The Journal of Risk and Insurance

 Vol. 86, No. 2, 297–313 (2019).
DOI: 10.1111/jori.12211

Moral Hazard, Risk Sharing, and the Optimal Pool Size

Frauke von Bieberstein
Eberhard Feess
José F. Fernando
Florian Kerzenmacher
Jörg Schiller

Abstract

We examine the optimal size of risk pools with moral hazard. In risk pools, the effective share of the own loss borne is the sum of the direct share (the retention rate) and the indirect share borne as residual claimant. In a model with identical individuals with mixed risk-averse utility functions, we show that the effective share required to implement a specific effort increases in the pool size. This is a downside of larger pools as it, *ceteris paribus*, reduces risk sharing. However, we find that the benefit from diversifying the risk in larger pools always outweighs the downside of a higher effective share. We conclude that, absent transaction costs, the optimal pool size converges to infinity. In our basic model, we restrict attention to binary effort levels, but we show that our results extend to a model with continuous effort choice.

Introduction

Formal risk pools such as mutual insurance arrangements, partnerships of lawyers, farmers, and physicians benefit from risk sharing, but are encumbered by free riding (moral hazard). In contrast to traditional insurance arrangements where risks are

---

Frauke von Bieberstein is at the University of Bern, IOP, Engehaldenstr. 4, Bern 3012, Switzerland. Bieberstein can be contacted via e-mail: vonbieberstein@iop.unibe.ch. Eberhard Feess is at the Frankfurt School of Finance & Management, Sonnemannstr. 9-11, Frankfurt 60314, Germany. Feess can be contacted via e-mail: e.feess@fs.de. José F. Fernando is at the Universidad Complutense de Madrid, Facultad de Matemáticas, Departamento de Álgebra, Plaza de Ciencias, 3, Madrid 28040, Spain. Fernando can be contacted via e-mail: josefer@mat.ucm.es. Florian Kerzenmacher is at the Frankfurt School of Finance & Management, Sonnemannstr. 9-11, Frankfurt 60314, Germany. Kerzenmacher can be contacted via e-mail: f.kerzenmacher@fs.de. Jörg Schiller is at the Universitaet Hohenheim, Chair in Insurance and Social Systems, Fruwirthstr. 48, Stuttgart 70593, Germany. Schiller can be contacted via e-mail: j.schiller@uni-hohenheim.de.
We are grateful to Harris Dellas, Marc Möller, Martin Nell, Richard Peter, Harris Schlesinger, and two anonymous referees for valuable comments.

transferred to an insurance company and its stockholders, the members of risk pools are the residual claimants of the transferred risks. Our article addresses the following question: In a world with independent risks, risk-averse participants, moral hazard, and perfect enforceability of contracts, does the optimal size of the risk pool converge to infinity if the risk-sharing arrangement is properly designed?

As a first intuition, the answer would be a straightforward "yes" due to better risk sharing in larger pools. However, the issue is more involved in case of moral hazard. In risk pools, part of the own loss is borne directly as a retention rate, and an additional share indirectly as residual claimant. We will refer to the sum of these two parts as the *effective share*. As the part of the own loss borne as residual claimant decreases in the pool size, the retention rate needs to increase in order to keep effort incentives constant. For the special case with linear marginal utility such as quadratic utility functions, we show that it suffices to increase the retention rate to an extent that keeps the effective share constant. It is then, indeed, straightforward to show that the utility increases in the pool size due to better risk sharing when effort incentives are kept constant.

For individuals with mixed risk-averse utility functions where higher-order derivatives weakly alternate in sign (see, e.g., Caballé and Pomansky, 1996),[1]however, implementing the high effort requires that the effective share increases in the pool size. In other words, a larger part of the own loss needs to be borne by each individual. To see the reason, consider the case with a binary effort choice. A binding incentive compatibility constraint (ICC) for choosing the high effort requires that the expected utility difference without own loss and with own loss is constant in the pool size, and equal to the cost difference of high and low efforts. Due to the diversification effect of larger pools (mean-preserving contraction), extremely low income levels become less likely even in case with own loss. For individuals with mixed risk-averse utility functions, this *ceteris paribus* decreases the incentive to avoid the own loss and thereby also decreases the incentive to choose the high effort. This incentive-reducing impact of larger pools needs to be balanced by a higher effective share. This yields a countervailing effect to the benefits of risk sharing in larger pools, so that it is *ex ante* unclear whether larger pools are superior.

Our main result is that the benefit from improved risk sharing in larger pools always dominates. The intuition is that the higher effective share is only needed because the expected utility in case with an own loss increases faster than in the case without own loss. Thus, the higher effective share just redistributes a part of the utility gain from larger pools to the case without own loss, so that the expected utility difference between the two cases remains constant. This ensures that the ICC is binding and that expected utility increases in both states of the world. In our main model, we consider only two effort levels, but we show that the superiority of larger pools carries over to the case of continuous effort. The intuition follows from the fact that our finding does

[1]Mixed risk-averse utility functions include most of the commonly used von Neumann–Morgenstern utility functions (see Eeckhoudt and Schlesinger, 2006).

not hinge on the optimality of effort—if the same effort level is implemented for two pools of different sizes, then expected utility is higher for the larger pool.

Our article is related to several strings of literature. First, our finding that the effective share required for incentive compatibility increases in the pool size relates to insights on how the expected utility of individuals with mixed risk aversion depends on the allocation of risks between different states of the world. Eeckhoudt and Schlesinger (2006) introduce the concept of risk apportionment to show how the optimal disaggregation of harms, which can be certain losses or random variables, between different states of nature depends on higher-order risk preferences, such as prudence (third-order risk attitude), temperance (fourth-order), and edginess (fifth-order). Deck and Schlesinger (2014) extend the scope of risk apportionment and provide experimental evidence that most individuals are risk apportionate, which is equivalent to mixed risk aversion.

Building on Eeckhoudt, Schlesinger, and Tsetlin (2009) who analyze the impact of higher-order risk preferences on the allocation of risks, Ebert (2013) shows that higher-order risk preferences can be characterized by the statistical moments of the distribution of outcomes, where nth-degree risk aversion is equivalent to a preference for higher (lower) odd (even) moments. In particular, prudence is related to skewness preference, whereas temperance is related to kurtosis aversion. In our setting with risk pools, all derivatives of the utility function can play a role in whether the utility with or without own loss increases faster in the pool size when the effective share is kept constant. Mixed risk aversion ensures that the effects of all derivatives go in the same direction, so that for a constant effective share the expected utility increases faster in the pool size in case with an own loss. As discussed above, this requires an increase in the effective share, which, however, never outweighs the benefits from risk sharing in larger pools.

Next, our article relates to research on the optimal size of risk pools. As we focus on *formal* risk pools where losses are observable and risk transfers of each member can be specified in an explicit and perfectly enforceable contract, *informal* risk pools are an important risk-sharing arrangement in developing countries. In informal risk pools, adjusting the retention rate optimally to the pool size and enforcing *ex ante* agreements may be difficult or impossible (Bold, 2009). Many articles on informal partnerships confirm that moral hazard increases in the pool size and conclude that stable pools might hence be of limited size (see, e.g., Genicot and Ray, 2003; Bramoullé and Kranton, 2007). If the retention rate is not adjusted to larger pools, then effort incentives decrease in the pool size, and the optimal pool size is reached when the marginal benefits from better risk sharing are equal to the marginal costs from lower effort (see the simulations in Lee and Ligon, 2001). Our result is complementary, as it shows that larger pools are unambiguously superior when the retention rate is adjusted to ensure incentive compatibility.

Ligon and Thistle (2005) suggest adverse selection as an explanation for the fact that mutuals are often small compared to stock insurers. In a separating equilibrium, mutuals attract low-risk consumers and offer higher expected indemnities, but are smaller in size than stock insurers in order to be unattractive for high-risk consumers. Another reason for limited pool sizes are transaction costs, which are neglected in our model. In particular, adjusting retention rates optimally may be expensive when types are heterogeneous (see, e.g., Murgai et al., 2002). For the points we wish to make, however, transaction costs would not add much to the existing literature, taking transaction costs into account, the optimal pool size would be reached when, after accounting for the required increase in the effective share, the marginal benefit of improved risk diversification is equal to marginal transaction costs. Barigozzi et al. (2017) consider moral hazard in risk pools in an infinitely repeated game. Analogous to market games, the incentive to free ride increases in the number of participants, so that the efficient effort level may only be self enforcing in small pools.

As mutual insurance companies play an important role in life insurance (Zanjani, 2007) and in particular for property–casualty insurance,[2] partnerships of, for example, lawyers, farmers, and physicians can also be interpreted as risk pools. In the theoretical parts of their mainly empirical articles on moral hazard problems in partnerships, Gaynor and Gertler (1995) and Lang and Gordon (1995) restrict attention to quadratic utility functions, and neither consider higher-order utility effects or the optimal pool size. Examples for more specialized risk pools are Risk Retention Groups (RRG), which were formed in the United States during the liability insurance crises in the 1970s and 1980s.[3]

An important part of the literature discusses why mutual insurance companies and stock insurers coexist in the same insurance markets. Stock insurers generally offer contracts in which policyholders transfer risks for a fixed premium and diversify their risk on the market, while policyholders in mutuals are owners and thus residual claimants of the insurance pool. As this can be a serious downside of small mutuals, Smith and Stutzer (1995) show that mutuals can be superior to stock insurers in case of economy-wide aggregate risk and moral hazard problems. Mayers and Smith (2013) point out that mutuals may have limited access to capital markets, which may diminish the control of the management by owners. Laux and Mürmann (2010) argue that mutuals may have comparative advantages in raising external capital, when stock insurers face free-rider and commitment problems. Among others empirical studies, Cummins, Weiss, and Zi (1999) provide evidence for the theoretical finding that mutual insurance companies are more successful in personal lines that require less managerial discretion with respect to individualized pricing and underwriting (Mayers and Smith, 1988). Thus, whether mutuals or stock insurers are preferable depends on the specific market and firm situation. Ligon and Thistle (2008) show that mutual insurance arrangements are equivalent to a fairly priced stock insurance policy with the same coverage plus a zero mean background risk. When the pool size converges to

[2] According to the Federal Insurance Office, four mutual insurance groups, State Farm Mutual Automobile Insurance (1), Liberty Mutual Insurance (2), Nationwide Mutual Group (6), and USAA Insurance (10), were ranked under the top 10 property–casualty insurance providers in the United States in 2014.

[3] One important advantage of the RRG are lower regulation standards (Leverty, 2011). In 1981, the U.S. Congress passed the Products Liability Risk Retention Act to allow a new type of insurance vehicle, the Risk Retention Groups, to cover product liability exposures. In 1986, the Act was expanded to allow the RRG to cover all casualty risks except workers' compensation.

infinity, then insurance contracts of a mutual converge to contracts offered by a stock insurer.[4] Thus, neglecting all other issues such as transaction costs and heterogeneity of policyholders, stock insurers and infinitely large mutuals are equivalent and both superior to smaller mutuals.

The remainder of the article is organized as follows: "The Model for Two Effort Levels" section introduces the model for two effort levels. The "Stage 2: Individual Effort Choices" section derives the individual effort choices, and the "Stage 1: Optimal Choice of the Effective Share β_n" section derives the optimal effective shares. The "Impact of the Pool Size on Incentive Compatibility" section analyzes the impact of the pool size on incentive compatibility. The optimal pool size is derived in "The Impact of Pool Size on Expected Utility" section. The "Continuous Effort" section extends to continuous effort. The "Conclusion" section concludes.

THE MODEL FOR TWO EFFORT LEVELS

There are n identical risk-averse individuals with initial wealth W_0. Each individual i faces the risk of a loss $L < W_0$ and can exert unobservable effort $x_i \in \{0, 1\}$ at cost $C(x_i) = cx_i$ where $c > 0$. Choosing effort $x_i = 1$ reduces the loss probability from p_0 (associated with the effort $x_i = 0$) to p_1 where $0 < p_1 < p_0 < 1$.

Individual losses are assumed to be independent, which is a reasonable assumption for many risk pools such as risk sharing for accidents, liability for medical malpractice, or sharing contracts in law firms. We consider a strictly increasing and strictly concave analytic utility function $u(W)$ where $(-1)^l u^l \leq 0$ for all $l \geq 3$ (mixed risk aversion). The set of functions satisfying these conditions contains the usually used utility functions for risk aversion with $u' > 0$ and $u'' < 0$, including quadratic utility functions, logarithmic functions, and exponential utility functions. Effort costs are additively separable and W denotes the individual's final wealth.

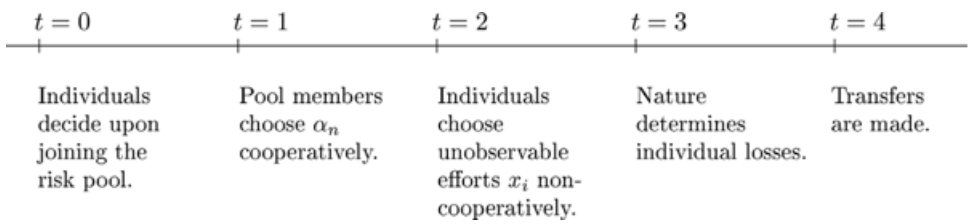
We consider the following game: at stage 0, each individual decides whether to join the risk pool or not. If an individual does not join the pool, no other insurance is available for the type of risk considered. At stage 1, the n individuals who joined the pool agree cooperatively on a retention rate $\alpha_n \in [0, 1)$ that maximizes an objective function that aggregates the expected utilities of the pool members.[5] As individuals and retention rates are identical, this also maximizes the expected utility of each single pool member. We restrict attention to retention rates that are independent of the number of losses (linear sharing rules). A retention rate α_n in stage 1 is feasible if and only if it is in the core, that is, if there is no coalition $\widetilde{n} < n$ that yields a higher payoff for all $\widetilde{n}$ members of the (sub) coalition.[6]

[4]We are grateful to an anonymous referee for pointing out this equivalence.

[5]We write $\alpha_n \in [0, 1)$ in order to allow for the full-insurance case, but we exclude no-insurance ($\alpha_n = 1$) as this is identical to the case where an individual does not participate in the pool.

[6]Observe that when the members of the pool agree on α_n, there is no private information as all pool members are identical, and because effort costs $C(x_i)$ and loss probabilities depending on the effort chosen are common knowledge. Furthermore, α_n depends only on losses and not (directly) of efforts chosen; that is, private information does not matter for the enforcement of

FIGURE 1
Timeline



When setting α_n, the pool members take into account that in stage 2, each pool member will choose the effort level that maximizes her individual utility; depending on n, α_n, and the anticipated effort choices of all other pool members. Thus, effort is chosen non-cooperatively and, due to the unobservability of the effort choice, the pool members cannot sign a contract contingent on effort. It follows that if the pool members want to implement a specific effort-level vector, they need to take *incentive compatibility* into account.

In stage 3, verifiable losses occur. Transfers are made through the risk pool in stage 4. Figure 1 summarizes the timeline of the game that we solve by backward induction.

Suppose for the moment that low effort maximizes utility even for $n = 1$, that is, for the no-insurance case. Then, low effort is *a fortiori* optimal for larger pools as the risk is maximum without insurance.[7] But then, there is no incentive problem, and pool members will optimally agree on $\alpha_n = 0$ irrespective of the pool size. Larger pools are then superior due to the pure insurance effect. To exclude this trivial solution where low effort is always optimal, we introduce the following assumption:

Assumption 1: High effort maximizes expected utility in the no-insurance case, that is,

$$(p_0 - p_1)\left[u\left(W_0\right) - u\left(W_0 - L\right)\right] > c.$$

Note that Assumption 1 does not exclude that low effort is optimal for larger pools, as losses can then be divided among all pool members. Assumption 1 is thus compatible with a setting where high effort is optimal for small pools, whereas low effort may be optimal for large pools.

the sharing rule agreed upon. Therefore, we can apply the basic concept of a core of a coalitional game with transferable payoffs (see, e.g., Osborne and Rubinstein, 1994, p. 258).

[7]Formally, this is implied by the proof of Lemma 1 below. Thus, for all pool sizes, incentives for high effort are maximum for $\alpha_n = 1$. But as this is identical to $n = 1$, there is no equilibrium with high effort for any n if low effort is optimal for $n = 1$ (note that there are no externalities for $\alpha = 1$, so that the individually rational behavior is also socially optimal).

In the following, we first assume that the members of the pool *always* want to implement high effort. We then extend to the case where low effort is superior for large pools.

STAGE 2: INDIVIDUAL EFFORT CHOICES

Following backward induction, we start with stage 2 on the pool members' effort choices. Each participant's ICC for choosing high effort in a symmetric Nash equilibrium is

$$\mathbb{E}\left[u\left(\alpha_n, x_i = 1, \mathbf{x}_{-i} = \mathbf{1}\right)\right] \geq \mathbb{E}\left[u\left(\alpha_n, x_i = 0, \mathbf{x}_{-i} = \mathbf{1}\right)\right] \tag{1}$$

$$\text{and } \alpha_n \in [0, 1), \tag{2}$$

where the effort level vector $\mathbf{x}_{-i} = \mathbf{1}$ means that all but i choose high effort.[8]

With α_n as share of the own loss directly borne by each individual, the remaining part $1 - \alpha_n$ is equally shared among all members of the pool. The *effective share* of the own loss borne by each individual is then the sum of the retention rate and the share indirectly borne via the redistribution in the pool, $\beta_n = \alpha_n + \frac{1-\alpha_n}{n}$. In what follows, we focus mostly on the effective share β_n.

Using β_n, we can write the ICC as

$$\begin{aligned} & p_1 \sum_{k=0}^{n-1} b(k; n-1, p_1) u\left(W_0 - \beta_n L - \left(\frac{1-\beta_n}{n-1}\right) kL\right) \\ & +(1-p_1) \sum_{k=0}^{n-1} b(k; n-1, p_1) u\left(W_0 - \left(\frac{1-\beta_n}{n-1}\right) kL\right) - c \\ \geq \quad & p_0 \sum_{k=0}^{n-1} b(k; n-1, p_1) u\left(W_0 - \beta_n L - \left(\frac{1-\beta_n}{n-1}\right) kL\right) \\ & +(1-p_0) \sum_{k=0}^{n-1} b(k; n-1, p_1) u\left(W_0 - \left(\frac{1-\beta_n}{n-1}\right) kL\right). \end{aligned} \tag{3}$$

Lemma 1: *Suppose that high effort is a Nash equilibrium for some $\widetilde{\beta}_n$. Then, any $\beta_n > \widetilde{\beta}_n$ also implements high effort as a Nash equilibrium.*

Proof: See the Online Appendix (von Bieberstein et al., 2017).

Lemma 1 follows immediately from the fact that the effort incentive increases in the part of the own loss that is effectively borne via the retention rate α_n and the redistribution in the pool.

[8] Bold letters denote vectors.

STAGE 1: OPTIMAL CHOICE OF THE EFFECTIVE SHARE β_n

Suppose that n individuals have joined the pool, and assume that the pool members' expected utility is maximized when they implement high effort. Then, the pool's optimization problem in stage 1 boils down to maximizing

$$\max_{\beta_n} \mathbb{E}\left[u\left(\beta_n, \mathbf{x} = \mathbf{1}\right)\right] \tag{4}$$

subject to the ICC

$$\mathbb{E}\left[u\left(\beta_n, x_i = 1, \mathbf{x}_{-i} = \mathbf{1}\right)\right] \geq \mathbb{E}\left[u\left(\beta_n, x_i = 0, \mathbf{x}_{-i} = \mathbf{1}\right)\right]. \tag{5}$$

Define $\beta_n^{\min}$ as the minimum effective share required for incentive compatibility. The following Lemma 2 expresses that when implementing high effort, the pool members agree upon the lowest possible retention rate $\alpha_n^{\min}$, and hence also the lowest possible effective share $\beta_n^{\min} = \alpha_n^{\min} + \frac{1-\alpha_n^{\min}}{n}$ that only just ensures incentive compatibility:[9]

Lemma 2: *Suppose high effort is optimal for pool size n. Then, subject to incentive compatibility, the pool members' expected utility is maximized for* $\beta_n^{\min}$.

Proof: See the Online Appendix (von Bieberstein et al., 2017).

Lemma 2 implies that the ICC is binding whenever $\alpha_n^{\min} > 0$. In the following, we will first restrict attention to this case; the case where $\alpha_n^{\min} = 0$ is discussed in Corollary 2. Given that the participation constraint and the ICC are fulfilled, existence of the symmetric Nash equilibrium with high effort in stage 2 is ensured. For stage 1, we assume cooperative behavior, so that the utility-maximizing retention rate $\alpha_n^{\min}$ is chosen.[10]

In the next section, we analyze how the minimum effective share required for incentive compatibility depends on the pool size. Then, we turn to the impact of the pool size on expected utility.

THE IMPACT OF THE POOL SIZE ON INCENTIVE COMPATIBILITY

To illustrate the importance of the higher derivatives of the utility function, we start with the special case of a quadratic utility function $u = W - AW^2 - cx_i$, where $A < \frac{1}{2W_0}$ ensures that the marginal utility of wealth is positive in the relevant wealth range. Obviously, $u''' = 0$ in this example, and higher-order risk preferences above the order of 2 do not matter.

Quadratic utility functions are convenient as the expected utility of final wealth can be represented by the expected value and the variance of this wealth

[9]Note that the existence of a $\alpha_n^{\min} \in [0, 1)$ is ensured by Assumption 1.

[10]The case in which all pool members choose to exert low effort is discussed in Corollary 1.

(Tobin, 1958; Baron, 1977):

$$\mathbb{E}[u] = \mathbb{E}[W] - A\left(\mathbb{E}[W]^2 + Var[W]\right) - cx_i. \tag{6}$$

Recall from Lemma 2 that when the pool members want to implement high effort as a Nash equilibrium, they will choose the effective share such that the ICC (3) is binding. In our case, the expected final wealth is composed of the initial wealth, reduced by two possible sources for losses: the part borne of the expected own loss, $p\beta_n L$, and the expected loss of all other members in the risk pool that has to be borne by redistribution, $p(1-\beta_n)L$. Given that high effort is chosen, the expected final wealth of each pool member is

$$\mathbb{E}[W] = W_0 - p_1\beta_n L - p_1(1-\beta_n)L - c = W_0 - p_1 L - c, \tag{7}$$

and hence independent of β_n. Similarly, as the individual risks are independent, the variance of the final wealth is composed of the variances of the two possible sources for losses:

$$\begin{aligned}\text{Var}[W] &= p_1(1-p_1)\beta_n^2 L^2 + p_1(1-p_1)(n-1)\left(\frac{1-\beta_n}{n-1}\right)^2 L^2 \\ &= p_1(1-p_1)L^2\left(\beta_n^2 + \frac{(1-\beta_n)^2}{n-1}\right).\end{aligned} \tag{8}$$

For a constant effective share β_n, we get $\frac{\partial \text{Var}[\cdot]}{\partial n} = -\left(p_1 - p_1^2\right) L^2 \frac{(1-\beta_n)^2}{(n-1)^2} < 0$. This expresses the benefit of larger pools that the variance, and hence the risk borne by each individual, *ceteris paribus* decreases in the pool size.

The crucial question is then how the effective share β_n required for incentive compatibility depends on the pool size n. As part of the proof of Proposition 1, we will show that for a quadratic utility function, $\beta_n^{\min}$ is independent of n. Thus, the retention rate increases in the pool size just to the extent that the effective share remains the same. The reason is that the marginal utility decreases at a constant rate when the income decreases, so that the changes in marginal utilities in response to changes in wealth differences are identical at all levels of wealth. This implies that when the pool size increases and β is kept constant, the utilities in case with and without own loss changes at the same degree.

For the general case with $u''' \geq 0$, we get the following proposition:

Proposition 1: *Assume a strictly increasing and strictly concave analytic utility function $u(W)$, where $u' > 0$, $u'' < 0$ and $(-1)^l u^l \leq 0$ for all $l \geq 3$. Suppose $\alpha_n^{\min}$ is strictly positive for all n. Then, (i) $\frac{\partial \beta_n^{\min}}{\partial n} = 0$ for $u''' = 0$, and (ii) $\frac{\partial \beta_n^{\min}}{\partial n} > 0$ for $u''' > 0$.*

Proof: See the Online Appendix (von Bieberstein et al., 2017).

Proposition 1 says that the effective share of the own loss borne required for incentive compatibility increases in the pool size, so that it is not straightforward that larger pools lead to a higher expected utility. The reason why the effective share $\beta_n^{\min}$ increases in the pool size with mixed risk aversion is that if β is kept constant, the expected utility in the case with an own loss increases faster in n than the expected utility without own loss. For partial insurance, the income level is lower with an own loss compared to no own loss, and this leads to a higher increase in utility due to risk sharing in pools. This effect depends on the derivatives of the utility function, and all effects go in the same direction if the derivatives of the utility function are alternating in sign. Thus, mixed risk aversion is a sufficient condition for $\frac{\partial \beta_n^{\min}}{\partial n} > 0$, and this leads to a countervailing effect to the benefit from risk sharing.

THE IMPACT OF POOL SIZE ON EXPECTED UTILITY

We have just seen that $\beta_n^{\min}$ increases in the pool size for mixed risk-averse individuals with $u''' > 0$, which reduces the benefits of larger pools. Nevertheless, the benefits from risk sharing in larger pools always dominate.

Proposition 2: *Assume a strictly increasing and strictly concave analytic utility function $u(W)$ where $u' > 0$, $u'' < 0$, and $(-1)^l u^l \leq 0$ for all $l \geq 3$. Suppose high effort is optimal for all n and $\beta_n^{\min}$ is chosen for n. Then, the effort vector implemented by $\beta_n^{\min}$ is in the core, all individuals join the pool, and the expected utility of each pool member is strictly increasing in n.*

Proof: See the Online Appendix (von Bieberstein et al., 2017).

For an intuition of the proposition, let us first stick to our assumption that high effort is optimal for all pool sizes. The case where low effort is optimal for all pool sizes larger than some pool size $\widehat{n}$ will then emerge as a simple corollary to Proposition 2. As discussed before, the negative side effect of larger pools is that for individuals with mixed risk aversion, $\beta_n^{\min}$ increases in n. This, however, follows from the fact that individuals with mixed risk-averse utility functions benefit strongly from larger pools in case of an own loss. Thus, it is the benefit from larger pools itself that induces the negative side effect of an increase in the required effective share, which explains why the side effect can never dominate. Observe that the fact that the expected utility of all pool members increases in n implies that the allocation is in the core: the maximum utility any coalition $\widetilde{n} < n$ can reach is by agreeing on $\beta_{\widetilde{n}}^{\min}$, and this utility is lower than the one for the grand coalition of all n pool members.

To further sharpen the intuition for the superiority of larger pools, it is instructive to consider the situation from a more formal point of view. When the high effort level is implemented for two pool sizes $n+1$ and n, then the utility comparison for these two pool sizes can be written as follows:

$$p_1\mathbb{E}[u_L(\beta_{n+1}^{\min})] + (1-p_1)\mathbb{E}[u_0(\beta_{n+1}^{\min})] > p_1\mathbb{E}[u_L(\beta_n^{\min})] + (1-p_1)\mathbb{E}[u_0(\beta_n^{\min})], \quad (9)$$

where $\mathbb{E}[u_L(\beta_n^{\min})]$ denotes the expected utility in case of own loss in a pool of n participants, and $\mathbb{E}[u_0(\beta_n^{\min})]$ denotes the expected utility in case without own loss. For later reference, note that the utility comparison for the two pool sizes holds whenever the two effort levels implemented via the effective share are identical; that is, in what follows, we do not make use of the fact that high effort is efficient.

Rearranging gives

$$\mathbb{E}[u_0(\beta_{n+1}^{\min})] - p_1 \underbrace{\mathbb{E}[u_0(\beta_{n+1}^{\min}) - u_L(\beta_{n+1}^{\min})]}_{\Delta u(\beta_{n+1}^{\min})} > \mathbb{E}[u_0(\beta_n^{\min})] - p_1 \underbrace{\mathbb{E}[u_0(\beta_n^{\min}) - u_L(\beta_n^{\min})]}_{\Delta u(\beta_n^{\min})}. \quad (10)$$

In this expression, the first part on either side of the utility comparison is the expected utility in the case without own loss, whereas the second parts consisting of $\Delta u(\beta_{n+1}^{\min})$ and $\Delta u(\beta_n^{\min})$ denote the differences in expected utility without own loss and with own loss for risk pool sizes $n+1$ and n, respectively. The advantage of this representation is that the binding ICC can be written as $(p_0 - p_1)\Delta u(\beta_{n+1}^{\min}) = c$ for pool size $n+1$ and $(p_0 - p_1)\Delta u(\beta_n^{\min}) = c$ for pool size n. Thereby, $(p_0 - p_1)\Delta u(\beta_{n+1}^{\min})$ and $(p_0 - p_1)\Delta u(\beta_n^{\min})$ capture the marginal benefits from high effort, which always equal the marginal costs c in case of binding ICCs. As a consequence, the marginal benefit is constant in n so that the utility comparison becomes:

$$\mathbb{E}[u_0(\beta_{n+1}^{\min})] > \mathbb{E}[u_0(\beta_n^{\min})], \quad (11)$$

meaning that it is only the difference in the expected utilities from the share of the losses from *other* pool members that matters for the utility comparison of the two pool sizes. This shows immediately that the expected utility in larger pools increases both for $u''' = 0$ *and* for $u''' > 0$: for $u''' = 0$, β_n remains constant, and it is only the better diversification of the risk from the losses of the other pool members that matters. For $u''' > 0$, in addition β_n increases in the pool size (recall Proposition 1), and this decreases the share of the losses borne from other pool members and hence also increases expected utility.

So far, we have assumed that implementing the high effort is always optimal, irrespective of the pool size. The following corollary expresses that our result on the optimality of larger pools is independent of this assumption:

Corollary 1: *Suppose that high effort is optimal for some pool sizes, but low effort for other pool sizes. Then, the utility of each pool member is still strictly increasing in n.*

Proof: See the Online Appendix (von Bieberstein et al., 2017).

To see the intuition for Corollary 1, recall that the whole argument for the superiority of larger pools discussed after Proposition 2 is independent of whether high effort is efficient or not—it extends to *all* cases where high effort is implemented for two pools of different size. First, we show that the argument also holds when comparing two pools that both implement low effort. Second, and a little more intricate, Proposition 2 implies that whenever the high effort is implemented for some pool size $n_1 < \widehat{n}$ and some pool size $n_2 \geq \widehat{n}$, then the utility of each pool member is higher for pool size

n_2. Importantly, this is even the case if low effort were optimal for the pool of size n_2. With low effort, the pool members would clearly agree on $\alpha_n = 0$, as this maximizes risk sharing. Thus, when there are pool sizes where $\alpha_n = 0$ and low effort is optimal, the comparison of the utilities from two pool sizes is as follows:

$$\mathbb{E}\left[u\left(n = n_2, \alpha_{n_2} = 0, \mathbf{x} = \mathbf{0}\right)\right] > \mathbb{E}\left[u\left(n = n_2, \alpha_{n_2}^{\min}, \mathbf{x} = \mathbf{1}\right)\right] > \mathbb{E}\left[u\left(n = n_1, \alpha_{n_1}^{\min}, \mathbf{x} = \mathbf{1}\right)\right].$$

Given that the larger pool is superior even when the pool members (suboptimally) implement the high effort, it follows by definition of optimality that the larger pool is also superior if the low effort is implemented. Finally, given that Proposition 2 carries over to the case of low effort, the same kind of argument applies if low effort is efficient for the smaller pool, and high effort for the larger pool.

So far, we have assumed that $\alpha_n^{\min} > 0$, so that the ICC is binding. The following corollary covers the case of full insurance where $\alpha_n^{\min} = 0$:

Corollary 2: *Suppose that $\alpha_n^{\min} = 0$ (full-insurance) is optimal for some n. Then, the expected utility of each pool member is still strictly increasing in n.*

Proof: See the Online Appendix (von Bieberstein et al., 2017).

The proofs of Propositions 1 and 2 rely on the assumption that effort incentives are kept constant when increasing the pool size. In particular, the ICCs are binding for pool sizes n and $n+1$. If the ICC is binding for $\alpha_n^{\min} = 0$, the optimality of increasing n follows directly from these propositions. If the ICC is slack for $\alpha_n^{\min} = 0$, the pool members can still implement the same effort incentives when increasing the pool size, irrespective of whether this is optimal or suboptimal. The resulting expected utility will be higher for the larger pool according to Proposition 2. If this is suboptimal for the larger pool, expected utility would be even higher for optimal effort incentives.

CONTINUOUS EFFORT

We now show that our main result of the superiority of larger pools carries over to the case with continuous effort. Assume that effort costs $c(x)$ are convex in effort x; $c'(x) > 0$ and $c''(x) > 0$. The loss probability $p(x) > 0$ is decreasing in effort at a decreasing rate, $p'(x) < 0$ and $p''(x) > 0$, where $p(0) < 1$. Furthermore, following, for example, Ligon and Thistle (2008), suppose that $\lim_{x \downarrow 0} p'(x) = -\infty$. These assumptions ensure that effort chosen by each individual is always positive.

Expected utility is given as

$$\mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})] = p(x_i) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \left[u\left(W_0 - \beta_n L - \left(\frac{1-\beta_n}{n-1}\right)kL\right)\right]$$
$$+ (1 - p(x_i)) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \left[u\left(W_0 - \left(\frac{1-\beta_n}{n-1}\right)kL\right)\right] - c(x_i). \quad (12)$$

Denote $x_i^*(\beta_n, \mathbf{x}_{-i})$ as the utility-maximizing effort of individual i given the pool size n, the effective share β_n, and the vector of efforts exerted by all other members, denoted by $\mathbf{x}_{-i}$:

$$x_i^* := \underset{x_i}{\operatorname{argmax}}\ \mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})]. \tag{13}$$

Note first that our standard assumptions on effort costs and loss probabilities ensure that the utility-maximizing effort level of individual i is always positive and increasing in the effective share β_n. Thus, the members of the pool can increase efforts by agreeing on higher β, and the *highest* implementable effort is bounded above by the no-insurance case, $\alpha = \beta = 1$. For further reference, denote $x_n^{\max}$ as the effort level in the Nash equilibrium for $\beta_n \to 1$. In the following, we restrict attention to symmetric equilibria, and denote the incentive compatible equilibrium vector as $\mathbf{x}_n^*(\beta_n)$.

We can then show that for any pool size, there is an interior solution for the utility-maximizing effective share β_n^*, where

$$\beta_n^* := \underset{\beta_n}{\operatorname{argmax}}\ \mathbb{E}[u(\beta_n, \mathbf{x}_n^*(\beta_n))]. \tag{14}$$

Proposition 3: *For all pool sizes n, any expected utility maximizing effective share β_n^* is strictly above $\frac{1}{n}$ and strictly below* 1.

Proof: See the Online Appendix (von Bieberstein et al., 2017).

Proposition 3 expresses that, for any pool size, neither no-insurance ($\alpha_n = \beta_n = 1$) nor full insurance ($\beta_n = \frac{1}{n}$, i.e., $\alpha_n = 0$) can be optimal. Full insurance is dominated because of the incentive effect, and no-insurance due to the benefit of risk sharing.

To derive our main proposition on the superiority of larger pools, we make use of two facts.

First, as proven by Lee and Ligon (2001),[11] the lowest implementable effort, which we denote by $x_n^{\min}$, decreases in the pool size. The reason is that for any pool size, the lowest implementable effort $x_n^{\min}$ is reached when maximizing risk sharing, that is, for $\alpha = 0$. And as the effective share is then just $\beta_n = \alpha_n + \frac{1-\alpha_n}{n} = \frac{1}{n}$ and thus decreasing in n, the impact of the redistribution on the effort choice decreases in the pool size.

Second, we show as part of the proof of Proposition 4 that any effort level that is implementable for small pool sizes is also implementable for larger pools by adjusting the effective share β_n accordingly. This implies immediately that our main result carries over to the case with continuous effort: for our discrete model, we have shown that whenever the same effort level is implemented for pool size n_1 and any pool size

[11] Lee and Ligon (2001) restrict their attention to $\alpha = 0$, but this is the case to be analyzed when considering the lowest implementable effort.

$n_2 > n_1$, expected utility is higher for pool size n_2. And as the proof was independent of whether the high effort is efficient or not, the fact that any effort level that is implementable for n_1 is also implementable for any pool size $n_2 > n_1$, it carries over to the continuous case.

Recall that in the discrete version, we have made use of the fact that $\beta_{n+1}^{\min}$ and $\beta_n^{\min}$ only just implement the high effort for pool sizes $n+1$ and n, respectively. As the ICCs are hence binding for those minimum effective shares, we could conclude that $\Delta u(\beta_{n+1}^{\min}) = \Delta u(\beta_n^{\min})$. As any $\beta_n > \beta_n^{\min}$ *a fortiori* implements high effort in the discrete version, any effort $x \in (x_n^{\min}, x_n^{\max})$ in the continuous version is implemented by a unique β_n. Thus, for any effort $x^* \in (x_n^{\min}, x_n^{\max})$, there exists a unique β_n that implements x^* for pool size n, and a unique β_{n+1} that implements x^* for pool size $n+1$, as given by the first-order condition for both pool sizes. When comparing the utilities with different pool sizes, we can again make use of the fact that β_{n+1} and β_n implement a given effort x^* for pool sizes $n+1$ and n, respectively. From the ICCs, we know that $\Delta u(\beta_{n+1}) = -\frac{c'(x^*)}{p'(x^*)}$ and $\Delta u(\beta_n) = -\frac{c'(x^*)}{p'(x^*)}$. From the identity of the right-hand sides, it then follows immediately that $\Delta u(\beta_{n+1}) = \Delta u(\beta_n)$, and the utility comparison again reduces to $\mathbb{E}[u_0(\beta_{n+1})] > \mathbb{E}[u_0(\beta_n)]$. We hence get:

Proposition 4: *Suppose the effective share is chosen to maximize the pool members' utility, taking incentive compatibility into account. Then, with continuous effort, increasing the pool size increases the pool members' expected utility, that is,* $\frac{\partial \mathbb{E}[u(\beta_n^*, x_n^*(\beta_n^*), n]}{\partial n} > 0$.

Proof: See the Online Appendix (von Bieberstein et al., 2017).

The optimal effort level will generally be different for different pool sizes—intuitively, avoiding risk becomes less important when a higher pool size allows for better risk sharing.[12] However, it is sufficient to show that a larger pool size would lead to a higher expected utility *even when the members of the pool suboptimally* agreed on an effective share $\widehat{\beta}_n$ that implements the effort that is optimal for the lower pool size $\widetilde{n}$, but not for the pool size n considered. But if the larger pool size leads to higher expected utility even in this case, it *a fortiori* leads to higher utility when the utility-maximizing effective share β_n is chosen instead. Note that the utility maximizing effective share β_n^* the grand coalition of all members agree upon is always in the core; that is, there is no coalition that has an incentive to block the grand coalition. The reason is that the best any coalition $\widetilde{n} < n$ can do is to agree on the utility-maximizing $\beta_{\widetilde{n}}^*$, and we know from Proposition 4 that this leads to a lower utility.

[12]As this may lead to the result that low effort is optimal for large pools in the model with discrete effort choice, our assumption $\lim_{x \downarrow 0} p'(x) = -\infty$ ensures that positive effort is always optimal in the continuous case.

Conclusion

We extend the literature on risk pools, such as partnerships and mutual insurance arrangements, to the optimal pool size in case of moral hazard. We assume that n individuals with mixed risk-averse utility functions, for which the derivatives are (weakly) alternating in sign, agree on the retention rate that maximizes their utility, thereby taking the ICC for the effort choice into account. Our main result is that, neglecting transaction costs, the optimal pool size converges to infinity. This holds both for binary and for continuous effort. In reality, transaction costs may increase in pool sizes, and these costs are neglected in our model. Thus, from a practical perspective, our result shows that the optimal pool size equilibrates the benefits from better risk sharing with transaction costs at the margin, whereas the residual claimant principle has no impact on the optimal pool size if the retention rate is optimally adjusted.

Starting with binary effort choices, we first consider quadratic utility functions where higher-order risk preferences such as prudence or temperance, do not matter. For this special case, incentive compatibility for the choice of the high effort requires the same *effective* share $\beta_n^{\min}$ of the own loss for all pool sizes, where the effective share is defined as the sum of the retention rate and the share of the own loss borne as residual claimant. The fact that $\beta_n^{\min}$ is constant in n implies that the risk from the losses of the other pool members has no impact on effort incentives. Then, the pool size only influences the degree of risk sharing, and each individual's expected utility is strictly increasing in the number of policyholders.

For individuals with mixed risk-averse utility functions, however, the incentive to choose high effort is lower for larger pools, even when the effective share β_n is the same. The reason is that due to the diversification effect of larger pools extremely low income levels become less likely even in the case with an own loss. For individuals with mixed risk-averse utility functions, this reduces the utility-decreasing impact of the own loss and thus decreases incentives to choose the high effort.

As a consequence, we find that the minimum effective share for incentive compatibility increases in n. As this is an interesting insight in itself, our main result is that the benefit from larger pools always dominates the negative impact of a higher effective share. Thus, given that retention rates are optimally adjusted and contracts are enforceable, the optimal pool size is infinite, even in the case with moral hazard and prudent individuals.

For the continuous effort case, we can prove that an optimal retention rate strictly between 0 and 1 always exists, that is, neither full- nor no-insurance is optimal. In the Online Appendix (von Bieberstein et al., 2017), we provide a sufficient condition for the optimal retention rate to be unique. However, we do not assume this to hold as our proof that each participant's utility increases in the pool size does not require the optimal retention rate to be unique. To see this, recall first that we have shown that for any retention rate, the effort chosen by the participants is unique. Now suppose that there are multiple retention rates that lead to different effort levels, but still to the same expected utility—more risk sharing with lower effort may be equally good as less risk sharing with higher effort. As each optimal retention rate leads to the same utility, each of these retention rates also leads to a higher utility for larger compared to smaller pools. Thus, it does not matter for the proof of the superiority of larger pools on which of those retention rates the participants agree upon in the cooperative game at stage 1.[13] This follows from the proof that any effort vector that can be implemented for n can also be implemented for $\widetilde{n} > n$, and that this effort vector then leads to higher utility even when it is suboptimal for $\widetilde{n}$.

As we are interested in the impact of the pool size on utility when the retention rate is optimally adjusted, we neglect other important issues such as heterogeneous individuals, transaction costs, and externalities. If there are externalities of the pool members' activities on third parties that cannot be internalized, then it may well be socially optimal to restrict the maximum pool size. The same result arises when the externality depends on the pool members' precaution effort: when the effort the pool members coordinate upon via the retention rate decreases in the pool size, then it may be socially optimal to restrict the pool size in order to increase the effort. Furthermore, it needs to be mentioned that we restrict attention to a monopolistic risk pool. The optimality of an infinite pool size in our model highlights the benefits of a monopoly for risk pooling, but various frictions may allow for competition among pools. For instance, some risk pools may prefer to attract only low-risk consumers and/or certain profession members (partnerships and RRG). In addition, risk pools may have limited sales force resources or other capacity constraints preventing them from covering the entire market. Additionally, some consumer groups may be inert due to significant switching costs, and may hence be reluctant to leave their actual pool. As it is thus interesting to analyze competition between mutual insurers or different risk pools, this is beyond the scope of this article and therefore left to future research.

REFERENCES

Baron, D. P., 1977, On the Utility Theoretic Foundation of Mean-Variance Analysis, *Journal of Finance*, 32(5): 1683-1697.

Barigozzi, F., R. Bourlès, D. Henriet, and G. Pignataro, 2017, Pool Size and the Sustainability of Optimal Risk-Sharing Agreements, *Theory and Decision*, 82(2): 273-303.

Bold, T., 2009, Implications of Endogenous Group Formation for Efficient Risk-Sharing, *Economic Journal*, 119(536): 562-591.

Bramoullé, Y., and R. Kranton, 2007, Risk-Sharing Networks, *Journal of Economic Behavior & Organization*, 64(3): 275-294.

Caballé, J., and A. Pomansky, 1996, Mixed Risk Aversion, *Journal of Economic Theory*, 71(2): 485-513.

Cummins, J. D., M. A. Weiss, and H. Zi, 1999, Organizational Form and Efficiency: The Coexistence of Stock and Mutual Property-Liability Insurers, *Management Science*, 45(9): 1254-1269.

Deck, C., and H. Schlesinger, 2006, Consistency of Higher Order Risk Preferences, *Econometrica*, 82(5): 1913-1943.

[13]The proof of the superiority of larger pools implies that each of these retention rates is in the core. Without loss of generality, we could, for example, assume that they agree upon the lowest retention rate that maximizes utility.

Ebert, S., 2013, Moment Characterization of Higher-Order Risk Preferences, *Theory and Decision*, 74(2): 267-284.

Eeckhoudt, L., and H. Schlesinger, 2006, Putting Risk in Its Proper Place, *American Economic Review*, 96(1): 280-289.

Eeckhoudt, L., H. Schlesinger, and I. Tsetlin, 2009, Apportioning of Risks via Stochastic Dominance, *Journal of Economic Theory*, 144(3): 994-1003.

Gaynor, M., and P. Gertler, 1995, Moral Hazard and Risk Spreading in Partnerships, *RAND Journal of Economics*, 26(4): 591-613.

Genicot, G., and D. Ray, 2003, Group Formation in Risk-Sharing Arrangements, *Review of Economic Studies*, 70(1): 87-113.

Knoblauch, A., 2008, Closed-Form Expressions for the Moments of the Binomial Probability Distribution, *SIAM Journal on Applied Mathematics*, 69(1): 197-204.

Lang, K., and P.-J. Gordon, 1995, Partnerships as Insurance Devices: Theory and Evidence, *RAND Journal of Economics*, 26(4): 614-629.

Laux, C., and A. Mürmann, 2010, Financing Risk Transfer Under Governance Problems: Mutual Versus Stock Insurers, *Journal of Financial Intermediation*, 19(3): 333-354.

Lee, W., and J. A. Ligon, 2001, Moral Hazard in Risk Pooling Arrangements, *Journal of Risk and Insurance*, 68(1): 175-190.

Leverty, J. T., 2011, The Cost of Duplicative Regulation: Evidence From Risk Retention Groups, *Journal of Risk and Insurance*, 79(1): 105-127.

Ligon, J. A., and P. Thistle, 2005, The Formation of Mutual Insurers in Markets With Adverse Selection, *Journal of Business*, 78(2): 529-555.

Ligon, J. A., and P. Thistle, 2008, Moral Hazard and Background Risk in Competitive Insurance Markets, *Economica*, 75(300): 700-709.

Mayers, D., and C. W. Smith, 1988, Ownership Structure Across Lines of Property-Casualty Insurance, *Journal of Law and Economics*, 3(2): 351-378.

Mayers, D., and C. W. Smith, 2013, On the Choice of Organizational Form: Theory and Evidence From the Insurance Industry, in: G. Dionnes, ed., *Handbook of Insurance*, 2nd edition (New York: Springer).

Murgai, R., P. Winters, E. Sadoulet, and A. de Janvry, 2002, Localized and Incomplete Mutual Insurance, *Journal of Development Economics*, 67(2): 245-274.

Osborne, M. J., and A. Rubinstein, 1994, *A Course in Game Theory* (Cambridge, MA: MIT Press).

Smith, B. D., and M. Stutzer, 1995, A Theory of Mutual Formation and Moral Hazard With Evidence From the History of the Insurance Industry, *Review of Financial Studies*, 8(2): 545-577.

Tobin, J., 1958, Liquidity Preference as Behavior Towards Risk, *Review of Economic Studies*, 25(2): 65-86.

von Bieberstein, F., E. Feess, J. F. Fernando, F. Kerzenmacher, and J. Schiller, 2017, Supplementary Material to 'Moral Hazard, Risk Sharing, and the Optimal Pool Size', *Journal of Risk and Insurance*. http://dx.doi.org/10.1111/jori.12211

Zanjani, G., 2007, Regulation, Capital, and the Evolution of Organizational Form in US Life Insurance, *American Economic Review*, 97(3): 973-983.

Supporting Information

Additional supporting information may be found in the online version of this article at the publisher's website.

Online Appendix

Proof of Lemma 1

Proof. First, define the function $f(\beta_n)$ as

$$f(\beta_n) := \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] - \frac{c}{p_0 - p_1}, \quad (15)$$

so that the (ICC) can be written as $f(\beta_n) \geq 0$. By definition of $\widetilde{\beta}_n$, $f(\widetilde{\beta}_n) \geq 0$ holds. The first derivative of f with respect to β_n is:

$$f'(\beta_n) := L \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u'\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\frac{k}{n-1} + u'\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(1 - \frac{k}{n-1}\Big)\Big], \quad (16)$$

which is strictly positive as $u'(W) > 0$ for each W. Therefore, for any $\beta_n > \widetilde{\beta}_n$ we have $f(\beta_n) > f(\widetilde{\beta}_n) \geq 0$, so that the ICC holds. ■

Proof of Lemma 2

Proof. For the case where high effort is optimal, we need to prove that the minimum β_n, which ensures incentive compatibility, maximizes expected utility. Expected utility for any individual i with pool size n is:

$$\begin{aligned}\mathbb{E}[u(\beta_n, x_i)] = p \sum_{k=0}^{n-1} b(k; n-1, p) u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) \\ + (1-p) \sum_{k=0}^{n-1} b(k; n-1, p) u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - cx_i.\end{aligned} \quad (17)$$

We prove that $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} < 0$ in the relevant range. For the relevant range, recall that $\beta_n = \alpha_n + \frac{1-\alpha_n}{n}$, where $\alpha_n \in [0, 1)$. Thus, $\beta_n \in [\frac{1}{n}, 1)$. We show that $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} = 0$ for $\beta_n = \frac{1}{n}$, and $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} < 0 \; \forall \; \beta_n > \frac{1}{n}$. The first and second derivatives of the expected utility with respect to β_n are

$$\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} = L\Big(p\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(-1+\frac{k}{n-1}\Big) \\ + (1-p)\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(\frac{k}{n-1}\Big)\Big) \tag{18}$$

and

$$\frac{\partial^2 \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n^2} = L^2\Big(p\sum_{k=0}^{n-1} b(k;n-1,p)u''\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(-1+\frac{k}{n-1}\Big)^2 \\ + (1-p)\sum_{k=0}^{n-1} b(k;n-1,p)u''\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(\frac{k}{n-1}\Big)^2\Big), \tag{19}$$

respectively. As $u_i'' < 0$, $\frac{\partial^2 \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n^2} < 0$, i.e. $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n}$ is strictly decreasing.

Next, we show that $\frac{\partial \mathbb{E}[u(\frac{1}{n}, x_i)]}{\partial \beta_n} = 0$. Substituting β_n for $\frac{1}{n}$ in equation (18), we get

$$\begin{aligned} \frac{\partial \mathbb{E}[u(\frac{1}{n}, x_i)]}{\partial \beta_n} &= L\Big(p\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0 - \frac{k+1}{n}L\Big)\Big(-1+\frac{k}{n-1}\Big) \\ &+ (1-p)\sum_{k=0}^{n-1} b(k;n-1,p)u'\Big(W_0 - \frac{k}{n}L\Big)\Big(\frac{k}{n-1}\Big)\Big) \\ &= L\Big(-\sum_{k=0}^{n-1}\binom{n-1}{k}p^{k+1}(1-p)^{n-1-k}u'\Big(W_0 - \frac{k+1}{n}L\Big)\Big(\frac{n-1-k}{n-1}\Big) \\ &+ \sum_{k=0}^{n-1}\binom{n-1}{k}p^{k}(1-p)^{n-k}u'\Big(W_0 - \frac{k}{n}L\Big)\Big(\frac{k}{n-1}\Big)\Big). \end{aligned} \tag{20}$$

Observe that $\frac{n-1-k}{n-1} = 0$ for $k = n-1$ and $\frac{k}{n-1} = 0$ for $k = 0$. Thus,

$$\begin{aligned}
\frac{\partial \mathbb{E}[u(\frac{1}{n}, x_i)]}{\partial \beta_n} &= L\Big(-\sum_{k=0}^{n-2} \frac{(n-1)!}{(n-1-k)!k!} p^{k+1}(1-p)^{n-1-k} u'\Big(W_0 - \frac{k+1}{n}L\Big)\Big(\frac{n-1-k}{n-1}\Big) \\
&+ \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} p^k (1-p)^{n-k} u'\Big(W_0 - \frac{k}{n}L\Big)\Big(\frac{k}{n-1}\Big)\Big) \\
&= \frac{L}{n-1}\Big(-\sum_{k=0}^{n-2} \frac{(n-1)!}{(n-1-k-1)!k!} p^{k+1}(1-p)^{n-1-k} u'\Big(W_0 - \frac{k+1}{n}L\Big) \\
&+ \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!(k-1)!} p^k (1-p)^{n-k} u'\Big(W_0 - \frac{k}{n}L\Big)\Big) \\
&\overset{\text{in the first addend}}{\underset{(k=\ell-1)}{=}} \frac{L}{n-1}\Big(-\sum_{\ell=1}^{n-1} \frac{(n-1)!}{(n-1-\ell)!(\ell-1)!} p^{\ell}(1-p)^{n-\ell} u'\Big(W_0 - \frac{\ell}{n}L\Big) \\
&+ \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)!(k-1)!} p^k (1-p)^{n-k} u'\Big(W_0 - \frac{k}{n}L\Big)\Big) = 0,
\end{aligned} \tag{21}$$

where, in the last step, we substitute k for $\ell - 1$ in the first addend in order to demonstrate equality between the two addends. As $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n}$ is strictly decreasing, it follows that $\frac{\partial \mathbb{E}[u(\beta_n, x_i)]}{\partial \beta_n} < 0 \ \forall \beta_n > \frac{1}{n}$. As the expected utility is decreasing in β_n for a given effort, pool members will agree on $\beta_n^{\min}$. ■

Proof of Proposition 1

Proof. We prove that $\frac{\partial \beta_n^{\min}}{\partial n} = 0$ if $u''' = 0$, while $\frac{\partial \beta_n^{\min}}{\partial n} > 0$ if $u''' > 0$. For ease of notation, we here use β_n instead of $\beta_n^{\min}$. We proceed as follows.

Step 1. We prove Lemma 3 that is required for the proofs of Propositions 1 and 2, as well as Corollary 1.

Step 2. We rewrite the ICC under the assumption that u is an analytic function.

Step 3. We compare the ICCs for pool sizes n and $n+1$ for $u''' = 0$ and prove part (i) of Proposition 1.

Step 4. We compare the ICCs for pool sizes n and $n+1$ for $u''' > 0$ and prove part (ii) of Proposition 1.

Step 1. Proof of Lemma 3

The following Lemma states a property of binomial distributions that will be used in later proofs.

Lemma 3. *Define* $\mathbb{E}[B(n,p)^\ell]$ *as the* ℓ*-th moment (about* 0*) of the binomial distribution* $B(n,p)$*. Then:*

$$\left(\frac{n-1}{n}\right)^\ell \leq \frac{\mathbb{E}[B(n-1,p)^\ell]}{\mathbb{E}[B(n,p)^\ell]} \tag{22}$$

for all $\ell \geq 1$*, where the condition holds with equality for* $\ell = 1$ *and the RHS is strictly larger than the LHS for* $\ell > 1$*.*

Proof. As $\mathbb{E}[B(n,p)^\ell]$ is the ℓ-th moment (about 0) of $B(n,p)$, we can write:

$$n^\ell \mathbb{E}[B(n-1,p)^\ell] = \sum_{j=0}^{n-1} \binom{n-1}{j} n^\ell j^\ell p^j q^{n-1-j} = \sum_{j=1}^{n-1} \binom{n-1}{j} n^\ell j^\ell p^j q^{n-1-j}$$

$$(n-1)^\ell \mathbb{E}[B(n,p)^\ell] = \sum_{k=0}^{n} \binom{n}{k} (n-1)^\ell k^\ell p^k q^{n-k} = \sum_{k=1}^{n} \binom{n}{k} (n-1)^\ell k^\ell p^k q^{n-k},$$

where $q := 1-p$. Thus, condition (22) is equivalent to

$$\begin{aligned}\Gamma_\ell(n) &:= n^\ell \mathbb{E}[B(n-1,p)^\ell] - (n-1)^\ell \mathbb{E}[B(n,p)^\ell] \\ &= \sum_{j=1}^{n-1} \binom{n-1}{j} n^\ell j^\ell p^j q^{n-1-j} - \sum_{k=1}^{n} \binom{n}{k} (n-1)^\ell k^\ell p^k q^{n-k} \geq 0.\end{aligned} \tag{23}$$

For simplicity, we denote $a_j := \binom{n-1}{j} n^\ell j^\ell$ and $b_k := \binom{n}{k}(n-1)^\ell k^\ell$, so we have

$$\Gamma_\ell(n) = \sum_{j=1}^{n-1} a_j p^j q^{n-1-j} - \sum_{k=1}^{n} b_k p^k q^{n-k}.$$

As $p + q = 1$,

$$a_j p^j q^{n-1-j} = a_j (p+q) p^j q^{n-1-j} = a_j p^{j+1} q^{n-1-j} + a_j p^j q^{n-j} = a_j p^{j+1} q^{n-(j+1)} + a_j p^j q^{n-j}$$

and we deduce

$$\begin{aligned}\Gamma_\ell(n) = \sum_{j=1}^{n-1} a_j p^j q^{n-1-j} - \sum_{k=1}^{n} b_k p^k q^{n-k} &= \sum_{j=1}^{n-1} (a_j p^{j+1} q^{n-(j+1)} + a_j p^j q^{n-j}) - \sum_{k=1}^{n} b_k p^k q^{n-k} \\ &= (a_1 - b_1) p q^{n-1} + \sum_{k=2}^{n-1} (a_{k-1} + a_k - b_k) p^k q^{n-k} + (a_{n-1} - b_n) p^n.\end{aligned}$$

To prove that $\Gamma_\ell(n) \geq 0$ it is enough to show that each coefficient

$$a_1 - b_1 \geq 0, \quad a_{n-1} - b_n \geq 0, \quad a_{k-1} + a_k - b_k \geq 0 \quad \text{for } k = 2, \ldots, n-1,$$

while some coefficient is strictly larger than 0. First, we approach the simpler coefficients:

$$\begin{aligned}
a_1 - b_1 &= \binom{n-1}{1} n^\ell - \binom{n}{1}(n-1)^\ell \\
&= (n-1)n^\ell - n(n-1)^\ell = (n-1)n(n^{\ell-1} - (n-1)^{\ell-1}) \geq 0, \\
a_{n-1} - b_n &= \binom{n-1}{n-1} n^\ell (n-1)^\ell - \binom{n}{n}(n-1)^\ell n^\ell = n^\ell(n-1)^\ell - (n-1)^\ell n^\ell = 0,
\end{aligned}$$

where $\ell \geq 1$. The difference between a_1 and b_1 is equal to zero for $\ell = 1$ and strictly positive for $\ell > 1$. Thus, it only remains to determine what happens with $a_{k-1} + a_k - b_k$ for $k = 2, \ldots, n-1$. We expand them in a clearer way

$$a_{k-1} + a_k - b_k = \binom{n-1}{k-1} n^\ell (k-1)^\ell + \binom{n-1}{k} n^\ell k^\ell - \binom{n}{k}(n-1)^\ell k^\ell.$$

As $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we deduce that

$$\begin{aligned}
a_{k-1} + a_k - b_k =& \binom{n-1}{k-1}(n^\ell(k-1)^\ell - (n-1)^\ell k^\ell) + \binom{n-1}{k}(n^\ell k^\ell - (n-1)^\ell k^\ell) \\
=& \frac{(n-1)!}{(k-1)!(n-1-(k-1))!}(n^\ell(k-1)^\ell - (n-1)^\ell k^\ell) \\
&+ \frac{(n-1)!}{k!(n-1-k)!}(n^\ell k^\ell - (n-1)^\ell k^\ell) \\
=& \frac{(n-1)!}{k!(n-k)!}\big(k(n^\ell(k-1)^\ell - (n-1)^\ell k^\ell) + (n-k)(n^\ell k^\ell - (n-1)^\ell k^\ell)\big).
\end{aligned}$$

Consequently, to prove that $a_{k-1} + a_k - b_k \geq 0$ we must show that

$$k(n^\ell(k-1)^\ell - (n-1)^\ell k^\ell) + (n-k)(n^\ell k^\ell - (n-1)^\ell k^\ell) \geq 0. \tag{24}$$

Or equivalently that

$$n^\ell k(k-1)^\ell + n^{\ell+1}k^\ell + (n-1)^\ell k^{\ell+1} - (n-1)^\ell k^{\ell+1} - n(n-1)^\ell k^\ell - n^\ell k^{\ell+1} \geq 0.$$

The previous inequality can be simplified as

$$k^{\ell-1}(n^\ell - (n-1)^\ell) - n^{\ell-1}(k^\ell - (k-1)^\ell) \geq 0. \tag{25}$$

For $\ell = 1$ this is again, equal to zero. Thus, we are reduced to prove equation (25) for $k = 2, \ldots, n-1$. We will successively reduce equation (25) until we achieve a family of inequalities easy to be checked. We begin by using the well-known identity

$$x^\ell - y^\ell = (x-y)\sum_{j=0}^{\ell-1} x^j y^{\ell-1-j}$$

to write

$$n^\ell - (n-1)^\ell = \sum_{j=0}^{\ell-1} n^j (n-1)^{\ell-1-j} \quad \text{and} \quad k^\ell - (k-1)^\ell = \sum_{j=0}^{\ell-1} k^j (k-1)^{\ell-1-j}.$$

Thus, equation (25) reduces to prove the following

$$\begin{aligned} k^{\ell-1}\Big(\sum_{j=0}^{\ell-1} n^j (n-1)^{\ell-1-j}\Big) - n^{\ell-1}\Big(\sum_{j=0}^{\ell-1} k^j (k-1)^{\ell-1-j}\Big) \\ = \sum_{j=0}^{\ell-1} (k^{\ell-1} n^j (n-1)^{\ell-1-j} - n^{\ell-1} k^j (k-1)^{\ell-1-j}) \geq 0. \quad (26) \end{aligned}$$

It is sufficient to show that each addend of the previous sum is ≥ 0. Namely,

$$k^{\ell-1} n^j (n-1)^{\ell-1-j} - n^{\ell-1} k^j (k-1)^{\ell-1-j} \geq 0 \quad \forall\, j = 0, \ldots, \ell-1. \quad (27)$$

Let us rewrite these addends in a clearer way

$$\begin{aligned} &(k^{\ell-1} n^j (n-1)^{\ell-1-j} - n^{\ell-1} k^j (k-1)^{\ell-1-j}) \\ &= k^j n^j (k^{\ell-1-j} (n-1)^{\ell-1-j} - n^{\ell-1-j} (k-1)^{\ell-1-j}) \\ &= k^j n^j ((k(n-1))^{\ell-1-j} - (n(k-1))^{\ell-1-j}). \quad (28) \end{aligned}$$

As $0 \leq j \leq \ell - 1$, to prove that inequalities (27) hold, it is enough to check that

$$\begin{aligned} &(k(n-1))^{\ell-1-j} \geq (n(k-1))^{\ell-1-j} \quad \forall\, k = 2, \ldots, n-1 \\ &\iff k(n-1) \geq n(k-1) \quad \forall\, k = 2, \ldots, n-1. \quad (29) \end{aligned}$$

But this is trivially true because

$$k(n-1) - n(k-1) = kn - k - nk + n = n - k \geq 0 \quad \forall\, k = 2, \ldots, n-1.$$

Going backwards and putting all the reductions together we conclude that equation

(25) holds and consequently also equation (23) (or equivalently (22)) holds, as wanted.

Step 2. Rewriting the ICC under the analytic assumption

We assume that u is an analytic function (that is, it coincides with its Taylor expansion as a series centered at W_0):

$$u(W) = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)}{\ell!}(W - W_0)^{\ell}. \tag{30}$$

Observe that

$$u(W_0 - \beta L) = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)}{\ell!}(W_0 - \beta L - W_0)^{\ell} = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!}\beta^{\ell}. \tag{31}$$

We use (31) to obtain an alternative expression for the ICC (3) in terms of the Taylor expansion of u. For simplicity, we denote temporarily $z_{k,n} := (\frac{1-\beta_n}{n-1})k$ and $w_{k,n} := \beta_n + (\frac{1-\beta_n}{n-1})k$ and we rewrite the ICC as follows

$$\frac{c}{p_0 - p_1} = \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u\Big(W_0 - z_{k,n}L\Big) - u\Big(W_0 - w_{k,n}L\Big)\Big] \tag{32}$$

and we obtain using (31)

$$\frac{c}{p_0 - p_1} = \sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!}\Big[\sum_{k=0}^{n-1} b(k; n-1, p_1)(z_{k,n}^{\ell} - w_{k,n}^{\ell})\Big]. \tag{33}$$

Observe that $z_{k,n}^0 - w_{k,n}^0 = 0$ and for $\ell \geq 1$

$$z_{k,n}^{\ell} - w_{k,n}^{\ell} = \Big(\frac{1-\beta_n}{n-1}\Big)^{\ell} k^{\ell} - \Big(\beta_n + \Big(\frac{1-\beta_n}{n-1}\Big)k\Big)^{\ell} = -\sum_{j=0}^{\ell-1}\binom{\ell}{j}\beta_n^{\ell-j}\Big(\frac{1-\beta_n}{n-1}\Big)^{j} k^{j}.$$

Therefore, for $\ell \geq 1$

$$\sum_{k=0}^{n-1} b(k; n-1, p_1)(z_{k,n}^{\ell} - w_{k,n}^{\ell}) = -\sum_{j=0}^{\ell-1}\binom{\ell}{j}\beta_n^{\ell-j}\Big(\frac{1-\beta_n}{n-1}\Big)^{j}\mathbb{E}[B(n-1, p_1)^j],$$

where $B(n-1, p_1)$ is the involved binomial distribution. We rewrite the ICC as:

$$\frac{c}{p_0 - p_1} = \sum_{\ell=1}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell+1} L^{\ell}}{\ell!}\Big[\sum_{j=0}^{\ell-1}\binom{\ell}{j}\beta_n^{\ell-j}(1-\beta_n)^j \frac{\mathbb{E}[B(n-1, p_1)^j]}{(n-1)^j}\Big], \tag{34}$$

which will be our reference in what follows in this section. By the ICC (34), it holds that β_n is a positive root of the equation

$$\sum_{\ell=1}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell+1}L^{\ell}}{\ell!} \Big[\sum_{j=0}^{\ell-1} \binom{\ell}{j} \beta_n^{\ell-j}(1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j}\Big] - \frac{c}{p_0 - p_1} = 0. \tag{35}$$

Step 3. Comparing the ICCs (35) for β_n and β_{n+1} for $u''' = 0$ to prove part (i) of Proposition 1

We compare the ICCs for pool sizes n and $n+1$ for $u''' = 0$. As the ICC (35) is binding for β_n in case of a pool size of n, and also binding for β_{n+1} in case of a pool size of $n+1$, we deduce that

$$\sum_{\ell=1}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell+1}L^{\ell}}{\ell!} \Big[\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j}(1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} - \beta_{n+1}^{\ell-j}(1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j}\Big)\Big] = 0. \tag{36}$$

Consider a utility function, where the derivatives $u^{(\ell)}$ are equal to zero for $\ell > 2$ (such as a quadratic utility function), then for $\ell > 2$ the addends of the sum are zero. For $\ell = 1$ and $\ell = 2$, $\frac{\mathbb{E}[B(n-1,p_1)^{\ell-1}]}{(n-1)^{\ell-1}} = \frac{\mathbb{E}[B(n,p_1)^{\ell-1}]}{n^{\ell-1}}$ according to the proof of Lemma 3 (Step 1), and it is easy to see, that the LHS is equal to zero, if $\beta_n = \beta_{n+1}$. This proves part (i) of Proposition 1.

Step 4. Comparing the ICCs (35) for β_n and β_{n+1} for $u''' > 0$ to prove part (ii) of Proposition 1

In the following, we prove the desired inequality for Proposition 1:

$$\beta_n < \beta_{n+1}, \tag{37}$$

for a function u with higher order derivative $u^{\ell} \neq 0$, for some $\ell > 2$. Assume by contradiction that $\beta_n \geq \beta_{n+1}$. As $u'(W_0) > 0$, we know that the first addend $u'(W_0)L(\beta_n - \beta_{n+1})$ of the sum (36) is weakly positive, as well as its second addend $\frac{-u''(W_0)L^2}{2}(\beta_n^2 - \beta_{n+1}^2 + 2\beta_n(1-\beta_n)p_1 - 2\beta_{n+1}(1-\beta_{n+1})p_1)$. As $\frac{u^{(\ell)}(W_0)(-1)^{\ell+1}L^{\ell}}{\ell!} \geq 0$ for all $\ell \geq 1$ and the inequality is strictly positive for some $\ell > 2$, there exists some $\ell > 2$ such that

$$\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} - \beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big) < 0 \qquad (38)$$

because otherwise equality (36) does not hold. Rewrite (38) as

$$\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} \Big) < \sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big).$$

By Lemma 3 (Step 1) we know that

$$\frac{\mathbb{E}[B(n,p_1)^j]}{n^j} < \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j}$$

for all $n \geq 1$ and all $j > 1$. For $\ell > 2$ we deduce

$$\begin{aligned}\sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big) &< \sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{\mathbb{E}[B(n-1,p_1)^j]}{(n-1)^j} \Big) \\ &\leq \sum_{j=0}^{\ell-1} \binom{\ell}{j} \Big(\beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{\mathbb{E}[B(n,p_1)^j]}{n^j} \Big). \qquad (39)\end{aligned}$$

As $\beta_n \geq \beta_{n+1}$, we have $1-\beta_n \leq 1-\beta_{n+1}$, so

$$(1-\beta_n)^\ell \frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell} \leq (1-\beta_{n+1})^\ell \frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell}. \qquad (40)$$

By (39) and (40) it holds

$$\mathbb{E}\Big[\sum_{j=0}^{\ell} \binom{\ell}{j} \Big(\beta_n^{\ell-j} (1-\beta_n)^j \frac{B(n,p_1)^j}{n^j} \Big) \Big] < \mathbb{E}\Big[\sum_{j=0}^{\ell} \binom{\ell}{j} \Big(\beta_{n+1}^{\ell-j} (1-\beta_{n+1})^j \frac{B(n,p_1)^j}{n^j} \Big) \Big].$$

Thus, we conclude

$$\mathbb{E}\Big[\Big(\beta_n + \frac{(1-\beta_n) B(n,p_1)}{n} \Big)^\ell \Big] < \mathbb{E}\Big[\Big(\beta_{n+1} + \frac{(1-\beta_{n+1}) B(n,p_1)}{n} \Big)^\ell \Big]. \qquad (41)$$

We rewrite the previous inequality as

$$\sum_{k=0}^{n} b(k;n,p_1) \Big(\beta_n + \frac{(1-\beta_n)k}{n} \Big)^\ell < \sum_{k=0}^{n} b(k;n,p_1) \Big(\beta_{n+1} + \frac{(1-\beta_{n+1})k}{n} \Big)^\ell. \qquad (42)$$

For $k = n$ it holds

$$b(n; n, p_1)\Big(\beta_n + \frac{(1-\beta_n)n}{n}\Big)^\ell = p_1^n = b(n; n, p_1)\Big(\beta_{n+1} + \frac{(1-\beta_{n+1})n}{n}\Big)^\ell,$$

so (42) is equivalent to

$$\sum_{k=0}^{n-1} b(k; n, p_1)\Big(\beta_n + \frac{(1-\beta_n)k}{n}\Big)^\ell < \sum_{k=0}^{n-1} b(k; n, p_1)\Big(\beta_{n+1} + \frac{(1-\beta_{n+1})k}{n}\Big)^\ell. \tag{43}$$

Necessarily, there exists at least one $k = 0, \dots, n-1$ such that

$$\Big(\beta_n + \frac{(1-\beta_n)k}{n}\Big)^\ell < \Big(\beta_{n+1} + \frac{(1-\beta_{n+1})k}{n}\Big)^\ell \quad \rightsquigarrow \quad \beta_n + \frac{(1-\beta_n)k}{n} < \beta_{n+1} + \frac{(1-\beta_{n+1})k}{n}.$$

because otherwise strict inequality (43) does not hold. Thus,

$$(\beta_n - \beta_{n+1}) + \frac{(\beta_{n+1} - \beta_n)k}{n} = (\beta_n - \beta_{n+1})\Big(1 - \frac{k}{n}\Big) < 0.$$

As $k = 0, \dots, n-1$, we conclude that $\beta_n < \beta_{n+1}$, which contradicts our assumption that $\beta_n \geq \beta_{n+1}$.

Thus, it holds that $\beta_n < \beta_{n+1}$ for all $n \geq 1$ and functions u with higher order derivatives $u^\ell \neq 0$, for $\ell > 2$. Together with Step 3 above that shows that $\beta_n = \beta_{n+1}$ for $u''' = 0$, this proves Proposition 1. ■

Proof of Proposition 2

Proof. We prove that each pool member's utility increases in the pool size, when high effort is always implemented. First, we demonstrate that the expected utility for pool size $n+1$ is higher than for pool size n. Then, we prove that the expected utility profile for pool size n is in the core.

Expected utility increasing in n

According to Lemma 2, $\beta_n^{\min}$ will be chosen for pool size n, while $\beta_{n+1}^{\min}$ will be chosen for pool size $n+1$. For ease of notation, we again use β_n instead of $\beta_n^{\min}$. As ICC is binding for both pool sizes, it suffices to prove that the expected utility without own loss is increasing in n, following equation (10):

$$\sum_{k=0}^{n} b(k;n,p_1)u\left(W_0 - \left(\frac{1-\beta_{n+1}}{n}\right)kL\right) > \sum_{k=0}^{n-1} b(k;n-1,p_1)u\left(W_0 - \left(\frac{1-\beta_n}{n-1}\right)kL\right) \quad (44)$$

We first rewrite the utility comparison under the analytic assumption. Using the Taylor expansion of u, we know:

$$\begin{aligned}
&\sum_{k=0}^{n-1} b(k;n-1,p_1)u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\\
&= \sum_{k=0}^{n-1} b(k;n-1,p_1)\sum_{\ell=0}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^\ell L^\ell}{\ell!}\Big(\frac{1-\beta_n}{n-1}\Big)^\ell k^\ell\\
&= \sum_{\ell=0}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^\ell L^\ell}{\ell!}\Big(\frac{1-\beta_n}{n-1}\Big)^\ell \sum_{k=0}^{n-1} b(k;n-1,p_1)k^\ell\\
&= \sum_{\ell=0}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^\ell L^\ell}{\ell!}(1-\beta_n)^\ell\frac{\mathbb{E}[B(n-1,p_1)^\ell]}{(n-1)^\ell},
\end{aligned} \quad (45)$$

where $B(n-1,p_1)$ is the involved binomial distribution. Thus, the utility comparison is equivalent to the following condition (which will be our reference in what follows in this section):

$$\sum_{\ell=1}^{\infty}\frac{u^{(\ell)}(W_0)(-1)^\ell L^\ell}{\ell!}\Big((1-\beta_{n+1})^\ell\frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell} - (1-\beta_n)^\ell\frac{\mathbb{E}[B(n-1,p_1)^\ell]}{(n-1)^\ell}\Big) > 0. \quad (46)$$

A sufficient condition under which inequality (46) holds for all $n \geq 1$ is that each addend in the infinite sum (46) is ≥ 0 and at least one addend is strictly larger than 0, that is,

$$\frac{u^{(\ell)}(W_0)(-1)^\ell L^\ell}{\ell!}\Big((1-\beta_{n+1})^\ell\frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell} - (1-\beta_n)^\ell\frac{\mathbb{E}[B(n-1,p_1)^\ell]}{(n-1)^\ell}\Big) \geq 0 \quad (47)$$

for all $\ell \geq 1$, where the inequality is strict for some $\ell \geq 1$.

According to the assumption of mixed-risk aversion, $\frac{u^{(\ell)}(W_0)(-1)^\ell L^\ell}{\ell!} \leq 0$ for all $\ell \geq 1$, and inequalities (47) are equivalent to

$$(1-\beta_{n+1})^\ell\frac{\mathbb{E}[B(n,p_1)^\ell]}{n^\ell} \leq (1-\beta_n)^\ell\frac{\mathbb{E}[B(n-1,p_1)^\ell]}{(n-1)^\ell} \quad \forall \ell \geq 1. \quad (48)$$

Clearly, the previous inequalities are equivalent to the following one

$$\frac{\mathbb{E}[B(n-1,p_1)^\ell]}{\mathbb{E}[B(n,p_1)^\ell]} \geq \Big(\frac{n-1}{n}\Big)^\ell\Big(\frac{1-\beta_{n+1}}{1-\beta_n}\Big)^\ell \quad \forall \ell \geq 1. \quad (49)$$

Observe that if $\ell = 1$, we deduce the necessary condition $\beta_n \leq \beta_{n+1}$, so $(\frac{1-\beta_{n+1}}{1-\beta_n})^\ell \leq 1$ for $\ell \geq 1$. Then, condition (49) is satisfied as $\left(\frac{n-1}{n}\right)^\ell \leq \frac{\mathbb{E}[B(n-1,p_1)^\ell]}{\mathbb{E}[B(n,p_1)^\ell]}$ for all $\ell \geq 1$ (where the inequality is strict for $\ell > 1$) according to Lemma 3 (see Step 1 of the proof of Proposition 1).

Expected utility profile in the core

It remains to be shown that the expected utility profile $\mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$ is in the core. This requires that there is no coalition $\widetilde{n} < n$ such that $\mathbb{E}[u(\beta_{\widetilde{n}}, \mathbf{x}_{\widetilde{n}})] > \mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$. Suppose $\mathbb{E}[u(\beta_{\widetilde{n}}, \mathbf{x}_{\widetilde{n}})] > \mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$ exists. From Lemma 2, we know that $\mathbb{E}[u(\beta_{\widetilde{n}}, \mathbf{x}_{\widetilde{n}})]$ is maximized for $\beta_{\widetilde{n}} = \beta_{\widetilde{n}}^{\min}$. However, we have just proved that $\mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)] > \mathbb{E}[u(\beta_{\widetilde{n}}^{\min}, \mathbf{x}_{\widetilde{n}})]$ $\forall \widetilde{n} < n$. Thus, a contradiction, which proves that the expected utility profile $\mathbb{E}[u(\beta_n^{\min}, \mathbf{x}_n)]$ is in the core. Observe that $\mathbb{E}[u(\beta_{\widetilde{n}}, x_{\widetilde{n}})] < \mathbb{E}[u(\beta_n^{\min}, x_n)]$ is sufficient to ensure that all individuals join the pool. ■

Proof of Corollary 1

Proof. We have to consider three cases. First, consider the case where high effort is optimal for some pool size $n_1 < \widehat{n}$ and low effort is optimal for some pool size $n_2 \geq \widehat{n}$. As argued in the main text, Proposition 2, that considers the case of high effort, directly applies because members of the pool with size n_2 could decide to suboptimally implement the high effort. According to Proposition 2 they would still be better off compared to the members of the pool with size n_1. But then, by definition, their utility would be even higher when they implement the optimal low effort.

Second, we have to consider the case where both pools want to implement the low effort, that is not covered by Proposition 2. We prove that a larger pool size also increases utility when low effort is optimal. Suppose low effort is optimal for pool size n. In this case, $\mathbb{E}[u(\alpha_n = 0, x_i = 0, \mathbf{x}_{-i} = \mathbf{0})] > \mathbb{E}[u(\alpha, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})]$ for all $\alpha \in [0, 1]$. Recall from Lemma 2 that, for given effort, expected utility is strictly decreasing in β. Thus, if low effort is implemented, then $\alpha_n = 0$ is optimal.

In case of $\alpha_n = 0$, for every loss an individual's wealth shrinks by $\frac{L}{n}$, irrespectively of whether the loss was incurred by individual i or any other member of the risk pool.

Therefore, a pool member's expected utility, given low effort, can be written as:

$$\begin{aligned}\mathbb{E}\left[u\left(\alpha_n=0, x_i=0, \mathbf{x}_{-i}=\mathbf{0}\right)\right] =& p_0 \sum_{k=0}^{n-1} b(k; n-1, p_0) u\left(W_0 - \frac{1+k}{n} L\right) \\ &+ (1-p_0) \sum_{k=0}^{n-1} b(k; n-1, p_0) u\left(W_0 - \frac{k}{n} L\right) \\ =& \sum_{k=0}^{n} b(k; n, p_0) u\left(W_0 - \frac{k}{n} L\right).\end{aligned} \tag{50}$$

Rewriting the utility comparison for pool sizes $n+1$ and n under the analytic assumption gives

$$\sum_{\ell=1}^{\infty} \frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!} \left(\left(\frac{1}{n+1}\right)^{\ell} \mathbb{E}[B(n+1,p_0)^{\ell}] - \left(\frac{1}{n}\right)^{\ell} \mathbb{E}[B(n,p_0)^{\ell}]\right) > 0. \tag{51}$$

As $\frac{u^{(\ell)}(W_0)(-1)^{\ell} L^{\ell}}{\ell!}$ is (weakly) negative, the infinite sum is positive, if

$$\left(\frac{1}{n+1}\right)^{\ell} \mathbb{E}[B(n+1,p_0)^{\ell}] \leq \left(\frac{1}{n}\right)^{\ell} \mathbb{E}[B(n,p_0)^{\ell}] \quad \forall \ell \geq 1, \tag{52}$$

while the inequality is strict for some $\ell \geq 1$. Note that this condition can be rearranged to:

$$\left(\frac{n}{n+1}\right)^{\ell} \leq \frac{\mathbb{E}[B(n,p_0)^{\ell}]}{\mathbb{E}[B(n+1,p_0)^{\ell}]}, \tag{53}$$

which is true according to Lemma 3 (see Step 1 of the proof of Proposition 1).

Third, consider the case where low effort is optimal for some pool size $n_1 < \widehat{n}$ and high effort is optimal for some pool size $n_2 \geq \widehat{n}$. Given that we have just shown that Proposition 2 carries over to the case of low effort, the same argument as in the first case applies: Members of the larger pool could suboptimally implement the low effort and would still be better off compared to the members of the smaller pool. But then, their utility would be even higher when they implement the optimal high effort.

Analogously to the proof of Proposition 2, the expected utility profile is in the core for all three cases considered here. ■

Proof of Corollary 2

Proof. So far, we have assumed that there exists a $\beta_n^{\min} \in (\frac{1}{n}, 1)$ such that the ICC is binding. Note that $\beta_n^{\min} < 1$ is ensured by Assumption 1. We will now consider the case of $\alpha_n^{\min} = 0$, which is equivalent to $\beta_n^{\min} = \frac{1}{n}$, and prove that each pool member's expected utility is still increasing in n. Recall that our previous proofs do not rely on the fact that $\beta_n^{\min} > \frac{1}{n}$, but it is assumed that the effort incentives are kept constant in n. Therefore, we will show that if the ICC is binding for $\beta_n^{\min} = \frac{1}{n}$, the optimality of increasing the pool size follows directly from Propositions 1 and 2.

The case to be analyzed here is the one where $\beta_n^{\min} = \frac{1}{n}$ and the ICC is slack:

$$\mathbb{E}\left[u\left(\beta_n = \frac{1}{n}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1}\right)\right] > \mathbb{E}\left[u\left(\beta_n = \frac{1}{n}, x_i = 0, \mathbf{x}_{-i} = \mathbf{1}\right)\right], \tag{54}$$

which can be rearranged to:

$$f\Big(\beta_n = \frac{1}{n}\Big) = \sum_{k=0}^{n-1} b(k; n-1, p_1)\Big[u\Big(W_0 - \frac{k}{n}L\Big) - u\Big(W_0 - \frac{1+k}{n}L\Big)\Big] - \frac{c}{p_0 - p_1} > 0. \tag{55}$$

Now, we have to consider two cases when increasing the pool size from n to $n+1$, as the effort incentives may change. If $\beta_{n+1}^{\min} = \frac{1}{n+1}$, the high effort is efficient for both pool sizes and the optimality of the larger pool follows from the proof of Corollary 1, where we just have to substitute p_1 for p_0. If, on the other hand, the ICC is violated for $\beta_{n+1} = \frac{1}{n+1}$, we have to compare the expected utilities from pool size n and $\beta_n^{\min}$, and from pool size $n+1$ and $\beta_{n+1}^{\min} > \frac{1}{n+1}$.

From the proof of Lemma 1 we know that $f(\beta_n = 1) > f(\beta_n = \frac{1}{n}) > 0$. Furthermore, $\beta_n = 1$ is equal to no-insurance and $f(\beta = 1)$ is independent of n. Hence, $f(\beta_{n+1} = 1) > f(\beta_n = \frac{1}{n}) > 0 = f(\beta_{n+1} = \beta_{n+1}^{\min})$. Therefore, there exists a $\widehat{\beta}_{n+1}$ such that $f(\beta_{n+1} = \widehat{\beta}_{n+1}) = f(\beta_n = \frac{1}{n})$, where $\frac{1}{n+1} < \widehat{\beta}_{n+1} < 1$. In other words, when increasing the pool size to $n+1$, $\widehat{\beta}_{n+1}$ keeps the effort incentives constant. It follows from the proof of Proposition 1 that $\widehat{\beta}_{n+1} \geq \beta_n^{\min} = \frac{1}{n}$.

From the proof of Proposition 2 we know that $\mathbb{E}[u(\beta_{n+1} = \widehat{\beta}_{n+1}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] > \mathbb{E}[u(\beta_n = \frac{1}{n}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})]$. Finally, it follows from Lemma 2 that $\mathbb{E}[u(\beta_{n+1} = \beta_{n+1}^{\min}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] > \mathbb{E}[u(\beta_{n+1} = \widehat{\beta}_{n+1}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})]$. Therefore, we have:

$$\mathbb{E}[u(\beta_{n+1} = \beta_{n+1}^{\min}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] > \mathbb{E}[u(\beta_n = \frac{1}{n}, x_i = 1, \mathbf{x}_{-i} = \mathbf{1})] \tag{56}$$

and the expected utility is increasing in n. In accordance with the proof of Proposition 2, the expected utility profile is in the core. ■

Proof of Lemma 4

The following Lemma will be used in the proofs of Propositions 3 and 4.

Lemma 4. *(i)* $x_i^* > 0 \; \forall n, \; \beta_n,$ *and* $\mathbf{x}_{-i}$. *(ii)* $\frac{\partial x_i^*}{\partial \beta_n} > 0 \; \forall n, \; \beta_n,$ *and* $\mathbf{x}_{-i}$.

Part (i)

Proof. We show that, in the continuous version, our standard assumptions ensure that individuals will always exert a strictly positive effort. Given $\mathbf{x}_{-i}$ and n members of the risk pool, the expected utility of individual i is:

$$\begin{aligned}\mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})] = p(x_i) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1} \Big) kL \Big) \\ + (1 - p(x_i)) \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1} \Big) kL \Big) \Big] - c(x_i)\end{aligned}$$

Define $V(x_i) := \mathbb{E}[u(\beta_n, x_i, \mathbf{x}_{-i})]$ as the expected utility as a function of x_i. The first order condition for choosing the utility maximizing effort level by individual i is $V'(x_i) = 0$, i.e.

$$\sum_{k=0}^{n-1} b(k; n-1, p(x_{-i})) \Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1} \Big) kL \Big) - u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1} \Big) kL \Big) \Big] + \frac{c'(x_i)}{p'(x_i)} = 0. \tag{57}$$

As $\lim_{x \downarrow 0} \frac{c'(x_i)}{p'(x_i)} = 0$ and $\beta_n > 0$, we have $V'(x_i = 0) > 0$. Next, note that x_i appears on $V'(x_i)$ only in the part $\frac{c'(x_i)}{p'(x_i)}$, so that $V''(x_i) = \frac{c''(x_i)p'(x_i) - c'(x_i)p''(x_i)}{p'(x_i)^2} < 0$, where the sign follows from $c''(x_i) > 0$, $p'(x_i) < 0$, $c'(x_i) \geq 0$ and $p''(x_i) < 0$. Thus, there is an interior solution for x_i^* where $x_i^* > 0$. ■

Part (ii)

Proof. We prove that, for a given pool size, effort is strictly increasing in β_n. Write $F := V'(x_i)$ to get

$$F(x_i, \beta_n) = \sum_{k=0}^{n-1} b(k; n-1, p(x_{-i}))\Big[u\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - u\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] + \frac{c'(x_i)}{p'(x_i)}, \quad (58)$$

where the incentive compatible effort x_i^* is implicitly defined by $F(x_i^*, \beta_n) = 0$. From the implicit function theorem, $\frac{dx_i^*}{d\beta_n} = -\frac{\frac{\partial F}{\partial \beta_n}}{\frac{\partial F}{\partial x_i^*}}$, where

$$\begin{aligned}\frac{\partial F}{\partial \beta_n} =& L\sum_{k=0}^{n-1} b(k; n-1, p(x_{-i}))\Big[u'\Big(W_0 - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\frac{k}{n-1} \\ &+ u'\Big(W_0 - \beta_n L - \Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big(1 - \frac{k}{n-1}\Big)\Big]\end{aligned} \quad (59)$$

and

$$\frac{\partial F}{\partial x_i^*} = \frac{c''(x_i^*)p'(x_i^*) - c'(x_i^*)p''(x_i^*)}{p'(x_i^*)^2}. \quad (60)$$

We have: $\frac{\partial F}{\partial \beta_n} > 0$ because $u'(W) > 0$ for each W and $\frac{k}{n-1} \in [0, 1]$. In addition, $\frac{\partial F}{\partial x_i^*} < 0$ because $c'(x_i), p''(x_i), c''(x_i) > 0$ and $p'(x_i) < 0$ for each x_i. Thus, $\frac{dx_i^*}{d\beta_n} > 0$ for each $\beta_n \in [\frac{1}{n}, 1]$. ■

Proof of Proposition 3

Proof. A sufficient condition that the expected utility achieves a maximum at $\beta_n^* \in (\frac{1}{n}, 1)$ is that the total derivative of the expected utility with respect to β_n is strictly negative at $\beta_n = 1$, i.e. $\frac{d\mathbb{E}[u(\beta_n, x_n^*(\beta_n))]}{d\beta_n}|_{\beta_n=1} < 0$, and strictly positive at $\beta_n = \frac{1}{n}$, i.e. $\frac{d\mathbb{E}[u(\beta_n, x_n^*(\beta_n))]}{d\beta_n}|_{\beta_n=\frac{1}{n}} > 0$. For ease of notation we continue to use x instead of $x_n^*(\beta_n)$. The total derivative of the expected utility with respect to β_n can be written as:

$$\frac{d\mathbb{E}[u(\beta_n, x)]}{d\beta_n} = \frac{\partial \mathbb{E}[u(\beta_n, x)]}{\partial \beta_n} + \frac{\partial \mathbb{E}[u(\beta_n, x)]}{\partial x^*}\frac{dx}{d\beta_n} \quad (61)$$

First, we will demonstrate that $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n} < 0$ at $\beta_n = 1$. Then, we will prove that $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n} > 0$ at $\beta_n = \frac{1}{n}$.

Analysis of the first derivative at $\beta_n = 1$

From Lemma 4 part (ii), we know that $\frac{dx}{d\beta_n} > 0$. Furthermore,

$$\begin{aligned}\frac{\partial\mathbb{E}[u(\beta_n,x)]}{\partial\beta_n}|_{\beta_n=1} =& L\Big[p(x)\sum_{k=0}^{n-1} b(k;n-1,p(x))u'(W_0-L)(\frac{k}{n-1}-1)] \\ &+ (1-p(x))\sum_{k=0}^{n-1} b(k;n-1,p(x))u'(W_0)\frac{k}{n-1}\Big] < 0\end{aligned} \tag{62}$$

(cf. proof of Lemma 2). When considering the effect of adjusting β_n, the pool members will anticipate that in equilibrium all individuals will choose the same effort. Therefore, as opposed to the derivation of the ICC (from the perspective of a single individual) we need to consider all efforts when calculating the derivative of the expected utility with respect to the effort (chosen by all members of the risk pool in equilibrium). However, at $\beta_n = 1$ the effect of the other $n-1$ members of the pool does not matter, as all individuals cover their potential losses themselves. Therefore, according to the ICC:

$$\frac{\partial\mathbb{E}[u(\beta_n,x)]}{\partial x}|_{\beta_n=1} = -p'(x)[u(W_0)-u(W_0-L)] - c'(x) = 0 \tag{63}$$

Hence, $\frac{d\mathbb{E}[u(\beta_n,x)]}{d\beta_n}|_{\beta_n=1} < 0$ and there exists an expected utility maximizing $\beta_n^* < 1$.

Analysis of the first derivative at $\beta_n = \frac{1}{n}$

From the proof of Lemma 2 we know that $\frac{\partial\mathbb{E}[u(\beta_n,x)]}{\partial\beta_n}|_{\beta_n=\frac{1}{n}} = 0$. Therefore, as $\frac{dx}{d\beta_n} > 0$, the sign of the total derivative of the expected utility with respect to β_n at $\beta_n = \frac{1}{n}$ is determined by the partial derivative with respect to x :

$$\frac{\partial \mathbb{E}[u(\beta_n,x)]}{\partial x} = p'(x)\Big[(1-p(x))\sum_{k=0}^{n-1} b'(k;n-1,p(x))u\Big(W_0-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) + p(x)\sum_{k=0}^{n-1} b'(k;n-1,p(x))u\Big(W_0-\beta_n L-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] + h(\beta_n,x), \tag{64}$$

where from the ICC we know that x is implicitly defined by:

$$h(\beta_n,x) := p'(x)\sum_{k=0}^{n-1} b(k;n-1,p(x))\Big[u\Big(W_0-\beta_n L-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big) - u\Big(W_0-\Big(\frac{1-\beta_n}{n-1}\Big)kL\Big)\Big] - c'(x) = 0. \tag{65}$$

Therefore, at $\beta_n = \frac{1}{n}$, equation (64) reduces to:

$$\frac{\partial \mathbb{E}[u(\beta_n,x)]}{\partial x}\Big|_{\beta_n=\frac{1}{n}} = p'(x)\sum_{k=0}^{n} b'(k;n,p(x))u\Big(W_0-\frac{kL}{n}\Big) \tag{66}$$

As $p'(x) < 0$, we have to show that

$$\sum_{k=0}^{n} b'(k;n,p(x))u\Big(W_0-\frac{k}{n}L\Big) < 0, \tag{67}$$

where

$$b'(k;n,p) = \binom{n}{k} p^{k-1}(1-p)^{n-k-1}(k-np). \tag{68}$$

Observe that $\mu = np$ is the mean of the binomial distribution $X = B(n,p)$. So we can rewrite inequality (67) as

$$\frac{1}{p(1-p)}\sum_{k=0}^{n} b(k;n,p)(k-\mu)u\Big(W_0-\frac{k}{n}L\Big) = \frac{1}{p(1-p)}\mathbb{E}\Big[(X-\mu)u\Big(W_0-\frac{X}{n}L\Big)\Big] < 0.$$

As u is an analytic function,

$$\mathbb{E}\Big[(X-\mu)u\Big(W_0-\frac{X}{n}L\Big)\Big] = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell u^{(\ell)}(W_0)L^\ell}{\ell! n^\ell}\mathbb{E}[(X-\mu)X^\ell].$$

For $\ell = 0$, we have $\mathbb{E}[(X-\mu)] = 0$, so we have to prove

$$\sum_{\ell=1}^{\infty} \frac{(-1)^\ell u^{(\ell)}(W_0) L^\ell}{\ell! n^\ell} \mathbb{E}[(X-\mu)X^\ell] < 0$$

As $(-1)^\ell u^{(\ell)}(W_0) < 0$ for each $\ell \geq 1$, it is enough to show that $\mathbb{E}[(X-\mu)X^\ell] > 0$ for each $\ell \geq 1$. According to Knoblauch (2008, Thm. 2.2), the ℓ-th raw moment of X can be written as

$$\mathbb{E}[X^\ell] = \sum_{i=0}^{\ell} b_{\ell,i} p^i n^{\underline{i}} \tag{69}$$

where $n^{\underline{i}} := \binom{n}{i} i! = n(n-1)\cdots(n-i+1)$ and $b_{\ell,i}$ is defined recursively as follows:

$$\begin{aligned} b_{0,i} &:= \delta_{i0}, \\ b_{\ell,i} &:= i b_{\ell-1,i} + b_{\ell-1,i-1}, \end{aligned} \tag{70}$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise, as well as $b_{\ell,i} = 0$ for $\ell < 0, i < 0$ or $i > \ell$. In particular $b_{\ell,\ell} = 1$ for $\ell \geq 0$. Observe that $n \cdot n^{\underline{i}} = n^{\underline{i+1}} + i n^{\underline{i}}$. Consequently, for $\ell \geq 1$

$$\begin{aligned} \mathbb{E}[(X-\mu)X^\ell] = \mathbb{E}[X^{\ell+1}] - np\mathbb{E}[X^\ell] &= \sum_{j=0}^{\ell+1} b_{\ell+1,j} p^j n^{\underline{j}} - np \sum_{i=0}^{\ell} b_{\ell,i} p^i n^{\underline{i}} \\ &= \sum_{j=0}^{\ell+1} (j b_{\ell,j} + b_{\ell,j-1}) p^j n^{\underline{j}} - \sum_{i=0}^{\ell} b_{\ell,i} p^{i+1} n \cdot n^{\underline{i}} \\ &= \sum_{j=0}^{\ell} j b_{\ell,j} p^j n^{\underline{j}} + \sum_{j=1}^{\ell+1} b_{\ell,j-1} p^j n^{\underline{j}} - \sum_{i=0}^{\ell} b_{\ell,i} p^{i+1} n^{\underline{i+1}} - \sum_{i=0}^{\ell} b_{\ell,i} p^{i+1} i n^{\underline{i}} \\ &= \sum_{j=1}^{\ell} j b_{\ell,j} p^j (1-p) n^{\underline{j}} > 0. \end{aligned} \tag{71}$$

We conclude that $\frac{d\mathbb{E}[u(\beta_n, x)]}{d\beta_n}|_{\beta_n = \frac{1}{n}} > 0$ and there exists an expected utility maximizing $\beta_n^* > \frac{1}{n}$. ■

Proof of Proposition 4

Proof. First of all, we prove that any effort that is implementable for pool size n_1 is also implementable for pool size $n_2 > n_1$. Let $x^*_{n_1}(\beta_{n_1})$ be the incentive compatible effort level for $\beta_{n_1} \in (\frac{1}{n_1}, 1)$. From *part (ii)* of Lemma 4, we know that the effort is continuous and strictly increasing in β_n. From Lee and Ligon (2001), we know that $x^*_{n_1}(\frac{1}{n_1}) > x^*_{n_2}(\frac{1}{n_2})$. As $x^*_{n_1}(1) = x^*_{n_2}(1)$ and $\beta_{n_1} < 1$, there exists a $\beta_{n_2} \in (\frac{1}{n_2}, 1)$ such that $x^*_{n_1}(\beta_{n_1}) = x^*_{n_2}(\beta_{n_2})$. Observe that β_{n_2} is not necessarily optimal for pool size n_2.

In order to prove that the expected utility is increasing in the pool size, let $\beta^*_{n_1}$ be the optimal effective share for pool size n_1 and let $x_{n_1} = x^*_{n_1}(\beta^*_{n_1})$ be the corresponding incentive compatible effort level. From the proof of Proposition 2 for our binary model, we know that the utility for each pool member is strictly increasing in the pool size whenever the *same* effort is implemented. Thus, $\mathbb{E}[u(\beta_{n_2}, x^*_{n_1}(\beta^*_{n_1}), n_2)] > \mathbb{E}[u(\beta^*_{n_1}, x^*_{n_1}(\beta^*_{n_1}), n_1)]$ for each $n_2 > n_1$. Finally, from the definition of optimality, it follows that $\mathbb{E}[u(\beta^*_{n_2}, x^*_{n_2}(\beta^*_{n_2}), n_2)] \geq \mathbb{E}[u(\beta_{n_2}, x^*_{n_1}(\beta^*_{n_1}), n_2)] > \mathbb{E}[u(\beta^*_{n_1}, x^*_{n_1}(\beta^*_{n_1}), n_1)]$. It immediately follows that the expected utility profile $\mathbb{E}[u(\beta^*_{n_2}, x^*_{n_2}(\beta^*_{n_2}), n_2)]$ is in the core. ■

Uniqueness of β^*_n

For a sufficient condition of uniqueness of the optimal retention rate, define $p[x^*_n(\beta_n)]$ as the accident probability for the incentive compatible effort implemented via the effective share β_n. We can then prove that a sufficient condition for uniqueness (proof available on request) is that

(i) $\frac{d^2 p[x^*_n(\beta_n)]}{d\beta^2} = \frac{\partial^2 p}{\partial (x^*_n)^2}\left(\frac{\partial x^*_n}{\partial \beta_n}\right)^2 + \frac{\partial p}{\partial x^*_n}\frac{\partial^2 x^*_n}{\partial \beta^2_n} \geq 0$ and

(ii) $\frac{u'(W_0 - L)}{u'(W_0)} - \frac{2 - p[x^*_n(\beta_n = 1)]}{1 - p[x^*_n(\beta_n = 1)]} \geq 0$.

Condition *(i)* expresses that the accident probability that is implemented via β_n decreases at a (weakly) decreasing rate. As $\frac{\partial^2 p}{\partial (x^*_n)^2} > 0$, $\left(\frac{\partial x^*_n}{\partial \beta^*_n}\right)^2 > 0$ and $\frac{\partial p}{\partial x^*_n} < 0$, a sufficient condition is that $\frac{\partial^2 x^*_n}{\partial \beta^2_n} \leq 0$, i.e. that the incentive compatible effort increases at a decreasing rate in β_n.

Intuitively, *(ii)* holds if the marginal utility of income with loss is large compared to the marginal utility without loss (i.e. if $\frac{u'(W_0 - L)}{u'(W_0)}$ is high), and if the accident probability in case the effort incentive is maximized, $p[x^*_n(\beta_n = 1)]$ is low (i.e. if $\frac{2 - p[x^*_n(\beta_n = 1)]}{1 - p[x^*_n(\beta_n = 1)]}$ is low).