Math. Ann. 322, 49–67 (2002) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080100264

Positive semidefinite germs in real analytic surfaces⋆

José F. Fernando

Received: 9 January 2001 / Published online: 24 September 2001 –

Abstract. We find all real analytic surface germs in $\mathbb{R}^3$ on which every positive semidefinite function germ is a sum of squares (in fact, of two squares) of analytic function germs.

1 Introduction and statement of the result

In the study of positive semidefinite functions (= *psd*) and sums of squares (= *sos*) one main problem is to know whether every positive semidefinite function is a sum of squares. As is well known, the interest on questions of this type stems from the famous Hilbert's 17th problem, and has been one of the streamlines of research in real algebra and real geometry. The history of psd's and sos's is long and rich, and we refer the reader to [BoCoRo] and [ChDLR]. One particular case which is receiving more attention lately is that of local rings, see for instance [Sch2]. Here real algebra and the techniques of real spectra appear in essential ways. In fact, psd's are defined over arbitrary commutative rings by means of the theory of the real spectrum.

In this paper, we will be dealing with the most geometric illustration of local rings: real analytic germs. Our goal is to determine when psd = sos for *analytic* function germs. Of course, this is always true for *meromorphic* function germs (see for instance [Rz2]), but in the analytic setting things are very different. For curve germs we know that it is true only for unions of independent lines; this is easy to see for curves in the plane, and in any case follows from a general result due to Scheiderer ([Sch2]). In higher dimensions only five germs are known to have that property: the plane, the Brieskorn singularity, the union of two planes, Whitney's umbrella and the cone, and in fact, very few more are expected to appear in the list (see [FeRz]).

J.F. Fernando**

Depto. Geometría y Topología, F. Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain (e-mail: josefer@eucmos.sim.ucm.es)

⋆ This paper is based on the author's doctoral dissertation, written under the supervision of Prof. J. Ruiz

** Partially supported by DGICYT, PB98-0756-C02-01

Before further comments, we need some notation and terminology. Let X be an analytic set germ (at the origin of $\mathbb{R}^n$); we denote by $\mathcal{O}(X)$ the ring of germs of analytic functions on X and by $\mathcal{M}(X)$ its total ring of fractions. For instance, $\mathcal{O}(\mathbb{R}^n)$ is the ring $\mathbb{R}\{x\}$ of convergent power series in $x = (x_1, \dots, x_n)$, and $\mathcal{M}(\mathbb{R}^n)$ is the field of fractions $\mathbb{R}(\{x\})$ of $\mathbb{R}\{x\}$. As $X \subset \mathbb{R}^n$ we have $\mathcal{O}(X) = \mathbb{R}\{x\}/I$, where I is the ideal of all analytic function germs vanishing on X. A germ $f \in \mathcal{O}(X)$ is *positive semidefinite* or *psd* if it is ≥ 0 on X; we denote by $\mathcal{P}(X)$ the set of all psd's of X. We will denote by $\Sigma(X)$ (resp. $\Sigma_2(X)$) the set of all sums of squares (resp. of 2 squares) of elements of $\mathcal{O}(X)$.

As was said before, we can introduce these notions for an arbitrary commutative ring A: $\mathcal{P}(A)$ is the set of all $f \in A$ such that $f(\alpha) \geq 0$ for every prime cone $\alpha \in \operatorname{Spec}_r(A)$. The fact that $\mathcal{P}(A) = \mathcal{P}(X)$ for $A = \mathcal{O}(X)$ is a consequence of the Artin-Lang property for germs (see [AnBrRz]). Also, we set $\Sigma(X)$ (resp. $\Sigma_2(X)$) for the set of all sums of squares (resp. of 2 squares) of elements of A.

We have the following general result for dimension ≥ 3.

Theorem 1.1. ([Sch1]) *Let A be a local regular ring of dimension ≥ 3. Then $\Sigma(A) \neq \mathcal{P}(A)$.*

From this we easily deduce:

Corollary 1.2. *Let $X \subset \mathbb{R}^n$ have dimension $d \geq 4$. Then $\Sigma(X) \neq \mathcal{P}(X)$.*

Proof. Indeed, if X_d is an irreducible component of X of dimension d, the curve selection lemma gives a curve germ $\gamma \subset X$ not contained in $\operatorname{Sing} X$ and the ideal of γ is a prime ideal $\mathfrak{p} \subset \mathcal{O}(X)$ of height $d-1$, such that $A = \mathcal{O}(X)_{\mathfrak{p}}$ is local regular of dimensión $d-1 \geq 3$. By 1.1 there is $f/g \in \mathcal{P}(A) \setminus \Sigma(A)$ with $g \notin \mathfrak{p}$.

Let $h \in \mathcal{O}(X)$ be an equation for the union of all irreducible components of X which do not contain γ (i.e. those distinct from X_d) plus the singular locus of X_d. The element $F = h^2 g^2 f/g$ is clearly in $\mathcal{P}(A)$, and consequently, positive in every ordering of $cf(A)$, which is in fact the field of meromorphic functions of X_d. Thus, $F \geq 0$ on $X_d \setminus \operatorname{Sing} X$ and 0 on any other irreducible component of X; we conclude $F \in \mathcal{P}(X)$. But $F \notin \Sigma(X)$, because otherwise, since $hg \notin \mathfrak{p}$ is a unit in A, we would have $f/g \in \Sigma(A)$. □

It is thought that this should also be true for dimension $= 3$, and consequently the germs for which $\Sigma = \mathcal{P}$ should be looked for in dimension 2. However, we know many examples with $\Sigma \neq \mathcal{P}$ and very few with $\Sigma = \mathcal{P}$ (the five surface germs already mentioned). Then, one starts by searching the surface germs in $\mathbb{R}^3$ with $\mathcal{P} = \Sigma$. In this framework, our main result is the following:

Theorem 1.3. *The singular surface germs $X \subset \mathbb{R}^3$ with $\mathcal{P}(X) = \Sigma(X)$ are exactly the following*

(i) $z^2 - x^3 - y^5 = 0$ *(Brieskorn's singularity)*
(ii) $z^2 - x^3 - xy^3 = 0$
(iii) $z^2 - x^3 - y^4 = 0$
(iv) $z^2 - x^2 = 0$ *(Two transversal planes)*
(v) $z^2 - x^2 - y^2 = 0$ *(Cone)*
(vi) $z^2 - x^2 - y^k = 0,\ k \geq 3$ *(Deformations of two planes)*
(vii) $z^2 - x^2y = 0$ *(Whitney's umbrella)*
(viii) $z^2 - x^2y + y^3 = 0$
(ix) $z^2 - x^2y - (-1)^k y^k = 0,\ k \geq 4$ *(Deformations of Whitney's umbrella)*

Furthermore, in all these cases $\mathcal{P}(X) = \Sigma_2(X)$.

The first part of this statement, that is, $\mathcal{P}(X) = \Sigma(X)$ puts X in the list, was partially proved in [Rz3]. Indeed, there it is shown that X is in the list if $\mathcal{P}(X) = \Sigma_2(X)$. We extend this to any number of squares in section §2.

The second part of the statement 1.3, and the hardest, is to prove $\mathcal{P} = \Sigma_2$ for all germs in the list. This is done in [Rz3] for Brieskorn's singularity, the two planes, and Whitney's umbrella, and in [FeRz] for the cone. The proof for Brieskorn's singularity is a transcription of the old argument for the plane, using the fact that the singularity and its complexification are both factorial (a property that characterizes this singularity). The proof for the other three surface germs are particular of each case, although that of the cone contains some hints for the more systematic method we develop in this paper. Indeed, we will prove $\mathcal{P} = \Sigma_2$ for all germs in the list, starting always from the factorial situation. To that end, we reduce the problem to a mixed polynomial case, by means of M. Artin's Approximation Theorem ([Ku et al], [M.Ar]) and a density result for psd's (section §3). Then, by blowings-up we go from each surface in the list to either the plane or Brieskorn's singularity (sections §4-§6) to settle the matter up to a universal denominator. The conclusion follows by clearing that denominator by some standard equations with sos's.

2 Generation of the list

Given an ideal $I \subset \mathbb{R}\{x\}$, let $\omega(I(X))$ stand for the minimal order of a series in I. We have the following general remark:

Lemma 2.1. *Let* $X \subset \mathbb{R}^n$ *verify* $\mathcal{P}(X) = \Sigma(X)$. *Then* $\omega(I(X)) = 2$.

Proof. To start with, we choose a series $F \in I(X)$ of order $r > 0$. After a linear change we may assume

$$F = x_n^r + a_{r-1}x_n^{r-1} + \cdots + a_1 x_n + a_0,$$

where $a_j \in \mathbb{R}\{y\}$, $y = (x_1, \ldots, x_{n-1})$ and $\omega(a_j) \geq r - j$ for $0 \leq j \leq r - 1$. We claim that there is $M > 0$ such that $|a_j| < M\|y\|^{r-j}$.

Indeed, say $f = a_1, \dots, a_{r-1}$ has order $\geq s$. Since $\omega(f^2) \geq 2s$ we can write $f^2 = \sum_{|\nu|=2s} a_\nu(y) y^\nu$, and so, near the origin we have

$$|f(y)|^2 \leq \sum_{|\nu|=2s} c_\nu |y|^\nu = \sum_{|\nu|=2s} c_\nu \|y\|^{2s} |z|^\nu = \|y\|^{2s} \sum_{|\nu|=2s} c_\nu |z|^\nu$$

with $c_\nu = 1 + |a_\nu(0)|$ and $z = y/\|y\|$. The function $\sum_{|\nu|=2r} c_\nu |z|^\nu$ is bounded on $\|z\| = 1$, say by $M > 0$, and we conclude $\|f\|^2 < M\|y\|^{2s}$, hence $\|f\| < M\|y\|^s$. This shows our claim.

Now, for every integer $k \geq 1$ we consider the quadratic form

$$g_k = k^2(x_1^2 + \cdots + x_{n-1}^2) - x_n^2,$$

and will prove that for k large g_k is psd in X.

In fact, otherwise, X would contain a sequence $x^{(k)} = (y^{(k)}, x_n^{(k)}) \to 0$ such that $g_k(x^{(k)}) < 0$, that is,

$$0 \leq k\rho_k < |x_n^{(k)}|, \quad \text{where} \ \ \rho_k = \|y^{(k)}\|.$$

Since $F \in I(X)$, we have $F(x^{(k)}) = 0$, and consequently

$$|x_n^{(k)}|^r = |\sum_{j=0}^{r-1} a_j(y^{(k)})(x_n^{(k)})^j| \leq \sum_{j=0}^{r-1} |a_j(y^{(k)})||x_n^{(k)}|^j \leq$$
$$M \sum_{j=0}^{r-1} \rho_k^{r-j} |x_n^{(k)}|^j < M \sum_{j=0}^{r-1} \frac{|x_n^{(k)}|^r}{k^{r-j}} = M|x_n^{(k)}|^r \sum_{j=0}^{r-1} \frac{1}{k^{r-j}}.$$

But $|x_n^{(k)}|^r > 0$, and we get $1 < M\left(\frac{1}{k} + \cdots + \frac{1}{k^r}\right)$, a contradiction.

Once we know that $g_k \in \mathcal{P}(X)$ for k large, let us see that $g_k \notin \Sigma(X)$ if $\omega(I(X)) \geq 3$. To that end, suppose g_k is an sos in $\mathcal{O}(X)$. Then

$$g_k = h_1^2 + \cdots + h_s^2 + h,$$

with $h_i \in \mathbb{R}\{x\}$, $h \in I(X)$ and $\omega(h) \geq 3$. Whence, equating initial forms in the above expression, we find $a_1, \dots, a_r \in \mathbb{R}[x_1, \dots, x_n]$ such that

$$g_k = a_1^2 + \cdots + a_r^2,$$

which is impossible. □

From the last result we deduce that if $\mathcal{P}(X) = \Sigma(X)$ holds for $X \subset \mathbb{R}^3$, say $X : f(x, y, z) = 0$, then $\omega(f) = 2$. Hence, after a change of coordinates, $f = z^2 - F(x, y)$ with $\omega(F) \geq 2$. Under this conditions, the arguments in [Rz3, §4] work for the condition $\mathcal{P} = \Sigma$ and not only for $\mathcal{P} = \Sigma_2$, and we get the list of 1.3.

3 Polynomial reduction

Given an analytic surface germ $X \subset \mathbb{R}^3$ with equation $z^2 = F(x, y)$, where F is a polynomial, we consider the algebraic surface S_X defined by the same equation $z^2 = F(x, y)$ and denote by $\mathcal{P}(S_X)$ the set of all polynomials $P(x, y) + zQ(x, y)$ which are ≥ 0 everywhere on S_X. To reduce the study of $\mathcal{P}(X)$ to that of $\mathcal{P}(S_X)$ we need the following density result:

Lemma 3.1. *Let $Z \subset \mathbb{R}^2$ be a closed semianalytic set germ and $f \in \mathcal{O}(\mathbb{R}^2)$. If $f|_{Z\setminus\{0\}} > 0$, the same holds true for every $g \equiv f \mod (x, y)^r$ with r large enough.*

The germs that verify the condition in the statement are the *positive definite* or *pd* germs on Z; we denote by $\mathcal{P}^+(Z)$ the set of all pd's on Z.

Proof. Since Z is a finite union of closed half-branches and connected slices between them, we can suppose $Z \setminus \{0\}$ connected. Let $f \in \mathcal{P}^+(Z)$. We have $Z \cap \{f = 0\} = \{0\}$, and there exists a separating polynomial $\varphi \in \mathbb{R}[x, y]$ such that $\varphi|_{Z\setminus\{0\}} > 0$ and $\varphi|_{\{f=0\}\setminus\{0\}} < 0$ ([Rz1]).
Claim: If $g \equiv f \mod (x, y)^r$ for r large enough, then $\{g = 0\} \setminus \{0\} \subset \{\varphi < 0\}$.

Assume this for the moment. Then, $\{g = 0\} \cap Z = \{0\}$, and, $Z \setminus \{0\}$ being connected, g has constant sign on $Z \setminus \{0\}$. To see that this sign is positive, we choose a half-branch $\gamma \subset Z$, say $\gamma : x = x(t),\ y = y(t),\ t > 0$. Since $f \in \mathcal{P}^+(Z)$, we have

$$0 < f(x(t), y(t)) = a_s t^s + \cdots,\ a_s > 0.$$

By the condition on g, $g(x(t), y(t)) \equiv f(x(t), y(t)) \mod t^r$, and for $r > s$, we get

$$g(x(t), y(t)) = a_s t^s + \cdots .$$

As $a_s > 0$, $g \in \mathcal{P}^+(Z)$. Thus, it remains to prove the claim.

To do that, by a linear change, we make f is regular of order m with respect to y, so that $f = UP$, where $P \in \mathbb{R}\{x\}[y]$ is a Weierstrass polynomial of degree m and $U \in \mathbb{R}\{x, y\}$ a unit of $\mathbb{R}\{x, y\}$.

Next, if $g \equiv f \mod (x, y)^r$ and r is large enough, g is also regular of order m, and $g = VQ$, where $Q \in \mathbb{R}\{x\}[y]$ is a Weierstrass polynomial of degree m and $V \in \mathbb{R}\{x, y\}$ a unit. Hence

$$Q = V^{-1}UP + V^{-1}(g - f) \equiv V^{-1}UP \mod (x, y)^r,$$

from which we can deduce that

$$Q \equiv P \mod (x, y)^{r-1}.$$

Indeed, consider the homogeneous components of $P, Q, W = V^{-1}U$:

$$P = p_m + p_{m+1} + \cdots + p_{m+k} + \cdots ,$$
$$Q = q_m + q_{m+1} + \cdots + q_{m+k} + \cdots ,$$
$$W = w_0 + w_1 + \cdots + w_k + \cdots ,$$

where

$$p_m = y^m + a_{m-1}(x)y^{m-1} + \cdots + a_1(x)y + a_0(x),$$
$$q_m = y^m + b_{m-1}(x)y^{m-1} + \cdots + b_1(x)y + b_0(x),$$

and $\partial_y(p_{m+k}),\ \partial_y(q_{m+k}) \leq m-1$ for $k \geq 1$. As $Q \equiv WP \mod (x,y)^r$, for all $k = 0, \dots, r-m-1$ we have:

$$q_{m+k} = \sum_{i+j=m+k} p_i w_j .$$

We must see that $q_{m+k} = p_{m+k}$ for $k = 0, \dots, r-m-1$, that $w_0 = 1$ and that $w_k = 0$ for $k = 1, \dots, r-m-1$. For $k = 0$ we have $q_m = p_m w_0$, and therefore $w_0 = 1$ and $q_m = p_m$. Assume now $q_{m+j} = p_{m+j}$ for $j = 0, \dots, k < r-m-1$ and $w_j = 0$ for $j = 1, \dots, k$. Then:

$$q_{m+k+1} = p_{m+k+1}w_0 + p_{m+k}w_1 + \cdots + p_m w_{k+1} = p_{m+k+1} + p_m w_{k+1},$$

but, if $w_{k+1} \neq 0$,

$$m \leq \partial_y(p_m w_{k+1}) = \partial_y(q_{m+k+1} - p_{m+k+1}) \leq \max\{\partial_y(q_{m+k+1}), \partial_y(p_{m+k+1})\} \leq m-1,$$

which is impossible, hence $w_{k+1} = 0$ and $q_{m+k+1} = p_{m+k+1}$.

Thus we write $P - Q = h \in (x,y)^{r-1}$, where $h \in \mathbb{R}\{x\}[y]$ has degree $< m$.

Finally, the germ $g = 0$ is a union of real half-branches, say $\gamma_i : x = \varepsilon_i t^p,\ y = g_i(t),\ t > 0$, where $\varepsilon_i = \pm 1$, and each Puiseux series $\xi = g_i(t^{1/p})$ is a root of the polynomial $Q(\varepsilon_i t, y)$ (we can take $p = m!$, [Ch]). Then there is a root ζ of the polynomial $P(\varepsilon_i t, y)$ such that $\omega(\zeta - \xi) \geq r/mp$. Indeed, suppose

$$P(\varepsilon_i t, y) = (y - \zeta_1) \cdots (y - \zeta_m),$$

with $\omega(\zeta_j - \xi) < r/mp$ for all j. Since

$$h(\varepsilon_i t, \xi) = P(\varepsilon_i t, \xi) - Q(\varepsilon_i t, \xi) = P(\varepsilon_i t, \xi) = (\xi - \zeta_1) \cdots (\xi - \zeta_m),$$

we get

$$r/p \leq \omega\big(h(\varepsilon_i t, \xi)\big) = \omega\big((\xi - \zeta_1) \cdots (\xi - \zeta_m)\big) < r/p,$$

a contradiction. Now, if ζ has some non-real coefficient, choosing r large, ξ would have the same non-real coefficient, which is not the case. Thus we conclude that

ζ defines a real half-branch $\sigma : x(t) = \varepsilon_i t^p, y(t) = f_i(t),\ t > 0$, contained in $\{f = 0\} \setminus \{0\} \subset \{\varphi < 0\}$. Hence

$$\varphi(\varepsilon_i t^p, f_i(t)) = a_i t^{s_i} + \cdots ,\ a_i < 0.$$

Finally, the condition $\omega(\zeta - \xi) \geq r/mp$ means $f_i \equiv g_i \mod t^{r/m}$, so that

$$\varphi(\varepsilon_i t^p, g_i(t)) \equiv \varphi(\varepsilon_i t^p, f_i(t)) \mod t^{r/m},$$

and for r large, $\varphi(\varepsilon_i t^p, g_i(t)) = a_i t^{s_i} + \cdots$, that is, $\gamma_i \subset \{\varphi < 0\}$. This completes the proof. □

Remarks 3.2. Let $X \subset \mathbb{R}^3$ be a germ $z^2 = F(x, y)$, with $F \in \mathbb{R}[x, y]$. Then the ring $\mathcal{O}(X)$ is a free $\mathbb{R}\{x, y\}$-module of rank 2: every analytic function germ on X can be written $f(x, y) + zg(x, y)$, with $f, g \in \mathbb{R}\{x, y\}$. Furthermore, psd's are given by ([Rz3]):

$$\mathcal{P}(X) = \{f + zg :\ f \in \mathcal{P}(F \geq 0), f^2 - Fg^2 \in \mathcal{P}(\mathbb{R}^2)\}.$$

For pd's we only have $\mathcal{P}^+(X) \supset \{f + zg :\ f \in \mathcal{P}^+(F \geq 0), f^2 - Fg^2 \in \mathcal{P}^+(\mathbb{R}^2)\}$, but as a converse we can prove the following:

If $f + zg \in \mathcal{P}(X)$ then $(f + (x^2 + y^2)^m)^2 - Fg^2 \in \mathcal{P}^+(\mathbb{R}^2)$ for m large enough.

Indeed, since $f + (x^2 + y^2)^m \pm zg \in \mathcal{P}^+(X)$, then

$$(f + (x^2 + y^2)^m)^2 - Fg^2 \in \mathcal{P}(\mathbb{R}^2) \cap \mathcal{P}^+(X)$$

and therefore $(f + (x^2 + y^2)^m)^2 - Fg^2 \in \mathcal{P}(\mathbb{R}^2) \cap \mathcal{P}^+(F \geq 0)$. Thus,

$$\{(f + (x^2 + y^2)^m)^2 - Fg^2 = 0\} \subset \{F < 0\} \cup \{0\},$$

and we obtain

$$\{(f + (x^2 + y^2)^m)^2 - Fg^2 = 0\} = \{f + (x^2 + y^2)^m = 0, g = 0\}.$$

But, $\{f + (x^2 + y^2)^m = 0\} \cap \{g = 0\} \neq \{0\}$ for finitely many m's, and so $(f + (x^2 + y^2)^m)^2 - Fg^2 \in \mathcal{P}^+(\mathbb{R}^2)$ for m large.

Now we can prove the polynomial reduction:

Theorem 3.3. *Let $X \subset \mathbb{R}^3$ be an analytic set germ of equation $z^2 - F(x, y) = 0$ with $F \in \mathbb{R}[x, y]$. If $\mathcal{P}(S_X) \subset \Sigma_2(X)$, then $\mathcal{P}(X) = \Sigma_2(X)$.*

Proof. Suppose $\mathcal{P}(S_X) \subset \Sigma_2(X)$ and let $h = f + zg \in \mathcal{P}(X)$. We first prove that for every $m \geq 1$ there exist $P, Q \in \mathbb{R}[x, y]$ such that $h_m = P + zQ \in \mathcal{P}(S_X)$ and $\omega(h - h_m) \geq m$.

We start with the function $\varphi_m = f + (x^2 + y^2)^m + zg \in \mathcal{P}^+(X)$ which has the following properties:

- $f + zg - \varphi_m = (x^2 + y^2)^m \in (x, y)^{2m}$
- $\varphi_m(x, y, -z) \in \mathcal{P}^+(X)$
- $f + (x^2 + y^2)^m \in \mathcal{P}^+(F \geq 0)$
- $(f + (x^2 + y^2)^m)^2 - Fg^2 \in \mathcal{P}^+(\mathbb{R}^2)$ for m large (by 3.2).

Now, by the density lemma (3.1), there exists $r > 2m$ such that:

$$f + (x^2 + y^2)^m + (x, y)^r \subset \mathcal{P}^+(F \geq 0) \qquad \text{(i)}$$

$$(f + (x^2 + y^2)^m)^2 - Fg^2 + (x, y)^r \subset \mathcal{P}^+(\mathbb{R}^2) \qquad \text{(ii)}$$

We consider the jet of degree $r - 1$ of φ_m,

$$\varphi_m^{r-1} = f_{r-1} + (x^2 + y^2)^m + zg_{r-2}$$

where f_{r-1}, g_{r-2} are the jets of degrees $r - 1, r - 2$ of f, g respectively, and it holds $\varphi_m^{r-1} \in \mathcal{P}^+(X)$.

Indeed, we only have to check that

$$f_{r-1} + (x^2 + y^2)^m \in \mathcal{P}^+(F \geq 0), \; (f_{r-1} + (x^2 + y^2)^m)^2 - Fg_{r-2}^2 \in \mathcal{P}^+(\mathbb{R}^2)$$

which follows from (i), (ii), because:

$$f + (x^2 + y^2)^m - \left(f_{r-1} + (x^2 + y^2)^m\right) = f - f_{r-1} \in (x, y)^r \qquad \text{(iii)}$$

$$\begin{aligned}&(f + (x^2 + y^2)^m)^2 - Fg^2 - \left((f_{r-1} + (x^2 + y^2)^m)^2 - Fg_{r-2}^2\right) \\ &= (f^2 - Fg^2) - (f_{r-1}^2 - Fg_{r-2}^2) + 2(x^2 + y^2)^m(f - f_{r-1}) \in (x, y)^r \qquad \text{(iv)}\end{aligned}$$

Since $\varphi_m^{r-1} \in \mathcal{P}^+(X)$, there is $\varepsilon > 0$ such that $\varphi_m^{r-1}(x, y, z) > 0$ for $(x, y, z) \in S_X$, $0 < \|(x, y, z)\| < \varepsilon$. Now, if $z^2 = F(x, y)$ and $\|(x, y, z)\| \geq \varepsilon$:

$$\begin{aligned}|\varphi_m^{r-1}(x, y, z)| &= \Big|\sum_{i+j+k\leq r-1} a_{ijk}x^i y^j z^k\Big| \leq \sum_{i+j+k\leq r-1} |a_{ijk}||x|^i|y|^j|z|^k \\ &\leq \sum_{i+j+k\leq r-1} |a_{ijk}|\|(x, y, z)\|^{i+j+k} \leq \sum_{i+j+k\leq r-1} \frac{|a_{ijk}|}{\varepsilon^{2r-(i+j+k)}}\|(x, y, z)\|^{2r} \\ &\leq M\|(x, y, z)\|^{2r} = M(x^2 + y^2 + z^2)^r\end{aligned}$$

for some $M > 0$.

Therefore, $h_m = \varphi_m^{r-1} + M(x^2 + y^2 + F)^r \in \mathcal{P}(S_X)$ and

$$h_m - (f + zg) = M(x^2 + y^2 + F)^r + \varphi_m^{r-1} - (f + zg) \in (x, y)^{2m}.$$

Thus, $h_m \in \mathcal{P}(S_X)$ and, since $\mathcal{P}(S_X) \subset \Sigma_2(X)$, there are $\alpha_m, \beta_m, q_m \in \mathbb{R}\{x, y, z\}$ such that $h_m = \alpha_m^2 + \beta_m^2 + (z^2 - F)q_m$. But $h \equiv h_m \mod (x, y, z)^m$, hence

$$h \equiv \alpha_m^2 + \beta_m^2 + (z^2 - F)q_m \mod (x, y, z)^m.$$

Consequently, by M. Artin's Approximation Theorem ([Ku et al], [M.Ar]), we deduce that the equation

$$h = \alpha^2 + \beta^2 + (z^2 - F)q$$

has a solution $\alpha, \beta, q \in \mathbb{R}\{x, y, z\}$, and $h \in \Sigma_2(X)$. □

We close this section with a result which will be useful later.

Lemma 3.4. *Let $P \in \mathbb{R}\{x, y\}[z]$ be a Weierstrass polynomial of degree m and $F, A_1, A_2, \dots, A_k \in \mathbb{R}\{x, y\}[z]$ polynomials of degree $\leq m-1$ such that $F = A_1^2 + A_2^2 + \cdots + A_k^2 + QP$ where $Q \in \mathbb{R}\{x, y, z\}$. Then $Q \in \mathbb{R}\{x, y\}[z]$ and $\partial_z(Q) \leq m-2$.*

Proof. We divide the polynomial $A_1^2 + A_2^2 + \cdots + A_k^2$ of degree $\leq 2m-2$ by P in $\mathbb{R}\{x, y\}[z]$ to obtain

$$A_1^2 + A_2^2 + \cdots + A_k^2 = Q_1 P + R_1$$

where $Q_1, R_1 \in \mathbb{R}\{x, y\}[z]$ and $\partial_z(Q_1) \leq m-2, \partial_z(R_1) \leq m-1$. Thus $F = (Q_1 + Q)P + R_1$, which is a Weierstrass division. But, since $\partial_y(F) < \partial_y(P)$ this division must be trivial, and $Q = -Q_1 \in \mathbb{R}\{x, y\}[z]$, $\partial_z(Q) \leq m-2$. □

4 The blowings-up of Brieskorn's singularity

Here we will prove $\mathcal{P} = \Sigma_2$ for $X : z^2 - x^3 - xy^3 = 0$ and $Y : z^2 - x^3 - y^4 = 0$. As we have shown in the preceding section (3.3), it suffices to see that:

Theorem 4.1. $\mathcal{P}(S_X) \subset \Sigma_2(X)$ *and* $\mathcal{P}(S_Y) \subset \Sigma_2(Y)$

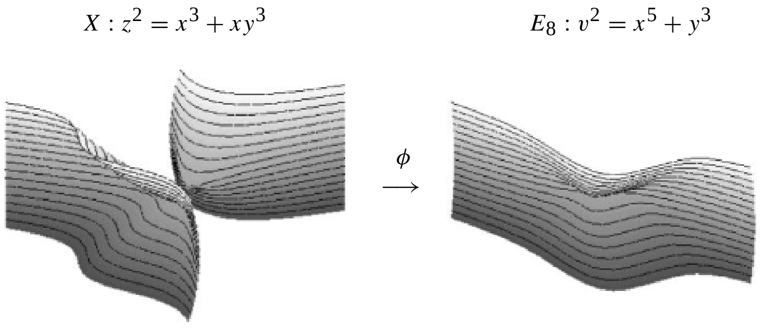
We treat X first.

Proof. Consider the biregular map

$$\begin{aligned} \phi : \{z^2 - x^3 - xy^3 = 0\} \setminus \{x = 0\} &\longrightarrow \{v^2 - x^5 - u^3 = 0\} \setminus \{x = 0\} \\ (x, y, z) &\longmapsto (x, xy, xz) = (x, u, v) \end{aligned}$$

with inverse

$$\begin{aligned} \psi : \{v^2 - x^5 - u^3 = 0\} \setminus \{x = 0\} &\longrightarrow \{z^2 - x^3 - xy^3 = 0\} \setminus \{x = 0\} \\ (x, u, v) &\longmapsto \left(x, \frac{u}{x}, \frac{v}{x}\right) \end{aligned}$$



Now take $T = P + zQ \in \mathcal{P}(S_X)$ and consider

$$T \circ \psi = P\Big(x, \frac{u}{x}\Big) + \frac{v}{x} Q\Big(x, \frac{u}{x}\Big) = \frac{F(x,u) + vG(x,u)}{x^{2r}}$$

where $F + vG \in \mathbb{R}[x, u, v]$, $r \geq 0$. Clearly, $F + vG \geq 0$ on $v^2 - x^5 - u^3 = 0$, $x \neq 0$, and by continuity, on $v^2 - x^5 - u^3 = 0$. Since this is Brieskorn's singularity, for which $\mathcal{P} = \Sigma_2$, there exist $\alpha, \beta, q \in \mathbb{R}\{x, u, v\}$ such that

$$x^{2r}(P + zQ) \circ \psi = F + vG = \alpha^2 + \beta^2 + q(v^2 - x^5 - u^3),$$

and so, in $\mathbb{R}\{x, y, z\}$

$$\begin{aligned} x^{2r}(P + zQ)(x, y, z) = \alpha^2(x, xy, xz) + \beta^2(x, xy, xz) \\ + q(x, xy, xz)x^2(z^2 - x^3 - xy^3) \end{aligned}$$

We divide $\alpha(x, xy, xz)$, $\beta(x, xy, xz)$ and $q(x, xy, xz)$ by $z^2 - x^3 - xy^3$ and apply 3.4 to obtain

$$x^{2r}(P + zQ) = (\alpha_0 + z\alpha_1)^2 + (\beta_0 + z\beta_1)^2 - (z^2 - x^3 - xy^3)q_0 \qquad \text{(i)}$$

where $\alpha_i, \beta_i, q_0 \in \mathbb{R}\{x, y\}$. The final step is to get rid of the denominator x^{2r}, which will be done, up to iteration, if $x^2|q_0$ and $x|\alpha_i, \beta_i$. To show this, comparing coefficients in (i) we get:

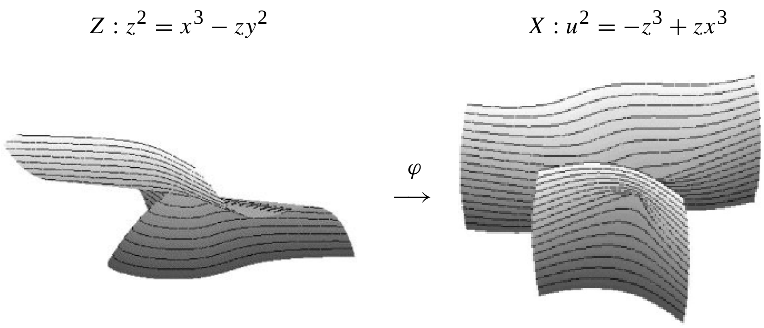
(0) $x^{2r}P = \alpha_0^2 + \beta_0^2 + q_0(x^3 + xy^3)$
(1) $x^{2r}Q = 2(\alpha_0\alpha_1 + \beta_0\beta_1)$
(2) $q_0 = \alpha_1^2 + \beta_1^2$

Now from (0) we obtain $x|\alpha_0^2 + \beta_0^2$, hence $x|\alpha_0, \beta_0$ and $x|q_0$. Finally, from (2), we deduce that $x|\alpha_1^2 + \beta_1^2$ and so $x|\alpha_1, \beta_1$ and $x^2|q_0$. □

Once X is solved, we treat Y similarly.

Proof. We consider the analytic germ $Z : z^2 + zy^2 - x^3 = 0$ and the invertible polynomial map $\varphi(x, y, z) = (x, \sqrt{2}y, z - y^2)$. Since $\varphi(S_Y) = S_Z$ and $\varphi(Y) = Z$, our problem translates into showing that $\mathcal{P}(S_Z) \subset \Sigma_2(Z)$. To do that we consider the biregular map:

$$\begin{aligned} \phi : \{z^2 + zy^2 - x^3 = 0\} \setminus \{z = 0\} &\longrightarrow \{u^2 + z^3 - zx^3 = 0\} \setminus \{z = 0\} \\ (x, y, z) &\longmapsto (x, z, zy) = (x, z, u) \end{aligned}$$



with inverse

$$\begin{aligned} \psi : \{u^2 + z^3 - zx^3 = 0\} \setminus \{x = 0\} &\longrightarrow \{z^2 + zy^2 - x^3 = 0\} \setminus \{x = 0\} \\ (x, z, u) &\longmapsto \left(x, \frac{u}{z}, z\right) \end{aligned}$$

Now take $P \in \mathcal{P}(S_Z)$, and consider

$$P \circ \psi = P\left(x, \frac{u}{z}, z\right) = \frac{F(x, u, z)}{z^{2r}}$$

where $F(x, z, u) \in \mathbb{R}[x, z, u], r \geq 0$. Since $F \geq 0$ on $u^2 + z^3 - zx^3 = 0, x \neq 0$, by continuity $F \geq 0$ on $u^2 + z^3 - zx^3 = 0$. But this surface is analytically equivalent to X, already discussed, hence there exist $\alpha, \beta, q \in \mathbb{R}\{x, z, u\}$ such that

$$z^{2r} P \circ \psi = F = \alpha^2 + \beta^2 + q(u^2 + z^3 - zx^3)$$

and so, we have in $\mathbb{R}\{x, y, z\}$

$$z^{2r} P(x, y, z) = \alpha^2(x, z, zy) + \beta^2(x, z, zy) + q(x, z, zy)z(zy^2 + z^2 - x^3)$$

We divide $\alpha(x, z, zy)$, $\beta(x, z, zy)$ and $q(x, z, zy)$ by $x^3 - z^2 - zy^2$ and apply 3.4 to get

$$\begin{aligned} z^{2r}(p_0 + p_1 x + p_2 x^2) &= (\alpha_0 + \alpha_1 x + \alpha_2 x^2)^2 \\ &\quad + (\beta_0 + \beta_1 x + \beta_2 x^2)^2 - (q_0 + q_1 x)(x^3 - z^2 - zy^2) \quad \text{(i)} \end{aligned}$$

where $\alpha_i, \beta_i, p_i, q_i \in \mathbb{R}\{y,z\}$. We eliminate the denominator z^{2r} by seeing that $z^2|q_i$ and $z|\alpha_i, \beta_i$. Comparing coefficients in (i) we obtain:

(0) $z^{2r}p_0 = \alpha_0^2 + \beta_0^2 + q_0(z^2 + zy^2)$
(1) $z^{2r}p_1 = 2(\alpha_0\alpha_1 + \beta_0\beta_1) + q_1(z^2 + zy^2)$
(2) $z^{2r}p_2 = \alpha_1^2 + \beta_1^2 + 2(\alpha_0\alpha_2 + \beta_0\beta_2)$
(3) $q_0 = 2(\alpha_1\alpha_2 + \beta_1\beta_2)$
(4) $q_1 = \alpha_2^2 + \beta_2^2$

Now, from (0) we deduce $z|\alpha_0^2+\beta_0^2$, and therefore $z|\alpha_0, \beta_0$, so that $z|q_0$. Applying this to (2) we see that $z|\alpha_1^2 + \beta_1^2$, hence $z|\alpha_1, \beta_1$ and, thus, in view of (1) $z|q_1$. From (4) we get $z|\alpha_2^2+\beta_2^2$, hence $z|\alpha_2, \beta_2$ y $z^2|q_1$. Finally, from (3) we conclude $z^2|q_0$. □

Thus, we have completed the proof of 4.1.

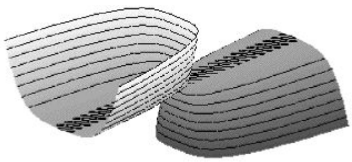
5 The two planes and its deformations

We show here that the germs $X_k : z^2 - x^2 - y^k = 0$, $k \geq 2$, have all the property $\mathcal{P} = \Sigma_2$. In addition, we will deduce the same for the two planes $z^2 - x^2 = 0$. As we know, by polynomial reduction, it is enough to prove:

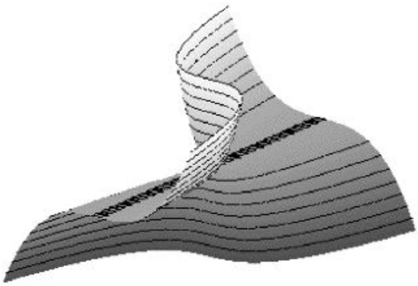
Theorem 5.1. $\mathcal{P}(S_{X_k}) \subset \Sigma_2(X_k)$.

Proof. Consider the germ $Y_k : y^k - uv = 0$ and the linear isomorfism $\varphi(x,y,z) = (z-x, y, z+x) = (u,y,v)$.

$Y_{2\ell} : y^{2\ell} = uv$



$Y_{2\ell+1} : y^{2\ell+1} = uv$



Since $\varphi(S_{X_k}) = S_{Y_k}$ and $\varphi(X_k) = Y_k$, we will see that $\mathcal{P}(S_{Y_k}) \subset \Sigma_2(Y_k)$. For this we consider the biregular map:

$$\phi : \mathbb{R}^2 \setminus \{v=0\} \longrightarrow S_{Y_k} \setminus \{v=0\}$$
$$(y,v) \longmapsto \left(\frac{y^k}{v}, y, v\right) = (u,y,v)$$

Now, take $P \in \mathcal{P}(S_{Y_k})$ and

$$P \circ \phi(y, v) = P\Big(\frac{y^k}{v}, y, v\Big) = \frac{Q(y, v)}{v^{2r}}$$

where $r \geq 0$, and $Q(y, v) \in \mathbb{R}[y, v]$ is ≥ 0 on $\mathbb{R}^2 \setminus \{v = 0\}$, hence on $\mathbb{R}^2$. Since $\mathcal{P}(\mathbb{R}^2_o) = \Sigma_2(\mathbb{R}^2_o)$, there are $\alpha, \beta \in \mathbb{R}\{y, v\}$ such that

$$v^{2r} P \circ \phi = Q = \alpha^2 + \beta^2$$

and in $\mathbb{R}\{u, y, v\}$ we obtain

$$v^{2r} P(u, y, v) = \alpha^2(y, v) + \beta^2(y, v) + (y^k - uv)q(u, y, v).$$

We divide P, α, β by $y^k - uv$ and apply 3.4 to get

$$v^{2r}(p_0 + p_1 y + \cdots + p_{k-1} y^{k-1}) = (\alpha_0 + \alpha_1 y + \cdots + \alpha_{k-1} y^{k-1})^2 + (\beta_0 + \beta_1 y + \cdots + \beta_{k-1} y^{k-1})^2 - (y^k - uv)(q_0 + q_1 y + \cdots + q_{k-2} y^{k-2}) \tag{i}$$

where $\alpha_i, \beta_i, p_i, q_i \in \mathbb{R}\{u, v\}$. Again, we end by showing that $v^2 | q_0, \ldots, q_{k-2}$, $v | \alpha_i, \beta_i$ for $0 \leq i \leq k-1$. Comparing coefficients in (i) we find:

$$\begin{array}{ll}
(1) & v^{2r} p_0 - uvq_0 = \alpha_0^2 + \beta_0^2 \\
(2) & v^{2r} p_1 - uvq_1 = 2\alpha_0\alpha_1 + 2\beta_0\beta_1 \\
 & \vdots \\
(\ell) & v^{2r} p_\ell - uvq_\ell = \sum_{i+j=\ell} (\alpha_i\alpha_j + \beta_i\beta_j) \\
 & \vdots \\
(k-1) & v^{2r} p_{k-1} - uvq_{k-1} = \sum_{i+j=k-1} (\alpha_i\alpha_j + \beta_i\beta_j) \quad (\text{where } q_{k-1} = 0) \\
(k) & q_0 = \sum_{i+j=k} (\alpha_i\alpha_j + \beta_i\beta_j) \\
 & \vdots \\
(k+\ell) & q_l = \sum_{i+j=k+\ell} (\alpha_i\alpha_j + \beta_i\beta_j) \\
 & \vdots \\
(2k-2) & q_{k-2} = \alpha_{k-1}^2 + \beta_{k-1}^2
\end{array}$$

Now, we see that $v | \alpha_\ell, \beta_\ell, q_\ell$ for $\ell = 0, \ldots, k-1$. For $\ell = 0$ we have $v^{2r} p_0 - uvq_0 = \alpha_0^2 + \beta_0^2$, hence $v | \alpha_0^2 + \beta_0^2$, and we deduce $v | \alpha_0, \beta_0$ and $v | q_0$. Next, let $\ell < k$ and suppose $v | \alpha_0, \beta_0, \ldots, \alpha_{\ell-1}, \beta_{\ell-1}, q_0, \ldots, q_{\ell-1}$. If $2\ell \leq k-1$ then:

$$v^{2r} p_{2\ell} - uvq_{2\ell} = \sum_{i+j=2\ell, i \neq j} (\alpha_i\alpha_j + \beta_i\beta_j) + \alpha_\ell^2 + \beta_\ell^2,$$

and since $v | \alpha_0, \beta_0, \ldots, \alpha_{\ell-1}, \beta_{\ell-1}$, we get $v | \alpha_\ell, \beta_\ell$. If $2\ell > k-1$ then:

$$q_{2\ell-k} = \sum_{i+j=2\ell, i \neq j} (\alpha_i\alpha_j + \beta_i\beta_j) + \alpha_\ell^2 + \beta_\ell^2$$

But $\ell \leq k-1$ implies $2\ell-k \leq \ell-1$ and by induction hypothesis $v|q_{2\ell-k}$. Since, also by induction hypothesis, $v|\alpha_0, \beta_0, \dots, \alpha_{\ell-1}, \beta_{\ell-1}$ we deduce again $v|\alpha_l, \beta_l$. Finally, from (ℓ) we see that $v|q_\ell$, which completes the induction. Once we know this, from $(k), \dots, (2k-2)$ we conclude that $v^2|q_\ell$ if $0 \leq \ell \leq k-2$. □

We finish the section with a limit argument that gives the property $\mathcal{P} = \Sigma_2$ for the two planes:

Corollary 5.2. $\mathcal{P}(z^2 - x^2 = 0) = \Sigma_2(z^2 - x^2 = 0)$

Proof. Let $f + zg \in \mathcal{P}(z^2 - x^2 = 0)$ and $m \geq 1$. By 3.2, $f \geq 0$ on $x^2 \geq 0$, that is, on $\mathbb{R}^2$, hence

$$f + (x^2 + y^2)^m \in \mathcal{P}^+(\mathbb{R}^2).$$

Again by 3.2, $f^2 - x^2 g^2 \geq 0$ on $\mathbb{R}^2$, so that

$$(f + (x^2 + y^2)^m)^2 - x^2 g^2 \in \mathcal{P}^+(\mathbb{R}^2).$$

Now, by the 3.1, for large $r \geq 2m$ we have $(f + (x^2 + y^2)^m)^2 - x^2 g^2 + (x, y)^r \subset \mathcal{P}^+(\mathbb{R}^2)$, and we consider the germ $X_{2r} : z^2 = x^2 + y^{2r}$, on which $f + (x^2 + y^2)^m + zg$ is ≥ 0. For, $f + (x^2 + y^2)^m \in \mathcal{P}^+(\mathbb{R}^2) = \mathcal{P}^+(x^2 + y^{2r} \geq 0)$, and

$$\begin{aligned}(f + (x^2 + y^2)^m)^2 - (x^2 + y^{2r})g^2 &= (f + (x^2 + y^2)^m)^2 - x^2 g^2 - y^{2r} g^2 \\ &\in (f + (x^2 + y^2)^m)^2 - x^2 g^2 + (x, y)^r \subset \mathcal{P}^+(\mathbb{R}^2).\end{aligned}$$

Then, since $\mathcal{P} = \Sigma_2$ holds for X_{2r}, there exist $\alpha_m, \beta_m, q_m \in \mathbb{R}\{x, y, z\}$ such that:

$$f + (x^2 + y^2)^m + zg = \alpha_m^2 + \beta_m^2 - (z^2 - x^2 - y^{2r})q_m,$$

and

$$f + zg \equiv \alpha_m^2 + \beta_m^2 - (z^2 - x^2)q_m \quad \mod (x, y)^{2m}.$$

This valid for every m, M.Artin's Approximation Theorem gives $\alpha, \beta, q \in \mathbb{R}\{x, y, z\}$ such that $f + zg = \alpha^2 + \beta^2 - (z^2 - x^2)q$. We are done. □

6 Whitney's umbrella and its deformations

In this section we will prove $\mathcal{P} = \Sigma_2$ for

$$X_k : z^2 - x^2 y + y^{2k+1} = 0,\ k \geq 1, \quad \text{and} \quad Y_k : z^2 - x^2 y - y^{2k} = 0,\ k \geq 2,$$

and deduce in the limit $\mathcal{P} = \Sigma_2$ for Whitney's umbrella.

As usual, we prove:

Theorem 6.1. *$\mathcal{P}(S_{X_k}) \subset \Sigma_2(X_k)$ and $\mathcal{P}(S_{Y_k}) \subset \Sigma_2(Y_k)$.*

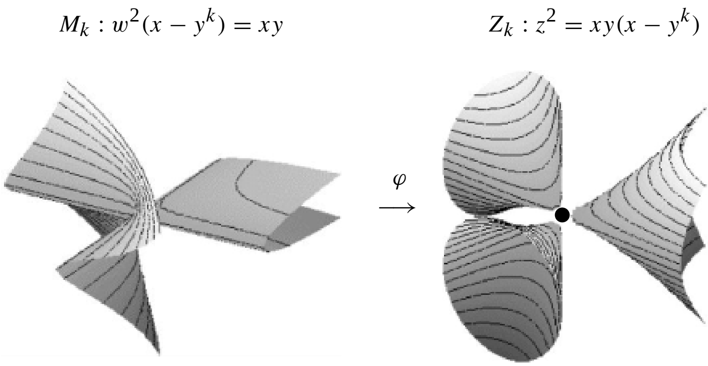
First we treat the odd case.

Proof. Let $Z_k : \; z^2 - xy(x - y^k) = 0$ and consider the invertible polynomial map $\varphi(x, y, z) = (x + y^k, \sqrt[k]{2}y, \sqrt[2k]{2}z)$. Since $\varphi(S_{Z_k}) = S_{X_k}$ and $\varphi(Z_k) = X_k$, the condition $\mathcal{P}(S_{X_k}) \subset \Sigma_2(X_k)$ is equivalent to $\mathcal{P}(S_{Z_k}) \subset \Sigma_2(Z_k)$, and we will see the latter. Consider the algebraic surface $M_k : \; w^2(x - y^k) - xy = 0$ and the biregular map:

$$\begin{aligned} \phi : M_k \setminus \{x - y^k = 0\} &\longrightarrow S_{Z_k} \setminus \{x - y^k = 0\} \\ (x, y, w) &\longmapsto (x, y, w(x - y^k)) = (x, y, z) \end{aligned}$$

with inverse:

$$\begin{aligned} \psi : S_{Z_k} \setminus \{x - y^k = 0\} &\longrightarrow M_k \setminus \{x - y^k = 0\} \\ (x, y, z) &\longmapsto \Big(x, y, \frac{z}{x - y^k}\Big) \end{aligned}$$



Now take $T = P + zQ \in \mathcal{P}(S_{Z_k})$; the polynomial $T \circ \phi = P(x, y) + w(x - y^k)Q(x, y)$ is ≥ 0 en M_k. Consider the biregular map:

$$\begin{aligned} \chi : \mathbb{R}^2 \setminus \{w^2 - y = 0\} &\longrightarrow M_k \setminus \{y = w = 0\} \\ (y, w) &\longmapsto \Big(\frac{w^2 y^k}{w^2 - y}, y, w\Big) \end{aligned}$$

and

$$T \circ \phi \circ \chi = P\Big(\frac{w^2 y^k}{w^2 - y}, y\Big) + w\Big(\frac{w^2 y^k}{w^2 - y} - y^k\Big) Q\Big(\frac{w^2 y^k}{w^2 - y}, y\Big) = \frac{F(y, w)}{(w^2 - y)^{2r}}$$

where $2r \geq \partial_x(T \circ \phi)$, and $F_k(y, w) \in \mathbb{R}[y, w]$ is ≥ 0 on $w^2 - y \neq 0$, hence on $\mathbb{R}^2$.

Here we need the fact that a psd polynomial $H \in \mathbb{R}[x, y]$ is a sum of two squares in $\mathbb{R}\{x\}[y]$. A proof of this can be the following. Such an H is positive in every total ordering of $\mathbb{R}(\{x\})[y]$. Indeed, otherwise there is a homomorphism from $\mathbb{R}(\{x\})[y]$ into the field of real Puiseux series, say $x \mapsto \varepsilon t$, $y \mapsto h(t^{1/p})$, $\varepsilon = \pm 1$, such that $H(\varepsilon t, h(t^{1/p})) > 0$ (Lang's homomorphism theorem) and specializing at $t > 0$ small enough we get a point at which H is negative. Hence H is an sos in $\mathbb{R}(\{x\})(y)$, which has Pythagoras number 2 ([ChDLR]), hence a sum of 2 squares. But then it is a sum of two squares in $\mathbb{R}\{x\}[y]$ (see for instance [ChLRR]).

Consequently, applying this to $H = F$ we find $F_1, F_2 \in \mathbb{R}\{y\}[w]$, such that

$$(w^2 - y)^{2r}(T \circ \phi \circ \chi) = F_1^2 + F_2^2,$$

and therefore

$$(w^2 - y)^{2r} T \circ \phi = F_1^2 + F_2^2 + \big((w^2 - y)x - w^2 y^k\big) q(x, y, w)$$

where $q \in \mathbb{R}[x, y, w]$ (it comes from division in $\mathbb{R}[x, y, w]$). Now, we compose with $\phi^{-1} = \psi$:

$$\left(\left(\frac{z}{x - y^k}\right)^2 - y\right)^{2r} T = F_1^2\left(y, \frac{z}{x - y^k}\right) + F_2^2\left(y, \frac{z}{x - y^k}\right) + \left(\frac{z^2}{x - y^k} - xy\right) q\left(x, y, \frac{z}{x - y^k}\right),$$

and multiply by a large power of $(x - y^k)^2$ to get

$$(x - y^k)^{2m}\big(z^2 - y(x - y^k)^2\big)^{2r} T = \alpha^2 + \beta^2 + (z^2 - xy(x - y^k))q'$$

where $\alpha, \beta, q' \in \mathbb{R}\{x, y, z\}$. Dividing $\alpha, \beta, \big(z^2 - y(x - y^k)^2\big)^{2r} T$ by $z^2 - xy(x - y^k)$ and applying 3.4, we obtain

$$\begin{aligned}(x - y^k)^{2m}((x - y^k)y^{k+1})^{2r}(P + zQ) &= (x - y^k)^{2m+2r} y^{2r(k+1)}(P + zQ)\\ &= (A_0 + zA_1)^2 + (B_0 + zB_1)^2 - (z^2 - xy(x - y^k))p_0 \quad \text{(i)}\end{aligned}$$

for some $A_i, B_i, p_0 \in \mathbb{R}\{x, y\}$. Next, we multiply (i) by $(x - y^k)^{2rk} y^{2m}$ and get

$$\begin{aligned}((x - y^k)y)^{2m+2r(k+1)}(P + zQ) &= ((x - y^k)y)^{2n}(P + zQ)\\ &= (\alpha_0 + z\alpha_1)^2 + (\beta_0 + z\beta_1)^2 - (z^2 - xy(x - y^k))q_0 \quad \text{(ii)}\end{aligned}$$

where $\alpha_i, \beta_i, q_0 \in \mathbb{R}\{x, y\}$ and $n = m + r(k + 1)$. Once again it remains to show that $(x - y^k)y | \alpha_i, \beta_i$ and $((x - y^k)y)^2 | q_0$. But, comparing coefficients in (ii) we see:

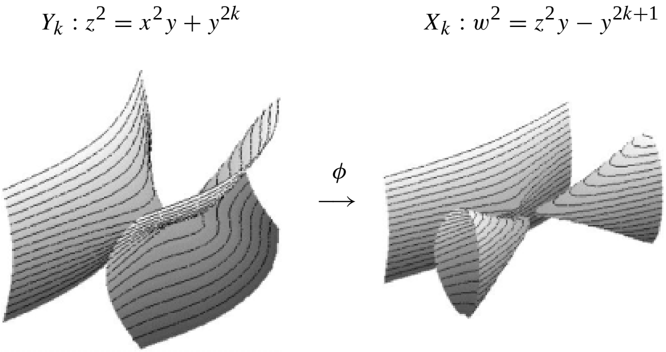
(0) $((x - y^k)y)^{2n} P = \alpha_0^2 + \beta_0^2 + xy(x - y^k)q_0$
(1) $((x - y^k)y)^{2n} Q = 2(\alpha_0\alpha_1 + \beta_0\beta_1)$
(2) $q_0 = \alpha_1^2 + \beta_1^2$

From (0) we get $(x - y^k)y|\alpha_0^2 + \beta_0^2$, hence $(x - y^k)y|\alpha_0, \beta_0$ and $(x - y^k)y|q_0$. Finally, by (2), we have $(x - y^k)y|\alpha_1^2 + \beta_1^2$ and therefore $(x - y^k)y|\alpha_1, \beta_1$ and $((x - y^k)y)^2|q_0$. We are done. $\square$

Now we solve the even case:

Proof. Consider the birregular map:

$$\begin{aligned}\phi : \{z^2 - x^2y - y^{2k}\} \setminus \{y = 0\} &\longrightarrow \{w^2 - z^2y + y^{2k+1} = 0\} \setminus \{y = 0\}\\ (x, y, z) &\longmapsto (z, y, xy) = (z, y, w)\end{aligned}$$



The inverse of ϕ is:

$$\begin{aligned}\psi : \{w^2 - z^2y + y^{2k+1} = 0\} \setminus \{y = 0\} &\longrightarrow \{z^2 - x^2y - y^{2k}\} \setminus \{y = 0\}\\ (z, y, w) &\longmapsto \left(\frac{w}{y}, y, z\right)\end{aligned}$$

Let $T = P + zQ \in \mathcal{P}(S_{Y_k})$, and consider

$$T \circ \psi = P\left(\frac{w}{y}, y\right) + zQ\left(\frac{w}{y}, y\right) = \frac{F(w, y) + zG(w, y)}{y^{2r}}$$

where $r \geq 0$, and $F + zG \in \mathbb{R}[x, y, w]$ is ≥ 0 on $w^2 - z^2y + y^{2k+1} = 0$, $y \neq 0$, hence on $w^2 - z^2y + y^{2k+1} = 0$. Since this is X_k, there exist $\alpha, \beta, q \in \mathbb{R}\{x, u, v\}$ such that

$$y^{2r}(P + zQ) \circ \psi = F + zG = \alpha^2 + \beta^2 + q(w^2 - z^2y + y^{2k+1})$$

and so we have in $\mathbb{R}\{x, y, z\}$

$$y^{2r}(P + zQ) = \alpha^2(z, y, xy) + \beta^2(z, y, xy) + q(z, y, xy)((xy)^2 - z^2y + y^{2k+1}).$$

We divide $\alpha(z, y, xy)$, $\beta(z, y, xy)$ and $q(z, y, xy)$ by $z^2 - x^2y - y^{2k}$ and apply 3.4 to get

$$y^{2r}(P + zQ) = (\alpha_0 + z\alpha_1)^2 + (\beta_0 + z\beta_1)^2 - (z^2 - x^2y - y^{2k})q_0 \qquad \text{(i)}$$

where $\alpha_i, \beta_i, q_0 \in \mathbb{R}\{x, y\}$. Typically, we end by seeing that $y^2|q_0, y|\alpha_i, \beta_i$. Comparing coefficients in (i) gives:

(0) $y^{2r}P = \alpha_0^2 + \beta_0^2 + q_0(x^2y + y^{2k})$
(1) $y^{2r}Q = 2(\alpha_0\alpha_1 + \beta_0\beta_1)$
(2) $q_0 = \alpha_1^2 + \beta_1^2$

Now from (0) we get $y|\alpha_0^2 + \beta_0^2$, hence $y|\alpha_0, \beta_0$ and $y|q_0$. By (2), $y|\alpha_1^2 + \beta_1^2$, so that $y|\alpha_1, \beta_1$ and $y^2|q_0$. The proof of the even case is thus complete. □

We conclude with the proof that $\mathcal{P} = \Sigma_2$ for Whitney's umbrella.

Corollary 6.2. $\mathcal{P}(z^2 - x^2y = 0) = \Sigma_2(z^2 - x^2y = 0)$

Proof. Let $f + zg \in \mathcal{P}(z^2 - x^2y = 0)$. By 3.2, for large m we have:

$$f + (x^2 + y^2)^m \in \mathcal{P}^+(x^2y \geq 0) = \mathcal{P}^+(\{y \geq 0\} \cup \{x = 0\}) \qquad \text{(i)}$$
$$(f + (x^2 + y^2)^m)^2 - x^2yg^2 \in \mathcal{P}^+(\mathbb{R}^2) \qquad \text{(ii)}$$

By 3.1, there exists $r \geq 1$ such that

$$(f + (x^2 + y^2)^m)^2 - x^2yg^2 + (x, y)^r \subset \mathcal{P}^+(\mathbb{R}^2). \qquad \text{(iii)}$$

We consider the germ $Y_k : z^2 = x^2y + y^{2k}$ with $k \geq r, 2(m + 1)$, and $f + (x^2 + y^2)^m + zg$ which is ≥ 0 on Y_k.

Indeed, by (iii)

$$(f + (x^2 + y^2)^m)^2 - (x^2y + y^{2k})g^2 \in \mathcal{P}^+(\mathbb{R}^2),$$

and by (i)

$$f + (x^2 + y^2)^m \in \mathcal{P}^+(\{y \geq 0\}).$$

Again by (i),

$$f(0, y) + y^{2m} = y^{2s}u(y),$$

where $u \in \mathbb{R}\{y\}$, $u(0) > 0$ and $s \leq m$. Therefore,

$$f + (x^2 + y^2)^m = y^{2s}u(y) + xh(x, y) \geq y^{2s}u(y) - |x||h(x, y)| \geq y^{2s}u(y) - c|x|,$$

where $h \in \mathbb{R}\{x, y\}$ and $c = |h(0, 0)| + 1$. Now, if $x^2 + y^{2k-1} \leq 0$ we have

$$|x| \leq |y|^{k-1} \leq |y|^{2m+1} \leq |y|^{2s+1} = -y^{2s+1}$$

and then

$$f + (x^2 + y^2)^m \geq y^{2s}(u(y) + y) \geq 0.$$

All of this means that $f + (x^2 + y^2)^m + zg \in \mathcal{P}(Y_k) = \Sigma_2(Y_k)$, hence there exist $\alpha_m, \beta_m, q_m \in \mathbb{R}\{x, y, z\}$ such that:

$$f + (x^2 + y^2)^m + zg = \alpha_m^2 + \beta_m^2 - (z^2 - x^2 y - y^{2k})q_m$$

and so,

$$f + zg = \alpha_m^2 + \beta_m^2 - (z^2 - x^2 y)q_m \qquad \mathrm{mod}\ (x, y)^{2m}.$$

Since this holds for every m, M. Artin's Approximation Theorem gives $\alpha, \beta, q \in \mathbb{R}\{x, y, z\}$ such that $f + zg = \alpha^2 + \beta^2 - (z^2 - x^2 y)q$. $\square$

References

[AnBrRz] C. Andradas, L. Bröcker, J.M. Ruiz: Constructible sets in real geometry. Ergeb. Math. **33**. Berlin Heidelberg New York: Springer Verlag, 1996

[M.Ar] M. Artin: On the solution of analytic equations, Invent. Math. **5**, 227–291 (1968)

[BoCoRo] J. Bochnak, M. Coste, M.F. Roy: Real Algebraic Geometry, Ergeb. Math. **36** Berlin Heidelberg New York: Springer-Verlag, 1998

[Ch] A. Chenciner: Courbes algebriques planes, Publ. Math. Univ. Paris VII, 1979

[ChDLR] M.D. Choi, Z.D. Dai, T.Y.Lam, B. Reznick: The Pythagoras number of some affine algebras and local algebras, J. reine Angew. Math. **336**, 45-82 (1982)

[ChLRR] M.D. Choi, T.Y.Lam, B. Reznick, A. Rosenberg: Sums of squares in some integral domains, J. Algebra **65**, 234–256 (1980)

[FeRz] J.F. Fernando, J.M. Ruiz: Positive semidefinite germs on the cone, Pacific. J. Math. (to appear)

[Ku et al] H. Kurke, T. Mostowski, G. Pfister, D. Popescu, M. Roczen: Die Approximationseigenschaft lokaler Ringe, Lecture Notes in Math. **634**. Berlin: Springer-Verlag, 1978

[Rz1] J.M. Ruiz: A note on a separation problem, Arch. Math. **43**, 422–426 (1984)

[Rz2] J.M. Ruiz: On Hilbert's 17th problem and real nullstellensatz for global analytic functions, Math. Z. **190**, 447–459 (1985)

[Rz3] J.M. Ruiz: Sums of two squares in analytic rings, Math. Z. **230**, 317–328 (1999)

[Sch1] C. Scheiderer: Sums of squares of regular functions on real algebraic varieties, Trans. A.M.S. **352**(3), 1039–1069 (1999)

[Sch2] C. Scheiderer: On sums of squares in local rings, Preprint Univ. Duisburg 2000