Math. Z. (2007) 257:13–21
DOI 10.1007/s00209-007-0107-6

Mathematische Zeitschrift

On the Pythagoras numbers of real analytic curves

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Received: 15 June 2006 / Accepted: 11 November 2006 /
Published online: 6 February 2007

Abstract We show that the Pythagoras number of a real analytic curve is the supremum of the Pythagoras numbers of its singularities, or that supremum plus 1. This includes cases when the Pythagoras number is infinite.

Keywords Pythagoras number · Sum of squares · Analytic curve germ · Global analytic curve

Mathematics Subject Classification (2000) Primary 14P99; Secondary 11E25 · 32B10 · 32S05

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1 Statements

The problem of representing a positive semidefinite (= psd) function as a sum of squares (= sos), and of how many, is a very old matter appealing the specialists from widespread different areas. The quantitative question is formulated for any ring of functions A in terms of the *Pythagoras number* $p(A)$, which is the smallest p such that every sum of squares of A is a sum of p squares. This p may well be infinity, and even deciding this can be very difficult. We refer to the beautiful paper [3] for a survey of what was known on the topic by the 1980s. Later advances can be found in [12], [13] and [7]. In this note we address the problem for real analytic curves.

To start with, we consider the local case. Let X_x be a (reduced) real analytic curve germ (at a point $x \in \mathbb{R}^n$), $\mathcal{O}(X_x)$ its ring of analytic function germs, and $\mathcal{M}(X_x)$ its ring of meromorphic function germs. One immediately sees that every psd *meromorphic* function germ is the square of *one meromorphic* function germ, which consequently leads to concentrate on *analytic* function germs. The question whether every psd analytic function germ is an sos of analytic function germs (psd = sos, in short) is also solved: this holds true if and only if X_x is a union of *independent* non-singular branches (which means that the tangent lines of those branches are independent [13, 3.9]). In fact, our results show that *being an sos is a local matter, plus some control condition on the number of squares needed locally*.

Now, we turn to the computation of the Pythagoras number $p(\mathcal{O}(X_x))$, which is always finite ([11]). Curve germs with minimum Pythagoras number 1 are completely characterized: they are the so-called *Arf* curve germs ([2]), of relevance in other contexts. Here, for instance, the above mentioned unions of independent non-singular branches are Arf; hence, the curve germs for which psd = sos have all Pythagoras number 1. On the other hand, the Pythagoras number of a curve germ can be arbitrarily large ([9]). This was first proved through an explicit construction of curve germs with large embedding dimension, but later it has been discovered that curve germs with large Pythagoras number are ubiquitous ([6]): *Every semianalytic germ Z_x of dimension $d \geq 3$ contains (punctured) irreducible curve germs with arbitrarily large Pythagoras number.* If $d \leq 2$, Z_x contains curve germs with as large as possible Pythagoras number, the bound being the Pythagoras number of the analytic closure of Z_x. These local constructions are of importance to produce examples in the global analytic case. All in all, there is a good systematic knowledge of $p(\mathcal{O}(X_x))$, even from an algorithmic viewpoint.

Consider now the global case. Let $X \subset \mathbb{R}^n$ be a real analytic curve, $\mathcal{O}(X)$ its ring of global analytic functions and $\mathcal{M}(X)$ its ring of global meromorphic functions. We denote by $\mathcal{O}_X$ the (reduced) sheaf of germs of analytic functions on X, which is a coherent sheaf, because analytic curves are always coherent; we have:, $\mathcal{O}(X) = \Gamma(\mathcal{O}_X, X)$ and $\mathcal{O}(X_x) = \mathcal{O}_{X,x}$ for $x \in X$. Then, every psd global meromorphic function is an sos of global meromorphic functions; moreover, $p(\mathcal{M}(X)) = 2$ if some irreducible component of X is compact, $p(\mathcal{M}(X)) = 1$ otherwise (see [8] for X non-singular; the general case can be deduced by normalization). Thus we look at analytic functions. What is known so far is that for non-singular X (that is, when X is a disjoint union of circles and lines), every psd analytic function on X is an sos of analytic functions, and $p(\mathcal{O}(X)) = p(\mathcal{M}(X))$ ([8]). The goal of this short note is to analyse the behaviour of psd analytic fuctions and sos of analytic functions in the presence of singularities, which is far more involved.

The main concern is the computation of the Pythagoras number $p(\mathcal{O}(X))$, but before that, and to make the discussion complete, let us say that the property that every psd analytic function on X is a sos of analytic functions (psd = sos for X, in short) holds very rarely, as shows the following result:

Proposition 1.1 *Let X be a real analytic curve. The following conditions are equivalent:*

(i) *The property psd = sos holds for X.*
(ii) *The property psd = sos holds for every germ X_x, $x \in X$.*
(iii) *Every germ X_x, $x \in X$, is a union of non-singular independent branches.*

If that is the case, then $p(\mathcal{O}(X)) \leq 2$.

Thus we see that for real analytic curves the property psd = sos is local, and when it holds all singularities are highly special. However, it is remarkable that while being psd is (trivially) a local property, being an sos needs not. In fact:

There are real analytic curves $X \subset \mathbb{R}^n$ $(n \geq 3)$ and psd global analytic functions f on X, such that every germ f_x, $x \in X$, is an sos of analytic function germs on X_x, without f being an sos of global analytic functions.

However, the only local obstruction is that the number of squares needed for the germs f_x, $x \in X$, is not uniformly bounded. This leads to our main result:

Theorem 1.2 *Let X be a real analytic curve in an open subset Ω of $\mathbb{R}^n$. Then*

$$p(\mathcal{O}(X)) = \sup_{x \in X}\{p(\mathcal{O}(X_x))\} + \varepsilon,$$

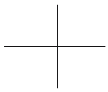
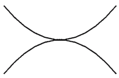
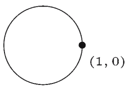
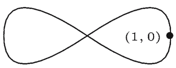
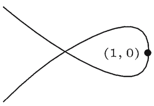
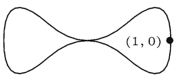
where $\varepsilon = 0$ or 1.

Note that since $p(\mathcal{O}(X_x)) = 1$ for every regular point $x \in X$, this computes, up to a margin of 1, the Pythagoras number of X in terms of the Pythagoras numbers of its singularities.

An interesting consequence of this result is that

There are analytic curves $X \subset \mathbb{R}^n$ $(n \geq 3)$, with infinite *Pythagoras number: $p(\mathcal{O}(X)) = +\infty$.*

Example 1.3 Consider the following nine curves $X_i \subset \mathbb{R}^2$ $(1 \leq i \leq 9)$, which are singular or non-singular, compact or non compact, irreducible or reducible. Note that since $p(\mathcal{O}(\mathbb{R}^2)) = 2$ ([1]), all of them have Pythagoras number $p \leq 2$. We look at their Pythagoras numbers p, to their excesses ε as in Theorem 1.2, and to whether the property psd = sos holds or not for them, and find many different possibilities.

curve $X_i \subset \mathbb{R}^2$		$p(\mathcal{O}(X_i))$	ε	psd = sos	an sos which is not a square
	$X_1 : y = 0$ line	1	0	YES	—
	$X_2 : y^3 = x^2$ cusp	1	0	NO	—
	$X_3 : xy = 0$ independent lines	1	0	YES	—
	$X_4 : y^2 = x^4$ two parabolas	1	0	NO	—
	$X_5 : y^3 = x^4$ singularity E_6	2	0	NO	$x^2 + y^2$
	$X_6 : x^2 + y^2 = 1$ circle	2	1	YES	$(x-1)^2 + y^2$
	$X_7 : y^2 = x^2 - x^4$ transversal eight	2	1	YES	$(x-1)^2 + y^2$
	$X_8 : y^2 = x^2 - x^3$ node	2	1	YES	$(x-1)^2 + y^2$
	$X_9 : y^2 = x^4 - x^6$ tangent eight	2	1	NO	$(x-1)^2 + y^2$

Note that the curve X_5 has Pythagoras number 2 because its singularity (at $(0, 0)$) has this Pythagoras number. For the curves X_i with $6 \leq i \leq 9$, which have also Pythagoras number 2, the situation is different because their singularities have all Pythagoras number 1 (since they are Arf). To find a function f which is a sum of two squares in $\mathcal{O}(X_i)$ but not a square, one proceeds as follows. Consider a polynomial f of degree 2 which is a sum of two squares of polynomials and only vanishes at a regular point x of X_i *which is contained in a loop of* X_i (see the previous pictures). Next, if f has a square root $h \in \mathcal{O}(X_i)$, then h cannot change sign because it only vanishes at x and this point does not disconnect X_i. Now, using a regular parametrization of X_i at x, one sees that h must have order 1 at x and consequently, must change sign at that point, a contradiction.

2 Proofs

We begin by the

Proof of Proposition 1.1. First, we see that (i) implies (ii). We assume psd = sos for X, and will prove psd = sos for the germ X_x at an arbitrary point $x \in X \subset \mathbb{R}^n$; we can suppose $x = 0$. Pick a psd analytic function germ $f_x \in \mathcal{O}(X_x)$, and let us see that f_x is an sos. First notice that f_x is the restriction to X_x of an analytic function germ on $\mathbb{R}^n_x$, which will be denoted f_x too. As every branch of X_x can be parametrized, to be psd depends essentially on finitely many terms of the Taylor expansion of f_x ([9]), but this requires some care, as f_x can vanish on some branches of the germ X_x. We proceed as follows. Every branch on which f_x does not vanish can be parametrized, say by $\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t)))$, and since f_x is psd, we have

$$f_x(\alpha(t)) = c_\alpha t^{2r_\alpha} + \cdots, \quad c_\alpha > 0.$$

Consequently, for m large, for all q-jets of f_x

$$g_q = \sum_{|\nu|<2q} a_\nu x_1^{\nu_1} \cdots x_n^{\nu_n}$$

with $q \geq m$, we have

$$g_q(\alpha(t)) = c_\alpha t^{2r_\alpha} + \cdots$$

Concerning the branches of X_x on which f_x vanishes, we choose for each a parametrization $\beta(t) = (\beta_1(t), \dots, \beta_n(t)))$, and denote by p the smallest t-order of all $\beta_k(t)$'s. Then, since $f_x(\beta(t)) \equiv 0$, there is $q \geq m$ such that

$$g_q(\beta(t)) = c_\beta t^{s_\beta} + \cdots, \quad \text{with } s_\beta > 2pm.$$

Finally, consider the global analytic function

$$h = g_q + (x_1^2 + \cdots + x_n^2)^m + \lambda(x_1^2 + \cdots + x_n^2)^q.$$

By construction, $h \equiv f_x \mod \mathfrak{m}_x^{2m}$, and we claim that h is psd on X for a suitable $\lambda > 0$.

Indeed, to check it we evaluate on every branch:

(1) On the α's, already g_q is psd: $g_q(\alpha(t)) = c_\alpha t^{2r_\alpha} + \cdots \geq 0$. Hence $h(\alpha(t)) \geq 0$.
(2) On the β's, we look at $h^* = g_q + (x_1^2 + \cdots + x_n^2)^m$, and get

$$h^*(\beta(t)) = (c_\beta t^{s_\beta} + \cdots) + (\beta_1(t)^2 + \cdots + \beta_n(t)^2)^m.$$

Here the order of the second addend is $2pm$, hence $< s_\beta$, so that for $|t|$ small the sign is that of the sum of squares. Thus, $h^*(\beta(t)) \geq 0$ and we deduce $h(\beta(t)) \geq 0$.

Next we find λ. As h is psd on X_x, we can choose $\eta > 0$ small enough such that h is psd on $X \cap \{\|x\| < \eta\}$. Consequently we only care about $X \cap \{\|x\| \geq \eta\}$. But there we have:

$$|g_q| = \left| \sum_{|\nu|\leq 2q} a_\nu x_1^{\nu_1} \cdots x_n^{\nu_n} \right| \leq \sum_{|\nu|\leq 2q} |a_\nu| |x_1|^{\nu_1} \cdots |x_n|^{\nu_n}$$
$$\leq \sum_{|\nu|\leq 2q} |a_\nu| \|x\|^{|\nu|} \leq \sum_{|\nu|\leq 2q} |a_\nu| \|x\|^{2q} \frac{1}{\eta^{2q-|\nu|}} = \lambda \|x\|^{2q},$$

with $\lambda = \sum_{|\nu|\le 2q} |a_\nu| \frac{1}{\eta^{2q-|\nu|}}$. Thus, $g_q + \lambda(x_1^2 + \cdots + x_n^2)^q$ is ≥ 0 on $X \cap \{\|x\| \ge \eta\}$, thus $h \ge 0$ there too. This completes the proof of the claim.

Now, by (i), h is an sos on X, hence, and sos on the germ X_x. But the Pythagoras number of $\mathcal{O}(X_x)$ is finite, say p, hence we have

$$h_x = h_1^2 + \cdots + h_p^2 \quad \text{in} \quad \mathcal{O}(X_x).$$

Consequently, we have

$$f_x \equiv h_1^2 + \cdots + h_p^2 \mod \mathfrak{m}_x^{2m},$$

where $\mathfrak{m}_x$ stands for the maximal ideal of $\mathcal{O}(X_x)$. This means that in $\mathcal{O}(X_x)$ the equation

$$f_x = z_1^2 + \cdots + z_p^2$$

has approximate solutions of every order m. By M. Artin's approximation, we conclude that it has some exact solution, and f_x is an sos in $\mathcal{O}(X_x)$. Thus we have proved the first implication of the proposition.

Next, note that the fact that (ii) implies (iii) is the local result quoted at the beginning of Sect. 1 ([13, 3.9]).

Finally, we prove that (iii) implies (i). We assume (iii) and will show that every psd global analytic function f on X is a sum of 2 squares of global analytic functions. Since every germ X_x is a union of independent nonsingular branches, it has the property psd = sos and $p(\mathcal{O}(X_x)) = 1$. Thus, for every $x \in X$ there is a well defined square root $\sqrt{f_x} \in \mathcal{O}(X_x)$ and those square roots define well a locally principal sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}^2 = f\mathcal{O}_X$. But then $\mathcal{J}$ is generated by two global analytic functions g, h ([4]).

Thus for every $x \in X$ we have $(g,h)\mathcal{O}(X_x) = \sqrt{f_x}\mathcal{O}(X_x)$, and there exist analytic function germs $\zeta_x, \xi_x, \lambda_x, \mu_x \in \mathcal{O}(X_x)$ such that

$$g_x = \zeta_x\sqrt{f_x}, \quad h_x = \xi_x\sqrt{f_x}, \quad \sqrt{f_x} = \lambda_x g_x + \mu_x h_x.$$

Hence

$$\sqrt{f_x} = (\lambda_x\zeta_x + \mu_x\xi_x)\sqrt{f_x} \quad \text{on} \quad X_x.$$

We claim that if $f_x \ne 0$ on X_x, then either $\zeta_x(x) \ne 0$ or $\xi_x(x) \ne 0$. Indeed, if $f_x \ne 0$ on an irreducible component Y_x of X_x, then $\lambda_x\zeta_x + \mu_x\xi_x \equiv 1$ on Y_x, and the claim follows by evaluating at x. Consequently, $u_x = \zeta_x^2 + \xi_x^2$ is a unit in $\mathcal{O}(X_x)$ and $(g^2+h^2)_x = u_x f_x$.

Now, let $X_1 \subset X$ be the union of the irreducible components of X over which f is not identically 0 and $X_2 \subset X$ the union of the others. From the fact that for every $x \in X$ with $f_x \ne 0$ there exists a unit $u_x \in \mathcal{O}(X_x)$ such that $u_x f_x = (g^2 + h^2)_x$ in $\mathcal{O}(X_x)$, we deduce that there exist an open neighborhood Ω of X_1, so that the function $\alpha = \frac{f}{g^2+h^2}$ is strictly positive and analytic in $X \cap \Omega$. Clearly, $\sqrt{\alpha}g$, $\sqrt{\alpha}h$ vanish on $X_2 \cap \Omega$, hence both can be extended by zero on X_2, which gives a representation of f as a sum of 2 squares on the whole $X = X_1 \cup X_2$. $\square$

Next, we prove the following:

Proposition 2.1 *There are real analytic curves $X \subset \mathbb{R}^n$ $(n \ge 3)$ and psd global analytic functions f on X, which are not sos of global analytic functions, but every germ f_x, $x \in X$, is an sos of analytic function germs on X_x.*

Proof Indeed, this is a straightforward argument based on analytic approximation and Cartan's Theorems A and B. Choose a discrete set of points $x_k \in \mathbb{R}^n$. We can draw through x_k a small representative of an irreducible curve germ X_{x_k} with Pythagoras number $p_k > k$ (these germs exist as explained in the introduction); we denote by $\mathfrak{m}_k$ de maximal ideal of $\mathcal{O}(X_{x_k})$. Next, we connect smoothly each representative with the next to obtain a smooth curve Y off the x_k's, and such that $Y_{x_k} = X_{x_k}$ for all k. Then we approximate Y by a real analytic curve X whose singular points are the x_k's, and whose germs at them are (equivalent to) the X_{x_k}'s. Now for each k we choose an analytic germ f_{x_k} which is a sum of squares in X_{x_k} but not a sum of $p_k - 1 \geq k$ squares. Finally, for any chosen integers m_k's we find an analytic function g on X such that

$$g_{x_k} - f_{x_k} \in \mathfrak{m}_k^{m_k} \quad \text{for all } k.$$

Again by the results in [9] concerning irreducible curve germs, for m_k large enough, g_{x_k} and f_{x_k} behave the same concerning sos. Whence g_{x_k} is an sos of analytic function germs (hence psd), but not a sum of $p_k - 1 \geq k$ squares, and consequently g is not either. We conclude that g is psd but cannot be an sos of global analytic functions. □

Now we turn to the main result Theorem 1.2.

Proof of Theorem 1.2. First, let x_k ($k \geq 1$) be the singular points of X, and denote as usual by $\mathfrak{m}_k$ the maximal ideal of the ring $\mathcal{O}(X_{x_k})$; set $p(\mathcal{O}(X_{x_k})) = p_k$. Fix k. We have an sos of analytic function germs $f_{x_k} = \sum_i f_{i,x_k}^2$ which is not a sum of $p_k - 1$ squares in $\mathcal{O}(X_{x_k})$. Then there are global analytic functions g_i such that

$$g_{i,x_k} - f_{i,x_k} \in \mathfrak{m}_k^{m_k}.$$

Now we consider $g = \sum_i g_i^2$, and have

$$g_{x_k} - f_{x_k} = \sum_i (g_{i,x_k} + f_{i,x_k})(g_{i,x_k} - f_{i,x_k}) \in \mathfrak{m}_k^{m_k}.$$

By M. Artin's approximation, for m_k large, g_{x_k} is not a sum of $p_k - 1$ squares in $\mathcal{O}(X_{x_k})$, which implies g is not either. Thus, $p(\mathcal{O}(X)) \geq p_k$ for all k.

After this, we must prove the inequality

$$p(\mathcal{O}(X)) \leq \sup_{x \in X} p(\mathcal{O}(X_x)) + 1,$$

which clearly follows from the assertion:
$(*)$ *Let $f \in \mathcal{O}(X)$ be an analytic function on X such that at every point x the germ f_x is a sum of p squares in $\mathcal{O}(X_x)$. Then f is a sum of $p + 1$ squares in $\mathcal{O}(X)$.*

To prove this, we will use the coherent $\mathcal{O}_X$-module $\mathcal{M} = \mathcal{O}_X / f^2 \mathcal{O}_X$. We are going to define global cross sections ξ of $\mathcal{M}$. The support of $\mathcal{M}$ consists of the zeros of f, and to look at them, we consider once again the union X_1 (resp. X_2) of the irreducible components of X on which f does not vanish (resp. does vanish) identically. The zeros of f split into some isolated ones $y_\ell \in X_1$, and all the points in X_2. Note that there are maybe non-isolated zeros through which some irreducible components are in X_2 and some others not: these zeros z_k are singular points of X, and important for the sequel.

By hypothesis, at each zero $x = y_\ell, z_k$ there are p analytic germs $\alpha_{i,x} \in \mathcal{O}_{X,x}$ such that $f_x = \sum_{i=1}^{p} \alpha_{i,x}^2$ in the local ring $\mathcal{O}_{X,x}$. We define our global cross-section $\xi_i \in \Gamma(\mathcal{M}, X)$ by

$$\xi_{i,x} = \begin{cases} \alpha_{i,x} \bmod f^2\mathcal{O}_{X,x} & \text{for } x = y_\ell, z_k, \\ 0 \bmod f^2\mathcal{O}_{X,x} & \text{otherwise.} \end{cases}$$

Notice that this cross-section is well-defined, because at a point $x = z_k$, the function $\alpha_{i,x}$ vanishes identically on the branches of X_x on which f vanishes.

Now, by Cartan's Theorem B, the homomorphism $\mathcal{O}_X = \Gamma(\mathcal{O}_X, X) \to \Gamma(\mathcal{M}, X)$ is surjective. Hence, there is a global analytic function $a_i \in \mathcal{O}(X)$ such that

$$\begin{cases} a_{i,x} - \alpha_{i,x} \in f^2\mathcal{O}_{X,x} & \text{for all } x = y_\ell, z_k, \text{ and} \\ a_{i,x} \in f^2\mathcal{O}_{X,x} & \text{otherwise.} \end{cases}$$

In particular, a_i vanishes on the zeros of f. Moreover, for each $x = y_\ell, z_k$ there is an analytic germ $\lambda_{i,x} \in \mathcal{O}_{X,x}$ with

$$a_{i,x} = \alpha_{i,x} + \lambda_{i,x} f^2 \quad \text{in } \mathcal{O}_{X,x}.$$

Hence,

$$\begin{aligned} \sum_i a_{i,x}^2 &= \sum_i (\alpha_{i,x} + \lambda_{i,x} f^2)^2 \\ &= \sum_i \alpha_{i,x}^2 + 2\sum_i \alpha_{i,x}\lambda_{i,x} f^2 + \sum_i (\lambda_{i,x} f^2)^2 \\ &= f + 2\sum_i \alpha_{i,x}\lambda_{i,x} f^2 + \sum_i \lambda_{i,x}^2 f^4 \\ &= f\Big(1 + 2\sum_i \alpha_{i,x}\lambda_{i,x} f + \sum_i \lambda_{i,x}^2 f^3\Big) = f(1+\gamma_x) = \theta_x f, \end{aligned}$$

where the germ γ_x vanishes at x, and the germ $\theta_x = 1 + \gamma_x$ is a unit in $\mathcal{O}_{X,x}$.

Next, denote $A = \sum_{i=1}^p a_i^2$. Then, $u = \sqrt{\frac{f}{A+f^2}}$ is a well defined strictly positive analytic function on a neighborhood W of X_1. This is because:

- f and $A + f^2$ are both positive off $f^{-1}(0)$, and
- for all $x = y_\ell, z_k$, the quotient $\dfrac{f_x}{A_x + f_x^2} = \dfrac{1}{\theta_x + f_x}$ is a positive unit in $\mathcal{O}_{X,x}$.

Thus, on $X \cap W$ we have $f = \sum_{i=1}^p (ua_i)^2 + (uf)^2$. But, all ua_i's and uf vanish on X_2; hence, the equality holds in fact on the whole of X, if we extend the functions ua_i's and uf by zero to $X_2 \backslash W$. Therefore, f is a sum of $p+1$ squares in $\mathcal{O}(X)$, as wanted. □

Finally, we deduce as an easy corollary:

Proposition 2.2 *There are real analytic curves $X \subset \mathbb{R}^n$ $(n \geq 3)$ with infinite Pythagoras number.*

Proof For, as in the proof of 2.1, we can glue a discrete family of irreducible singularities X_{x_k} with Pythagoras numbers $p(\mathcal{O}(X_{x_k})) = p_k \to +\infty$, to obtain a real analytic curve $X \subset \mathbb{R}^n$ with Pythagoras number $\sup_k\{p_k\} = +\infty$. □

Acknowledgments F. Acquistapace and F. Broglia supported by Italian GNSAGA of INdAM and MIUR, J. F. Fernando and J. M. Ruiz by Spanish GEOR MTM2005-02865 and Grupos UCM 910444.

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