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On the one dimensional polynomial and regular images of $\mathbb{R}^n$

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ABSTRACT

In this work we present a full geometric characterization of the 1-dimensional polynomial and regular images of $\mathbb{R}^n$. In addition, given a polynomial image S of $\mathbb{R}^n$, we compute the smallest positive integer $p := \mathrm{p}(S)$ such that S is a polynomial image of $\mathbb{R}^p$. Analogously, given a regular image S' of $\mathbb{R}^n$, we determine the smallest positive integer $r := \mathrm{r}(S')$ such that S' is a regular image of $\mathbb{R}^r$.

1. Introduction

A map $f := (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a *polynomial map* if each of its components $f_i \in \mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$. A subset S of $\mathbb{R}^m$ is a *polynomial image* of $\mathbb{R}^n$ if there exists a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $S = f(\mathbb{R}^n)$. Let S be a subset of $\mathbb{R}^m$; we define

$$\mathrm{p}(S) := \begin{cases} \text{smallest } p \geqslant 1 & \text{such that } S \text{ is a polynomial image of } \mathbb{R}^p, \\ +\infty & \text{otherwise.} \end{cases}$$

More generally, a map $f := (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a *regular map* if each component f_i is a regular function of $\mathbb{R}(\mathtt{x}) := \mathbb{R}(\mathtt{x}_1, \dots, \mathtt{x}_n)$, that is, each $f_i = \frac{g_i}{h_i}$ is a quotient of polynomials such that the zero set of h_i is empty. Analogously, a subset S of $\mathbb{R}^m$ is a *regular image* of $\mathbb{R}^n$ if it is the image $S = f(\mathbb{R}^n)$ of $\mathbb{R}^n$ under a regular map f and we define the invariant

$$\mathrm{r}(S) := \begin{cases} \text{smallest } r \geqslant 1 & \text{such that } S \text{ is a regular image of } \mathbb{R}^r, \\ +\infty & \text{otherwise.} \end{cases}$$

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Obviously $\mathrm{r}(S) \leqslant \mathrm{p}(S)$ and by Tarski's Theorem (see [1, 2.8.8]) the dimension $\dim S$ of S is less than or equal to both of them. Of course the inequalities $\dim S \leqslant \mathrm{r}(S) \leqslant \mathrm{p}(S)$ can be strict and it may happen that the second invariant is finite while the third is infinite, even if $S \subset \mathbb{R}$ (see Lemma 3.1).

A celebrated theorem of Tarski–Seidenberg [1, 1.4] says that the image of any polynomial map (and more generally of a regular map) $f: \mathbb{R}^m \to \mathbb{R}^n$ is a semialgebraic subset S of $\mathbb{R}^n$, that is, it can be written as a finite boolean combination of polynomial equations and inequalities, which we will call a *semialgebraic* description. By *elimination of quantifiers* S is semialgebraic if it has a description by a first order formula *possibly with quantifiers*. Such a freedom gives easy semialgebraic descriptions for topological operations: interiors, closures, borders of semialgebraic sets are again semialgebraic.

In an *Oberwolfach* week [7] Gamboa proposed to characterize the semialgebraic sets of $\mathbb{R}^m$ that are polynomial images of $\mathbb{R}^n$ for some $n \geqslant 1$. The interest of polynomial (and also regular) images is far from discussion since there are many problems in Real Algebraic Geometry for that such sets can be reduced to the case $S = \mathbb{R}^n$ (see [2,3] or [5,6] for further comments). Examples of such problems are

- optimization of polynomial (and/or regular) functions on S,
- characterization of the polynomial (or regular) functions that are positive semidefinite on S (Hilbert's 17th problem and Positivestellensatz).

As we have already pointed out in [2], there are some straightforward properties that a regular image $S \subset \mathbb{R}^m$ must satisfy: it has to be pure dimensional, connected, semialgebraic and its Zariski closure has to be irreducible. Furthermore, S must be by [4, 3.1] *irreducible* in the sense that its ring $\mathcal{N}(S)$ of Nash functions on S is an integral domain. Recall here that a *Nash function* on an open semialgebraic subset $U \subset \mathbb{R}^m$ is an analytic function that satisfies a non-trivial polynomial equation, that is, there exists $P \in \mathbb{R}[\mathtt{x}, \mathtt{y}]$ such that $P(x, f(x)) = 0$ for all $x \in U$. Now the ring $\mathcal{N}(S)$ of Nash functions on S is the collection of all functions on S that admit a Nash extension to an open semialgebraic neighborhood U of S in $\mathbb{R}^m$ and it is endowed with the usual sum and product (for further details see [4]).

In this work we focus our attention on the one dimensional case and present a full geometric characterization of the polynomial and regular one dimensional images of $\mathbb{R}^n$; in fact, we compute the exact value of the invariants p and r for all of them. We will see in this work that in the one dimensional case the only three possible values for both invariants p and r are $1, 2$ or $+\infty$. In fact, all possibilities with the restriction $1 \leqslant \mathrm{r} \leqslant \mathrm{p} \leqslant +\infty$ are attained except for the pair $\mathrm{r} = 1$ and $\mathrm{p} = 2$, which is not attainable (see Theorems 1.1 and 1.3, Propositions 1.2 and 1.4, Corollary 1.5 and Lemma 3.1 to complete the picture). We provide the following table illustrating the situation.

S	$\mathbb{R}$ or $[0, +\infty)$	–	$[0, 1)$	$(0, +\infty)$	$(0, 1)$	Any non-rational algebraic curve
$\mathrm{r}(S)$	1	1	1	2	2	$+\infty$
$\mathrm{p}(S)$	1	2	$+\infty$	2	$+\infty$	$+\infty$

We recall that the study of one dimensional polynomial images of $\mathbb{R}^n$ was partially and naively approached before in our previous work [3, §2] but without presenting any conclusive result.

Notations and terminology. Before stating our main results whose proofs are developed in Section 3 after the preparatory work of Section 2, we recall some preliminary standard notations and terminology. We write $\mathbb{K}$ to refer indistinctly to $\mathbb{R}$ or $\mathbb{C}$ and denote the hyperplane of infinity of the projective space $\mathbb{K}\mathbb{P}^m$ with $\mathsf{H}_\infty(\mathbb{K}) := \{x_0 = 0\}$, which contains $\mathbb{K}^m$ as the set $\mathbb{K}\mathbb{P}^m \setminus \mathsf{H}_\infty(\mathbb{K}) = \{x_0 = 1\}$. If $m = 1$, we denote the point of infinity of the projective line $\mathbb{K}\mathbb{P}^1$ with $\{p_\infty\} := \{x_0 = 0\}$.

For each $n \geqslant 1$ denote the complex conjugation with

$$\sigma_n : \mathbb{C}\mathbb{P}^n \to \mathbb{C}\mathbb{P}^n, \qquad z = (z_0 : z_1 : \cdots : z_n) \mapsto \bar{z} = (\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n}).$$

Clearly, $\mathbb{RP}^n$ is the set of fixed points of σ_n. A set $A \subset \mathbb{CP}^n$ is called *invariant* if $\sigma(A) = A$. It is well-known that if $Z \subset \mathbb{CP}^n$ is an invariant non-singular (complex) projective variety, then $Z \cap \mathbb{RP}^n$ is a non-singular (real) projective variety. We also say that a rational map $h: \mathbb{CP}^n \dashrightarrow \mathbb{CP}^m$ is *invariant* if $h \circ \sigma_n = \sigma_m \circ h$. Of course, h is invariant if its components can be chosen as homogeneous polynomials with real coefficients; hence, by restriction it provides a real rational map $h|_{\mathbb{RP}^n} : \mathbb{RP}^n \dashrightarrow \mathbb{RP}^m$.

Given a semialgebraic set $S \subset \mathbb{R}^m \subset \mathbb{RP}^m \subset \mathbb{CP}^m$, we denote its Zariski closure in $\mathbb{KP}^m$ with $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{KP}^m}(S)$. Obviously, $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathbb{RP}^m = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)$ and $\mathrm{Cl}^{\mathrm{zar}}(S) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S) \cap \mathbb{R}^m$ is the *Zariski closure of S in $\mathbb{R}^m$*. We denote the set of points of S that have local dimension k with $S_{(k)}$.

Recall that a *complex rational curve* is the image of $\mathbb{CP}^1$ under a birational (and hence regular) map while a *real rational curve* is a real projective irreducible curve C such that $C_{(1)}$ is the image of $\mathbb{RP}^1$ under a birational (and hence regular) map (see Lemma 2.1).

Main results. We begin with a geometrical characterization of the 1-dimensional polynomial images of Euclidean spaces (that is, those with $\mathrm{p} = 1, 2$, see also [3, 2.1-2]) and then determine those with $\mathrm{p} = 1$.

Theorem 1.1. *Let $S \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set. Then the following assertions are equivalent*:

(i) *S is a polynomial image of $\mathbb{R}^n$ for some $n \geqslant 1$.*
(ii) *S is irreducible, unbounded and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ is an invariant rational curve such that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathsf{H}_\infty(\mathbb{C})$ is a singleton $\{p\}$ and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)_p$ is irreducible.*

In particular, if that is the case, then $\mathrm{p}(S) \leqslant 2$.

Proposition 1.2. *Let $S \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set that is a polynomial image of $\mathbb{R}^n$ for some $n \geqslant 1$. Then $\mathrm{p}(S) = 1$ if and only if S is closed in $\mathbb{R}^m$.*

The counterpart of the previous results in the regular setting consists of the full geometric characterization of the 1-dimensional regular images of Euclidean spaces and the description of those with $\mathrm{r} = 1$.

Theorem 1.3. *Let $S \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set. Then the following assertions are equivalent*:

(i) *S is a regular image of $\mathbb{R}^n$ for some $n \geqslant 1$.*
(ii) *S is irreducible and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)$ is a rational curve.*

In particular, if that is the case, then $\mathrm{r}(S) \leqslant 2$.

Proposition 1.4. *Let $S \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set that is a regular image of $\mathbb{R}^n$ for some $n \geqslant 1$. Then $\mathrm{r}(S) = 1$ if and only if either*

(i) *$\mathrm{Cl}_{\mathbb{RP}^m}(S) = S$ or*
(ii) *$\mathrm{Cl}_{\mathbb{RP}^m}(S) \setminus S = \{p\}$ is a singleton and the analytic closure of the germ S_p is irreducible.*

Corollary 1.5. *There is no 1-dimensional semialgebraic set $S \subset \mathbb{R}^m$ with $\mathrm{p}(S) = 2$ and $\mathrm{r}(S) = 1$.*

Proof. Suppose that there exists a semialgebraic set $S \subset \mathbb{R}^m$ with $\dim S = 1$, $\mathrm{p}(S) = 2$ and $\mathrm{r}(S) = 1$. By Theorem 1.1 and Proposition 1.2 we deduce that S is unbounded and not closed in $\mathbb{R}^m$. Thus, $\mathrm{Cl}_{\mathbb{RP}^m}(S) \setminus S$ has at least two elements: one point in $\mathsf{H}_\infty(\mathbb{R})$ because S is unbounded and another one in $\mathbb{R}^m$ since S is not closed in $\mathbb{R}^m$. But by Proposition 1.4 $\mathrm{Cl}_{\mathbb{RP}^m}(S) \setminus S$ is either empty or a singleton, which is a contradiction. $\square$

2. Main tools

In this section we present the main tools used to prove the results presented in this article. We will use usual concepts of (complex) Algebraic Geometry such as: rational map, regular map, normalization, etc. and refer the reader to [8,9] for further details. We recall the following useful and well-known fact concerning the regularity of rational maps defined on a non-singular curve (see [8, 7.1]) that will be used several times.

Lemma 2.1. *Let $Z \subset \mathbb{CP}^n$ be a non-singular projective curve and $F: Z \dashrightarrow \mathbb{CP}^m$ a rational map. Then F can be (uniquely) extended to a regular map $F': Z \to \mathbb{CP}^m$. Moreover, if Z, F are invariant, then also F' is invariant.*

Normalization of an algebraic curve. A main tool will be the normalization $(\widetilde{X}, \Pi)$ of an either affine or projective algebraic curve X, both in the real and in the complex case. The normalization is birationally equivalent to X and therefore unique up to a biregular homeomorphism; furthermore, if X is an invariant complex algebraic curve, we may assume that also $\widetilde{X}$ and π are invariant. To prove this, one can construct $(\widetilde{X}, \pi)$ as the desingularization of X via a finite chain of suitable invariant blowing-ups. Recall that all fibers of $\Pi: \widetilde{X} \to X$ are finite and if $x \in X$ is a non-singular point, then the fiber of x is a singleton. Moreover, if X is complex, then the cardinal of the fiber of a point $x \in X$ coincides with the number of irreducible components of the germ X_x. If $X \subset \mathbb{R}^m$ is an affine algebraic curve, $Y := \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(X)$ and $(\widetilde{Y} \subset \mathbb{CP}^k, \Pi)$ is an invariant normalization of Y, we have

- $(\widetilde{Z} := \widetilde{Y} \cap \mathbb{RP}^k, \Pi|_{\widetilde{Z}})$ is the normalization of $Z := \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(X)$ and $\Pi(\widetilde{Z}) = Z_{(1)}$,
- $(\widetilde{X} := \widetilde{Y} \cap \mathbb{R}^k, \pi := \Pi|_{\widetilde{X}})$ is the normalization of X and $\pi(\widetilde{X}) = X_{(1)}$. $\square$

The following two results are crucial to prove the Main results stated in the Introduction.

Lemma 2.2. *Let $f: \mathbb{R} \to \mathbb{R}^m$ be a non-constant rational map and $S := f(\mathbb{R})$. Then*

(i) *f can be (uniquely) extended to an invariant regular map $F: \mathbb{CP}^1 \to \mathbb{CP}^m$ such that $F(\mathbb{CP}^1) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$.*
(ii) *$\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ is an invariant rational curve and if $(\mathbb{CP}^1, \Pi)$ is an invariant normalization of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$, then there exists an invariant surjective regular map $\widetilde{F}: \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $F = \Pi \circ \widetilde{F}$.*
(iii) *If f is polynomial, then we may choose Π and $\widetilde{F}$ such that $\pi := \Pi|_{\mathbb{R}}$ and $\widetilde{f} := \widetilde{F}|_{\mathbb{R}}$ are polynomial. In particular, $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathsf{H}_\infty(\mathbb{C})$ is a singleton p and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)_p$ is irreducible.*

Proof. (i) Observe that f can be naturally extended to an invariant rational map

$$F := (F_0 : F_1 : \cdots : F_m) : \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^m$$

where $F_i \in \mathbb{R}[\mathtt{x}_0, \mathtt{x}_1]$ are homogeneous polynomials of the same degree d. In fact, such an extension is by Lemma 2.1 regular and unique. As $S = f(\mathbb{R})$, we deduce that $F(\mathbb{CP}^1) \subset \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ contains by [8, 2.31] a non-empty Zariski open subset of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$. Since F is proper and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ is irreducible, we conclude by [8, 2.33] $F(\mathbb{CP}^1) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$.

(ii) Let $(\widetilde{Y} \subset \mathbb{CP}^k, \Pi)$ be a σ-invariant normalization of $Y := \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$. Now the composition $\Pi^{-1} \circ F: \mathbb{CP}^1 \dashrightarrow \widetilde{Y}$ defines an invariant rational map that can be extended to an invariant surjective regular map $\widetilde{F}: \mathbb{CP}^1 \to \widetilde{Y}$ such that $F = \Pi \circ \widetilde{F}$. Observe that $\widetilde{Y}$ is by [8, 7.6, 7.20] a smooth curve of arithmetic genus 0, that is, a smooth rational curve (see [8, 7.17]); hence, we may take $\widetilde{Y} = \mathbb{CP}^1$. Thus, $(\mathbb{RP}^1, \Pi|_{\mathbb{RP}^1})$ is the normalization of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)$ and $\Pi(\mathbb{RP}^1) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)}$.

(iii) If f is polynomial, then $F_0 := \mathtt{x}_0^d$. Write $\Pi := (\Pi_0, \dots, \Pi_m)$ and $\widetilde{F} := (\widetilde{F}_0, \widetilde{F}_1)$ where $\Pi_i, \widetilde{F}_j \in \mathbb{R}[\mathtt{x}_0, \mathtt{x}_1]$ are homogeneous polynomials and let us check that we may assume $\Pi_0 = \lambda \mathtt{x}_0^e$ and $\widetilde{F}_0 = \mu \mathtt{x}_0^\ell$ for some positive integers e, ℓ such that $d = e\ell$; hence, $\pi := \Pi|_{\mathbb{R}}$ and $\widetilde{f} := \widetilde{F}|_{\mathbb{R}}$ are polynomial.

Indeed, observe first that $\widetilde{F}$ is not constant because it is surjective. Factorize

$$\Pi_0 = \prod_{i=1}^{e} (a_i \mathtt{x}_1 - b_i \mathtt{x}_0) \in \mathbb{C}[\mathtt{x}_0, \mathtt{x}_1]$$

where $a_i, b_i \in \mathbb{C}$ and $(a_i, b_i) \neq (0,0)$ for $i = 1, \dots, m$. Let us check that all factors $a_i \mathtt{x}_1 - b_i \mathtt{x}_0$ are proportional. Denote $\mathtt{p}_i := \widetilde{F}_i(1, \mathtt{x}_1)$ and observe

$$\prod_{i=1}^{e} (a_i \mathtt{p}_1 - b_i \mathtt{p}_0) = \Pi_0(\mathtt{p}_0, \mathtt{p}_1) = F_0(1, \mathtt{x}_1) = 1;$$

hence, all factors in the previous expression are non-zero constants $c_i \in \mathbb{C}$. Suppose that two of the pairs (a_i, b_i) are not proportional, for instance, (a_1, b_1) and (a_2, b_2). Then $(\mathtt{p}_0, \mathtt{p}_1)$ is the unique solution of the linear system

$$\begin{cases} a_1 \mathtt{x}_1 - b_1 \mathtt{x}_0 = c_1, \\ a_1 \mathtt{x}_2 - b_2 \mathtt{x}_0 = c_2 \end{cases}$$

and so $\mathtt{p}_0, \mathtt{p}_1 \in \mathbb{C}$, which contradicts the fact that $\widetilde{F}$ is not constant. Thus, we may write $\Pi_0 = \pm(a\mathtt{x}_1 - b\mathtt{x}_0)^e$ where $a, b \in \mathbb{R}$ and $(a,b) \neq (0,0)$. Consider an invariant change of coordinates $\Psi : \mathbb{CP}^1 \to \mathbb{CP}^1$ that transforms $(a : b)$ into $(0 : 1)$ and define $\Pi' := \Pi \circ \Psi$. Of course, $(\mathbb{CP}^1, \Pi')$ is an invariant normalization of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ with $\Pi'_0 = \lambda \mathtt{x}_0^e$. Define $\widetilde{F}'$ as the regular extension of $(\Pi')^{-1} \circ F$ to $\mathbb{CP}^1$; in particular, $F = \Pi' \circ \widetilde{F}'$. Since $\lambda(\widetilde{F}'_0)^e = \mathtt{x}_0^d$, we conclude $\widetilde{F}'_0 = \mu \mathtt{x}_0^\ell$.

Finally, we have $\Pi^{-1}(\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathsf{H}_\infty(\mathbb{C})) = \{(0 : 1)\}$ and so we deduce that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathsf{H}_\infty(\mathbb{C}) = \{p\}$ is a singleton and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)_p$ is irreducible. □

Lemma 2.3. *Let $f := (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant rational map such that its image $f(\mathbb{R}^n)$ has dimension 1. Then*

(i) *f factors through $\mathbb{R}$, that is, there exist a rational function $g \in \mathbb{R}(\mathtt{x})$ and a rational map $h : \mathbb{R} \to \mathbb{R}^m$ such that $f = h \circ g$.*
(ii) *If f is moreover a polynomial map, we may also assume that g and h are polynomial.*

Proof. Let $\mathbb{F} := \mathbb{R}(f_1, \dots, f_m)$ be the smallest subfield of the field of rational functions $\mathbb{R}(\mathtt{x})$ in n variables that contains $\mathbb{R}$ and $f_1, \dots, f_m$. Note that $\mathrm{tr.deg}(\mathbb{F}|\mathbb{R}) = \dim(\mathrm{im}\, f) = 1$, so we may assume $f_1 \notin \mathbb{R}$. Thus, by Lüroth's Theorem there exists a rational function $g \in \mathbb{R}(\mathtt{x}) \setminus \mathbb{R}$ such that $\mathbb{F} = \mathbb{R}(g)$. Since $f_i \in \mathbb{F} = \mathbb{R}(g)$, we have $f_i = \frac{P_i(g)}{Q_i(g)}$ for some coprime polynomials $P_i, Q_i \in \mathbb{R}[\mathtt{t}]$. Now the rational map $h := (\frac{P_1}{Q_1}, \dots, \frac{P_m}{Q_m}) : \mathbb{R} \to \mathbb{R}^m$ satisfies $f = h \circ g$ and so (i) holds.

Suppose next that f is moreover polynomial. Following [11, Lemma 2] (see also [10, Lemma 2, pp. 710–711]),

(2.3.1). *We may assume that the Lüroth's generator g of $\mathbb{F}$ is in fact polynomial.*

By Bezout's Lemma we can write now $1 = P_i A_i + Q_i B_i$ for some $A_i, B_i \in \mathbb{R}[\mathtt{t}]$. Substituting the variable $\mathtt{t}$ by g we get the polynomial identity

$$1 = P_i(g)A_i(g) + Q_i(g)B_i(g) = Q_i(g)f_iA_i(g) + Q_i(g)B_i(g) = Q_i(g)\big(f_iA_i(g) + B_i(g)\big);$$

hence, $Q_i(g)$ is a non-zero constant and so the polynomials $h_i := \frac{P_i(\mathtt{t})}{Q_i(g)}$ fit our situation.

For the sake of completeness let us include the elementary proof of [11, Lemma 2] that shows statement (2.3.1). Let $g_0 \in \mathbb{R}(\mathtt{x}) \setminus \mathbb{R}$ be a Lüroth's generator of $\mathbb{F}$. Since the extension $\mathbb{F}|\mathbb{R}$ has transcendence degree 1, we may assume $F := f_1 \in \mathbb{R}[\mathtt{x}] \setminus \mathbb{R}$. Let $R, S \in \mathbb{R}[\mathtt{t}] \setminus \{0\}$ and $P, Q \in \mathbb{R}[\mathtt{x}] \setminus \{0\}$ be pairs of relatively prime polynomials such that

$$F = \frac{R(g_0)}{S(g_0)} \quad \text{and} \quad g_0 = \frac{P}{Q}.$$

Consequently, we deduce

$$F = \frac{Q^r R(P/Q)}{Q^s S(P/Q)} Q^{s-r}$$

where $r := \deg(R)$ and $s := \deg(S)$. Notice that the polynomials Q, $Q^r R(P/Q)$ and $Q^s S(P/Q)$ are pairwise relatively prime; once this is shown, it follows directly that $H := Q^s S(P/Q) \in \mathbb{R}$ and $s - r \geqslant 0$ by using the fact that $\mathbb{R}[\mathtt{x}]$ is a UFD.

Indeed, using $R, S \in \mathbb{R}[\mathtt{t}]$, it is straightforward to show

$$\gcd\big(Q, Q^r R(P/Q)\big) = \gcd\big(Q, Q^s S(P/Q)\big) = \gcd(P, Q) = 1.$$

By Bezout's Lemma we find polynomials $A_1, A_2 \in \mathbb{R}[\mathtt{t}]$ of degrees $k_i := \deg(A_i)$ such that $1 = A_1R + A_2S$. Substituting $\mathtt{t} \rightsquigarrow P/Q$ and multiplying the expression with Q^ℓ where $\ell := \max\{\deg(A_1)+\deg(R), \deg(A_2)+\deg(S)\}$, we get

$$Q^\ell = Q^{\ell-k_1-r}\big(Q^{k_1}A_1(P/Q)\big)\big(Q^r R(P/Q)\big) + Q^{\ell-k_2-r}\big(Q^{k_2}A_2(P/Q)\big)\big(Q^r S(P/Q)\big)$$

and so $\gcd(Q^r R(P/Q), Q^s S(P/Q))$ divides Q^ℓ; hence,

$$\gcd\big(Q^r R(P/Q), Q^s S(P/Q)\big) = \gcd\big(Q^r R(P/Q), Q^s S(P/Q), Q^\ell\big) = 1.$$

Factorize $S = \alpha(\mathtt{t} - \xi_1)\cdots(\mathtt{t} - \xi_s)$ where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\xi_i \in \mathbb{C}$. We have

$$H = Q^s S(P/Q) = \alpha(P - \xi_1 Q)\cdots(P - \xi_s Q),$$

so $(P - \xi_i Q) = \gamma_i \in \mathbb{C}$ for $1 \leqslant i \leqslant s$. If any two ξ_i's were distinct, for instance, $\xi_1 \neq \xi_2$, we would get $(\xi_2 - \xi_1)Q = \gamma_1 - \gamma_2 \in \mathbb{C}$; hence, $Q \in \mathbb{R}[\mathtt{x}] \cap \mathbb{C} = \mathbb{R}$ and $P \in \mathbb{R}[\mathtt{x}] \cap \mathbb{C} = \mathbb{R}$, which contradicts the fact that $g_0 = P/Q \in \mathbb{R}(\mathtt{x}) \setminus \mathbb{R}$. Thus, $S = \alpha(\mathtt{t} - \xi)^s$ where $\alpha, \xi \in \mathbb{R}$ and $s \geqslant 0$.

If $s = 0$, we may assume $Q = 1$ and $g := g_0 = P \in \mathbb{R}[\mathtt{x}]$. If $s > 0$, then $P - \xi Q = \gamma \in \mathbb{R}$ and so $g_0 = P/Q = \xi + \gamma/Q$; hence, $\mathbb{F} = \mathbb{R}(g_0) = \mathbb{R}(\xi + \gamma/Q) = \mathbb{R}(\gamma/Q) = \mathbb{R}(g)$ where $g := Q \in \mathbb{R}[\mathtt{x}]$. □

We finish this section with an elementary crucial example.

Example 2.4. Let us show that $\mathbb{S}^1$ and $\mathbb{RP}^1$ are regular images of $\mathbb{R}$. Since $\mathbb{RP}^1$ is the image of $\mathbb{S}^1$ via the canonical projection $\pi : \mathbb{S}^1 \to \mathbb{RP}^1$, it is enough to prove that $\mathbb{S}^1$ is a regular image of $\mathbb{R}$. To that end, we may take for instance the regular map

$$f : \mathbb{R} \to \mathbb{S}^1, \quad t \mapsto \left(\left(\frac{t^2-1}{t^2+1}\right)^2 - \left(\frac{2t}{t^2+1}\right)^2, 2\left(\frac{t^2-1}{t^2+1}\right)\left(\frac{2t}{t^2+1}\right)\right).$$

Observe that the previous map is the composition of the inverse of the stereographic projection of $\mathbb{S}^1$ from $(1,0)$ with

$$g:\mathbb{C}\equiv\mathbb{R}^2\to\mathbb{C}\equiv\mathbb{R}^2,\quad z=x+\sqrt{-1}y\equiv(x,y)\mapsto z^2\equiv\big(x^2-y^2,2xy\big).$$

3. Proofs of the main results

The purpose of this section is to prove Theorems 1.1 and 1.3 and Propositions 1.2 and 1.4. We begin with the case $m=1$, that is, $S:=I$ is an interval of $\mathbb{R}$.

Lemma 3.1. *Let $I\subset\mathbb{R}$ be an interval. Then*

(i) $\mathrm{p}(I)<+\infty$ *if and only if I is unbounded. Moreover, if such is the case, then* $\mathrm{p}(I)\leqslant 2$ *and* $\mathrm{p}(I)=2$ *if and only if $I\subsetneq\mathbb{R}$ is open.*
(ii) $\mathrm{r}(I)\leqslant 2$ *and* $\mathrm{r}(I)=2$ *if and only if $I\subsetneq\mathbb{R}$ is open.*

Proof. (i) If $f:\mathbb{R}\to\mathbb{R}$ is a non-constant polynomial map, the image of f is either $\mathbb{R}$ or a proper closed unbounded interval; hence, if $I\subsetneq\mathbb{R}$ is open, then $\mathrm{p}(I)\geqslant 2$. On the other hand, if $\mathrm{p}(I)=n<+\infty$ and $g:\mathbb{R}^n\to\mathbb{R}$ is a polynomial map such that $g(\mathbb{R}^n)=I$, we take $x_0\in\mathbb{R}^n$ with $g(x_0)\neq g(0)$ and consider the non-constant polynomial map $h:\mathbb{R}\to\mathbb{R}$, $t\mapsto g(tx_0)$; hence, $h(\mathbb{R})\subset I$ is unbounded.

To finish it is enough to prove that the interval $[0,+\infty)$ is a polynomial image of $\mathbb{R}$ while $(0,+\infty)$ is a polynomial image of $\mathbb{R}^2$. To that end, consider the polynomial maps

$$f_1:\mathbb{R}\to\mathbb{R},\quad t\mapsto t^2\quad\text{and}\quad f_2:\mathbb{R}^2\to\mathbb{R},\quad (x,y)\mapsto(xy-1)^2+x^2.$$

(ii) For the second part observe that a regular map $f:\mathbb{R}\to\mathbb{R}$ can be extended regularly to a map $F:\mathbb{RP}^1\to\mathbb{RP}^1$ by Lemma 2.1. Thus, the image of F is either $\mathbb{RP}^1$ or a proper closed interval J of $\mathbb{RP}^1$. If $F(p_\infty)=p_\infty$, then $I=J\setminus\{p_\infty\}$ is an unbounded closed interval of $\mathbb{R}$. On the other hand, if $F(p_\infty)=c\in\mathbb{R}$, then $J=[a,b]$ is a bounded closed interval of $\mathbb{R}$ and I is either equal to J (if $F^{-1}(c)$ is not a singleton) or $J\setminus\{c\}$ (if $F^{-1}(c)$ is a singleton). As I is connected, it is either $[a,b]$ or one of the half-open bounded intervals $[a,b)$ or $(a,b]$. Thus, if $I\subsetneq\mathbb{R}$ is open, then $\mathrm{r}(I)\geqslant 2$.

To finish the proof and in view of (i), it is enough to notice that the intervals $[0,1]$ and $(0,1]$ are regular images of $\mathbb{R}$ via the regular maps

$$f_3:\mathbb{R}\to\mathbb{R},\quad t\mapsto\frac{t}{1+t^2}+\frac{1}{2},\qquad f_4:\mathbb{R}\to\mathbb{R},\quad t\mapsto\frac{1}{1+t^2}$$

while the interval $(0,1)$ is a regular image of $\mathbb{R}^2$ via the regular map

$$f_5:\mathbb{R}^2\to\mathbb{R},\quad (x,y)\mapsto\frac{(xy-1)^2+x^2}{1+(xy-1)^2+x^2}.$$

The concrete details are left to the reader. □

Proof of Theorem 1.1. (i) $\Longrightarrow$ (ii) We know that S is unbounded and by [4, 3.1] S is irreducible. Since $\mathrm{p}(S)<+\infty$, there exists a regular map $f:\mathbb{R}^n\to\mathbb{R}^m$ such that $f(\mathbb{R}^n)=S$. By Lemma 2.3 there exist polynomial maps $h:\mathbb{R}\to\mathbb{R}^m$ and $g:\mathbb{R}^n\to\mathbb{R}$ satisfying $f=h\circ g$; notice that the Zariski closures of $f(\mathbb{R}^n)$ and $h(\mathbb{R})$ coincide. By an application of Lemma 2.2 to the polynomial map h we conclude that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ is an invariant rational curve such that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)\cap\mathsf{H}_\infty(\mathbb{C})=\{p\}$ is a singleton and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)_p$ is irreducible.

(ii) $\Longrightarrow$ (i) Let $\Pi := (\Pi_0 : \cdots : \Pi_m) : \mathbb{CP}^1 \to \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ be an invariant normalization of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$; in particular, $\pi(\mathbb{RP}^1) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)}$. Since $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathsf{H}_\infty(\mathbb{C}) = \{p\}$ is a singleton and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)_p$ is irreducible, we may assume

$$\Pi^{-1}\big(\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S) \cap \mathsf{H}_\infty(\mathbb{C})\big) = \big\{(0:1)\big\};$$

hence, $\Pi_0 = \mathtt{t}_0^d$ for some $d \geqslant 1$. Therefore $\pi := \Pi|_{\mathbb{R}} : \mathbb{R} \equiv \mathbb{RP}^1 \setminus \{p_\infty\} \to \mathbb{R}^m$ is a polynomial map and since S is irreducible and 1-dimensional, $S \subset \pi(\mathbb{R}) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)} \setminus \mathsf{H}_\infty(\mathbb{R})$. Moreover, since $(\mathbb{R}, \pi)$ is the normalization of $\mathrm{Cl}^{\mathrm{zar}}(S)$, there exists by [4, 3.5] an interval $I \subset \mathbb{R}$ such that $\pi(I) = S$; in fact, since S is unbounded, also I is unbounded. By Lemma 3.1 I and therefore S are polynomial images of $\mathbb{R}^2$. $\square$

Proof of Proposition 1.2. If $\mathrm{p}(S) = 1$, there exists a non-constant polynomial map $f : \mathbb{R} \to \mathbb{R}^m$ such that $f(\mathbb{R}) = S$. Since f is proper, S is closed in $\mathbb{R}^m$.

Conversely, as we have seen in the proof of (ii) $\Longrightarrow$ (i) in Theorem 1.1, there exists a polynomial map $\pi : \mathbb{R} \to \mathbb{R}^m$ such that $(\mathbb{R}, \pi)$ is the normalization of $\mathrm{Cl}^{\mathrm{zar}}(S)$. Thus, by [4, 3.5] there exists an interval $I \subset \mathbb{R}$ such that $\pi(I) = S$. Since S is unbounded and closed, such an interval can be chosen unbounded and closed. Thus, by Lemma 3.1 I and therefore S are polynomial images of $\mathbb{R}$. $\square$

Proof of Theorem 1.3. (i) $\Longrightarrow$ (ii) By [4, 3.1] S is irreducible. Let now $f : \mathbb{R}^n \to \mathbb{R}^m$ be a regular map such that $f(\mathbb{R}^n) = S$. By Lemma 2.3 there exist a rational function $g \in \mathbb{R}(\mathtt{x})$ and a rational map $h := (\frac{h_1}{h_0}, \ldots, \frac{h_m}{h_0}) : \mathbb{R} \to \mathbb{R}^m$ such that $f = h \circ g$. Now we deduce by Lemma 2.2 that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ is a rational curve.

(ii) $\Longrightarrow$ (i) Let $\pi : \mathbb{RP}^1 \to \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)$ be the normalization of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)$; recall $\pi(\mathbb{RP}^1) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)}$. If $S = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)}$, then S is by Example 2.4 a regular image of $\mathbb{R}$. On the other hand, if $S \neq \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)}$, we may assume that the image of the infinite point p_∞ of $\mathbb{RP}^1$ under π belongs to $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)_{(1)} \setminus S$. By [4, 3.5] there exists now an interval $I \subset \mathbb{R} = \mathbb{RP}^1 \setminus \{p_\infty\}$ such that $\pi(I) = S$. By Lemma 3.1 we conclude that I and therefore S are regular images of $\mathbb{R}^2$. $\square$

Proof of Proposition 1.4. Suppose first $\mathrm{r}(S) = 1$. Let $f : \mathbb{R} \to \mathbb{R}^m$ be a regular map such that $f(\mathbb{R}) = S \subset \mathrm{Cl}^{\mathrm{zar}}(S)$. By Lemma 2.2 f can be extended to a surjective regular map $F : \mathbb{CP}^1 \to \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)$ and we may decompose $F = \Pi \circ \widetilde{F}$ where $\widetilde{F} : \mathbb{CP}^1 \to \mathbb{CP}^1$ is an invariant surjective regular map and $(\mathbb{CP}^1, \Pi)$ is an invariant normalization of $\mathbb{CP}^1$; we may assume $p_\infty \in \Pi^{-1}(\mathsf{H}_\infty(\mathbb{C}))$. Since f is a regular map,

$$\varnothing = F^{-1}\big(\mathsf{H}_\infty(\mathbb{C})\big) \cap \mathbb{R} = (\widetilde{F})^{-1}\big(\Pi^{-1}\big(\mathsf{H}_\infty(\mathbb{C})\big)\big) \cap \mathbb{R}.$$

As $p_\infty \in \Pi^{-1}(\mathsf{H}_\infty(\mathbb{R}))$, we deduce that the image of $\widetilde{f} := \widetilde{F}|_{\mathbb{R}}$ is contained in $\mathbb{R}$ and so $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ is a regular map such that $f = \pi \circ \widetilde{f}$ where $\pi := \Pi|_{\mathbb{R}}$. By Lemma 3.1 we may assume $\widetilde{f}(\mathbb{R}) = \mathbb{R}$, $[0, \infty)$, $[0,1]$ or $[0,1)$.

If $\widetilde{f}(\mathbb{R}) = [0,1]$, then we obtain $\mathrm{Cl}_{\mathbb{RP}^m}(S) = S$. Otherwise let $q := p_\infty$ if $\widetilde{f}(\mathbb{R}) = \mathbb{R}$ or $[0, \infty)$ and $q := 1$ if $\widetilde{f}(\mathbb{R}) = [0,1)$. Observe that $J := \widetilde{f}(\mathbb{R}) \cup \{q\}$ is a closed subset of $\mathbb{RP}^1$; hence, its image $S \cup \{\pi(q)\}$ under π is a closed subset of $\mathbb{RP}^m$ and so $\mathrm{Cl}_{\mathbb{RP}^m}(S) = S \cup \{\pi(q)\}$. Thus, $\mathrm{Cl}_{\mathbb{RP}^m}(S) \setminus S$ is either empty or a singleton.

Suppose now $\mathrm{Cl}_{\mathbb{RP}^m}(S) \setminus S = \{p := \pi(q)\}$; hence, $\pi^{-1}(p) \cap \widetilde{f}(\mathbb{R}) = \varnothing$ because $S = f(\mathbb{R}) = \pi(\widetilde{f}(\mathbb{R}))$ and so $\pi^{-1}(p) \cap J = \{q\}$. Thus, $S_p = \pi((\widetilde{f}(\mathbb{R}))_q)$ and we conclude that the analytic closure of the germ S_p is irreducible.

Conversely, by Theorem 1.3 and [4, 3.5] there exists a connected subset $I \subset \mathbb{RP}^1$ such that $\pi(I) = S$ where $(\mathbb{RP}^1, \pi)$ is the normalization of $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(S)$. In fact, I is the unique 1-dimensional connected component of $\pi^{-1}(S)$. We distinguish two possibilities:

Case 1. $\mathrm{Cl}_{\mathbb{RP}^m}(S) = S$. Then S is closed in $\mathbb{RP}^m$ and so I is either $\mathbb{RP}^1$ or a compact interval contained in $\mathbb{RP}^1$ that we may assume equal to $[0,1]$.

Case 2. $\mathrm{Cl}_{\mathbb{RP}^m}(S) \setminus S = \{p\}$ is a singleton and the analytic closure of the germ S_p is irreducible. Observe $\mathrm{Cl}_{\mathbb{RP}^m}(S) = \pi(\mathrm{Cl}_{\mathbb{RP}^1}(I))$ and since the analytic closure of the germ S_p is irreducible, we deduce that $(\pi|_{\mathrm{Cl}_{\mathbb{RP}^1}(I)})^{-1}(p) = \{a\}$ is a singleton. Thus, $I = \mathrm{Cl}_{\mathbb{RP}^1}(I) \setminus \{a\}$ and we may assume either $I = [0, 1)$ or $I = \mathbb{R}$.

In both cases we conclude by Lemma 3.1 and Example 2.4 that S is a regular image of $\mathbb{R}$, as required. □

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