Math. Z. 244, 725–752 (2003) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-003-0519-x

Analytic surface germs with minimal Pythagoras number

José F. Fernando[⋆]

Depto. Algebra, F. Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain (e-mail: josefer@mat.ucm.es)

Received: 2 July 2002 / Published online: 16 May 2003 –

Abstract. We determine all complete intersection surface germs whose Pythagoras number is 2, and find that they are all embedded in $\mathbb{R}^3$ and have the property that every positive semidefinite analytic function germ is a sum of squares of analytic function germs. In addition, we discuss completely these properties for *mixed* surface germs in $\mathbb{R}^3$. Finally, we find in higher embedding dimension three different families with these same properties.

Mathematical Subject Classification (2000): 11E25, 14P15.

1. Introduction

In the investigation of *sums of squares* in analytic surface germs, the property that every *positive semidefinite* function germ is a sum of squares (in short $\mathcal{P} = \Sigma$) has appeared closely connected to the minimal value of the *Pythagoras number* p. Here we will refer always to the *analytic* Pythagoras number, that is, the smallest integer $p \geq 1$ such that every sum of squares of *analytic* function germs is a sum of p squares of *analytic* function germs. This invariant is always finite for surface germs ([Fe1]) and infinite for germs of higher dimension ([Fe3]).

Back to our properties $\mathcal{P} = \Sigma$ and $p = 2$, they were first compared in [Rz2], where a small list of candidates for them was produced. Later, in [Fe2], we saw that in fact the list gave all *unmixed* surface germs in $\mathbb{R}^3$ with $\mathcal{P} = \Sigma$, and all had $p = 2$. In this paper we single out the invariant p, and look for surface germs with $p = 2$. Our main result is proved in Section 2:

Theorem 1.1. *The complete intersection germs of dimension* ≥ 2 *with* $p[X] = 2$ *are exactly the following*

[⋆] Partially supported by DGICYT, BFM2002-04797 and HPRN-CT-2001-00271

(i) $z^2 - x^3 - y^5 = 0$ *(Brieskorn's singularity).*
(ii) $z^2 - x^3 - xy^3 = 0$.
(iii) $z^2 - x^3 - y^4 = 0$.
(iv) $z^2 - x^2 = 0$ *(two transversal planes).*
(v) $z^2 - x^2 - y^2 = 0$ *(cone).*
(vi) $z^2 - x^2 - y^k = 0,\ k \geq 3$ *(deformations of two planes).*
(vii) $z^2 - x^2y = 0$ *(Whitney's umbrella).*
(viii) $z^2 - x^2y + y^3 = 0$.
(ix) $z^2 - x^2y - (-1)^k y^k = 0,\ k \geq 4$ *(deformations of Whitney's umbrella).*

Since we already know that all the germs of the list above have $p = 2$, the essential goal here is the converse, that is, *every complete intersection with* $p = 2$ *belongs to the list*. This theorem together with [Fe2] shows that the properties $\mathcal{P} = \Sigma$ and $p = 2$ are equivalent for *unmixed* surface germs in $\mathbb{R}^3$. Mixed surface germs in $\mathbb{R}^3$ are unions of surface germs with some irreducible components of dimension 1. These are exactly the surface germs in $\mathbb{R}^3$ which are not complete intersections, and very few of them have the properties under consideration. Namely, we prove

Theorem 1.2. *The mixed surface germs in* $\mathbb{R}^3$ *with* $p = 2$ *are either*

(a) the union of a plane and a transversal line, and then $\mathcal{P} = \Sigma$, *or*

(b) the union of a plane and a transversal singular planar curve, and then $\mathcal{P} \neq \Sigma$.

This requires an extremely careful analysis that we present in Section 3. With this results we close completely the case of surface germs in $\mathbb{R}^3$. Let us say here that although very predictable, as we naively stated at the *2001 Rennes International Congress of Real Analytic and Algebraic Geometry*, the actual proofs are far from easy. The main difficulty is due to the fact that sums of squares which are not sums of two squares are very rare (not generic) in the mixed surface cases that one is lead to analize.

The proofs of Theorem 1.1 (Section 2) and Theorem 1.2 (Section 3) run in the following way. First, we remark that for the germs satisfying the conditions of the statements the property $p = 2$ holds true. Hence, we only have to prove the converse. To do this we proceed in three steps:

(1) We obtain general order restrictions for a minimal system of generators of the ideal of a germ X which has $p = 2$. For an analytic surface germ X which satisfy these restrictions but not the conditions of the statement the analysis is more delicate.

(2) By means of *classification of singularities* we obtain simplified equation(s) of X. This together with some preliminary technical lemmas will be crucial to achieve our goal, because they simplify the computations involved.

(3) For the equation(s) of X obtained in the previous step we construct an analytic function germ G which is a sum of three but not two squares in the ring of analytic function germs of X. To prove that the chosen G is not a sum of two squares we always argue by way of contradiction. Unfortunately, heavy technical computations are needed in each case.

In higher embedding dimension there are many more possibilities. First of all, it is easy to produce *reducible* surface germs with $\mathcal{P} = \Sigma$ and $p = 2$: If $X, Y \subset \mathbb{R}^3$

have the properties, then $Z = (X \times \{0\}) \cup (\{0\} \times Y) \subset \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ has them too. Note that every non-unit $f \in \mathbb{R}\{x_1, x_2, x_3, y_1, y_2, y_3\} = \mathbb{R}\{x, y\}$ can be written over Z as $f(x, 0) + f(0, y)$ (all products $x_i y_j$'s vanish on Z), so that $f \in \mathcal{P}(Z)$ if and only if $f(x, 0) \in \mathcal{P}(X)$, $f(0, y) \in \mathcal{P}(Y)$. Therefore, if $f \in \mathcal{P}(Z)$, $f \equiv f(x, 0) + f(0, y) \equiv a_1^2 + a_2^2 + b_1^2 + b_2^2 \equiv (a_1 + b_1)^2 + (a_2 + b_2)^2$, and we conclude $\mathcal{P} = \Sigma$ and $p = 2$ for Z. Hence, we concentrate on *irreducible* germs and we study the following examples in Section 4.

Example 1.3. The *Veronese cones* $X_n \subset \mathbb{R}^{n+1}, n \geq 2$ (cones over the rational normal curve), which are the surface germs given by the equations

$$F_{ij} = x_i x_j - x_{i-1} x_{j+1} = 0, \quad 1 \leq i \leq j \leq n-1,$$

and whose complexifications are parametrized by $\gamma(z, w) = (z^n, z^{n-1}w, \dots, zw^{n-1}, w^n)$, (see [Ha]). It is easy to prove that X_n has multiplicity n and embedding dimension $n + 1$. For these surface germs $\mathcal{P} = \Sigma$ and $p = 2$, which we shortly denote by $\mathcal{P} = \Sigma_2$ (Th. 4.1). These X_n's are not complete intersections, but they are at least normal, hence Cohen-Macaulay (but not Gorenstein). In particular, $X_2 \subset \mathbb{R}^3$ is the usual cone $x_1^2 = x_0 x_2$, already settled in [FeRz],[Fe2].

Example 1.4. The *generalized Whitney umbrellas* $Y_n \subset \mathbb{R}^{n+1}$, $n \geq 2$, which are the analytic closures of the set germs parametrized by

$$\varphi_n : (s, t) \mapsto (s, st, \dots, st^{n-1}, t^n) = (x_0, x_1, \dots, x_{n-1}, x_n).$$

It is not difficult to check that the ideal of Y_n is generated by the polynomials

$$x_i x_j - x_0 x_\ell x_n^q : \quad i + j = qn + \ell \quad \text{and} \quad \begin{matrix} 1 \leq i \leq j \leq n-1 \\ 0 \leq \ell \leq n-1, \end{matrix}$$

and Y_n consists of the union of the image of φ_n and the x_n-axis. Again we find multiplicity n and embedding dimension $n+1$. These surface germs have also $\mathcal{P} = \Sigma_2$ (Th. 4.4). However, these Y_n's are not complete intersections. In fact, they are neither normal (x_1/x_0 is integral over $\mathcal{O}(Y_n)$) nor Gorenstein (by Stanley's Criterion, [Ei, 21.14], [St]). On the positive, they are Cohen-Macaulay: $\operatorname{depth}(Y_n) \leq \dim(Y_n) = 2$ and $\{x_0, x_n\}$ is a regular sequence. The first umbrella $Y_2 \subset \mathbb{R}^3$ is the classical Whitney umbrella $x_1^2 = x_0^2 x_2$, for which we already knew $\mathcal{P} = \Sigma_2$ ([Rz2]).

Example 1.5. A family of irreducible surface germs $Z_n \subset \mathbb{R}^{n+1}$, $n \geq 3$, parametrized by

$$\phi_n : (s, t) \mapsto (x_0, \dots, x_n) = (s, st, \dots, st^{n-2}, t^{n-1}, t^n),$$

with $p = 2$ and $\mathcal{P} \neq \Sigma$ (Th. 4.5), and also multiplicity n and embedding dimension $n + 1$. The surface germ Z_n is given by the equations

$$\begin{cases} x_k^{n-1} - x_0^{n-1} x_{n-1}^k & k = 1, \dots, n-2 \\ x_0 x_n^k - x_k x_{n-1}^k & k = 1, \dots, n-2 \\ x_n^{n-1} - x_{n-1}^n. \end{cases}$$

These Z_n's cannot be complete intersections by Th. 1.1, but in fact, it is not difficult to verify that they are not even Cohen-Macaulay and, therefore, not Gorenstein (the general hyperplane section given by $x_0 - x_{n+1} = 0$ contains an embedded point given by the ideal $(x_0, x_{n-1}, x_i x_j, x_i x_n, x_n^{n-1} : 1 \le i \le j \le n-2)$ and, hence, depth $Z_n <$ dim Z_n). Notice that for $n = 2$, $Z_2 \subset \mathbb{R}^3$ would be the regular germ $x_2 = x_1^2$ for which of course $\mathcal{P} = \Sigma_2$ ([BR]).

We finish here with several questions that arise naturally from the above results and examples:

Open questions. *(1)* Is there in higher embedding dimension any analytic germ with $\mathcal{P} = \Sigma$ and $p \ne 2$?

(2) Are there *very singular* surfaces (worse than our normal X_n's and our Cohen-Macaulay Y_n's) with $\mathcal{P} = \Sigma$ and $p = 2$?

(3) Are there *very regular* surfaces (better than our not Cohen-Macaulay Z_n's) with $p = 2$ and $\mathcal{P} \ne \Sigma$?

The author thanks Prof. J. Ruiz for many discussions during the preparation of this work.

2. Proof of the main result

The purpose of this section is to prove Theorem 1.1. Let X be an analytic set germ (at the origin of $\mathbb{R}^n$); we denote by $\mathcal{O}(X)$ the ring of germs of analytic functions on X. If $X \subset \mathbb{R}^n$ we have $\mathcal{O}(X) = \mathbb{R}\{x_1, \dots, x_n\}/\mathcal{J}(X)$, where $\mathcal{J}(X)$ is the ideal of all analytic function germs vanishing on X. We recall that a germ $f \in \mathcal{O}(X)$ is *positive semidefinite* or *psd* if it is ≥ 0 on X. We denote by $\mathcal{P}(X)$ the set of all psd's of X and by $\Sigma(X)$ (resp. Σ_q) the set of all sums of squares (resp. q squares) of elements of $\mathcal{O}(X)$. Morever, $p[X]$ stands for the Pythagoras number of $\mathcal{O}(X)$.

Lemma 2.1. *Let $a_1, a_2, b_1, b_2 \in \mathbb{R}\{x, y\}$. If $\omega(b_1) = 0$, there exist $\alpha_1, \alpha_2, \beta_1 \in \mathbb{R}\{x, y\}$ such that*

$$(a_1 + zb_1)^2 + (a_2 + zb_2)^2 = (\alpha_1 + z\beta_1)^2 + \alpha_2^2.$$

Proof. Just take $\alpha_1 = \dfrac{a_1 b_1 + a_2 b_2}{\sqrt{b_1^2 + b_2^2}}$, $\alpha_2 = \dfrac{a_1 b_2 - a_2 b_1}{\sqrt{b_1^2 + b_2^2}}$ and $\beta_1 = \sqrt{b_1^2 + b_2^2}$. □

Lemma 2.2. *Let $G \in \mathbb{R}\{y\}[u, v]$ be a polynomial such that $X : G(u, y, v) = 0$ is an analytic surface germ of $\mathbb{R}^3$. Suppose that there exist integers $\ell_1, \ell_2, m \ge 0$ and a polynomial $H(x, y) \in \mathbb{R}\{y\}[x]$ not divisible by y such that $G(xy^{\ell_1}, y, zy^{\ell_2}) = y^m(z^2 - yH(x, y))$. Let Y be the analytic surface germ of $\mathbb{R}^3$ of equation $z^2 - yH(x, y) = 0$. Then $p[Y] \le p[X]$.*

Proof. Let $\varepsilon > 0$ be such that the series $G \in \mathbb{R}\{y\}[u, v] \subset \mathbb{R}\{u, y, v\}$ converge on the set $S = \mathbb{R} \times (-\varepsilon, \varepsilon) \times \mathbb{R}$, and let $S_{X,\varepsilon} = \{(u, y, v) \in \mathbb{R} \times (-\varepsilon, \varepsilon) \times \mathbb{R} : G(u, y, v) = 0\}$. Note that $z^2 - yH(x, y) = 0$ also converges in S and consider the set $S_{Y,\varepsilon} = \{(x, y, z) \in \mathbb{R} \times (-\varepsilon, \varepsilon) \times \mathbb{R} : z^2 - yH(x, y) = 0\}$ and the biregular map φ:

$$\begin{aligned} \varphi : S_{X,\varepsilon} \setminus \{y = 0\} &\to S_{Y,\varepsilon} \setminus \{y = 0\} \\ (u, y, v) &\mapsto (x, y, z) = \left(\frac{u}{y^{\ell_1}}, y, \frac{v}{y^{\ell_2}} \right). \end{aligned}$$

After this preparation, let us see that $p[Y] \leq p(X) = p$.

First, let $f = \sum_i (c_i + zd_i)^2$ be a sum of squares in $\mathcal{O}(Y)$ such that c_i, $d_i \in \mathbb{R}[x, y]$ for all i. Consider the composition

$$f \circ \varphi = \frac{g(u, y, v)}{y^{2r}}$$

where $g \in \mathbb{R}[u, y, v]$ is clearly a sum of squares in $\mathcal{O}(X)$. Since $p[X] = p$, there exist $\alpha_1, \dots, \alpha_p, \gamma \in \mathbb{R}\{u, y, v\}$ such that $g = \alpha_1^2 + \cdots + \alpha_p^2 + G(u, y, v)\gamma$. Composing with φ^{-1} we obtain

$$\begin{aligned} y^{2r} f = \alpha_1(xy^{\ell_1}, y, zy^{\ell_2})^2 + \cdots + \alpha_p(xy^{\ell_1}, y, zy^{\ell_2})^2 \\ + y^m (z^2 - yH(x, y))\gamma(xy^{\ell_1}, y, zy^{\ell_2}) \end{aligned}$$

One can check that there exist $\alpha_{11}, \dots, \alpha_{1p}, \alpha_{21}, \dots, \alpha_{2p}, \gamma_1 \in \mathbb{R}\{x, y\}$ such that

$$\begin{aligned} y^{2r} f &\equiv y^{2r} \left(\sum_i (c_i^2 + yHd_i^2) + 2z \sum_i c_i d_i \right) \\ &= \sum_{j=1}^{p} (\alpha_{1j} + z\alpha_{2j})^2 - \gamma_1 (z^2 - yH(x, y)) \ (*) \end{aligned}$$

and comparing coefficients with respect to z we have

0) $y^{2r} \left(\sum_i (c_i^2 + yHd_i^2) \right) = \sum_{j=1}^{p} \alpha_{1j}^2 + yH\gamma_1$
1) $\sum_i c_i d_i = \sum_{j=1}^{p} \alpha_{1j}\alpha_{2j}$
2) $\gamma_1 = \sum_{j=1}^{p} \alpha_{2j}^2$.

If $r = 0$ we are done, so we can suppose $r > 0$. By *0)* $y | \alpha_{1j}$ for all j. Since y does not divide H, y divides γ_1. By *2)* $y | \alpha_{2j}$ for all j, hence $y^2 | \gamma_1$. Thus, the expression $(*)$ can be divided by y^2. We continue the argument until we end up with and expression of f as a sum of 2 squares in $\mathcal{O}(Y)$. This shows $p[Y] \leq p$ for any germ in $\mathcal{O}(Y)$ which has a polynomial as a representant. Finally, using M. Artin's Approximation Theorem ([Ar],[JP]) one deduces, by a standard argument ([Fe2]), that $p[Y] \leq p$. □

After these preliminary results we turn to our main result:

Proof of Theorem 1.1. We begin by remarking that in [Rz2, 1.1,2.1], the author actually proves that if $X \subset \mathbb{R}^n$ is a complete intersection of dimension ≥ 2 and $p[X] = 2$ then X is analytically equivalent to a surface germ in $\mathbb{R}^3$ of equation $z^2 = F(x, y)$. In what follows, we will see in several steps that:

A surface germ $X : z^2 - F(x, y) = 0$ with $p = 2$ is one of the surfaces of the list.

To achieve this, we suppose that $p = 2$ and obtain succesive restrictions on the series F. To start with, we get rid of order ≥ 4 series:

(2.3) First restriction. $\omega(F) \leq 3$.

Proof. Indeed, suppose $\omega(F) \geq 4$ and write $F = Q + H$ where $\omega(H) \geq 5$ and Q is either 0 or a homogeneous polynomial of degree 4. We will find an element of the type

$$G = (x^2 + az)^2 + (y^2 + bz)^2 + (xy + cz)^2.$$

which is not a sum of two squares in $\mathcal{O}(X)$. If such a G is a sum of two squares, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}\{x, y\}$ with $\omega(\beta_2) \geq 1$ (maybe $\beta_2 = 0$, see 2.1) such that

$$\begin{aligned} G &\equiv x^4 + y^4 + x^2y^2 + 2(ax^2 + by^2 + cxy)z + (a^2 + b^2 + c^2)F \\ &= (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma(z^2 - F) \end{aligned}$$

Comparing coefficients with respect to z, we get the following equations

0) $x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2)F = \alpha_1^2 + \alpha_2^2 + \gamma F$
1) $ax^2 + by^2 + cxy = \alpha_1\beta_1 + \alpha_2\beta_2$
2) $0 = \beta_1^2 + \beta_2^2 - \gamma$

From *0)* we deduce that $\omega(\alpha_1), \omega(\alpha_2) \geq 2$ and from *1)* that $\omega(\beta_1) = 0$. Thus, $\beta_1(0) = \lambda \neq 0$. If we compare initial forms in *0)* and *1)* we deduce that

0) $x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2)Q = \mathrm{In}(\alpha_1)^2 + \mathrm{In}(\alpha_2)^2 + \lambda^2 Q$
1) $ax^2 + by^2 + cxy = \mathrm{In}(\alpha_1)\lambda$.

Hence the following formula must hold:

$$\begin{aligned} &\lambda^2(x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2 - \lambda^2)Q) \\ &\quad = (ax^2 + by^2 + cxy)^2 + (ux^2 + vy^2 + wxy)^2 \quad (*) \end{aligned}$$

for some $\lambda, u, v, w \in \mathbb{R}$. We distinguish three different cases. In each case we will make a suitable choice of a, b, c to obtain $G \in \Sigma(X) \setminus \Sigma_2(X)$:

(2.3.1) *Case* $Q = 0$.
We take $a = 0, b = 1, c = 1$. Suppose that the corresponding G is in $\Sigma_2(X)$. We get

$$\lambda^2(x^4 + y^4 + x^2y^2) = (y^2 + xy)^2 + (ux^2 + vy^2 + wxy)^2.$$

Thus, $u^2 = \lambda^2 \neq 0, uw = 0, 2 + 2vw = 0$ which is impossible. This means that G is not a sum of two squares in $\mathcal{O}(X)$. $\square$

If $Q \neq 0$, after a linear change, we can suppose $Q = \varepsilon x^4 + q_3x^2y^2 + q_4xy^3 + q_5y^4$ where $\varepsilon = \pm 1$ and $q_3, q_4, q_5 \in \mathbb{R}$. Then:

(2.3.2) *Case* $q_4 = 0$.
A linear change allows us to suppose $Q = \varepsilon x^4 + q_3x^2y^2 + q_5y^4$ with $(q_3, q_5) \neq (\pm 1, 0), (\pm 0, 1)$. Consider the non zero polynomial in s, t:

$$P(s, t) = (\varepsilon + s^2 + t^2)(\varepsilon(\varepsilon - q_3)(\varepsilon - q_5) - s^2q_5(\varepsilon - q_3) - t^2q_3(\varepsilon - q_5)),$$

and take $a = 0$ and $b, c \neq 0$ such that $P(b, c) \neq 0$. Suppose that the corresponding G is in $\Sigma_2(X)$. Comparing coefficients in $(*)$ we obtain

$$\begin{array}{rl} x^4) & \lambda^2(1 + (b^2 + c^2 - \lambda^2)\varepsilon) = u^2 \\ y^4) & \lambda^2(1 + (b^2 + c^2 - \lambda^2)q_5) = b^2 + v^2 \\ x^2y^2) & \lambda^2(1 + (b^2 + c^2 - \lambda^2)q_3) = c^2 + 2uv + w^2 \\ x^3y) & 0 = uw \\ xy^3) & 0 = bc + vw. \end{array}$$

Since $bc \neq 0$ then $v, w \neq 0$ and $u = 0$. Thus, we deduce that $\lambda^2 = \varepsilon + b^2 + c^2$ and $w = -bc/v$. Plugging these values in the equations above we have

$$(\varepsilon + b^2 + c^2)(1 - \varepsilon q_5) - b^2 = v^2$$
$$((\varepsilon + b^2 + c^2)(1 - \varepsilon q_3) - c^2)v^2 = b^2c^2$$

Eliminating v^2, we conclude that $P(b, c)$ must be zero, against our choice. Hence, G is not a sum of two squares in $\mathcal{O}(X)$. □

(2.3.3) *Case* $q_4 \neq 0$.

After a linear change $Q = \varepsilon x^4 + q_3x^2y^2 + 2xy^3 + q_5y^4$. Take $c = 0, a = 0, b \neq 0$ if $\varepsilon(q_3 + q_5) - q_3q_5 \geq 0$ and $c = 0, a = 2b \neq 0$ if $\varepsilon(q_3 + q_5) < q_3q_5$. Suppose that the corresponding G is in $\Sigma_2(X)$. Comparing coefficients in $(*)$ we get

$$\begin{array}{rl} x^4) & \lambda^2(1 + (a^2 + b^2 - \lambda^2)\varepsilon) = a^2 + u^2 \\ y^4) & \lambda^2(1 + (a^2 + b^2 - \lambda^2)q_5) = b^2 + v^2 \\ x^2y^2) & \lambda^2(1 + (a^2 + b^2 - \lambda^2)q_3) = 2ab + 2uv + w^2 \\ x^3y) & 0 = uw \\ xy^3) & \lambda^2(a^2 + b^2 - \lambda^2) = vw. \end{array}$$

If $w = 0$ then $\lambda^2 = a^2 + b^2$ and we have $u^2 = b^2, v^2 = a^2, a^2 + b^2 = 2ab + 2uv$. Thus, $a^2 + b^2 = 2ab \pm 2ab$ which is impossible by our choice $a = 0 \neq b$ or $a = 2b \neq 0$. Hence, $w \neq 0, u = 0$. Substituting $u = 0$ and $\lambda^2(a^2 + b^2 - \lambda^2) = vw$ in the equations above we get

$$\lambda^2 + \varepsilon vw = a^2$$
$$\lambda^2 + q_5vw = b^2 + v^2$$
$$\lambda^2 + q_3vw = 2ab + w^2.$$

Now, substituting $\lambda^2 = a^2 - \varepsilon vw$ we obtain

$$a^2 - b^2 = v(v - (q_5 - \varepsilon)w)$$
$$0 = a(a - 2b) = w(w - (q_3 - \varepsilon)v).$$

Since $w \neq 0$, we have $w = (q_3 - \varepsilon)v$, $v \neq 0$. Thus, $a^2 - b^2 = v^2(\varepsilon(q_3 + q_5) - q_3q_5)$. But since this is impossible with our choice of a, b, we conclude G is not a sum of two squares in $\mathcal{O}(X)$. □

This completes the proof of **(2.3)**, and we can assume henceforth $\omega(F) \leq 3$. Concerning order 2 series we have:

(2.4) Second restriction. *If $\omega(F) = 2$, then X is equivalent to $z^2 - x^2 = 0$ or $z^2 - x^2 - y^k = 0$ for some $k \geq 2$.*

Proof. After a change of coordinates, we can suppose that the equation of X is $z^2 - x^2 = 0$ or of the type $z^2 + \varepsilon x^2 - y^k$ with $\varepsilon = \pm 1$, $k \geq 2$. If $k = 2$, $z^2 + \varepsilon x^2 - y^2 = 0$ is equivalent to $z^2 - x^2 - y^2 = 0$. Now, we prove that ε must be -1 for $k \geq 3$.

We claim that if $\varepsilon = 1$ the function germ $(z+x^2)^2+x^2+y^2$ is not a sum of two squares in $\mathcal{O}(X)$. Indeed, were it so, there would exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}\{x, y\}$ such that $\omega(\beta_2) \geq 1$ (maybe $\beta_2 = 0$, see 2.1) and

$$(z+x^2)^2+x^2+y^2 \equiv 2zx^2+x^4+y^2+y^k = (\alpha_1+z\beta_1)^2+(\alpha_2+z\beta_2)^2-\gamma(z^2+x^2-y^k)$$

Comparing coefficients with respect to z we get the equations

0) $x^4 + y^2 + y^k + \gamma(x^2 - y^k) = \alpha_1^2 + \alpha_2^2$
1) $x^2 = \alpha_1\beta_1 + \alpha_2\beta_2$
2) $0 = \beta_1^2 + \beta_2^2 - \gamma$.

If $\omega(\beta_1) \geq 1$ then, by *2)*, $\omega(\gamma) \geq 1$ and by *0)*, $\alpha_i = \lambda_i y + g_i$ where $\lambda_i \in \mathbb{R}$, $\lambda_1^2 + \lambda_2^2 = 1$, $g_i \in (x, y)^2$, which is impossible by *1)*. Hence, $\omega(\beta_1) = 0$. Eliminating α_1, γ in *0)* we get

$$\beta_1^2(x^4 + y^2 + y^k + (\beta_1^2 + \beta_2^2)(x^2 - y^k)) = (x^2 - \alpha_2\beta_2)^2 + \alpha_2^2\beta_1^2,$$

and computing a little we conclude that

$$\beta_1^2((\beta_1^2+\beta_2^2)x^2+y^2)+(\beta_1^2-1)x^4+\beta_1^2(1-\beta_1^2-\beta_2^2)y^k = \alpha_2((\beta_1^2+\beta_2^2)\alpha_2-2x^2\beta_2),$$

but this is impossible because the power series on the left is irreducible, since its initial form is $a^2x^2 + b^2y^2$ with $a, b \neq 0$. The proof of **(2.4)** is finished. □

Next we look at order 3 series and get:

(2.5) Third restriction. *If $\omega(F) = 3$, then X is equivalent to one of the following:*

$$\begin{cases} z^2 - x^2y - (-1)^k y^k = 0 \ (k \geq 3),\ z^2 - x^2y; \\ z^2 - x^3 + xy^3 = 0,\ z^2 - x^3 - y^4 = 0 \text{ or } z^2 - x^3 - y^5 = 0. \end{cases}$$

Proof. After a linear change, the initial form of F is x^2y, $x^2y \pm y^3$ or x^3. We study two cases:

(2.5.1) *If* $\mathrm{In}(F) = x^2y$ *or* $x^2y \pm y^3$, *then X is equivalent to* $z^2-x^2y-(-1)^k y^k = 0$ *($k \geq 3$) or* $z^2 - x^2y$.

After a change of coordinates (classification of singularities), we can suppose that F is one of the following power series: x^2y, $x^2y \pm y^k$, $k \geq 3$. If $F = x^2y - (-1)^k y^k$ we show that there exist a sum of squares G of analytic function germs on $X_k : z^2 = x^2y - (-1)^k y^k$ which is not sums of 2 squares in $\mathcal{O}(X_k)$.

First, we find G for $X_3 : z^2 - y(x^2 + y^2)$. Let $f = g_6 + z^2(x^2 + y^2)$ for a homogeneous polynomial $g_6 \in \mathbb{R}[x, y]$ of degree 6 which is a sum of squares but

$(x^2+y^2) \not| \ g_6$, e.g. x^6. If f was a sum of two squares, then there would exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}\{x, y\}$ such that

$$f \equiv g_6 + (x^2+y^2)^2 y = (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma(z^2 - (x^2+y^2)y),$$

and so, comparing coefficients with respect to z we have

0) $g_6 + (x^2+y^2)^2 y = \alpha_1^2 + \alpha_2^2 + (\beta_1^2+\beta_2^2)(x^2+y^2)y$,
1) $0 = \alpha_1\beta_1 + \alpha_2\beta_2$.

Comparing orders in *0)* we deduce that $\omega(\beta_1^2+\beta_2^2-(x^2+y^2)) \geq 3$. Thus, $\mathrm{In}(\beta_1^2+\beta_2^2) = x^2+y^2$ and we conclude that $\omega(\beta_i) = 1$. Hence, the series β_1, β_2 are relatively prime. By *1)*, there exist a series $d \in \mathbb{R}\{x, y\}$ such that $\alpha_1 = \beta_2 d, \alpha_2 = -\beta_1 d$. Plugging these in *0)* we get $g_6 = d^2(\beta_1^2+\beta_2^2) + y(x^2+y^2)q$ where $q \in \mathbb{R}\{x, y\}$ is a series of order ≥ 3. Comparing initial forms in this expression, we conclude that $(x^2+y^2)|g_6$, impossible by hypothesis.

Next, we prove, using 2.2, that $p[X_k] \geq 3$ if $k \geq 4$. Consider the equations

$$\begin{cases} F_k(u, y, v) = v^2 - u^2 y - y^k = 0 & \text{if } k = 2\ell+3 \\ F_k(u, y, v) = u^2 - v^2 y + y^k = 0 & \text{if } k = 2\ell+4. \end{cases}$$

X_k is the surface germ of equation $F_k(x, y, z) = 0$ if $k = 2\ell+3$ and $F_k(z, y, x)$ if $k = 2\ell+4$. It is enough to see that the surface germ $Z_k : \ F_k(u, y, v) = 0$ has $p[Z_k] \geq 3$ for all k. Take

$$\begin{cases} \ell_1 = \ell_2 = \ell, m = 2\ell & \text{if } k = 2\ell+3 \\ \ell_1 = \ell+1, \ell_2 = \ell, m = 2\ell+1 & \text{if } k = 2\ell+4. \end{cases}$$

On can check that $F_k(xy^{\ell_1}, y, zy^{\ell 2}) = (-1)^{k_0+1} y^m (z^2 - x^2 y - y^3)$. By 2.2, we conclude that $3 \leq p[X_3] \leq p[X_k]$, as wanted. □

After **(2.5.1)**, we see:

(2.5.2) *If* $\mathrm{In}(F) = x^3$ *then* X *is equivalent to* $z^2 - x^3 + xy^3 = 0$, $z^2 - x^3 - y^4 = 0$ *or* $z^2 - x^3 - y^5 = 0$.

Changing x by $-x$ if necessary, there exist a Weierstrass polynomial $P = x^3 + p_1(y)y^2x^2 + p_2(y)y^3x + p_3(y)y^4$, $(p_i \in \mathbb{R}\{y\})$ and a unit $U \in \mathbb{R}\{x, y\}$ such that $U(0, 0) > 0$ and $F = PU$. After the change $(x, y, z) \mapsto (x - p_1(y)y^2/3, y, \sqrt{U}z)$, we can suppose that the equation of X is of the type $z^2 - x^3 - a(y)y^3x - b(y)y^4$ for some $a, b \in \mathbb{R}\{y\}$. After this preparation we proceed in several steps:

(a) *If* $\omega(a) \geq 1$ *and* $\omega(b) \geq 2$ *then* $p[X] \geq 3$.
To prove this we are going to use 2.2. Let $G(u, y, v) = v^2 - F(u, y)$. If we take $\ell_1 = \ell_2 = 1, m = 2$ we have $G(xy^{\ell_1}, y, zy^{\ell_2}) = y^m(z^2 - yH(x, y))$ where $H = x^3 - a(y)yx - b(y)y$. By 2.2, we conclude that if $Y : \ z^2 - yH(x, y) = 0$, then $p[Y] \leq p[X]$. On the other hand, as we have seen in **(2.3)**, we have that $p[Y] \geq 3$ (because $\omega(yH) \geq 4$). Hence, $p[X] \geq 3$.

Next, we discuss the factorization of $F = x^3 + a(y)y^3x + b(y)y^4$:

(b) *If F is the product of three (possibly equal) irreducible factors then $p[X] \geq 3$.* Suppose $F = f_1 f_2 f_3$, where some or all the factors may coincide. Since the initial form of F is x^3, we can write $f_k = x + \lambda_k(x, y)$ where $\omega(\lambda_k) \geq 2$ and then

$$\begin{aligned} F &= (x + \lambda_1)(x + \lambda_2)(x + \lambda_3) \\ &= x^3 + x^2(\lambda_1 + \lambda_2 + \lambda_3) + x(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + (\lambda_1\lambda_2\lambda_3) \\ &= x^3 + a(y)y^3x + b(y)y^4 \end{aligned}$$

From this equality we deduce that

$$b(y)y^4 = F(0, y) = \lambda_1(0, y)\lambda_2(0, y)\lambda_3(0, y) \text{ has order} \geq 6,$$

$$a(y)y^3 = \frac{\partial F}{\partial x}(0, y) = \sum_{1 \leq i < j \leq 3} \lambda_i(0, y)\lambda_j(0, y)\left(1 + \frac{\partial \lambda_k}{\partial x}(0, y)\right) \text{ has order} \geq 4,$$

where $1 \leq k \leq 3$, $k \neq i, j$. Hence, $\omega(a) \geq 1, \omega(b) \geq 2$ and, by **(2.5.2.*a*)**, $p[X] \geq 3$.

(c) *If F is reducible and $p[X] = 2$, then $F = x^3 - xy^3$.* By the previous remark, $F = fg$ and f, g must be irreducible, say $\omega(f) = 2, \omega(g) = 1$ and we can suppose $\mathrm{In}(f) = x^2$, $\mathrm{In}(g) = x$. If f is semidefinite, it is a sum of two squares with initial form x^2. After multiplying by a suitable ortogonal matrix, we can suppose $f = (x + \mu_1(x, y))^2 + (\mu_2(x, y))^2$ and $g = x + \mu_3(x, y)$ with $\omega(\mu_k) \geq 2$. Thus,

$$F = (x + \mu_1(x, y) + i\mu_2(x, y))(x + \mu_1(x, y) - i\mu_2(x, y))(x + \mu_3(x, y)).$$

Proceeding similarly to **(2.5.2.*b*)** (we have again three irreducible factors although two of them are complex) we are in the hypothesis of **(2.5.2.*a*)** and $p[X] \geq 3$. Hence, if $p[X] = 2$, f should be irreducible and real. Thus, we can assume $F = (x^2 - y^k)(x + \mu(x, y))$, $k \geq 3$, $\omega(\mu) \geq 2$. By the Weierstrass Preparation Theorem there exist a series $\alpha \in \mathbb{R}\{y\}$ and a unit $U \in \mathbb{R}\{x, y\}$ such that $x + \mu(x, y) = (x + \alpha(y)y^2)U(x, y)$. Changing x by $-x$ (if necessary) we can suppose $U(0, 0) > 0$ and after a change $(x, y, z) \mapsto (x, y, \sqrt{U(x, y)}z)$, the equation of our germ is $z^2 - (x^2 - y^k)(x + \alpha(y)y^2)$.
For $k \geq 4$, $F = x^3 + \alpha(y)x^2y^2 - y^kx - y^{k+2}\alpha(y)$. After the change $x \mapsto x - \alpha(y)y^2/3$, we are again in the conditions of *(a)*. Hence, $p[X] \geq 3$.
Finally, for $k = 3$ we get $F = (x^2 - y^3)(x + \cdots)$ and by classification of singularities F is equivalent to $x^3 - xy^3$.

(d) *If F is irreducible then $F = x^3 + y^4$ or $x^3 + y^5$.* Suppose F irreducible. By classification of singularities we can transform F into $x^3 \pm y^4$ or $F = x^3 + xy^4a'(y) + y^5b'(y)$. Suppose first $F = x^3 + xy^4a'(y) + y^5b'(y)$. If $b'(0) = 0$, by **(2.5.2.*a*)**, $p[X] \geq 3$. If $b'(0) \neq 0$ then another change makes $F = x^3 + y^5$.

For $F = x^3 - y^4$ we see that $p[X] \geq 3$. Consider $G(u, y, v) = v^2 - u^3 + y^4$, and $\ell_1 = \ell_2 = 1, m = 2$. We have that $G(xy^{\ell_1}, y, vy^{\ell_2}) = y^2(z^2 + y^2 - x^3y)$ and by 2.2, we conclude that if $Y : z^2+y^2-x^3y = 0$, then $p[Y] \leq p[X]$. Now, we see that $Y : z^2 + y^2 - x^3y = z^2 + (y - x^3/2)^2 - x^6/4 = 0$ is equivalent to $Y' : z^2 + w^2 - x^6 = 0$ which has, as we proved in **(2.4)**, $p[Y'] \geq 3$. Hence $p(X) \geq p(Y) \geq 3$. □

Thus, we have proved **(2.5.2)**. Summing up, **(2.3)** says that $\omega(F) \leq 3$, **(2.4)** that if $\omega(F) = 2$, the germ $z^2 - F = 0$ is among *(iv)–(vi)* in the list of Th.1.1, and **(2.5)** that if $\omega(F) = 3$, the germ $z^2 - F = 0$ is among *(i)–(iii)* and *(vii)–(ix)* in that list. All together we conclude that if an unmixed surface germ $X \subset \mathbb{R}^3$ has $p = 2$ then X belongs to the list, as wanted. □

3. The mixed case

In this section we find all mixed surface germs $X \subset \mathbb{R}^3$ with Pythagoras number 2 and prove that only the simplest one of them has the property $\mathcal{P} = \Sigma$. First of all:

Proposition 3.1. *Let $X \subset \mathbb{R}^3$ be a mixed surface germ. Then $\mathcal{P}(X) = \Sigma(X)$ if and only if X is equivalent to the union of a plane and a transversal line. Furthermore, in this case, $p[X] = 2$.*

Proof. First, we prove that if X is the union of a plane π and a transversal line ℓ then every $f \in \mathcal{P}(X)$ is a sum of squares of analytic function germs. Indeed, after a change of coordinates the ideal of X is (zx, zy) and, every non unit f in $\mathcal{O}(X)$ can be written uniquely as $f_1(x, y) + f_2(z)$ where $f_1 \in \mathbb{R}\{x, y\}$, $f_2 \in \mathbb{R}\{z\}$. Note that $f(0, 0) = 0, g(0) = 0$. Now, $f = f_1(x, y) + f_2(z) \in \mathcal{P}(X)$ if and only if $f_1 \in \mathcal{P}(\pi)$ and $f_2 \in \mathcal{P}(\ell)$, or equivalently $f_1(x, y) = a(x, y)^2 + b(x, y)^2$ and $f_2(z) = c(z)^2$. Thus, $f = f_1 + f_2 \equiv (a + c)^2 + b^2$ in $\mathcal{O}(X)$.

Conversely, if $\mathcal{P}(X) = \Sigma(X)$, by [Fe2, 2.1], $\omega(\mathcal{J}(X)) = 2$. Let I (resp. J) be the ideal of the union of the components of X of dimension 2 (resp. 1). Then $\mathcal{J}(X) = I \cap J$. Moreover, since the ideal $I \subset \mathbb{R}\{x, y, z\}$ has height 1, it is principal, and we write $I = (\varphi)$ with $\varphi \in \mathbb{R}\{x, y, z\}$. One can check that $\mathcal{J}(X) = I \cdot J$; hence, $2 = \omega(\mathcal{J}(X)) = \omega(I) + \omega(J)$. Thus, $\omega(I) = \omega(J) = 1$ and we can suppose that $I = (z)$ and $J = (\psi_1, \psi_2)$ where $\psi_j \in \mathbb{R}\{x, y, z\}$ and $1 = \omega(\psi_1) \leq \omega(\psi_2)$.

We are to prove that after a change of coordinates $J = (x, y)$. To that end, we begin by proving that $\mathcal{Z}(J)$ is regular, hence irreducible. Indeed, suppose $\mathcal{Z}(J)$ singular. Then there exists a function f in $\mathcal{P}(\mathcal{Z}(J)) \setminus \Sigma(\mathcal{Z}(J))$ ([Sch]). We claim that $z^2 f \in \mathcal{P}(X)$ is not a sum of squares in $\mathcal{O}(X)$. If it were, there would exist $a_1, \ldots, a_p, b_1, b_2 \in \mathbb{R}\{x, y, z\}$ such that $z^2 f = a_1^2 + \cdots + a_p^2 + z\psi_1 b_1 + z\psi_2 b_2$. From this we see that $z | a_i$, say $a_i = z\alpha_i$. Hence, the function $z^2(f - \alpha_1^2 - \cdots - \alpha_p^2)$ vanishes on X, which means that $f - \alpha_1^2 - \cdots - \alpha_p^2 = 0$ on $\mathcal{Z}(J)$. Thus f is a sum of squares, contradiction. Therefore, J is generated by two elements of order 1 with independent initial forms. We can suppose $I = (z)$, $J = (x, \psi(y, z))$ where $\psi = y$ or $z - y^k$, $k \geq 2$.

If $\psi = y$ we are done. So we only have to discard the cases $\psi = z - y^k, k \geq 2$. If k is even, $f = z$ is psd on X but it is not a sum of squares. If k is odd we consider the function $f = zy$ which is psd on X but not a sum of squares. Suppose that there are $a_1, \dots, a_p, b_1, b_2 \in \mathbb{R}\{x, y, z\}$ such that $zy = a_1^2 + \cdots + a_p^2 + zxb_1 + z(z - y^k)b_2$. Comparing initial forms, there exist $\lambda, \mu \in \mathbb{R}$ such that the quadratic form $q = zy + \lambda z^2 + \mu zx$ is a sum of squares of linear forms. This is imposible because such a q is not psd in $\mathbb{R}^3$. Whence, q is not a sum of squares in $\mathcal{O}(X)$. □

Now, we characterize the mixed surface germs in $\mathbb{R}^3$ with Pythagoras number 2. First, note that if $Y \subset X$ then $p[Y] \leq p[X]$ (there exist an epimorphism from $\mathcal{O}(X)$ onto $\mathcal{O}(Y)$). Hence, if Y is a mixed surface germ contained in the union of two transversal planes, which has Pythagoras number 2 ([Rz2], [Fe2]), it also has Pythagoras number 2. The aim of the following theorem is to prove that there are no more mixed surface germs in $\mathbb{R}^3$ with Pythagoras number 2.

Theorem 3.2. *Let $X \subset \mathbb{R}^3$ be a mixed surface germ with Pythagoras number 2. Then X is contained in the union of two transversal planes.*

To prove this, we need the following preliminary results.

Lemma 3.3. *Let $\varphi \in \mathbb{R}\{t\}$ be a unit, $n \geq 1$. Consider the equation $\lambda(x)^n = \varphi(x\lambda(x))$ in $\mathbb{R}\{x\}$. If $\varphi(0)$ has an n-root in $\mathbb{R}$, then the previous equation has a solution in $\mathbb{R}\{x\}$.*

Proof. Let $b \in \mathbb{R}$ be such that $b^n = \varphi(0)$. The equation above is equivalent to $F(x, y) = (b + y)^n - (\varphi(x(b + y))) = 0$, which satisties:

$$F(0,0) = 0, \qquad \frac{\partial F}{\partial y}(0,0) = nb^{n-1} \neq 0.$$

By the Implicit Function Theorem ([JP, 3.3]) there exists a series $\psi \in \mathbb{R}\{x\}$ such that $\psi(0) = 0$ and $F(x, \psi(x)) = 0$. Taking $\lambda = b + \psi$ we are done. □

The next result, which is a consequence of the classification of singularities, is included for the sake of the reader.

Lemma 3.4. (Classification of singularities) *Let $f, g \in \mathbb{R}\{x, y\}$ be two power series such that $\omega(f) = 2, \omega(g) \geq 2$ and the ideal $I = (zf, z(z - 2g))$ is real radical. Then I is analytically equivalent to one of the following:*

(a) $(zxy, z(z - 2g'))$ *where* $\mathrm{In}(g') = x^2 + y^2$;
(b) $(zxy, z(z - 2g'))$ *where* $\mathrm{In}(g') = x^2 - y^2$;
(c) $(zxy, z(z - 2g'))$ *where* $\mathrm{In}(g') = y^2$;
(d) $(z(y^2 - x^k), z(z - x^2))$ *where* $k \geq 3$;
(e) $(z(y^2 - x^k), z(z - xy))$ *where* $k \geq 3$
(f) $(z(z - x(y + x^\ell)), z(y^2 - x^k))$ *where* $2 \leq \ell < k, 3 \leq k \neq 2\ell$;
(g) $(zy(y + x^\ell), z(z - x(y + b(x)x^\ell)))$ *where* $\ell \geq 2$, $b \in \mathbb{R}\{x\}$, $b(0) \neq 0, 1$;
(h) $(zy(y + x^\ell), z(z - x(y + \delta x^{\ell+n})))$ *where* $\delta = \pm 1$, $\ell \geq 2$, $n \geq 1$;
(i) $(z(x^2 - y^k), z(z - 2g'))$ *where* $k \geq 2$ *and* $g' = y^{\ell+1}(xa(y) + b(y))$ *for some* $\ell \geq 1$ *and* $a, b \in \mathbb{R}\{y\}$ *such that* $\omega(a^2 + b^2) = 0$ *and* $\omega(g') \geq 3$.

Proof. First, if $\operatorname{In}(f), \operatorname{In}(g)$ are linearly dependent quadratic forms then we can write $\operatorname{In}(g) = \lambda \operatorname{In}(f)$. Thus,

$$\begin{aligned} I &= (z(z-2g(x,y)), zf(x,y)) \\ &= (z(z-2(g(x,y)-\lambda f(x,y))), zf(x,y)) = (z(z-2h(x,y)), zf(x,y)). \end{aligned}$$

where $h = g(x,y) - \lambda f(x,y)$ has order ≥ 3. By classification of singularities and using the fact that I is real radical, we can suppose (after a change of coordinates) that $f = x^2 - y^k$, $k \geq 2$. Moreover, one can check, by means of the Weierstrass division theorem and the fact that I is real radical, that $h = (x^2 - y^k)q + g'$ where $q \in \mathbb{R}\{x, y\}$ and $g' = y^{\ell+1}(xa(y) + b(y))$ for some $\ell \geq 1$ and $a, b \in \mathbb{R}\{y\}$ such that $\omega(a^2 + b^2) = 0$ and $\omega(g') \geq 3$. Thus,

$$\begin{aligned} I = (z(z-2h(x,y)), zf(x,y)) &= (z(z-2(h(x,y)-qf(x,y))), zf(x,y)) \\ &= (z(z-2g'(x,y)), zf(x,y)) \quad \text{(case } (i)) . \end{aligned}$$

Hence, in what follows we assume that $\operatorname{In}(f), \operatorname{In}(g)$ are linearly independent quadratic forms. We proceed in several steps:

Step 1. If $\operatorname{In}(f)$ *is a quadratic form of rank 2 the ideal I is analytically equivalent to one of (a), (b) or (c).* Indeed, since I is real radical, we can suppose $f = xy$ ($\operatorname{In}(f)$ is a non definite quadratic form because I is a real radical ideal) and $\operatorname{In}(g) = ax^2 + bxy + cy^2$ with $a^2 + c^2 \neq 0$ (since $\operatorname{In}(f), \operatorname{In}(g)$ are linearly independent). We have $I = (z(z-2g), zf) = (z(z-2g) + 2bzf, zf) = (z(z-2g'), zf)$, where $g' = g - bf$ has initial form $\operatorname{In}(g') = ax^2 + cy^2$. After a linear change in the variables x, y, we can suppose $\operatorname{In}(g') = x^2 + y^2, x^2 - y^2$ or y^2.

Step 2. If $\operatorname{In}(f)$ *is a quadratic form of rank 1 the ideal I is analytically equivalent to one of (d), (e), (f), (g) or (h).* We can suppose that $\operatorname{In}(f) = y^2$ and $\operatorname{In}(g) = ax^2 + bxy + cy^2$ and $I = (z(z-2g), zf) = (z(z-2g+2cf), zf) = (z(z-2g_1), zf)$ where $g_1 = g - cf$ has initial form $ax^2 + bxy$, with $a^2 + b^2 \neq 0$. After a suitable change of coordinates of the type $(x, y, z) \mapsto (d_1 x, d_2 y, \pm z)$ we can assume $\operatorname{In}(g_1) = x^2, x^2 + 2xy$ or xy. In fact, if $\operatorname{In}(g_1) = x^2 + 2xy$ then $I = (z(z-2g_1), zf) = (z(z-2(g_1+f)), zf) = (z(z-2g_2), zf)$ where $g_2 = g_1 + f$ has initial form $(x+y)^2$. Hence, up to the change $x \mapsto x - y$, we can assume that $\operatorname{In}(f) = y^2$ and $\operatorname{In}(g_2) = x^2$ or xy. Therefore, we have two cases to study:

(a) $\operatorname{In}(g) = x^2$. Consider the generators of I: $z(z-2(g+f)), zf$. By Morse's Lemma ([Rz1]), after a suitable change of coordinates of the type $\varphi(x,y) = (x + \varphi_1, y + \varphi_2)$, $\omega(\varphi_i) \geq 2$, we have $g + f = x^2 + y^2$.

On the other hand, $f = PU$ where $P = y^2 + 2a(x)x^2y + b(x)x^3 \in \mathbb{R}\{x\}[y]$ is a Weierstrass polynomial of degree 2 and $U \in \mathbb{R}\{x, y\}$ is a unit. Thus,

$$I = (z(z-2(x^2+y^2)), zP) = (z(z-2(x^2+y^2-P)), zP) = (z(z-x^2u(x,y)), zP)$$

where $u(x,y) = 2 - 4a(x)y - 2b(x)x$ is a unit. Moreover, since I is a real radical ideal, changing x by $\pm x$, $P = \left(y + a(x)x^2\right)^2 - x^k w(x)$ where $k \geq 3$ and $w \in \mathbb{R}\{x\}$ is a unit with $w(0) > 0$. After the change $y \mapsto y - a(x)x^2$, we have

$I = (z(z - x^2 v(x,y)), z(y^2 - x^k w(x)))$, for a new unit $v(x,y) \in \mathbb{R}\{x,y\}$. Now, after the change $x\sqrt[k]{w(x)} \mapsto x$ we get $I = (z(z - x^2 v'(x,y)), z(y^2 - x^k))$, for yet a new unit $v'(x,y) \in \mathbb{R}\{x,y\}$. Finally, after the change $z \mapsto zv'$ we obtain $I = (z(y^2 - x^k), z(z - x^2))$, $k \geq 3$ (case *(d)*).

(b) $\mathrm{In}(g) = xy$. By classification of singularities, after a suitable change of coordinates of the type $\varphi(x,y) = (x + \varphi_1, y + \varphi_2)$, $\omega(\varphi_i) \geq 2$, we can suppose $g = xy$. On the other hand, $f = PU$ where $P = y^2 + 2a(x)x^2 y + b(x)x^3 \in \mathbb{R}\{x\}[y]$ is a Weierstrass polynomial of degree 2 and $U \in \mathbb{R}\{x,y\}$ is a unit. Thus, $I = (z(z - 2xy), zP)$. We recall that since I is the ideal of a real surface germ (changing x by $\pm x$) $P = \big(y + a(x)x^2\big)^2 - x^k w(x)$ where $k \geq 3$ and $w \in \mathbb{R}\{x\}$ is a unit with $w(0) > 0$. After the change $(y,z) \mapsto (y - a(x)x^2, 2z)$, we have $I = (z(z - x(y - a(x)x^2)), z(y^2 - x^k w(x)))$. Now, we proceed in the following way:

(b.1) If $a(x) = 0$, after the change $(y,z) \mapsto \sqrt{w(x)}(y,z)$ we get $I = (z(y^2 - x^k), z(z - xy))$ (case *(e)*).

(b.2) If $a(x) \neq 0$ then $a(x) = x^{\ell-2} v(x)$ where $\ell \geq 2$ and $v \in \mathbb{R}\{x\}$ is a unit. Here we distinguish two subcases:

(•) If $k \neq 2\ell$, consider a change of the type $(x,y,z) \mapsto (\lambda(x)x, \mu(x)y, \gamma(x)z)$, where $\lambda, \mu, \gamma \in \mathbb{R}\{x\}$ are units, such that after this change we have

$$\begin{aligned} I &= \Big(\gamma z(\gamma z - x\lambda(y\mu - v(x\lambda)x^\ell \lambda^\ell)), z\gamma(y^2\mu^2 - x^k \lambda^k w(x\lambda))\Big) \\ &= (z(z - x(y + x^\ell)), z(y^2 - x^k)), \quad k \geq 3, \ell \geq 2. \end{aligned}$$

That is, such that $\mu = -v(x\lambda)\lambda^\ell$, $\gamma = \lambda\mu$, $\mu^2 = \lambda^k w(x\lambda)$. This system of equations has a solution if an only if the equation $v(x\lambda)\lambda^{2\ell} = \lambda^k \omega(x\lambda)$ has a solution. But, since $k \neq 2\ell$, we get an equation of the kind $\lambda^n = \varphi(\lambda x)$ where $\varphi \in \mathbb{R}\{t\}$ is a unit, $n \geq 1$. This equation has a solution by 3.3.
Note that if $\ell \geq k$ then $I = (z(z - x(y + x^{\ell-k}y^2)), z(y^2 - x^k)) = (z(z - xy(1 + x^{\ell-k}y)), z(y^2 - x^k))$ and after the change $z \mapsto z(1 + x^{\ell-k}y)$ we get $I = (z(y^2 - x^k), z(z - xy))$ (case *(e)*). For $\ell < k$ we are in the case *(f)* of the statement.

(••) If $k = 2\ell$, such a change does not exist and we proceed as follows. After the change $xw(x) \mapsto x$ we get that $I = (z(z - x(y - x^\ell u(x))), z(y^2 - x^{2\ell}))$ for certain unit $u \in \mathbb{R}\{x\}$. If $u = \pm 1$ one can check that I is not a radical ideal, a contradiction. Therefore, $u \neq \pm 1$ and again we distinguish two further subcases:

★ If $u(0) \neq \pm 1$, after the change $(x,y,z) \mapsto (\frac{x}{\sqrt[\ell]{2}}, y + \frac{x^\ell}{2}, \frac{z}{\sqrt[\ell]{2}})$ we see that there exist a unit $b \in \mathbb{R}\{x\}$ with $b(0) \neq 0, 1$ such that $I = (z(z - x(y + x^\ell b(x))), zy(y + x^\ell))$ (case *(g)*).

★ If $u(0) = \delta = \pm 1$, we begin by making the change $y \mapsto y - \delta x^\ell$ and we get $I = (z(z - x(y - x^{l+n}c(x))), z(y^2 + 2y\delta x^\ell))$ for some unit $c \in \mathbb{R}\{x\}$ and $n \geq 1$. Considering a change of the type $(x,y,z) \mapsto (\lambda(x)x, \mu(x)y, \gamma(x)z)$ where $\lambda, \mu, \gamma \in \mathbb{R}\{x\}$ are units and proceeding as in the previous case **(•)**, we come to $I = (z(z - x(y + \delta x^{\ell+r})), zy(y + x^\ell))$, $\delta = \pm 1$ (case *(h)*). □

Lemma 3.5. *Let $X \subset \mathbb{R}^3$ be given by the equations:*

$$\text{(1)}\quad z(z+2g)=0, zf=0; \ \text{or}$$
$$\text{(2)}\quad z^2-g^2=0, (z-g)f=0$$

($f, g \in \mathbb{R}\{x, y\}$). Let $\varphi \equiv \sum_{i=1}^{r}(a_i+zb_i)^2 \in \mathcal{O}(X)$ with $a_i, b_i \in \mathbb{R}\{x, y\}$. Then, there exist $\varphi_1, \varphi_2, q_1, q_2 \in \mathbb{R}\{x, y\}$ such that either

$$\text{(1)}\ \varphi \equiv \varphi_1 + z\varphi_2 = \sum_{i=1}^{r}(a_i+zb_i)^2 + q_1 z(z+2g) + q_2 zf; \ \text{or}$$
$$\text{(2)}\ \varphi \equiv \varphi_1 + z\varphi_2 = \sum_{i=1}^{r}(a_i+zb_i)^2 + q_1(z^2-g^2) + q_2(z-g)f.$$

Proof. First, suppose $X : z(z+2g) = zf = 0$. Since $z(z+2g) \in \mathbb{R}\{x, y\}[z]$, by Weierstrass division, there exist $\varphi_1, \varphi_2 \in \mathbb{R}\{x, y\}$ such that $\varphi_1 + z\varphi_2 \equiv \varphi \equiv \sum_{i=1}^{r}(a_i+zb_i)^2$. Therefore

$$\varphi_1 + z\varphi_2 = \sum_{i=1}^{r}(a_i+zb_i)^2 + Q_1 z(z+2g) + Q_2 zf,$$

where $Q_1, Q_2 \in \mathbb{R}\{x, y, z\}$. Again by Weierstrass division, $Q_2 = Q_3(z+2g)+q_2$ where $q_2 \in \mathbb{R}\{x, y\}$ and $Q_3 \in \mathbb{R}\{x, y, z\}$. Thus:

$$\varphi_1 + z\varphi_2 = \sum_{i=1}^{r}(a_i+zb_i)^2 + (Q_1+Q_3)z(z+2g) + q_2 zf.$$

On the other hand, in the ring $\mathbb{R}\{x, y\}[z]$ we have $\sum_{i=1}^{r}(a_i+zb_i)^2 = -q_1 z(z+2g) + R$ where $\deg_z R \leq 1$, $\deg_z q_1 = 0$. By the uniqueness of the Weierstrass division $Q_1 + Q_3 = q_1 \in \mathbb{R}\{x, y\}$ as we wanted.

The case $X : z^2 - g^2 = (z-g)f = 0$ follows from the previous one by the change of coordinates $z \mapsto z+g$. □

Lemma 3.6. *Let $X \subset \mathbb{R}^3$ be the union of the analytic surface germ $z = 0$ and a curve germ $X_1 : f = 0, z+2g = 0$, ($f, g \in \mathbb{R}\{x, y\}$) such that $X_1 \cap \{z = 0\} = \{0\}$. Let $G \in \Sigma(X)$ be a sum of squares such that $G(x, y, 0)$ is a Weierstrass polynomial of degree 4 with respect to y. Let $\eta_1, \overline{\eta_1}, \eta_2, \overline{\eta_2}$ be the roots of Q. Write $\eta_{i1} = \eta_i$ and $\eta_{i,-1} = \overline{\eta_i}$ for $i = 1, 2$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Then we have:*

(a) $G \equiv \|\alpha+z\beta\|^2 \mod \mathcal{J}(X)$ where $\alpha = (y-\eta_{11}(x))(y-\eta_{2\varepsilon}(x))$, $\beta \in \mathbb{C}\{x, y\}$, $\varepsilon = \pm 1$.

(b) If $\gamma : \mathbb{R} \to \mathbb{R}^3, t \mapsto (t^n, \gamma_2, \gamma_3)$ is a parametrization of an irreducible component of X_1 then $G \circ \gamma = \|(\gamma_2 - \eta_{1\varepsilon}(t^n))(\gamma_2 - \eta_{2\varepsilon}(t^n)) + \gamma_3 \beta(t^n, \gamma_2)\|^2$.

Proof. First, since $X_1\cap\{z=0\}=\{0\}$, one can check that $\mathcal{J}(X)=(z(z+2g),zf)$. By 3.5 and using the fact that G is a sum of two squares in $\mathcal{O}(X)$, we deduce that there exist $a_i,b_i,q_1,q_2\in\mathbb{R}\{x,y\}$ such that $G=\sum_{i=1}^2(a_i+zb_i)^2+q_1z(z+2g)+q_2zf$.

For $z=0$, we have $Q(x,y)=G(x,y,0)=a_1^2+a_2^2$. Since $\mathbb{C}\{x,y\}$ is an UFD and Q is a Weierstrass polynomial of degree 4, there exist a Weierstrass polynomial $P\in\mathbb{C}\{x,y\}$ of degree 2 and a unit $U\in\mathbb{C}\{x,y\}$ such that $a=a_1+ia_2=PU$. Since $Q=P\overline{P}U\overline{U}$, by the uniqueness part of the Weierstrass preparation theorem, $U\overline{U}=1$. If $b=b_1+ib_2$ we have

$$\begin{aligned}(\alpha_1+zb_1)^2+(\alpha_2+zb_2)^2&=(\alpha+zb)\overline{(\alpha+zb)}=(PU+zb)\overline{U}U\overline{(PU+zb)}\\&=(P+zb\overline{U})\overline{(P+zb\overline{U})}=(P_1+z(b\overline{U})_1)^2+(P_2+z(b\overline{U})_2)^2.\end{aligned}$$

Since $Q=P\overline{P}$ and P is a Weierstrass polynomial then we can assume $P=(y-\eta_{11}(x))(y-\eta_{2\varepsilon}(x))$ for $\varepsilon=\pm1$. Taking $\alpha=P$ and $\beta=b\overline{U}$ we have *(a)*. Statement *(b)* follows directly from *(a)*. □

Remark 3.7. Under the hypotheses of the previous lemma, if $\mathcal{J}(X_1)=(f,z-2g)$ where $f,g\in\mathbb{R}\{x,y\}$ and $\omega(f(0,y))=2$ or $\omega(f(x,0))=2$ we can suppose that either $\beta=\lambda+y\mu$ for some $\lambda,\mu\in\mathbb{C}\{x\}$ or $\beta=\lambda+x\mu$ for some $\lambda,\mu\in\mathbb{C}\{y\}$. This is an straightforward consequence of the fact that $zf\in\mathcal{J}(X)$.

After all this preparation we proceed with the proof of 3.2.

Proof of Theorem 3.2. Let I (resp. J) be the ideal of the union of components of dimension 2 (resp. 1) of X. Then, $\mathcal{J}(X)=I\cap J$. Using $f=x^2+y^2+z^2$ one sees that if $p[X]=2$, then $\omega(\mathcal{J}(X))=2$. As in 3.1we can assume that $I=(z)$ and $J=(\psi_1,\psi_2)$ where $\psi_j\in\mathbb{R}\{x,y,z\}$ and $1=\omega(\psi_1)\le\omega(\psi_2)$.

If the initial form of a ψ_i has order 1 and is linearly independent to z, after a change of coordinates, $J=(x,\psi(y,z))$ and we are done. Thus, we can suppose $I=(z)$, $J=(z-2g(x,y),f(x,y))$ where $f,g\in\mathbb{R}\{x,y\}$ and $\omega(f),\omega(g)\ge2$. For all this situations we will find a function germ G in $\Sigma_3(X)\setminus\Sigma_2(X)$. We proceed in several steps:

(3.8) Step I. *If* $\omega(f)\ge3$ *then* $p[X]\ge3$. Replacing z by $z+g$, we can suppose that the equations of X are: $z^2-g^2=0$, $(z+g)f=0$.

Proof. Let $G=(Q_1+az)^2+(Q_2+bz)^2+(Q_3+cz)^2$ where $(a,b,c)\in\mathbb{R}^3\setminus\{0\}$ and Q_1,Q_2,Q_3 are quadratic forms. Suppose that such a G is a sum of two squares. By 3.5 there exist $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2\in\mathbb{R}\{x,y\}$ with $\omega(\beta_2)\ge1$ (maybe $\beta_2=0$, see 2.1) such that

$$\begin{aligned}G&\equiv Q_1^2+Q_2^2+Q_3^2+2(aQ_1+bQ_2+cQ_3)z+(a^2+b^2+c^2)g^2\\&=(\alpha_1+z\beta_1)^2+(\alpha_2+z\beta_2)^2-\gamma_1(z^2-g^2)-2\gamma_2(z+g)f\end{aligned}$$

where $\gamma_1,\gamma_2\in\mathbb{R}\{x,y\}$ by 3.5. Comparing coefficients with respect to z:

0) $Q_1^2+Q_2^2+Q_3^2+(a^2+b^2+c^2)g^2=\alpha_1^2+\alpha_2^2+\gamma_1g^2-2\gamma_2gf$
1) $aQ_1+bQ_2+cQ_3=\alpha_1\beta_1+\alpha_2\beta_2-\gamma_2f$
2) $0=\beta_1^2+\beta_2^2-\gamma_1$

From *0)* we see that $\omega(\alpha_1), \omega(\alpha_2) \geq 2$ and from *1)* we deduce that $\omega(\beta_1) = 0$, that is, $\beta_1(0) = \lambda \neq 0$. By *2)*, $\gamma_1(0) = \lambda^2$ and comparing initial forms in *0)* and *1)* we deduce that

0) $Q_1^2 + Q_2^2 + Q_3^2 + (a^2 + b^2 + c^2)g_2^2 = \mathrm{In}(\alpha_1)^2 + \mathrm{In}(\alpha_2)^2 + \lambda^2 g_2^2$
1) $aQ_1 + bQ_2 + cQ_3 = \mathrm{In}(\alpha_1)\lambda$

where g_2 is the homogeneous component of g of degree 2. If we take $Q_1 = x^2$, $Q_2 = y^2$, $Q_3 = xy$ we obtain

$$\begin{aligned}\lambda^2\left(x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2 - \lambda^2)g_2^2\right)\\ = (ax^2 + by^2 + cxy)^2 + (ux^2 + vy^2 + wxy)^2\end{aligned}$$

for suitable $u, v, w \in \mathbb{R}$. Thus, we obtain the same equations $(*)$ used in the step **(2.3)** of the proof of Th.1.1. As it was seen there, there exists values $a, b, c \in \mathbb{R}$ such that the previous equation is not solvable. Such a G is not a sum of two squares. □

(3.9) Step II. *If* $\omega(f) = 2$, $\omega(g) \geq 2$ *then* $p[X] \geq 3$.

Proof. By 3.4 it is enough to consider the following cases:

(3.9.1) *If* $\mathcal{J}(X) = (zxy, z(z-2g))$ *and* $\mathrm{In}(g) = x^2+y^2, x^2-y^2$ *or* y^2 *there exists an element of the type* $G = (Q_1 + az)^2 + (Q_2 + bz)^2 + (Q_3 + cz)^2 \in \Sigma(X)$ *with* $(a, b, c) \in \mathbb{R}^3 \setminus \{0\}$ *and the* Q_i*'s quadratic forms such that G is not a sum of two squares in* $\mathcal{O}(X)$.

Suppose that such a G is a sum of two squares. By 3.5 there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}\{x, y\}$ with $\omega(\beta_2) \geq 1$ (maybe $\beta_2 = 0$, see 2.1) such that

$$\begin{aligned}G &\equiv Q_1^2 + Q_2^2 + Q_3^2 + 2(aQ_1 + bQ_2 + cQ_3 + (a^2 + b^2 + c^2)g)z\\ &= (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma_1(z^2 - 2zg) - 2\gamma_2 zxy\end{aligned}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}\{x, y\}$ by 3.5. Comparing coefficients with respect to z and proceeding as in **(3.8)**, we obtain the following equation for suitable real numbers $\lambda \neq 0, \mu, u, v, w$:

$$\begin{aligned}\lambda^2(Q_1^2 + Q_2^2 + Q_3^2) = \left(aQ_1 + bQ_2 + cQ_3 - \mu xy + (a^2 + b^2 + c^2 - \lambda^2)\,\mathrm{In}(g)\right)^2\\ + (ux^2 + vy^2 + wxy)^2 \quad (\star).\end{aligned}$$

Now, we make specific choices for each possible initial form of g:

(i) *If* $\mathrm{In}(g) = x^2 + y^2$, *consider* $G = (3x^2 + 2y^2 - 2z)^2 + (3y^2 + x^2 - z)^2 + \left(\frac{21}{10}xy\right)^2$.

After simplifying equation $(\star)$ for this G we obtain

$$\lambda^2\left(10x^4 + \tfrac{2241}{100}x^2y^2 + 13y^4\right) = \left(\mu xy + (2+\lambda^2)(x^2+y^2)\right)^2 + (ux^2+vy^2+wxy)^2.$$

for suitable real numbers $\lambda \neq 0, \mu, u, v, w$. Comparing coefficients we get

$x^4)$ $10\lambda^2 = (2 + \lambda^2)^2 + u^2$
$y^4)$ $13\lambda^2 = (2 + \lambda^2)^2 + v^2$

$$\begin{array}{rl} x^2y^2) & \frac{2241}{100}\lambda^2 = 2(2+\lambda^2)^2 + \mu^2 + 2uv + w^2 \\ x^3y) & 0 = \mu(2+\lambda^2) + uw \\ xy^3) & 0 = \mu(2+\lambda^2) + vw. \end{array}$$

Substracting the two last equations we deduce $w(u-v) = 0$. If $w = 0$ then $\mu = 0$ and adding the two first equations and substracting the third we obtain $\frac{59}{100}\lambda^2 = (u-v)^2$. Combining this with the two first equations we conclude that

$$(u^2+4)(556960000u^4 - 4850867039u^2 + 13493610244) = 0.$$

But this equation has no real root. Thus, $w \neq 0$ and $u = v$. Substracting the two first equations we conclude $3\lambda^2 = 0$, a contradiction. Hence, G is not a sum of two squares in $\mathcal{O}(X)$.

(ii) *If $g = x^2-y^2$, consider $G = (3x^2-2y^2-2z)^2+(-3y^2+x^2-z)^2+(3xy)^2$.*
After simplifying equation ($\star$) for this G we obtain

$$\lambda^2\left(10x^4 - 9x^2y^2 + 13y^4\right) = \left((2+\lambda^2)(x^2-y^2)+\mu xy\right)^2 + (ux^2+vy^2+wxy)^2.$$

for suitable real numbers $\lambda \neq 0$, μ, u, v, w. Comparing coefficients we get

$$\begin{array}{rl} x^4) & 10\lambda^2 = (2+\lambda^2)^2 + u^2 \\ y^4) & 13\lambda^2 = (2+\lambda^2)^2 + v^2 \\ x^2y^2) & -9\lambda^2 = -2(2+\lambda^2)^2 + \mu^2 + 2uv + w^2 \\ x^3y) & 0 = \mu(2+\lambda^2) + uw \\ xy^3) & 0 = -\mu(2+\lambda^2) + vw. \end{array}$$

Adding the two last equations we obtain $w(u+v) = 0$. If $w = 0$ then $\mu = 0$. Adding the three first equations we deduce $14\lambda^2 = (u+v)^2$. Combining this with the two first equations we conclude that $(u^2+4)(3136u^4 - 26015u^2 + 58564) = 0$. But this equation has no real root, and so $w \neq 0$. Thus, $u = -v$ and substracting the two first equations we conclude $3\lambda^2 = 0$, a contradiction. Hence, G is not a sum of two squares in $\mathcal{O}(X)$.

(iii) *If $g = y^2$, consider $G = x^4 + (-3y^2+z)^2 + 9x^2y^2$.*
After simplifying equation ($\star$) for this G we obtain

$$\lambda^2\left(x^4 + 9y^4 + 9x^2y^2\right) = \left((2+\lambda^2)y^2 + \mu xy\right)^2 + (ux^2+vy^2+wxy)^2.$$

for suitable real numbers $\lambda \neq 0$, μ, u, v, w. Here we get

$$\begin{array}{rl} x^4) & \lambda^2 = u^2 \\ y^4) & 9\lambda^2 = (2+\lambda^2)^2 + v^2 \\ x^2y^2) & 9\lambda^2 = \mu^2 + 2uv + w^2 \\ x^3y) & 0 = uw \\ xy^3) & 0 = -\mu(2+\lambda^2) + vw. \end{array}$$

Since $\lambda \neq 0$, $u \neq 0$ and $w = 0$, hence $\mu = 0$. Thus, $v = \frac{9}{2}u$ and hence $9u^2 = (2+u^2)^2 + (\frac{9}{2}u)^2$ or equivalently $4u^4 + 61u^2 + 16 = 0$, a contradiction. Hence, G is not a sum of two squares in $\mathcal{O}(X)$.

(3.9.2) *If $\mathcal{J}(X) = (z(y^2-x^k), z(z-x^2))$, $k \geq 3$ there exists $G \in \Sigma(X) \setminus \Sigma_2(X)$.*

Just to apply 3.6 using the same notation, we interchange the variables x and y, and we obtain $\mathcal{J}(X) = (z(x^2-y^k), z(z-y^2)), k \geq 3$. We distinguish two subcases:

(i) *If k is odd we take $G = (y^2 + xy - z)^2 + (xy)^2 + (x^2)^2$.*

Note that $X = X_1 \cup \{z = 0\}$ where X_1 is the curve germ parametrized by $t \mapsto (t^k, t^2, t^4)$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let $\eta_{1\varepsilon} = (u_1 + \varepsilon i u_2)x$, $\eta_{2\varepsilon} = (v_1 + \varepsilon i v_2)y$ where $u_i, v_i \in \mathbb{R}$, $\varepsilon = \pm 1$ be the roots of the Weierstrass polynomial $G(x, y, 0)$. Since

$$\begin{aligned} y^4 + 2y^3x + 2y^2x^2 + x^4 = Q &= \|(y - \eta_{11})(y - \eta_{2\varepsilon})\|^2 \\ &= \|y^2 - (\eta_{11} + \eta_{2\varepsilon})xy + (\eta_{11}\eta_{2\varepsilon})x^2\|^2 \end{aligned}$$

we deduce that $\eta_{11} + \eta_{2\varepsilon} = -1 - ib_2$, $b_2 \in \mathbb{R}$. Write $\eta_{11}\eta_{2\varepsilon} = c = c_1 + ic_2 \in \mathbb{C}$. By 3.6, there exist $\beta \in \mathbb{C}\{x, y\}$ such that

$$\begin{aligned} 2t^{2k+4} + t^{4k} = G(t^k, t^2, t^4) &= \|t^4 + (1 + ib_2)t^{k+2} + ct^{2k} + t^4\beta(t^k, t^2)\|^2 \\ &= \|t^4(1 + \beta(t^k, t^2)) + (1 + ib_2)t^{k+2} + ct^{2k}\|^2. \end{aligned}$$

Hence $\omega(1 + \beta(t^k, t^2)) \geq k - 2$. Since k is odd, $k - 2$ is not in the semigroup of the curve germ (t^k, t^2) (see [JP]). Thus, in fact, $\omega(1 + \beta(t^k, t^2)) \geq k - 1$. Therefore, comparing initial forms in the previous equation we deduce $2 = 1 + b_2^2$ and hence we can suppose $b_2 = 1$. Thus, $(y^2 + xy)^2 + (xy)^2 + (x^2)^2 = (y^2 + xy + c_1x^2)^2 + (xy + c_2x^2)^2$ which transforms into $x^4(c_1^2 + c_2^2 - 1) + 2x^3y(c_1 + c_2) + 2x^2y^2c_1 = 0$, a contradiction.

(ii) *If $k = 2n$ is even $n \geq 2$ we take $G = (y^2 + xy - z(1 + y^{n-1}))^2 + (xy - zy^{n-1})^2 + (x^2)^2$.*

Note that $X = X_1 \cup \{z = 0\}$ where X_1 is the union of the curve germs parametrized by $t \mapsto (\varepsilon t^n, t^2, t^4)$, $\varepsilon = \pm 1$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Procedding as in **(3.9.2.*i*)** and applying 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{y\}$ such that

$$\begin{aligned} t^{4k} = G(t^n, t, t^2) &= \|t^2 + (1 + ib_2)t^{n+1} + ct^{2n} + t^2(\lambda(t) + t^n\mu(t))\|^2 \\ &= \|t^2(1 + \lambda(t) + (1 + ib_2)t^{n-1}) + t^{n+2}\mu(t) + ct^{2n}\|^2. \end{aligned}$$

Hence $\omega(1 + \lambda(t) + (1 + ib_2)t^{n-1}) \geq n$ and we can write $\lambda(t) = -1 - (1 + ib_2)t^{n-1} + g(t)t^n$, $g \in \mathbb{C}\{t\}$. On the other hand,

$$\begin{aligned} 8t^{2n+2} + t^{4n} = G(-t^n, t, t^2) &= \|t^2 - (1 + ib_2)t^{n+1} + ct^{2n} + t^2(\lambda(t) - t^n\mu(t))\|^2 \\ &= \| - 2(1 + ib_2)t^{n+1} - t^{n+2}\mu(t) + ct^{2n}\|^2. \end{aligned}$$

Comparing initial forms we get $8 = 4 + 4b_2^2$, and we can suppose $b_2 = 1$. In a similar way to **(3.9.2.*i*)**, we get a contradiction. □

(3.9.3) *If $\mathcal{J}(X) = (z(y^2 - x^k), z(z - xy))$, $k \geq 3$ there exists $G \in \Sigma(X) \setminus \Sigma_2(X)$.*

Take $G = (y^2 - x^k)^2 + (x^k - \varepsilon_k zx^{k/2-1})^2 + (xy - z)^2$, where $\varepsilon_k = 0$ if k is odd and 1 otherwise. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let

$$\eta_{1\varepsilon} = x^{k-1}g_1(x) + \varepsilon i(x + x^{k-1}g_2(x)),\ \ \eta_{2\varepsilon} = x^{2k-3}h_1 + \varepsilon i(\sqrt{2}x^{k-1} + x^{2k-3}h_2)$$

be the roots of the Weierstrass polynomial $G(x, y, 0)$, where $g_i, h_i \in \mathbb{R}\{x\}, \varepsilon = \pm 1$. By 3.6 and 3.7, $G = \|\alpha + z\beta\|^2 \mod (\mathcal{J}(X))$ where

$$\alpha = y^2 \pm \sqrt{2}x^k + yx^{k-1}\varphi_1(x) + x^{2k-2}\psi_1(x) + i(-xy + yx^{k-1}\varphi_2(x) + x^{2k-2}\psi_2(x))$$

for suitable series $\varphi_i, \psi_i \in \mathbb{R}\{x\}$ and $\beta = \lambda + y\mu$ for some $\lambda, \mu \in \mathbb{C}\{t\}$. We distinguish two subcases:

(i) *k odd.* In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the curve germ parametrized by $t \mapsto (t^2, t^k, t^{k+2})$. We have

$$t^{4k} = G(t^2, t^k, t^{k+2}) = \|(1 \pm \sqrt{2})t^{2k} + t^{3k-2}\rho(t) + t^{k+2}(-i + \lambda(t^2) + t^k\mu(t^2))\|^2$$

for some $\rho \in \mathbb{C}\{t\}$. Since k is odd, we conclude $1 = (1 \pm \sqrt{2})^2$, a contradiction.

(ii) *$k = 2n$ even.* In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the union of the curve germs parametrized by $t \mapsto (t, \varepsilon t^n, \varepsilon t^{n+2})$, $\varepsilon = \pm 1$. We have

$$0 = G(t, t^n, t^{n+1}) = \|(1 \pm \sqrt{2})t^{2n} + t^{3n-2}\rho(t) + t^{n+1}(-i + \lambda(t) + t^n\mu(t))\|^2$$

for some $\rho \in \mathbb{C}\{t\}$. Hence $\lambda(t) = i - (1 \pm \sqrt{2})t^{n-1} + t^n\theta$, $\theta \in \mathbb{C}\{t\}$. Moreover,

$$4t^{4n} = G(t, -t^n, -t^{n+1}) = \|(1 \pm \sqrt{2})t^{2n} + t^{3n-2}\rho'(t) - t^{n+1}(-i + \lambda(t) - t^n\mu(t))\|^2$$

for some $\rho' \in \mathbb{R}\{t\}$. Putting all together, we conclude that $4 = 4(1 \pm \sqrt{2})^2$, a contradiction. □

(3.9.4) *If $\mathcal{J}(X) = (z(z - x(y + x^\ell)), z(y^2 - x^k))$, $2 \le \ell < k$, $3 \le k \ne 2\ell$ there exists $G \in \Sigma(X) \setminus \Sigma_2(X)$.*

We distinguish four different situations:

(i) *If k is odd and $2\ell > k$, we take $G = y^4 + (x^{k-l-1}(xy - z))^2 + x^{2k}$.* In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the curve germ parametrized by $t \mapsto (t^2, t^k, t^{k+2} + t^{2\ell+2})$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let

$$\eta_{1\varepsilon} = x^{5\ell-2k}g_1 + \varepsilon i(x^\ell + x^{5\ell-2k}g_2), \quad \eta_{2\varepsilon} = x^{3\ell-k}h_1 + \varepsilon i(x^{k-l} + x^{3\ell-k}h_2)$$

be the roots of the Weierstrass polynomial $G(x, y, 0)$, where $g_i, h_i \in \mathbb{R}\{x\}, \varepsilon = \pm 1$. By 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{x\}$ such that

$$\begin{aligned}
3t^{4k} &\equiv G(t^2, t^k, t^{k+2} + t^{2\ell+2}) \\
&\equiv \|(t^k - it^{2\ell})(t^k + \varepsilon it^{2k-2\ell}) + (t^{k+2} + t^{2\ell+2})(\lambda(t^2) + t^k\mu(t^2))\|^2 \\
&\equiv \|(1 + \varepsilon)t^{2k} + i(\varepsilon t^{3k-2\ell} - t^{k+2\ell}) + (t^{k+2} + t^{2\ell+2})\lambda(t^2)\|^2 \mod (t^{4k+1}).
\end{aligned}$$

Since k is odd we get, comparing orders and initial forms, $\lambda(t^2) = -i\varepsilon t^{2(k-\ell)-2} + \cdots$. Hence, $3 = (1 + \varepsilon)^2 + (\varepsilon)^2$, a contradiction.

(ii) *If k is odd and $2\ell < k$, we take $G = (y^2 + x^{\ell+1} - z)^2 + (xy - x^{\ell+1} + z)^2 + (x^{\ell+1} - z)^2$.* In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the curve germ parametrized by $t \mapsto (t^2, t^k, t^{2\ell+2} + t^{k+2})$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let

$$\eta_{1\varepsilon} = x^\ell g_1 + \varepsilon i(x + x^\ell g_2), \quad \eta_{2\varepsilon} = -x^\ell + x^{2\ell-1}h_1 + \varepsilon i(\sqrt{2}x^\ell + x^{2\ell-1}h_2)$$

be the roots of the Weierstrass polynomial $G(x, y, 0)$, where $g_i, h_i \in \mathbb{R}\{x\}, \varepsilon = \pm 1$. By 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{x\}$ such that

$$\begin{aligned} 6t^{2k+4} - 2t^{3k+2} + t^{4k} &= G(t^2, t^k, t^{2\ell+2} + t^{k+2}) \\ &= \|(t^k - t^{2\ell} g_1(t^2) - i(t^2 + t^{2\ell} g_2(t^2)))(t^k + t^{2\ell} - t^{4\ell-2} h_1(t^2) \\ &\quad + \varepsilon i(\sqrt{2} t^{2\ell} + t^{4\ell-2} h_2(t^2))) + (t^{2\ell+2} + t^{k+2})(\lambda(t^2) + t^k \mu(t^2))\|^2. \end{aligned}$$

Since $k > 2\ell$, comparing initial forms, we deduce that $\lambda(0) = i - \varepsilon\sqrt{2}$. Moreover, using that k is odd and $\ell \geq 2$ we conclude that $6 = |-i + \lambda(0)|^2 = 2$, a contradiction.

(iii) *If $k = 2n$ is even and $\ell > n$, we take*

$$G = \left(y^2 - \frac{zx^{n-1}}{1 + x^{\ell-n}}\right)^2 + \left(x^{k-\ell-1}(xy - z) + \frac{zx^{n-1}}{1 + x^{\ell-n}}\right)^2 + \left(x^k - \frac{zx^{n-1}}{1 + x^{\ell-n}}\right)^2.$$

In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the union of the curve germs parametrized by $t \mapsto (t, \varepsilon t^n, \varepsilon t^{n+1} + t^{\ell+1})$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let $\eta_{1\varepsilon}, \eta_{2\varepsilon}$ be the roots of the Weierstrass polynomial $G(x, y, 0)$ which are the same obtained in **(3.9.4.*i*)**. By 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{x\}$ such that

$$\begin{aligned} 0 \equiv G(t, t^n, t^{n+1} + t^{\ell+1}) &\equiv \|(t^n - it^\ell)(t^n + \varepsilon i t^{2n-\ell}) \\ &\qquad + (t^{n+1} + t^{\ell+1})(\lambda(t) + t^n \mu(t))\|^2 \\ &\equiv \|(1 + \varepsilon)t^{2n} + i(\varepsilon t^{3n-\ell} - t^{n+\ell}) + (t^{n+1} + t^{\ell+1})(\lambda(t) + t^n \mu(t))\| \\ &\qquad \mod (t^{4n+1}). \end{aligned}$$

Comparing orders and initial forms, we get $\lambda(t) = -\varepsilon i t^{2n-\ell-1} - (1 + \varepsilon - i\varepsilon)t^{n-1} + \cdots$. Again by 3.6 we have

$$\begin{aligned} 12t^{4n} &\equiv G(t, -t^n, -t^{n+1} + t^{\ell+1}) \\ &\equiv \|(-t^n - it^\ell)(-t^n + \varepsilon i t^{2n-\ell}) + (-t^{n+1} + t^{\ell+1})(\lambda(t) - t^n \mu(t))\|^2 \\ &\equiv \|(2 + 2\varepsilon - 2\varepsilon i)t^{2n}\|^2 \mod (t^{4n+1}) \end{aligned}$$

which is impossible.

(iv) *If $k = 2n$ is even and $\ell < n$, we take*

$$\begin{aligned} G = &\left(y^2 + x^{2\ell} - z\frac{x^{\ell-1} + yx^{n-\ell-1}}{1 + x^{n-\ell}}\right)^2 + \left(x^\ell y - z\frac{x^{n-1}}{1 + x^{n-\ell}}\right)^2 \\ &+ (x^{2\ell} + x^{n+\ell} - zx^{\ell-1})^2. \end{aligned}$$

In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the union of the curve germs parametrized by $t \mapsto (t, \varepsilon t^n, \varepsilon t^{n+1} + t^{\ell+1})$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let

$$\begin{aligned} \eta_{1\varepsilon} &= x^{2n-\ell} g_1 + \varepsilon i(x^\ell + x^n + x^{2n-\ell} g_2), \\ \eta_{2\varepsilon} &= x^{2n-\ell} h_1 + \varepsilon i(\sqrt{2} x^\ell - \sqrt{2}/2 x^n + x^{2n-\ell} h_2) \end{aligned}$$

be the roots of the Weierstrass polynomial $G(x, y, 0)$, where $g_i, h_i \in \mathbb{R}\{x\}, \varepsilon = \pm 1$. By 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{x\}$ such that

$$\begin{aligned}
0 &\equiv G(t, t^n, t^{\ell+1} + t^{n+1}) \\
&\equiv \|(t^n - i(t^l + t^n))(t^n + \varepsilon i(\sqrt{2}t^\ell - \sqrt{2}/2t^n)) + (t^{\ell+1} + t^{n+1})(\lambda(t) + t^n\mu(t))\|^2 \\
&\equiv \|\varepsilon\sqrt{2}t^{2\ell} + (\varepsilon\sqrt{2}/2 + i(\varepsilon\sqrt{2} - 1))t^{n+l} + (1 - \varepsilon\sqrt{2}/2 - i(1 + \varepsilon\sqrt{2}/2))t^{2n} \\
&\quad + (t^{\ell+1} + t^{n+1})(\lambda(t) + t^n\mu(t))\|^2 \mod (t^{2\ell+2n+1}).
\end{aligned}$$

Hence, we get that $\lambda(t) = -\varepsilon\sqrt{2}t^{\ell-1} + t^{n-1}(\varepsilon\sqrt{2}/2 + i(1 - \varepsilon\sqrt{2})) + \cdots$. Again by 3.6 we have

$$\begin{aligned}
12t^{2\ell+2n} &\equiv G(t, -t^n, t^{\ell+1} - t^{n+1}) \equiv \|(-t^n - i(t^l + t^n)) \\
&\qquad \times (-t^n + \varepsilon i(\sqrt{2}t^\ell - \sqrt{2}/2t^n)) + (t^{\ell+1} - t^{n+1})(\lambda(t) + t^n\mu(t))\|^2 \\
&\equiv \|2(\varepsilon\sqrt{2} + i(1 - \varepsilon\sqrt{2}))t^{n+\ell}\|^2 \mod (t^{2\ell+2n+1}),
\end{aligned}$$

which is impossible. □

(3.9.5) *If $\mathcal{J}(X) = (zy(y+x^\ell), z(z-x(y+b(x)x^\ell)))$, $\ell \geq 2$, $b \in \mathbb{R}\{x\}$, $b(0) \neq 0, 1$ there exists $G \in \Sigma(X) \setminus \Sigma_2(X)$.*

We take $G = (y^2 - x^{2\ell})^2 + 2\left(x^\ell y - \frac{zx^{\ell-1}}{1-b(x)}\right)^2 + 3\left(x^{2\ell} + \frac{zx^{\ell-1}}{1-b(x)}\right)^2$. In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the union of the curve germs parametrized by $t \mapsto (t, -t^\ell, -t^{\ell+1}(1 - b(t)))$ and $t \mapsto (t, 0, t^{\ell+1}b(t))$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let $\eta_{1\varepsilon} = (1+\varepsilon i)x^\ell$, $\eta_{2\varepsilon} = (-1+i)x^\ell$ be the roots of the Weierstrass polynomial $G(x, y, 0)$. By 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{x\}$ such that

$$\begin{aligned}
0 &= G(t, -t^\ell, -t^{\ell+1}(1 - b(t))) \\
&= \|(-t^\ell - (1+i)t^\ell)(-t^\ell - (-1+\varepsilon i)t^\ell) - t^{\ell+1}(1 - b(t))(\lambda(t) - t^\ell\mu(t))\|^2 \\
&= \|t^{2\ell}\varepsilon(2i - 1) - t^{\ell+1}(1 - b(t))(\lambda(t) - t^\ell\mu(t))\|^2.
\end{aligned}$$

Hence, we deduce that $\lambda(t) = t^{\ell-1}\varepsilon(2i - 1)/(1 - b(0)) + \cdots$. Again by 3.6 we get

$$\begin{aligned}
&\frac{t^{4\ell}(4 - 2b(t) + 3b(t)^2)}{(b(t) - 1)^2} \\
&\quad = G(t, 0, t^{\ell+1}b(t)) = \|(-(1+i)t^\ell)(-(-1+\varepsilon i)t^\ell) + t^{\ell+1}b(t)\lambda(t)\|^2 \\
&\quad = \|t^{2\ell}(-1 - \varepsilon + (-1+\varepsilon)i) + t^{\ell+1}b(t)\lambda(t)\|^2.
\end{aligned}$$

Comparing initial forms, we conclude

$$\begin{aligned}
4 - 2b(0) + 3b(0)^2 &= |(-1 - \varepsilon + (-1+\varepsilon)i)(1 - b(0)) + b(0)\varepsilon(2i - 1)|^2 \\
&= 4 - 2b(0) - 2\varepsilon b(0) + 3b(0)^2 + 2\varepsilon b(0)^2.
\end{aligned}$$

Hence, $b(0) - b(0)^2 = 0$, a contradiction. □

(3.9.6) *If $\mathcal{J}(X) = (zy(y + x^\ell), z(z - x(y + \delta x^{\ell+n})))$, $\delta = \pm 1$, $\ell \geq 2$, $n \geq 1$ there exists $G \in \Sigma(X) \setminus \Sigma_2(X)$.*

We take $G = (y^2)^2 + 2(x^\ell y)^2 + (x^{2\ell+n} - \delta z x^{\ell-1})^2$. In this case $X = X_1 \cup \{z = 0\}$ where X_1 is the union of the curve germs parametrized by $t \mapsto (t, 0, \delta t^{\ell+n+1})$ and $t \mapsto (t, -t^\ell, -t^{\ell+1} + \delta t^{\ell+n+1})$. Suppose that G is a sum of two squares in $\mathcal{O}(X)$. Let

$$\eta_{1\varepsilon} = x^{\ell+2n} g_1 + \varepsilon i (x^\ell \sqrt{2} + x^{\ell+2n} g_2), \quad \eta_{2\varepsilon} = x^{\ell+3n} h_1 + \varepsilon i (\sqrt{2}/2\, x^{\ell+n} + x^{\ell+3n} h_2)$$

be the roots of the Weierstrass polynomial $G(x, y, 0)$, where $g_i, h_i \in \mathbb{R}\{x\}$, $\varepsilon = \pm 1$. By 3.6 and 3.7 there exist $\lambda, \mu \in \mathbb{C}\{x\}$ such that

$$\begin{aligned} 0 \equiv G(t, 0, \delta t^{\ell+n+1}) &\equiv \| i (t^\ell \sqrt{2}) \varepsilon i (\sqrt{2}/2\, t^{\ell+n}) + \delta t^{\ell+n+1} \lambda(t) \|^2 \\ &\equiv \| - \varepsilon t^{2\ell+n} + \delta t^{\ell+n+1} \lambda(t) \|^2 \mod (t^{4\ell+2n+1}). \end{aligned}$$

Hence, we get that $\lambda(t) = \varepsilon \delta t^{\ell-1} + \cdots$. Again by 3.6 we get

$$\begin{aligned} 4t^{4\ell} &\equiv G(t, -t^\ell, -t^{\ell+1} + \delta t^{\ell+n+1}) \\ &\equiv \| (-t^\ell - i t^\ell \sqrt{2})(-t^\ell - \varepsilon i \sqrt{2}/2\, t^{\ell+n}) - (t^{\ell+1} - \delta t^{\ell+n+1})(\lambda(t) - t^\ell \mu(t)) \|^2 \\ &\equiv \| (1 + \sqrt{2} i - \varepsilon \delta) t^{2\ell} \|^2 \mod (t^{4\ell+1}) \end{aligned}$$

and therefore, $2 = (1 - \varepsilon\delta)^2$, a contradiction. □

(3.9.7) *If $\mathcal{J}(X) = (z(x^2 - y^k), z(z - 2g))$, $k \geq 2$ and $g = y^{\ell+1}(xa(y) + b(y))$ for some $\ell \geq 1$ and $a, b \in \mathbb{R}\{y\}$ such that $\omega(a^2 + b^2) = 0$ and $\omega(g') \geq 3$, there exists $G \in \Sigma(X) \setminus \Sigma_2(X)$.*

Let $\psi = \sum_{i=1}^r (a_i + z b_i)^2$ ($a_i, b_i \in \mathbb{R}\{x, y\}$) be a sum of three squares which is not a sum of squares in $\mathcal{O}(X')$ of $X' : \ zf = 0, z(z + (xa(y) + b(y))y) = 0$. Such a ψ exists because X' is equivalent to a germ in one of the previous cases **(3.9.1)** to **(3.9.6)**. We claim that function germ in $\mathcal{O}(X)$ given by $\varphi = \sum_{i=1}^r (a_i y^\ell + z b_i)^2$ is in $\Sigma(X) \setminus \Sigma_2(X)$. Indeed, if $\varphi \in \Sigma_2(X)$ then

0) $\sum_i a_i^2 y^{2\ell} = \alpha_1^2 + \alpha_2^2$

1) $\sum_i a_i y^\ell b_i + \sum_i b_i^2 y^\ell (xa(y) + b(y)) y = y^\ell (xa(y) + b(y)) y (\beta_1^2 + \beta_2^2) + \gamma_2 f + \alpha_1 \beta_1 + \alpha_2 \beta_2.$

for certain $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}\{x, y\}$. From *0)* we deduce that $y^\ell | \alpha_j$, and hence, from *1)*, $y^\ell | \gamma_2$. Therefore we can write $\alpha_j = y^\ell \alpha'_j$, $\gamma_2 = y^\ell \gamma'_2$ and then we have

0) $\sum_i a_i^2 = (\alpha'_1)^2 + (\alpha'_2)^2$

1) $\sum_i a_i b_i + \sum_i b_i^2 (xa(y) + b(y)) y = (xa(y) + b(y)) y (\beta_1^2 + \beta_2^2) + \gamma'_2 f + \alpha'_1 \beta_1 + \alpha'_2 \beta_2.$

which means that $\psi \in \Sigma_2(X')$, a contradiction. Hence, φ is not a sum of two squares in $\mathcal{O}(X)$, and we are done. □

Thus, putting all together we conclude that, after a change of coordinates, X is contained in the union of two transversal planes. □

4. Examples in higher embedding dimension

In this section we discuss the examples X_n (Veronese cones), Y_n (generalized Whitney's umbrellas) and Z_n. We begin by proving that:

Theorem 4.1. $\mathcal{P}(X_n) = \Sigma_2(X_n)$.

First, we show:

Lemma 4.2. *For every $f \in \mathbb{R}\{x_0, x_1, \dots, x_n\}$ there exist $f_0, f_1, \dots, f_{n-1} \in \mathbb{R}\{x_n\}$ and $g \in \mathbb{R}\{x_0, x_1, \dots, x_n\}$ such that $f = f_0(x_n) + \sum_{i=1}^{n-1} f_i(x_n)x_i + x_0 g$ mod $\mathcal{J}(X_n)$.*

Proof. We consider for every $\nu = (\nu_1, \dots, \nu_{n-1})$ the homogeneous polynomial

$$G_\nu = x_1^{\nu_1} \cdots x_n^{\nu_{n-1}} - x_0^{d-1-k} x_n^k x_i$$

where $d = |\nu|$, $0 \le i < n$ and $\sum_{j=1}^{n-1} j\nu_j = nk + i$. Since

$$\begin{aligned} G_\nu \circ \gamma &= \prod_{j=1}^{n-1} (z^{n-j} w^j)^{\nu_j} - z^{n(d-1-k)} w^{nk} z^{n-i} w^i \\ &= z^{nd-nk-i} w^{nk+i} - z^{n(d-k)-i} w^{nk+i} = 0 \end{aligned}$$

we see that $G_\nu \in \mathcal{J}(X_n)$. For $\nu = (0, \dots, 1, \dots, 0)$ we get $x_i^n - x_n^i x_0^{n-i} \in \mathcal{J}(X_n)$. Therefore, we divide $f \in \mathbb{R}\{x_0, \dots, x_n\}$ succesively by these polynomials $x_i^n - x_n^i x_0^{n-i}$ until we obtain

$$f = \sum_{0 \le \nu_1, \dots, \nu_{n-1} < n} a_\nu(x_0, x_n) x_1^{\nu_1} \cdots x_{n-1}^{\nu_{n-1}} \qquad \text{mod } \mathcal{J}(X_n).$$

Furthermore, $G_\nu \in \mathcal{J}(X_n)$ means $x_1^{\nu_1} \cdots x_n^{\nu_{n-1}} = x_0^{d-1-k} x_n^k x_i \quad \text{mod } \mathcal{J}(X_n)$, and we obtain $b_0, b_1, \dots, b_{n-1} \in \mathbb{R}\{x_0, x_n\}$ such that $f = b_0(x_0, x_n) + \sum_{i=1}^{n-1} b_i(x_0, x_n)x_i \quad \text{mod } \mathcal{J}(X_n)$.

Finally, since $b_i(x_0, x_n) = f_i(x_n) + x_0 g_i(x_0, x_n)$, there exists $g \in \mathbb{R}\{x_0, x_1, \dots, x_n\}$ such that $f = f_0(x_n) + \sum_{i=1}^{n-1} f_i(x_n)x_i + x_0 g \quad \text{mod } \mathcal{J}(X_n)$. □

To prove that $\mathcal{P}(X_n) = \Sigma_2(X_n)$ we need the following polynomial reduction lemma.

Lemma 4.3. *For every $f \in \mathcal{P}(X_n)$ and every $k \ge 1$ there exists a polynomial f_k positive semidefinite on the algebraic surface S_n given by the same equations as X_n such that $\omega(f - f_k) > k$.*

Proof. We parametrize S_n as follows. If n is odd, we take the complex parametrization γ, which maps $\mathbb{R}^2$ over S_n. We write $\gamma_+ = \gamma|_{\mathbb{R}^2}$. If n is even γ_+ only parametrizes $S_n \cap \{x_0 \ge 0\}$ and we must use $\gamma_- = -\gamma_+$ to parametrize $S_n \cap \{x_0 < 0\}$.

Now, choose $k \geq 1$ and $f \in \mathcal{P}(X_n)$ and set $g_k = f + (x_0^2 + x_1^2 + \cdots + x_n^2)^k$. We claim that $g_k + (x_0, x_1, \ldots, x_n)^r \subset \mathcal{P}^+(X_n)$ for $r \geq 2k$ big enough.

Indeed, the germs $g_k \circ \gamma_+$ and $g_k \circ \gamma_-$ are positive definite in $\mathbb{R}^2$. In [Fe2, 3.1] we showed that if a function germ g is positive semidefinite in a semianalytic germ Z of $\mathbb{R}^2$ there exits an integer r such that all function germs in $g + \mathfrak{m}_2^r$ are also postive semidefinite in Z. Thus, in our case, there exist $r \geq 2k$ such that $g_k \circ \gamma_+ + (s,t)^{rn}, g_k \circ \gamma_- + (s,t)^{rn} \subset \mathcal{P}^+(\mathbb{R}^2)$ from which the claim follows.

We consider now the $(r-1)$-jet h_k of g_k, which, as we have seen, is positive definite in X_n. Therefore, there exists $\varepsilon > 0$ such that h_k is ≥ 0 in $S_n \cap B_\epsilon(0)$. On the other hand, if $y \in S_n \cap \mathbb{R}^{n+1} \setminus B_\varepsilon(0)$, then $\|y\| \geq \varepsilon$, and we deduce

$$\begin{aligned}|h_k(y)| &= \Big| \sum_{0 \leq |\nu| \leq r-1} a_\nu y^\nu \Big| \leq \sum_{0 \leq |\nu| \leq r-1} |a_\nu||y_0^{\nu_0}||y_1^{\nu_1}| \ldots |y_n^{\nu_n}| \\ &\leq \sum_{0 \leq |\nu| \leq r-1} |a_\nu| \|y\|^{|\nu|} \leq \sum_{0 \leq |\nu| \leq r-1} \frac{|a_\nu|}{\varepsilon^{2r-|\nu|}} \|y\|^{2r} \leq M \|y\|^{2r}\end{aligned}$$

Hence, the polynomial $f_k = h_k + M(x_0^2 + x_1^2 + \ldots + x_n^2)^r$ is ≥ 0 on S_n and $\omega(f - f_k) > k$. $\square$

Now we proceed with the

Proof of Theorem 4.1. From the previous lemmas and the M. Artin's Approximation Theorem, it suffices to prove that every polynomial f which is positive semidefinite on S_n is a sum of two squares of analytic function germs on X_n. To that end, we consider the biregular equivalence

$$\begin{aligned}\phi_n : \mathbb{R}^2 \setminus \{x_0 = 0\} &\to S_n \setminus \{x_0 = 0\} \\ (x_0, x_1) &\mapsto \Big(x_0, x_1, \frac{x_1^2}{x_0}, \ldots, \frac{x_1^k}{x_0^{k-1}}, \ldots, \frac{x_1^n}{x_0^{n-1}}\Big)\end{aligned}$$

whose inverse π is the obvious projection. Now, let

$$g = f \circ \phi_n = f\Big(x_0, x_1, \frac{x_1^2}{x_0}, \ldots, \frac{x_1^k}{x_{k-1}}, \ldots, \frac{x_1^n}{x_0^{n-1}}\Big) = \frac{P(x_0, x_1)}{x_0^{2r}},$$

where $r \geq 0$, and $P \in \mathbb{R}[x_0, x_1]$ is ≥ 0 on $x_0 \neq 0$, hence on $\mathbb{R}^2$. Since $\mathcal{P} = \Sigma_2$ in $\mathbb{R}\{x_1, x_2\}$, we have $x_0^{2r} g = P = a^2 + b^2, \quad a, b \in \mathbb{R}\{x_0, x_1\}$. Thus, composing with π we obtain

$$x_0^{2r} f = (a^2 + b^2) \quad \mod \mathcal{J}(X_n). \quad (*)$$

In view of 4.2, there exist power series $a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}\{x_n\}$ and $\alpha, \beta \in \mathbb{R}\{x_0, x_1, \ldots, x_n\}$ such that

$$a \equiv a_0(x_n) + \sum_{i=1}^{n-1} a_i(x_n) x_i + x_0 \alpha, \quad b \equiv b_0(x_n) + \sum_{i=1}^{n-1} b_i(x_n) x_i + x_0 \beta \quad \mod \mathcal{J}(X_n).$$

Hence,

$$x_0^{2r} f = \left(a_0(x_n) + \sum_{i=1}^{n-1} a_i(x_n)x_i + x_0\alpha \right)^2 + \left(b_0(x_n) + \sum_{i=1}^{n-1} b_i(x_n)x_i + x_0\beta \right)^2 \mod \mathcal{J}(X_n).$$

Substituting γ_+ we get

$$s^{2rn}(f \circ \gamma_+) = \left(a_0(t^n) + \sum_{i=1}^{n-1} a_i(t^n)s^{n-i}t^i + s^n(\alpha \circ \gamma_+) \right)^2 + \left(b_0(t^n) + \sum_{i=1}^{n-1} b_i(t^n)s^{n-i}t^i + s^n(\beta \circ \gamma_+) \right)^2$$

and counting orders respect to s

$$\operatorname{ord}_s \left(a_0(t^n) + \sum_{i=1}^{n-1} a_i(t^n)s^{n-i}t^i + s^n(\alpha \circ \gamma_+) \right) \geq rn,$$
$$\operatorname{ord}_s \left(b_0(t^n) + \sum_{i=1}^{n-1} b_i(t^n)s^{n-i}t^i + s^n(\beta \circ \gamma_+) \right) \geq rn.$$

Thus, we deduce that $a_i(t^n), b_i(t^n) = 0$ for $i = 0, \dots n-1$. Hence, $a_i, b_i = 0$ for $i = 0, \dots n-1$. Therefore, $x_0^{2r} f = x_0^2(\alpha^2 + \beta^2) \mod \mathcal{J}(X_n)$. Since $x_0 \notin \mathcal{J}(X_n)$ and this ideal is prime, we conclude $x_0^{2r-2} f = (\alpha^2 + \beta^2) \mod \mathcal{J}(X_n)$. We can begin again the argument from $(*)$ and, at the end, we will obtain $f \in \Sigma_2(X_n)$, as wanted. □

Next we turn to the generalized Whitney umbrellas:

Theorem 4.4. $\mathcal{P}(Y_n) = \Sigma_2(Y_n)$.

Proof. The parametrization $\varphi_n : (s,t) \mapsto (s, st, \dots, st^{n-1}, t^n) = (x_0, x_1, \dots, x_{n-1}, x_n)$ induces a homomorphism of rings

$$\begin{aligned} \varphi_n^* : \mathcal{O}(Z_n) &\to \mathbb{R}\{s,t\} \\ f &\mapsto f \circ \varphi_n. \end{aligned}$$

which is finite, injective and $(s)\ \mathbb{R}\{s,t\} \subset \operatorname{im} \psi_n$. The last remark, because $s^i t^j = s^{i-1}(st^r)t^{nq} = \psi_n(x_0^{i-1}x_r x_n)$ where $j = nq + r$, $0 \leq r < n$.

Let $f \in \mathcal{P}(Y_n)$ and consider $f \circ \varphi_n$. Since f is a psd in $\mathcal{O}(Y_n)$, $f \circ \varphi_n$ is psd in $\mathbb{R}\{s,t\}$. Thus, there exist $\alpha_1, \alpha_2, \in \mathbb{R}\{s,t\}$ and $\beta_1, \beta_2 \in \mathbb{R}\{t\}$ such that

$$f \circ \varphi_n \equiv (\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2.$$

It is clear that $C_n = Z_n \cap \{x_0 = 0\}$ is the line $x_0 = 0, \dots, x_{n-1} = 0$, which has Pythagoras number 1. Setting $g = f|_{C_n} \in \mathcal{P}(C_n)$, there exists $g_1 \in \mathbb{R}\{x_n\}$ such that $g \equiv g_1^2 \mod \mathcal{J}(C_n)$. Therefore, if $\gamma_1 = g_1 \circ \varphi_n(0, t)$, $i = 1, 2$, we deduce

$$\beta_1^2 + \beta_2^2 = f \circ \varphi_n(0, t) = f|_{C_n} \circ \varphi_n(0, t) = g_1^2 \circ \varphi_n(0, t) = \gamma_1^2,$$

and $\dfrac{\gamma_1}{\beta_1 + i\beta_2}, \dfrac{\gamma_1}{\beta_1 - i\beta_2}$ are two units in $\mathbb{C}\{t\}$ whose product is 1. Hence,

$$\begin{aligned}
&(\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2 \\
&\qquad = (\alpha_1 s + i\alpha_2 s + \beta_1 + i\beta_2)) \left(\frac{\gamma_1}{\beta_1 + i\beta_2}\right) \left(\frac{\gamma_1}{\beta_1 - i\beta_2}\right) \\
&\qquad\quad \times (\alpha_1 s - i\alpha_2 s + \beta_1 - i\beta_2)) \\
&\qquad = ((a_1 s + \gamma_1) + i a_2 s)((a_1 s + \gamma_1) - i a_2 s) \\
&\qquad = (a_1 s + \gamma_1)^2 + (a_2 s)^2,
\end{aligned}$$

with $a_1, a_2 \in \mathbb{R}\{s, t\}$. Now, using that $(s)\mathbb{R}\{s, t\} \subset \operatorname{im} \psi_n$ and that $\gamma_1(t) = g_1 \circ \varphi_n(0, t) = g_1 \circ \varphi_n(s, t)$, we conclude that there exist $h_1, h_2 \in \mathcal{O}(Y_n)$ such that

$$f \equiv (h_1 + g_1)^2 + h_2^2 \mod \mathcal{J}(Y_n).$$

Hence, $\mathcal{P}(Y_n) = \Sigma_2(Y_n)$. □

We finish with the surface germs Z_n:

Theorem 4.5. *The surface germs Z_n, $n \geq 3$, have $p = 2$ and $\mathcal{P} \neq \Sigma$.*

Proof. The parametrization $\phi_n : (s, t) \mapsto (s, st, \dots, st^{n-2}, t^{n-1}, t^n)$, defines the homomorphism of rings

$$\begin{aligned}
\phi_n^* : \mathcal{O}(Z_n) &\to \mathbb{R}\{s, t\} \\
f &\mapsto f \circ \phi_n.
\end{aligned}$$

which is finite, injective and $(s)\ \mathbb{R}\{s, t\} \subset \operatorname{im} \phi_n^*$. For this last fact, note that $s^i t^j = s^{i-1}(st^r)t^{(n-1)q} = \phi_n^*(x_0^{i-1} x_r x_{n-1})$ where $j = (n-1)q + r, 0 \leq r < n-1$, $i \geq 1$.

We now check that $p[Z_n] = 2$. Let $f \in \Sigma(Z_n)$ and consider $f \circ \phi_n$. Since f is a sum of squares in $\mathcal{O}(Z_n)$ then $f \circ \phi_n$ is a sum of squares in $\mathbb{R}\{s, t\}$ and there exist $\alpha_1, \alpha_2, \in \mathbb{R}\{s, t\}$ and $\beta_1, \beta_2 \in \mathbb{R}\{t\}$ such that $f \circ \phi_n = (\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2$. It is clear that $C_n = Z_n \cap \{x_0 = 0\}$ is the planar curve parametrized by $\phi_n(0, t) = (0, \dots, 0, t^{n-1}, t^n)$. This curve has ideal $\mathcal{J}(C_n) = (x_0, \dots, x_{n-2}, x_n^{n-1} - x_{n-1}^n)$ and Pythagoras number 2. Thus, for $g = f|_{C_n} \in \Sigma(C_n)$, we find $g_1, g_2 \in \mathbb{R}\{x_{n-1}, x_n\}$ such that $g \equiv g_1^2 + g_2^2 \mod \mathcal{J}(C_n)$. Hence, if $\gamma_i = g_i \circ \phi_n(0, t)$, $i = 1, 2$, we deduce

$$\beta_1^2 + \beta_2^2 = f \circ \phi_n(0, t) = f|_{C_n} \circ \phi_n(0, t) = \gamma_1^2 + \gamma_2^2,$$

and $\frac{\gamma_1 + i\gamma_2}{\beta_1 + i\beta_2}, \frac{\gamma_1 - i\gamma_2}{\beta_1 - i\beta_2}$ are two units in $\mathbb{C}\{t\}$ whose product is 1. Consequently,

$$\begin{aligned}(\alpha_1 s + \beta_1)^2 &+ (\alpha_2 s + \beta_2)^2 \\ &= (\alpha_1 s + i\alpha_2 s + \beta_1 + i\beta_2)) \left(\frac{\gamma_1 + i\gamma_2}{\beta_1 + i\beta_2}\right) \left(\frac{\gamma_1 - i\gamma_2}{\beta_1 - i\beta_2}\right) \\ &\qquad \times (\alpha_1 s - i\alpha_2 s + \beta_1 - i\beta_2)) \\ &= ((a_1 s + \gamma_1) + i(a_2 s + \gamma_2))((a_1 s + \gamma_1) - i(a_2 s + \gamma_2)) \\ &= (a_1 s + \gamma_1)^2 + (a_2 s + \gamma_2)^2,\end{aligned}$$

with $a_1, a_2 \in \mathbb{R}\{s,t\}$. Now, using that $(s)\ \mathbb{R}\{s,t\} \subset \operatorname{im}\psi_n$ and that $\gamma_i(t) = g_i \circ \phi_n(0,t) = g_i \circ \phi_n(s,t)$, we conclude that there exist $h_1, h_2 \in \mathcal{O}(Z_n)$ such that

$$f \equiv (h_1 + g_1)^2 + (h_2 + g_2)^2 \mod \mathcal{J}(Z_n).$$

Therefore, $p[Z_n] = 2$.

Finally, $\mathcal{P}(Z_n) \neq \Sigma(Z_n)$. Let $f = \begin{cases} x_{n-1} & \text{if } n \text{ is odd} \\ x_n & \text{if } n \text{ is even} \end{cases}$. Clearly, $f \in \mathcal{P}(Z_n)$ is not a sum of squares in $\mathcal{O}(Z_n)$. □

References

[Ar] Artin, M.: On the solution of analytic equations. Invent. Math. **5**, 227–291 (1968)

[BR] Bochnak, J., Risler, J.-J.: Le théorème des zéros pour les variétés analytiques réelles de dimension 2. Ann. Sc. Éc. Norm. Sup. 4^e serie **8**, 353–364 (1975)

[Ei] Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. New York Berlin Heidelberg: Springer Verlag, 1999

[Fe1] Fernando, J.F.: On the Pythagoras numbers of real analytic rings. J. Algebra **243**, 321–338 (2001)

[Fe2] Fernando, J.F.: Positive semidefinite germs in real analytic surfaces. Math. Ann. **322**(1), 49–67 (2002)

[Fe3] Fernando, J.F.: Sums of squares in real analytic rings. Trans. AMS **354**(5), 1909–1919 (2002)

[FeRz] Fernando, J.F., Ruiz, J.M.: Positive semidefinite germs on the cone. Pacific J. Math. **205**, 109–118 (2002)

[JP] de Jong, T., Pfister, G.: Local Analytic Geometry, Basic Theory and Applications. Advanced Lectures in Mathematics. Braunschweig/Wiesbaden: Vieweg, 2000

[Ha] Harris, J.: Algebraic Geometry, A First Course. Graduate Text in Math. **133**. Berlin Heidelberg New York: Springer Verlag, 1992

[Rz1] Ruiz, J.M.: The Basic Theory of Power Series. Advanced Lectures in Mathematics. Braunschweig Wiesbaden: Vieweg Verlag, 1993

[Rz2] Ruiz, J.M.: Sums of two squares in analytic rings. Math. Z. **230**, 317–328 (1999)

[Sch] Scheiderer, C.: On sums of squares in local rings. J. reine angew. Math. **540**, 205–227 (2001)

[St] Stanley, R.: Hilbert functions of gradded algebras. Adv. in Math. **28**, 57–83 (1978)