f_n)$: $R^m \to R^n$ is said to be *polynomial* if the functions $f_i \in R[x_1, \dots, x_m]$.

A celebrated theorem of Tarski–Seidenberg [2, 1.4] says that the image of any polynomial map $f : R^m \to R^n$ is a semialgebraic subset of R^n, i.e. it can be written as a finite union of subsets defined by a finite conjunction of polynomial equalities and inequalities. We study some kind of converse of this result.

In an *Oberwolfach* week [4], the second author proposed to characterize the semialgebraic sets of R^n which are polynomial images of R^m. In particular, the open ones deserve a special attention, in connection with the real Jacobian Conjecture [7].

First of all, we introduce some notation and terminology. Let S be a semialgebraic subset of R^n. We define the *exterior boundary* δS of S by $\delta S = \bar{S} \setminus S$ where $\bar{S}$ denotes the closure of S in R^n with respect to the usual topology. We will denote by $\bar{S}^{\text{zar}}$ the

* Corresponding author.
E-mail addresses: josefer@mat.ucm.es (J.F. Fernando), jmgamboa@mat.ucm.es (J.M. Gamboa).
[1] Partially supported by DGICYT, PB98-0756-C02-01.

PII: S0022-4049(02)00121-4

closure of S in R^n with respect to the Zariski topology. We say that a subset A of R^n is *irreducible* if its Zariski closure is an irreducible algebraic set.

Now we show some necessary conditions for a subset $S \subset R^n$ to be a polynomial image of R^m. For $m=n=1$ the problem is trivial: *the images of polynomial maps $R \to R$ are singletons, unbounded closed intervals and the whole R*. In the general case, by Tarski–Seidenberg Theorem [2], S is semialgebraic and semialgebraically connected. Moreover, S is irreducible and pure dimensional; this is an easy consequence of the identity principle for polynomials.

This problem can be also stated in other contexts: regular, Nash, analytic, etc. In fact, Shiota (private comm.) has proved the following result:

Theorem 1.1 (Shiota). *An irreducible, semialgebraic, connected and pure m-dimensional subset $X \subset \mathbb{R}^n$ is the image of $\mathbb{R}^m$ for some Nash map $f:\mathbb{R}^m \to \mathbb{R}^n$ if and only if there exists a Nash curve $\alpha:\mathbb{R} \to X$ which meets each connected component of the regular locus of X.*

As we will see immediately, some extra constrains appear in the polynomial case. Following Jelonek [6] and Delfs-Knebush [3, Section 9], we recall that a polynomial map $f:R^m \to R^n$ is *semialgebraically proper at a point* $p \in R^n$ if there exists an open neighbourhood K of p in R^n such that the restriction

$$\begin{array}{ccc} f^{-1}(K) & \to & K \\ x & \mapsto & f(x) \end{array}$$

is a semialgebraically proper map. We denote by $\mathcal{S}_f$ the set of points $p \in R^n$ at which f is not semialgebraically proper. A *parametric semiline of* R^n is a non-constant polynomial image of R; it is semialgebraically closed since every polynomial map $R \to R^n$ is semialgebraically proper [5]. For dimension 2, Jelonek proves:

Theorem 1.2 (Jelonek [6]). *Let $f:R^2 \to R^2$ be a dominant polynomial map. Then $\mathcal{S}_f$ is a finite union of parametric semilines.*

As easy consequences of this we state

Remark 1.3. Let $f:R^m \to R^n$ be a polynomial map and $S = f(R^m)$. Then:

(1) $\delta S \subset \mathcal{S}_f$. If $p \in \delta S \setminus \mathcal{S}_f$, then there exists an open neighbourhood K of p such that the restriction $f^{-1}(K) \to K$ of f is proper, and so its image $K \cap S$ is a closed subset of K. Hence, $p \in K \cap \bar{S} = K \cap \overline{K \cap S} = K \cap S$, a contradiction.

(2) Suppose $m = n = 2$, and let Γ be a one-dimensional irreducible component of $\overline{\delta S}^{\text{zar}}$. Then Γ is the Zariski closure of a parametric semiline of R^2. Indeed, f is a dominant map, and so, by Theorem 1.2, $\mathcal{S}_f$ is a finite union of parametric semilines $M_1, \ldots, M_s$ of R^2. Therefore, $\Gamma \subset \overline{\delta S}^{\text{zar}} \subset \overline{\mathcal{S}_f}^{\text{zar}} = \overline{M_1}^{\text{zar}} \cup \cdots \cup \overline{M_s}^{\text{zar}}$, and since Γ and the $\overline{M_i}^{\text{zar}}$'s are irreducible, we deduce $\Gamma = \overline{M_i}^{\text{zar}}$ for some i.

(3) Let $S \subset R^n$ be a polynomial image of R^m and $p:R^n \to R$ be a polynomial function which is nonconstant on S. Then $p(S) \subset R$ is unbounded.

Indeed, if $S = f(R^m)$ for some polynomial map $f: R^m \to R^n$, then for each point $a \in R^m$, $p(S)$ would contain the image $\varphi_a(R)$ of the polynomial map $\varphi_a(t) = p(f(ta))$. If $\varphi_a(R)$ were bounded for all a then, for each a, $\varphi_a(R)$ would be a point, say r_a. Therefore, given two points $a, b \in R^m$ we would have

$$p(f(a)) = r_a = \varphi_a(0) = \varphi_b(0) = r_b = p(f(b))$$

and so, p would be constant on S, a contradiction.

Thus, all linear projections of S are either a point or unbounded. In particular, S is also unbounded or it is a point.

The first examples one tries to realize as polynomial images are semialgebraic subsets of R^2:

Examples 1.4. (i) The exterior $S = \{u^2 + v^2 > 1\}$ of the closed unit disc is not a polynomial image of R^2, since the only irreducible component of its exterior boundary is the unit circle which, being bounded, is not a parametric semiline.

(ii) None of the sets $S_1 = \{uv < 1\}$ and $S_2 = \{u > 0, uv > 1\}$ is a polynomial image of R^2, because the common Zariski closure of their exterior boundaries is the hyperbola $uv = 1$ which is not a parametric semiline.

(iii) The punctured plane $S = R^2 \setminus \{(0,0)\}$ is the image of the polynomial map:

$$(x, y) \mapsto (xy - 1, (xy - 1)x^2 - y).$$

(iv) The open half-planes are polynomial images of R^2. For, it suffices to verify that the upper half plane $\mathbb{H}: v > 0$ is the image of the polynomial map

$$(x, y) \mapsto (y(xy - 1), (xy - 1)^2 + x^2).$$

Probably, this is the simplest polynomial map whose image is $\mathbb{H}$.

In fact, our main results are generalizations of the two last examples above:

Theorem 1.5. *Let $n \geqslant 2$. For every finite subset F of R^n, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n.*

Theorem 1.6. *Let $n \geqslant 2$. For any independent linear forms $l_1, \dots, l_r$ of R^n, the open semialgebraic set $\{l_1 > 0, \dots, l_r > 0\} \subset R^n$ is a polynomial image of R^n.*

Until now, the known open sets which are polynomial images of R^2 (see for instance [8]) have irreducible exterior boundary and they are *deformations* of the open upper half-plane $\{y > 0\}$. In [4,8], the authors outline the problem of finding out whether or not the open quadrant $Q = \{x > 0, y > 0\}$ is a polynomial image of R^2; note that the exterior boundary of Q is not irreducible. This is a crucial particular case of 1.6. The best known approach to the solution of the problem was given by the double quadratic transform

$$(x, y) \mapsto (x^4y^2, x^2y^4)$$

whose image is $Q \cup \{(0,0)\}$. In Section 3 we will prove that in fact:

Theorem 1.7. *The quadrant Q is a polynomial image of R^2.*

The proof consists of two parts. The first one is the choice of a good candidate to have the quadrant as image. In Section 3 we will give enlighting arguments to explain the reason of our choice. The second one is devoted to check that actually the image of the chosen map is the open quadrant. After some preparation the question is reduced to prove the nonexistence of real roots of some univariate polynomials on certain intervals, and to compare some rational functions on those intervals. Due to the high degree of the involved polynomials we have used a package which performs the Sturm algorithm [2, 1.2.10].

The particular case of the quadrant is the key to prove Theorem 1.6:

Proof of Theorem 1.6. It is clear that after a linear change of coordinates, we can suppose that $l_1 = x_1, \dots, l_r = x_r$ and then, we only have to prove that for every pair of positive integers $r \leqslant n$ the semialgebraic set $\{x_1 > 0, \dots, x_r > 0\} \subset R^n$ is a polynomial image of R^n. It is not difficult to see that this reduces to prove that:

(a) $\mathbb{H} = \{x_1 > 0\}$ and $Q = \{x_1 > 0, x_2 > 0\} \subset R^2$ are polynomial images of R^2, which is true by Example 1.4 (iv) and 1.7.

(b) $O = \{x_1 > 0, x_2 > 0, x_3 > 0\} \subset R^3$ is a polynomial image of R^3. To show this, we proceed as follows: let $H_1, H_2 : R^2 \to R^2$ be polynomial maps whose respective images are $\mathbb{H}$ and Q. Now consider the maps:

$$(H_1, \mathrm{id}_R) : R^3 = R^2 \times R \to R^3 = R^2 \times R$$
$$(\mathrm{id}_R, H_2) : R^3 = R \times R^2 \to R^3 = R \times R^2.$$

Then, O is the image of the map $H = (\mathrm{id}_R, H_2) \circ (H_1, \mathrm{id}_R) : R^3 \to R^3$. $\square$

The proofs of Theorems 1.5 and 1.7 are written just in the case that $R = \mathbb{R}$ is the field of real numbers. For both of them explicit polynomial maps with the prescribed image are found. Hence, the usual transfer trick (to see the coefficients of the involved polynomials as new variables, [2]) extends the results to arbitrary R.

2. Complementary set of a finite set

The purpose of this section is to prove Theorem 1.5.

Proof of Theorem 1.5. Let $F = \{p_1, \dots, p_k\}$. We can assume that each $p_j = (a_j, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ for some $a_j \in \mathbb{R}$.

Indeed, after a linear change of coordinates we have $p_j = (a_{1j}, \dots, a_{nj})$ such that $a_{1j} \neq a_{1l}$ if $j \neq l$. Then, there exists a polynomial $P_1 \in \mathbb{R}[T]$ such that $P_1(a_{1j}) = a_{nj}$, and so, if $x' = (x_1, \dots, x_{n-1})$, the polynomial map

$$h_1 : \mathbb{R}^n \to \mathbb{R}^n : (x', x_n) \mapsto (x', x_n + P_1(x_1))$$

is bijective and for $p'_j = (a_{1j}, \dots, a_{(n-1)j}, 0)$ it holds $h_1(p'_j) = p_j$. Analogously, let $P_2 \in \mathbb{R}[T]$ be such that $P_2(a_{1j}) = a_{(n-1)_j}$, and $p''_j = (a_{1j}, \dots, a_{(n-2)_j}, 0, 0)$. Then, the polynomial bijection

$$h_2 : \mathbb{R}^n \to \mathbb{R}^n : (x'', x_{n-1}, x_n) \mapsto (x'', x_{n-1} + P_2(x_1), x_n),$$

where $x'' = (x_1, \ldots, x_{n-2})$, satisfies $h_2(p''_j) = p'_j$. Thus, the polynomial bijection $h_1 \circ h_2$ maps p''_j to p_j. In this way we construct, inductively, a polynomial bijection $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(q_j) = p_j$ for $q_j = (a_{1j}, \vec{0})$. Therefore, if $G = \{q_1, \ldots, q_k\}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map such that $g(\mathbb{R}^n) = \mathbb{R}^n \setminus G$, then $h \circ g(\mathbb{R}^n) = \mathbb{R}^n \setminus F$. So, in what follows we suppose that $p_j = (a_j, \vec{0})$.

Now, let r be an integer such that $r \neq a_1 - a_j$ for $j = 1, \ldots, k$, $\sigma(x) = \sum_{j=3}^n x_j^2$ and

$$\rho(x) = \prod_{j=1}^{k} (x_1 x_2 - r + a_1 - a_j).$$

We claim that the image of the polynomial map $f = (f_1, \ldots, f_n)$ given by

$$f(x) = (x_1 x_2 - r + a_1, x_1^4 \rho(x) + x_1^2 \sigma(x) + x_2, x_3, \ldots, x_n)$$

is $\mathbb{R}^n \setminus F$.

Indeed, suppose first that there exists $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ such that $f(b) = p_\ell$ for some $\ell = 1, \ldots, k$. Then $f_1(b) = b_1 b_2 - r + a_1 = a_\ell$ and $f_i(b) = 0$ for $i = 2, \ldots, n$. Thus, $\rho(b) = 0$ and $\sigma(b) = 0$; hence $b_2 = 0$ and $r = a_1 - a_\ell$, which is impossible. So $\mathrm{im}(f) \subset \mathbb{R}^n \setminus F$. Conversely, let $u = (u_1, \ldots, u_n) \in \mathbb{R}^n \setminus F$. We have to solve the system of polynomial equations:

$$\begin{aligned} f_1(x) &= x_1 x_2 - r + a_1 = u_1, \\ f_2(x) &= x_1^4 \rho(x) + x_1^2 \sigma(x) + x_2 = u_2, \\ f_j(x) &= x_j = u_j, \quad j \geqslant 3. \end{aligned}$$

If $u_1 = a_1 - r$ then $f(0, u_2, \ldots, u_n) = u$. If $u_1 \neq a_1 - r$, substituting $x_2 = (u_1 - a_1 + r)/x_1$, $x_j = u_j$ for $j \geqslant 3$ in f_2, we see that x_1 must be a nonzero root of the polynomial

$$Q(T) = \prod_{j=1}^{k} (u_1 - a_j) T^5 + \sigma(u) T^3 - u_2 T + (r - a_1 + u_1)$$

which has odd degree (because $u \notin F$) and $Q(0) = r - a_1 + u_1 \neq 0$. Then, if b_1 is a real root of Q we have that

$$f\left(b_1, \frac{u_1 - a_1 + r}{b_1}, u_3, \ldots, u_n\right) = u. \qquad \square$$

3. The quadrant

Before entering into the proof of Theorem 1.7, we must point out that a main difference between polynomials in one or two variables is that the open interval $(0, +\infty)$ is a polynomial image of $\mathbb{R}^2$ but not of $\mathbb{R}$. In fact, $(0, +\infty)$ is the image of $\mathbb{R}^2$ by $f(x, y) = (xy - 1)^2 + x^2$. However, this polynomial is not useful to obtain Q as we see immediately.

Remark 3.1. There is not a polynomial map

$$f = (P_1, P_2) : \mathbb{R}^2 \to \mathbb{R}^2$$

such that $f(\mathbb{R}^2) = Q$ and $P_1(x, y) = (xy - 1)^2 + x^2$.

Otherwise, since for each $\lambda \geqslant 0$ the point $(\lambda^2, 0) \in \bar{Q}$, the Curve Selection Lemma [1, VIII.2.6] gives an analytic half-branch curve γ_λ: $(0, \delta_\lambda) \to \mathbb{R}^2$ such that

$$\lim_{t\to 0} P_1(\gamma_\lambda(t)) = \lambda^2 \quad \text{and} \quad \lim_{t\to 0} P_2(\gamma_\lambda(t)) = 0.$$

We can write $\gamma_\lambda(t) = (t^{n_\lambda} u_\lambda(t), t^{m_\lambda} v_\lambda(t))$ for some $n_\lambda, m_\lambda \in \mathbb{Z}$ and some units u_λ, v_λ in the ring $\mathbb{R}\{t\}$ of power series in one variable, and

$$P_1(\gamma_\lambda(t)) = \lambda^2 + t\xi_\lambda(t),$$

where $\xi_\lambda \in \mathbb{R}\{t\}$. Therefore,

$$\lambda^2 + t\xi(t) = (t^{n_\lambda + m_\lambda} u_\lambda(t) v_\lambda(t) - 1)^2 + t^{2n_\lambda} u_\lambda^2(t),$$

and since $(\lambda^2, 0) \notin Q$, taking orders with respect to t in the previous expression, it is not difficult to deduce that $n_\lambda > 0$ and $m_\lambda = -n_\lambda$. Hence we can reparametrize γ_λ as

$$\gamma_\lambda(s) = (\varepsilon_\lambda s^{n_\lambda}, s^{-n_\lambda} \eta_\lambda(s))$$

for some unit $\eta_\lambda \in \mathbb{R}\{s\}$ and $\varepsilon_\lambda = \pm 1$. Now,

$$P_1(\gamma_\lambda(s)) = (\varepsilon_\lambda \eta_\lambda(s) - 1)^2 + s^{2n_\lambda}$$

and so, $\lambda^2 = \lim_{s\to 0} P_1(\gamma_\lambda(s)) = (\varepsilon_\lambda \eta_\lambda(0) - 1)^2$. Without loss of generality we can assume that $\varepsilon_\lambda = 1$ and $\eta_\lambda(0) = 1 + \lambda$ for infinitely many values of λ. Let us write

$$P_2(x, y) = \sum_{0 \leqslant i+j \leqslant d} a_{ij} x^i y^j.$$

After substituting, for these λ's,

$$P_2(\gamma_\lambda(s)) = \sum_{0 \leqslant i+j \leqslant d} a_{ij} \eta_\lambda^j s^{(i-j)n_\lambda} = \sum_{-l \leqslant i-j \leqslant r} a_{ij} \eta_\lambda^j s^{(i-j)n_\lambda},$$

where $l \geqslant 0$ because $a_{00} = P_2(0,0) > 0$. Now, since $\lim_{s\to 0} P_2(\gamma_\lambda(s)) = 0$ and $\eta_\lambda(0) = 1 + \lambda$, it follows, step by step, that for each $0 \leqslant k \leqslant l$

$$\sum_{i-j=-k} a_{ij}(1+\lambda)^j = 0$$

for infinitely many λ's, and so each $a_{i,i+k} = 0$. In particular, for $k = 0$ we get $a_{00} = 0$, a contradiction.

Let us now look for a polynomial map $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $\Phi(\mathbb{R}^2) = Q$. The major difficulty to find such a map is to get that:

The closure of its image contains the positive half-axes. $(*)$

Using Theorem 1.5, to realize Q as a polynomial image of $\mathbb{R}^2$ it is enough to find a polynomial map $P = (F, G): \mathbb{R}^2 \to \mathbb{R}^2$ such that $P(\mathbb{R}^2)$ is the disjoint union of Q and a set of finite preimage.

If such a P exists it also must satisfy $(*)$. Then for every nonnegative numbers λ, μ there will exist analytic half-branch curve germs $\alpha_\lambda(s), \beta_\mu(s)$ which cannot be extended to 0 and such that

$$\lim_{s\to 0} P(\alpha_\lambda(s)) = (\lambda^2, 0) \quad \text{and} \quad \lim_{s\to 0} P(\beta_\mu(s)) = (0, \mu^2).$$

We can try parametrizations of the kind

$$\alpha_\lambda(s) = \left(s^{n_\lambda}, \frac{a_{\lambda 0} + a_{\lambda 1}s + \cdots}{s^{m_\lambda}}\right) \quad \text{and} \quad \beta_\mu(s) = \left(\frac{b_{\mu 0} + b_{\mu 1}s + \cdots}{s^{\ell_\mu}}, s^{k_\mu}\right),$$

and as remarked above, it is not difficult to see that $a_{\lambda 0}, b_{\mu 0}$ must be constants (except maybe for finitely many values of λ, μ). In view of this, we will take curves of the type:

$$\alpha_\lambda(s) = \left(s^{n_\lambda}, \frac{1 + a_{\lambda 1}s + \cdots}{s^{m_\lambda}}\right) \quad \text{and} \quad \beta_\mu(s) = \left(\frac{1 + b_{\mu 1}s + \cdots}{s^{\ell_\mu}}, s^{k_\mu}\right),$$

and in fact we choose the simplest ones, namely

$$\alpha_\lambda(s) = \left(s, \frac{1 + a_\lambda s}{s}\right) \quad \text{and} \quad \beta_\mu(s) = \left(\frac{1 + b_\mu s}{s}, s^3\right).$$

The following pair of polynomials

$$\begin{aligned} F &= (1 - x^3y + y - xy^2)^2 + (x^2y)^2 = F_1^2 + F_2^2 \\ G &= (1 - xy + x - x^4y)^2 + (x^2y)^2 = G_1^2 + G_2^2 \end{aligned}$$

have a good behaviour along these curves in the following sense:

(a) $F_1 \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$, $F_1 \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$, $F_1 \circ \alpha_\lambda(0) = 1 - a_\lambda$ and $F_1 \circ \beta_\mu(0) = 0$,
(b) $G_1 \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$, $G_1 \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$, $G_1 \circ \alpha_\lambda(0) = 0$ and $G_1 \circ \beta_\mu(0) = 1 - 3b_\mu$,
(c) $F_2 \circ \alpha_\lambda = G_2 \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$, $F_2 \circ \beta_\mu = G_2 \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$ and $F_2 \circ \alpha_\lambda(0) = G_2 \circ \alpha_\lambda(0) = F_2 \circ \beta_\mu(0) = G_2 \circ \beta_\mu(0) = 0$,

and therefore, the following properties hold:

(i) The polynomials F, G are nonnegative in $\mathbb{R}^2$,
(ii) $F^{-1}(0) = F_1^{-1}(0) \cap F_2^{-1}(0) = \{(0, -1)\}$ whose image by P is $\{(0, 1)\}$ and $G^{-1}(0) = G_1^{-1}(0) \cap G_2^{-1}(0) = \{(-1, 0)\}$ whose image by P is $\{(1, 0)\}$,
(iii) $P \circ \alpha_\lambda = (F \circ \alpha_\lambda, G \circ \alpha_\lambda) = ((1 - a_\lambda)^2 + \cdots, g_1(a_\lambda)s^2 + \cdots)$ and $P \circ \beta_\mu = (F \circ \beta_\mu, G \circ \beta_\mu) = (f_1(b_\mu)s^2 + \cdots, (1 - 3b_\mu)^2 + \cdots)$

for certain polynomials $g_1 \in \mathbb{R}[a_\lambda]$, $f_1 \in \mathbb{R}[b_\mu]$. The information above does not prove that the image of P is the open quadrant, but guarantees the necessary condition $(*)$.

Remark 3.2. From (i) and (ii) above it follows also that $P(\mathbb{R}^2) \subset Q \cup \{(1, 0), (0, 1)\}$ and that the preimage of $\{(1, 0), (0, 1)\}$ is the finite set $\{(-1, 0), (0, -1)\}$. Our next aim is to prove that the previous inclusion is, in fact, an equality. Next, if $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a map with image $\mathbb{R}^2 \setminus \{(-1, 0), (0, -1)\}$ (which exists by Theorem 1.5) the composition $\Phi = P \circ \varphi$ gives the desired result.

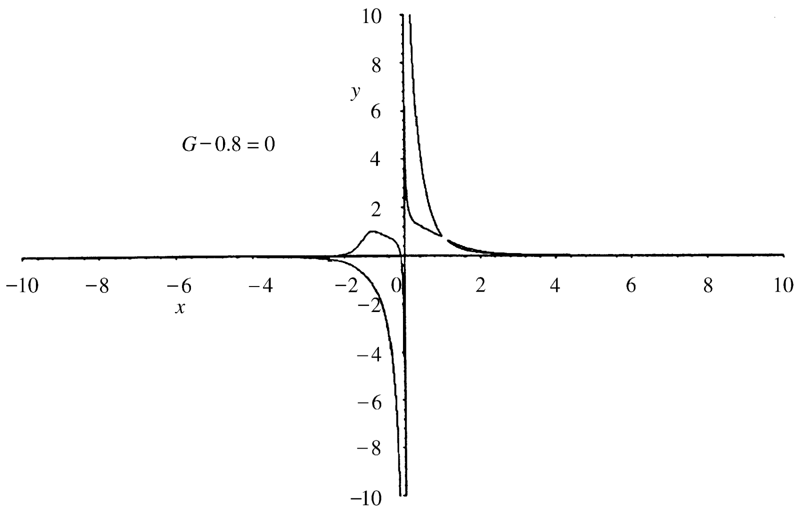
Proof of Theorem 1.7. To prove that $P(\mathbb{R}^2) \supset Q$ it is enough to show that for all $v > 0$ the image of the restriction $F : \{G = v\} \to \mathbb{R}$ contains the open interval $(0, +\infty)$. Let us fix from now on a positive real number v. We proceed in several steps:

Step 1: *Parametrization of the curve* $\{G-v=0\}$. Solving the equation $G-v=0$, which has degree 2 with respect to y, we obtain the roots

$$y^+(x,v)=\frac{1+x+x^3+x^4+\sqrt{\Delta(x,v)}}{x(x^2+(x^3+1)^2)},$$

$$y^-(x,v)=\frac{1+x+x^3+x^4-\sqrt{\Delta(x,v)}}{x(x^2+(x^3+1)^2)},$$

where $\Delta(x,v)=v(x^2+(x^3+1)^2)-x^2(x+1)^2$.



Let $D_v=\{x\in\mathbb{R}\colon \Delta(x,v)\geqslant 0,\ x\neq 0\}$ be its common domain and consider the functions

$$\begin{array}{rcl} \gamma_v^+\colon D_v & \to & \mathbb{R} \\ x & \mapsto & F(x,y^+(x,v)) \end{array} \qquad \begin{array}{rcl} \gamma_v^-\colon D_v & \to & \mathbb{R} \\ x & \mapsto & F(x,y^-(x,v)) \end{array}$$

Notice that the image $F(\{G-v=0\})$ is the union $\operatorname{im}\gamma_v^+\cup\operatorname{im}\gamma_v^-$.

Step 2: *Main properties of the functions* γ_v^+,γ_v^-. We show that:

(i) $\lim_{x\to\pm\infty}\gamma_v^+(x)=\lim_{x\to\pm\infty}\gamma_v^-(x)=0$.

(ii) $\lim_{x\to 0}\gamma_v^+(x)=+\infty$, $\lim_{x\to 0}\gamma_v^-(x)=\begin{cases}+\infty & \text{for } v\neq 1,\\ 4 & \text{for } v=1.\end{cases}$

In fact, a straightforward computation shows that there exist polynomials $A_1,A_2,B_1,B_2\in\mathbb{R}[x,v]$ and $C(x)=x^2(x^2+(x^3+1)^2)^4$ such that:

(a) $\gamma_v^+(x)=\dfrac{A_1(x,v)+B_1(x,v)\sqrt{\Delta(x,v)}}{C(x)}$, $\gamma_v^-(x)=\dfrac{A_2(x,v)+B_2(x,v)\sqrt{\Delta(x,v)}}{C(x)}$.

(b) $\deg_x(A_1)=\deg_x(A_2)=24$; $\deg_x(B_1)=\deg_x(B_2)=21$.

Moreover, $\deg_x(\Delta)=6$ and $\deg_x(C)=26$.

Firstly we analyse the behaviour of γ_v^+ and γ_v^- at infinity. Since

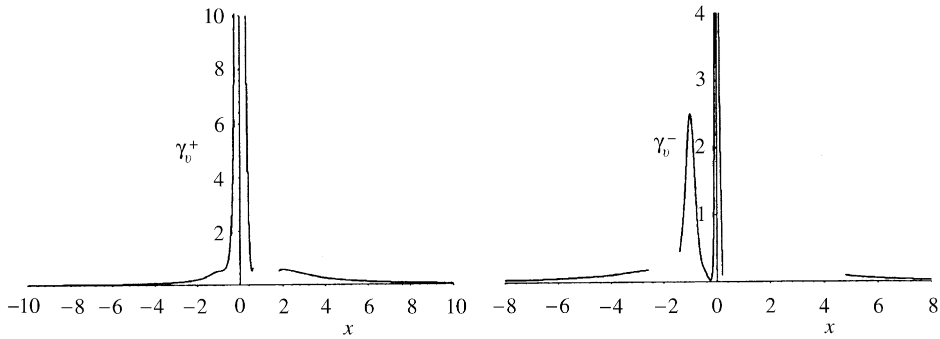
$$\Delta_v(x) = \Delta(x,v) = v((x^3+1)^2 + x^2) - x^2(x+1)^2$$

has even degree and positive leading coefficient as a polynomial in x then it is positive for $|x|$ large enough. Thus, from (a) and (b) we conclude (i).

Secondly, we study γ_v^+ and γ_v^- at the origin. Since $\Delta(0,v) = v > 0$ then $0 \in \overline{D_v}$. Moreover, from the explicit computation of A_i, B_i, it follows that

- $A_1(0,v) + B_1(0,v)\sqrt{\Delta(0,v)} = v(1+\sqrt{v})^2 > 0$,
- $A_2(0,v) + B_2(0,v)\sqrt{\Delta(0,v)} = v(1-\sqrt{v})^2 \geqslant 0$ and it is 0 if and only if $v = 1$.

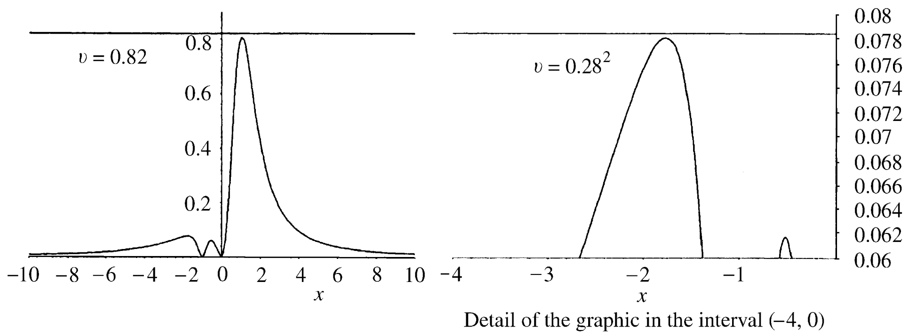
From this we conclude (ii). The precise finite value of $\lim_{x\to 0} \gamma_1^-(x) = 4$ is irrelevant, but to calculate it the explicit formulae of the A_i's, B_i's are needed.



To check that $\operatorname{im}\gamma_v^+ \cup \operatorname{im}\gamma_v^- \supset (0,+\infty)$ we must study the domain D_v. To that end we determine the union $D = \bigcup_{v>0} D_v$ which is the set $\{\Delta(x,v) \geqslant 0,\ x \neq 0\}$. For that, finding the value of v in the equation $\Delta(x,v) = 0$ we obtain the univariated function defined over the whole $\mathbb{R}$

$$v(x) = \frac{x^2(x+1)^2}{x^2 + (x^3+1)^2}$$

whose graph is:



Detail of the graphic in the interval (–4, 0)

We claim that $v(x) < 0.28^2$ for x in the interval $(-\infty, 0)$. Since $v(x)$ is continuous, to check this it is enough to show that for $v_0 = 0.28^2$ the polynomial $\Delta(x, v_0)$ has no negative root, which is verified using Sturm's algorithm. Furthermore, in view of the previous graphics this bound seems to be quite sharp. Therefore, we will treat differently the values $v \geqslant 0.28^2$ and $0 < v < 0.28^2$. In the first case we have already proved that $(-\infty, 0) \subset D_v$. Since γ_v^+ is continuous in this interval and from the limits computed in *Step* 2 we conclude:

$$\text{If } v \geqslant 0.28^2, \quad \operatorname{im}\gamma_v^+ \cup \operatorname{im}\gamma_v^- \supset \operatorname{im}\gamma_v^+ \supset (0,+\infty).$$

To end up with all reduces to check

Step 3: If $0 < v < 0.28^2$ then $\operatorname{im}\gamma_v^+ \cup \operatorname{im}\gamma_v^- \supset \operatorname{im}\gamma_v^- \supset (0,+\infty)$. For that it suffices, using *Step* 2, to prove that there exist negative real numbers $N_v < \delta_v$ such that

$$(-\infty, N_v] \cup [\delta_v, +\infty) \subset D_v \quad \text{and} \quad \gamma_v^-(N_v) > \gamma_v^-(\delta_v). \tag{$**$}$$

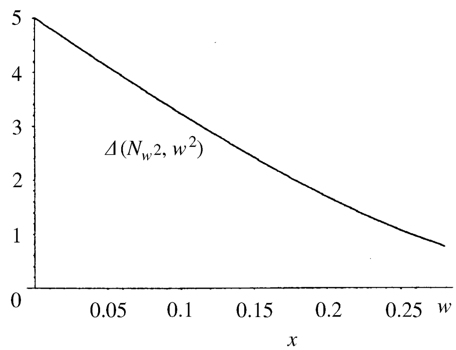
The existence of N_v, δ_v comes from a detailed analysis of the set D_v. We begin computing the roots of $\Delta_v(x)$ in the field of Puiseux series $\mathbb{C}(\{v^*\})$: these roots are power series in $\mathbb{C}(\{w\})$ where $w = v^{1/2}$, and between them we choose

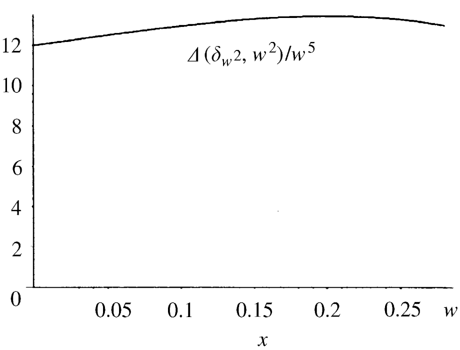
$$\eta_v = -\frac{1}{w} + 1 + w + w^2 + \frac{5}{2}w^3 + \cdots,$$
$$\xi_v = -w - w^2 - \frac{5}{2}w^3 - 6w^4 + \cdots$$

which are the most and the less negative roots of Δ_v in $\mathbb{R}(\{v^*\})$ with respect to the unique ordering of $\mathbb{R}(\{v^*\})$ that makes $v > 0$. In view of this we take

$$N_v = -\frac{1}{w} + 1 + w + w^2 = \eta_v - \left(\frac{5}{2}w^3 + \cdots\right) < \eta_v,$$
$$\delta_v = -w - w^2 - \frac{5}{2}w^3 = \xi_v - (-6w^4 + \cdots) > \xi_v.$$

It is not difficult to show that $-\infty < N_v < \delta_v < 0$ for $0 < v < 0.28^2$. To prove $(**)$ we will proceed as follows. First, we verify that $N_v, \delta_v \in D_v$; for that, we consider the polynomials $\Delta(N_{w^2}, w^2), \Delta(\delta_{w^2}, w^2)$ in the variable w and check that they are positive in (0,0.28) using the Sturm algorithm.

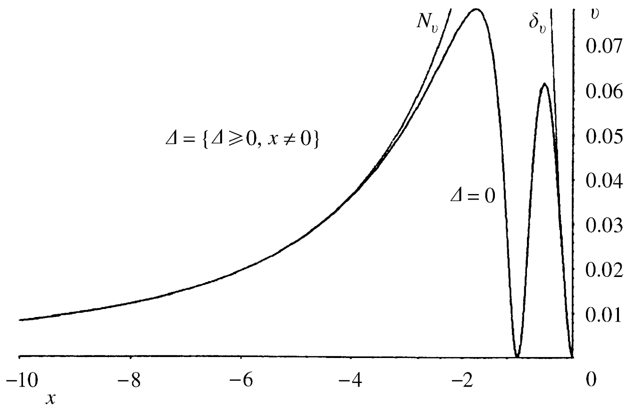




Next, we prove that $(-\infty, N_v] \cup [\delta_v, +\infty) \subset D_v$. To that end, we consider the semialgebraic set $D = \bigcup_{v>0} D_v = \{\Delta(x,v) \geqslant 0,\ x \neq 0\}$ whose boundary is the union of the axis $x=0$ and the curve given by the equation

$$v=\frac{x^2(x+1)^2}{x^2+(x^3+1)^2},$$

which is a graph over the axis $v=0$. Then, since the curves $\{(\delta_v,v)\colon 0<v<0.28^2\}\subset D$ and $\{(N_v,v)\colon 0<v<0.28^2\}\subset D$ are graphs over the vertical axis $x=0$, and for v small enough $\delta_v>\xi_v$, $N_v<\eta_v$, we conclude that the interior of D_v contains the intervals $[\delta_v,0)$ and $(-\infty,N_v]$ for $0<v<0.28^2$.



Finally, we must only check that $\gamma_v^-(N_v)>\gamma_v^-(\delta_v)$. We recall that $\gamma_v^-=(A_2+B_2\sqrt{\Delta})/C$ where $A_2,B_2,\Delta\in\mathbb{R}[x,v]$ and $C\in\mathbb{R}[x]$. Consider the polynomials

$$\begin{aligned}
f_1(w)&=A_2(N_{w^2},w^2)w^{24}, & f_2(w)&=A_2(\delta_{w^2},w^2),\\
g_1(w)&=B_2(N_{w^2},w^2)w^{21}, & g_2(w)&=B_2(\delta_{w^2},w^2),\\
q_1(w)&=\Delta(N_{w^2},w^2), & q_2(w)&=\Delta(\delta_{w^2},w^2),\\
h_1(w)&=C(N_{w^2})w^{26}, & h_2(w)&=C(\delta_{w^2}).
\end{aligned}$$

Then, we have to verify that for w in the interval (0,0.28) the function

$$\frac{f_1/w^{24}+(g_1/w^{21})\sqrt{q_1}}{h_1/w^{26}}-\frac{f_2+g_2\sqrt{q_2}}{h_2}>0,$$

or equivalently, that

$$\frac{w^2h_2f_1-f_2h_1}{h_1h_2}+\frac{w^5g_1\sqrt{q_1}}{h_1}-\frac{g_2\sqrt{q_2}}{h_2}>0.$$

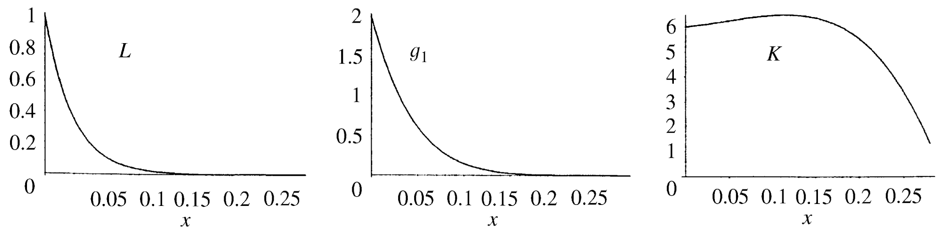
It is enough to check that the functions

$$\frac{w^2h_2f_1-f_2h_1}{h_1h_2},\quad \frac{w^5g_1\sqrt{q_1}}{h_1},\quad -\frac{g_2\sqrt{q_2}}{h_2}$$

are positive in the interval (0,0.28), and so it suffices to verify that the polynomials

$$L = \frac{w^2 h_2 f_1 - f_2 h_1}{w^4}, \quad g_1, \quad K = \frac{-g_2}{w^3}$$

are positive in (0,0.28).



To check this we use again Sturm's algorithm. Just the proof of the positiveness of L in (0,0.28) requires some care. To simplify it we write L as $L = L_1 + L_2 w^{55} + L_3 w^{101}$ where $L_1, L_2, L_3 \in \mathbb{R}[w]$ are polynomials of respective degrees 54,45,49; now, applying Sturm's algorithm to L_1, L_2, L_3 we observe that these three polynomials are positive in (0,0.28), which concludes the proof. □

4. Some consequences and open problems

We begin this section with some consequences of Theorem 1.7:

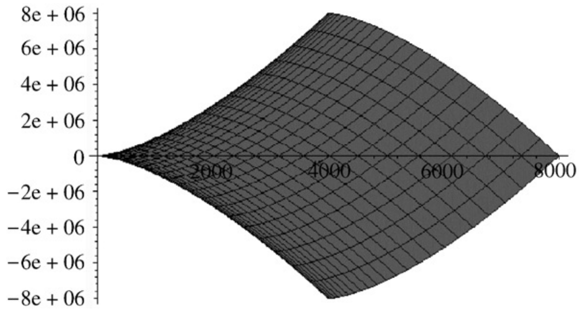
Corollary 4.1. *Let l_1, l_2 be independent linear forms of R^2. Then the complementary set of the closed semialgebraic set $\{l_1 \geqslant 0, l_2 \geqslant 0\}$ is a polynomial image of R^2.*

Proof. It is enough to consider the case $l_1 = x$, $l_2 = -y$. Let $G_1, G_2 : R^2 \to R^2$ be polynomial maps such that $G_1(R^2)$ is the open quadrant Q and, with *complex* notation, $G_2(z) = z^3$ where $z = x + \mathrm{i}y$. Then the composition $G = G_2 \circ G_1$ has the desired image. □

We can also produce examples of open polynomial images of R^2 whose exterior boundary is not piecewise linear.

Example 4.2. Let $f(x, y) = x^2 - y^2 + x^3$. The semialgebraic set $S = \{f(x, y) > 0, x > 0\}$ is a polynomial image of R^2. Consider first the parametrization $\alpha(t) = (t^2 - 1, t(t^2 - 1))$ of $\{f = 0\}$. Then, S is the image of the quadrant $Q' = \{u > 1, v < -1\}$ under the polynomial map

$$\begin{aligned} \psi:\ & R^2 \to R^2 \\ & (u, v) \mapsto \alpha(u) + \alpha(v). \end{aligned}$$



As we pointed out in the Introduction the problem discussed here can be formulated in several contexts. In particular, we recall that a function $f : R^m \to R$ is *regular* if there exist two polynomials $p, q \in R[x_1, \dots, x_m]$ such that q has no real zeros and $f = p/q$. A map $R^m \to R^n$ is *regular* if all of its components are regular functions.

It is easy to produce examples of semialgebraic subsets of R^n, which are regular images of R^n and not polynomial images of R^n.

Examples 4.3. (i) The images of regular functions $R \to R$ are all the nonopen intervals.

(ii) The open disc $\mathbb{D} = \{u^2 + v^2 < 1\}$ is a regular image of R^2. Indeed, let $P : R^2 \to R^2$ be a polynomial map whose image is the upper half-plane $\mathbb{H} = \{v > 0\}$. With *complex* notation, the *Möbius transform*

$$\begin{array}{rccl} \phi\colon & \mathbb{H} & \to & R^2 \\ & z = u + \mathrm{i}v & \mapsto & \dfrac{z - i}{z + i} \end{array}$$

maps $\mathbb{H}$ onto $\mathbb{D}$. Thus $\phi \circ P$ is a regular map whose image is $\mathbb{D}$.

(iii) In contrast with the polynomial case, the exterior S of the closed unit disc of R^2 is a regular image of R^2. For, let $G_1 : R^2 \to R^2$ be a polynomial map whose image is the upper half-plane $\mathbb{H} = \{v > 0\}$ and such that the fiber $F = G_1^{-1}(0, 1)$ is a finite set. By 1.5 there exists a polynomial map $G_2 : R^2 \to R^2$ whose image is $R^2 \setminus F$. The image of $\phi \circ G_1 \circ G_2$ (where ϕ is the one of the previous example) is the punctured disc $\mathbb{D} \setminus \{(0, 0)\}$. Finally, the inversion

$$\begin{array}{rccl} \rho\colon & \mathbb{D} \setminus \{(0,0)\} & \to & R^2 \\ & z = u + \mathrm{i}v & \mapsto & \dfrac{z}{\|z\|^2} \end{array}$$

is regular and has S as image.

(iv) The open band $\mathbb{B} = \{u > 0, -1 < v < 1\}$ is not a polynomial image of R^2 by 1.3 (3). However, it is the image of the quadrant $Q'' = \{x - y > 0,\ x + y > 0\}$ under the regular map

$$\begin{array}{rccl} \sigma\colon & Q'' & \to & R^2 \\ & (u, v) & \mapsto & \left(u, \dfrac{v}{u}\right). \end{array}$$

4.1. Some open questions

1. We have already seen that an open polynomial image of R^2 in R^2 is a pure dimensional, semialgebraically connected and semialgebraic set S such that $p(S)$ is unbounded for each nonconstant polynomial function p on S and whose $\overline{\delta S}^{\text{zar}}$ is a finite union of Zariski closures of parametric semilines. However the converse is note true. In a forthcoming paper we will prove, for example, that the set $S=\{x>0, y>0, x-y+4>0\}$ is not a polynomial image of R^2.

2. Analogously, it is not difficult to check that an open regular image of R^2 in R^2 is a pure dimensional, semialgebraically connected and semialgebraic set S such that its $\overline{\delta S}^{\text{zar}}$ is a finite union of real algebraic curves of genus zero. Is the converse true? Notice that in this case the set S above is the image of the band $\mathbb{B}$ under the map

$$\begin{aligned} \eta\colon\quad \mathbb{B} \quad &\to R^2 \\ (x,y) &\mapsto (2x,(x+y+1)(y+1)). \end{aligned}$$

Moreover, using this set S it is not difficult to verify that every euclidean polygon of 3, 4 or 5 vertices is a regular image of R^2. Is this true for polygons with more vertices?

3. Related with the previous questions it seems interesting to characterize those regular images of R^2 which are not polynomial images of R^2 (like the exterior of the closed unit disc or the open band).

Acknowledgements

Most of the computations involved in the elaboration of this paper has been performed by Maple. The authors thank M.E. Alonso, F.J. Cirre and J.M. Ruiz for helpful discussions during the preparation of this work.

References

[1] C. Andradas, L. Bröcker, J.M. Ruiz, Constructible sets in real geometry, Ergebnisse der Mathematik, Vol. 33, Springer, Berlin, Heidelberg, New York, 1996.
[2] J. Bochnak, M. Coste, M.-F. Roy, Géométrie algébrique réelle, Ergebnisse der Mathematik, Vol. 12, Springer, Berlin, Heidelberg, New York, 1987.
[3] H. Delfs, M. Knebusch, Semialgebraic topology over a real closed field II. Basic theory of semialgebraic spaces, Math. Z. 178 (2) (1981) 175–213.
[4] J.M. Gamboa, Reelle Algebraische Geometrie, Oberwolfach, June, 10th–16th, 1990.
[5] J.M. Gamboa, C. Ueno, Proper polynomial maps: the real case, Lecture Notes in Mathematics, Vol. 1524, Springer, Berlin, 1992, pp. 240–256. Real Algebraic Geometry (Rennes, 1991).
[6] Z. Jelonek, A geometry of polynomial transformations of the real plane, Bull. Polish Acad. Sci. Math. 48 (1) (2000) 57–62.
[7] S. Pinchuk, A counterexample to the real Jacobian Conjecture, Math. Z. 217 (1994) 1–4.
[8] J.M. Ruiz, Semialgebraic and semianalytic sets, Cahiers d'Histoire des Mathématiques, Sér. 2, No. 1, Institut Henri Poincaré, 1991, pp. 59–70.