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Locally injective semialgebraic mappings

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Abstract. We characterize locally injective semialgebraic maps between two semialgebraic sets in terms of the induced homomorphism between their rings of (continuous) semialgebraic functions.

Contents

1. Introduction
2. Preliminaries
3. Finiteness of semialgebraic homomorphisms
4. Locally injective semialgebraic mappings
References

1. Introduction

In this article we characterize locally injective semialgebraic maps $\pi : M \to N$ in terms of some finiteness properties of the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$, $f \mapsto f \circ \pi$ between their rings of continuous semialgebraic functions $\mathcal{S}(N)$ and $\mathcal{S}(M)$.

To lighten notations we denote $\operatorname{Spec_s}(M) := \operatorname{Spec}(\mathcal{S}(M))$ the set of prime ideals of $\mathcal{S}(M)$ endowed with the Zariski topology. Our previous articles [4,7–14] and [15] are devoted to study the relationship between π and its spectral counterpart

$$\operatorname{Spec_s}(\pi) : \operatorname{Spec_s}(M) \to \operatorname{Spec_s}(N),\ \mathfrak{p} \mapsto \varphi_\pi^{-1}(\mathfrak{p}),$$

and this article is a new step in our attempt to understand this relationship.

A subset $M \subset \mathbb{R}^m$ is said to be *basic semialgebraic* if it can be written as

$$M := \{x \in \mathbb{R}^m : \ f(x) = 0,\ g_1(x) > 0, \dots, g_\ell(x) > 0\}$$

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for some polynomials $f, g_1, \ldots, g_\ell \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_m]$. The finite unions of basic semialgebraic sets are called *semialgebraic sets*. A continuous function $f : M \to \mathbb{R}$ is said to be *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{n+1}$. Usually, semialgebraic function just means a function, non necessarily continuous, whose graph is semialgebraic. However, since all semialgebraic functions occurring in this article are continuous we will omit for simplicity the continuity condition when we refer to them. Likewise, a continuous mapping $\pi : M \to N$ between semialgebraic sets whose graph is semialgebraic will be called, simply, a *semialgebraic mapping*.

The sum and product of functions, defined pointwise, endow the set $\mathcal{S}(M)$ of semialgebraic functions on M with a natural structure of commutative ring whose unity is the semialgebraic function $\mathbf{1}_M$ with constant value 1. In fact $\mathcal{S}(M)$ is an $\mathbb{R}$-algebra, if we identify each real number r with the constant function which just attains this value. The most simple examples of semialgebraic functions on M are the restrictions to M of polynomials in n variables. Other relevant ones are the absolute value of a semialgebraic function, the distance function to a given semialgebraic set, the maximum and the minimum of a finite family of semialgebraic functions, the inverse and the k-root of a semialgebraic function whenever these operations are well-defined.

Each semialgebraic map $\pi : M \to N$ induces an $\mathbb{R}$-algebras homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$, $g \mapsto g \circ \pi$, and in this work we characterize the local injectivity of π in terms of finiteness conditions imposed to this homomorphism.

Note that each homomorphism $\varphi : \mathcal{S}(N) \to \mathcal{S}(M)$ of rings with unity is a homomorphism of $\mathbb{R}$-algebras. To prove this observe that $\varphi(\mathbf{1}_N) = \mathbf{1}_M$, so for every point $x \in M$ we have $\varphi(\mathbf{1}_N)(x) = \mathbf{1}_M(x) = 1$. Hence the map $\varphi_x : \mathbb{R} \to \mathbb{R}$, $r \mapsto \varphi(r)(x)$ is a field homomorphism with $\varphi_x(1) = 1$. Therefore φ_x is the identity, that is, $\varphi(r)(x) = r$ for every $x \in M$ and $r \in \mathbb{R}$, that is, $\varphi(r) = r$.

In Section 2 we introduce the terminology employed along the paper, whereas the relationship between finiteness conditions of π and φ_π are collected in Section 3. A semialgebraic map $\pi : M \to N$ is said to be *locally injective* if there exists a family $\mathcal{U} := \{U_i : i \in I\}$ of open semialgebraic subsets of M such that $M = \bigcup_{i \in I} U_i$ and each restriction $\pi|_{U_i} : U_i \to N$ is injective. We will see in Remark 4.6 that in case φ_π is a finite homomorphism, then π is locally injective and the family $\mathcal{U}$ above can be assumed to be finite. However, local injectivity does not implies injectivity. For example, $\pi : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$ is a non injective but locally injective semialgebraic map.

The main result of the paper is Theorem 4.5, where we prove that if $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is a finite homomorphism, then the maps

$$\operatorname{Spec_s}(\pi) : \operatorname{Spec_s}(M) \to \operatorname{Spec_s}(N) \ \text{ and } \ \pi : M \to N$$

are proper, separated, locally injective and their fibers are finite sets. As a consequence we deduce in Corollary 4.7 that a semialgebraic map $\pi : M \to N$ between compact semialgebraic sets M and N is locally injective if and only if $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is finite.

Next we characterize in Corollary 4.13 the fact that $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is integral, and to finish we obtain in Theorem 4.14 a result of a similar natura to

Corollary 4.7, where the finiteness of the homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is substituted by the surjectivity of the map π and the condition φ_π is simple.

2. Preliminaries

Let us fix some notations and recall some basic results concerning semialgebraic sets and functions. For each $f \in \mathcal{S}(M)$ and each semialgebraic subset $N \subset M$, we denote

$$\mathcal{Z}_N(f) := \{x \in N : f(x) = 0\} \quad \& \quad \mathcal{D}_N(f) := N \setminus \mathcal{Z}_N(f).$$

In addition, if $N \subset M$ we denote $\operatorname{Int}_M(N)$ and $\operatorname{Cl}_M(N)$, respectively, the interior and the closure of N in M.

Lemma 2.1. *Let N be a closed semialgebraic subset of a semialgebraic set M.*
(1) There exists a function $f \in \mathcal{S}(M)$ such that $N = \mathcal{Z}_M(f)$.
(2) The restriction homomorphism $\mathcal{S}(M) \to \mathcal{S}(N),\ f \mapsto f|_N$ is surjective.

Proof. (1) It suffices to choose $f := \operatorname{dist}(\cdot, N)$.
(2) This was proved by Delfs and Knebusch in [6, Thm. 3]. □

For our purposes it is useful to know that $\mathcal{S}(M)$ is a *Gelfand ring*, that is, each prime ideal in $\mathcal{S}(M)$ is contained in a unique maximal ideal. This was proved e.g. by Carral-Coste [5, Prop. 2]. We provide in Proposition 2.3 a distinct and more elementary result. First we need an auxiliary lemma.

Lemma 2.2. *Let M be a semialgebraic set and let $f, g \in \mathcal{S}(M)$ with $\mathcal{Z}_M(g) \subset \operatorname{Int}_M(\mathcal{Z}_M(f))$. Then, there exists $h \in \mathcal{S}(M)$ with $f = gh$ and $\mathcal{Z}_M(f) \subset \mathcal{Z}_M(h)$.*

Proof. The open semialgebraic subsets $U := M \setminus \mathcal{Z}_M(g)$ and $V := \operatorname{Int}_M(\mathcal{Z}_M(f))$ of M cover M and $f|_{U\cap V} \equiv 0$. Thus,

$$h : M \to \mathbb{R},\ x \mapsto \begin{cases} \frac{f(x)}{g(x)} & \text{if } x \in U, \\ 0 & \text{if } x \in V, \end{cases}$$

is a semialgebraic function satisfying $f = gh$ and $\mathcal{Z}_M(f) \subset \mathcal{Z}_M(h)$. □

Proposition 2.3. *(1) Let M be a semialgebraic set and let $\mathfrak{a}_1, \mathfrak{a}_2$ be two comaximal ideals in $\mathcal{S}(M)$. Then, its intersection $\mathfrak{a}_1 \cap \mathfrak{a}_2$ does not contain any prime ideal in $\mathcal{S}(M)$.*
(2) $\mathcal{S}(M)$ is a Gelfand ring.

Proof. The second part follows from the first one because two distinct maximal ideals are comaximal. For the first part suppose, by the way of contradiction, the existence of a prime ideal $\mathfrak{p}$ in $\mathcal{S}(M)$ contained in $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Consider the semialgebraic functions

$$G_1 : \mathbb{R} \to \mathbb{R},\ t \mapsto \begin{cases} 0 & \text{if } |t| \geq 2, \\ t+2 & \text{if } -2 < t < -1, \\ 1 & \text{if } -1 \leq t \leq 1, \\ 2-t & \text{if } 1 < t < 2, \end{cases} \quad \& \quad H : \mathbb{R} \to \mathbb{R},\ t \mapsto \begin{cases} 0 & \text{if } t \leq 1/3, \\ 6t-2 & \text{if } 2 < 6t < 3, \\ 1 & \text{if } t \geq 1/2, \end{cases}$$

and define $G_2 \in \mathcal{S}(\mathbb{R})$ by $G_2(t) = G_1(t+4)$. Notice that the product $G_1 G_2 \equiv 0$ and define $h := 5H - 4 \in \mathcal{S}(\mathbb{R})$.

The ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ being comaximal, there exist $f_1 \in \mathfrak{a}_1$ and $f_2 \in \mathfrak{a}_2$ such that $1 = f_1 + f_2$. Let $g_1, g_2 \in \mathcal{S}(M)$ be defined by

$$g_1 := G_1 \circ h \circ f_1 \quad \& \quad g_2 := G_2 \circ h \circ f_1,$$

whose product $g_1 g_2 = 0 \in \mathfrak{p}$ because $G_1 G_2 = 0$. Since $\mathfrak{p}$ is a prime ideal, either $g_1 \in \mathfrak{p}$ or $g_2 \in \mathfrak{p}$. Since $\mathfrak{p} \subset \mathfrak{a}_1 \cap \mathfrak{a}_2$ we obtain $g_1 \in \mathfrak{a}_2$ or $g_2 \in \mathfrak{a}_1$. We will get a contradiction by showing that both statements are false. To that end it suffices to check that $1 - g_1 \in \mathfrak{a}_2$ and $1 - g_2 \in \mathfrak{a}_1$. Thus it is enough to see that

$$1 - g_1 \in f_2 \mathcal{S}(M) \quad \& \quad 1 - g_2 \in f_1 \mathcal{S}(M). \tag{2.1}$$

Let us check first that

$$\mathcal{Z}_M(f_2) \subset \operatorname{Int}_M(\mathcal{Z}_M(1 - g_1)) \quad \& \quad \mathcal{Z}_M(f_1) \subset \operatorname{Int}_M(\mathcal{Z}_M(1 - g_2)). \tag{2.2}$$

To prove the first inclusion note that the open subset $U := f_1^{-1}((1/2, +\infty))$ of M contains $\mathcal{Z}_M(f_2)$. Hence, it suffices to show that $U \subset \mathcal{Z}_M(1 - g_1)$. Indeed, for every $x \in U$ we have $H(f_1(x)) = 1$, thus $h(f_1(x)) = 1$, which implies $g_1(x) = G_1(1) = 1$.

Analogously, $V := f_1^{-1}((-\infty, 1/3))$ is an open neighbourhood in M of $\mathcal{Z}_M(f_1)$, and it is enough to check that $V \subset \mathcal{Z}_M(1 - g_2)$. But $H(f_1(x)) = 0$ for every point $x \in V$, so $h(f_1(x)) = -4$. Consequently $g_2(x) = G_2(-4) = G_1(0) = 1$, which proves (2.2). Therefore, by Lemma 2.2, there exist $u, v \in \mathcal{S}(M)$ such that

$$\mathcal{Z}_M(1 - g_1) \subset \mathcal{Z}_M(u), \quad \mathcal{Z}_M(1 - g_2) \subset \mathcal{Z}_M(v),$$
$$1 - g_1 = f_2 u \in f_2 \mathcal{S}(M) \quad \& \quad 1 - g_2 = f_1 v \in f_1 \mathcal{S}(M),$$

which proves equalities (2.1). □

(2.4) Induced spectral morphisms Let $\pi : M \to N$ be a semialgebraic map. All through this article we denote

$$\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M),\ f \mapsto f \circ \pi.$$

This ring homomorphism induces a continuous map

$$\operatorname{Spec_s}(\pi) : \operatorname{Spec_s}(M) \to \operatorname{Spec_s}(N),\ \mathfrak{p} \mapsto \varphi_\pi^{-1}(\mathfrak{p})$$

where both spaces are endowed with the Zariski topology. Denote respectively $\beta_s M$ and $\beta_s N$ the subspaces of $\operatorname{Spec}_s(M)$ and $\operatorname{Spec}_s(N)$ consisting of the maximal ideals of $\mathcal{S}(M)$ and $\mathcal{S}(N)$. Let $\mathtt{r}_N : \operatorname{Spec}_s(N) \to \beta_s N$ be the retraction that maps each prime ideal in $\mathcal{S}(N)$ to the unique maximal ideal containing it. It is proved in [2, Prop. 1.6.2] and [19, Thm. 1.2] that $\mathtt{r}_N$ is a closed and continuous retraction. Thus, the composition

$$\beta_s\pi := \mathtt{r}_N \circ \operatorname{Spec}_s(\pi)|_{\beta_s M} : \beta_s M \to \beta_s N$$

is continuous too. Given a function $f \in \mathcal{S}(M)$ we will denote

$$\mathcal{Z}_{\operatorname{Spec}_s(M)}(f) := \{\mathfrak{p} \in \operatorname{Spec}_s(M) : f \in \mathfrak{p}\} \quad \& \quad \mathcal{Z}_{\beta_s M}(f) := \mathcal{Z}_{\operatorname{Spec}_s(M)}(f) \cap \beta_s M,$$
$$\mathcal{D}_{\operatorname{Spec}_s(M)}(f) := \{\mathfrak{p} \in \operatorname{Spec}_s(M) : f \notin \mathfrak{p}\} \quad \& \quad \mathcal{D}_{\beta_s M}(f) := \mathcal{D}_{\operatorname{Spec}_s(M)}(f) \cap \beta_s M.$$

For each $x \in M$ consider the maximal ideal $\mathfrak{m}_x := \{f \in \mathcal{S}(M) : f(x) = 0\}$ of $\mathcal{S}(M)$, and let $\mathtt{j}_M : M \hookrightarrow \beta_s M,\ x \mapsto \mathfrak{m}_x$. We study very detailed in [12, Prop. 4.7] the pair $(\beta_s M, \mathtt{j}_M)$, which is a Hausdorff compactification of M that we call the *semialgebraic Stone–Čech compactification* of M. Since $\mathtt{j}_N(N)$ and $\mathtt{j}_M(M)$ are respectively dense in $\beta_s N$ and $\beta_s M$, the map $\beta_s\pi$ is the unique continuous map making the following square commutative:

$$\begin{array}{ccc} M & \overset{\mathtt{j}_M}{\hookrightarrow} & \beta_s M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\beta_s\pi} \\ N & \overset{\mathtt{j}_N}{\hookrightarrow} & \beta_s N \end{array}$$

The difference $\beta_s M \setminus \mathtt{j}_M(M)$ is *the remainder* of the compactification $(\beta_s M, \mathtt{j}_M)$.

To finish this preliminary section we recall some elementary notions concerning finiteness of ring homomorphisms and fix some notations.

Definition 2.5. (1) Given two rings (commutative with unity) A and B, and a ring homomorphism $\varphi : A \to B$ we consider in B the structure of A-algebra given by the multiplication $a \cdot b := \varphi(a)b$ for every $a \in A$ and $b \in B$.
(2) For every ideal $\mathfrak{a}$ in A we will denote $\mathfrak{a}B$ the smallest ideal of B containing $\varphi(\mathfrak{a})$, without any explicit mention to the homomorphism φ.
(3) It is said that φ is *finite*, or that B is a *finite* A-algebra, if B is a finitely generated A-module, that is, if there exist finitely many elements $b_1, \dots, b_n \in B$ such that $B = Ab_1 + \cdots + Ab_n$.
(4) An element $b \in B$ is *integral* over A if there exists a monic polynomial $\mathtt{p(t)} \in A[\mathtt{t}]$ such that $\mathtt{p}(b) = 0$. If every element in B is integral over A then it is said that B is an *integral* A-algebra, or that φ is an *integral* homomorphism.
(5) It is said that B is a *finitely generated* A-algebra if there are $b_1, \dots, b_n \in B$ such that $B := A[b_1, \dots, b_n]$, where $A[b_1, \dots, b_n]$ is the image of the evaluation homomorphism

$$A[\mathtt{t}_1, \dots, \mathtt{t}_n] \to B,\ \mathtt{p}(\mathtt{t}_1, \dots, \mathtt{t}_n) \mapsto \mathtt{p}(b_1, \dots, b_n)$$

In the case $n = 1$, i.e. $B = A[b]$ for some $b \in B$, then it is said that φ is *simple* and that B is a *simple* A-algebra.

Recall (see e.g. [1, Cor. 5.2]), that B is a finite A-algebra if and only if B is an integral and a finitely generated A-algebra.

Remark 2.6. Consider the following commutative square of ring homomorphisms with unity,

$$\begin{array}{ccc} A_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 \end{array}$$

and suppose that the homomorphism $B_1 \to B_2$ is surjective. Then, it follows straightforwardly from the definitions that if B_1 is a finite (resp. integral, simple or finitely generated) A_1-algebra then B_2 is a finite (resp. integral, simple or finitely generated) A_2-algebra.

3. Finiteness of semialgebraic homomorphisms

Proposition 3.1. *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and let $\mathtt{j} : \mathbb{R} \hookrightarrow \mathcal{S}(M)$ be the ring homomorphism that maps each real number r to the constant function*

$$\mathtt{j}(r) : M \to \mathbb{R},\ x \mapsto r,$$

which endows $\mathcal{S}(M)$ with an structure of $\mathbb{R}$-algebra. Then, the following conditions are equivalent:
(1) M is a finite set.
(2) $\mathcal{S}(M)$ is a finite (equiv. integral, simple, finitely generated, noetherian, artinian) $\mathbb{R}$-algebra.

Proof. (1) $\Longrightarrow$ (2) Suppose first that $M := \{p_1, \dots, p_k\}$ is a finite set with k elements. Then the topology in M is discrete, and $\mathcal{S}(M) = \mathbb{R}^k$ is a finite $\mathbb{R}$-algebra. Thus it is also an integral and finitely generated $\mathbb{R}$-algebra. In addition, $\mathbb{R}^k$ is an artinian (and so noetherian) ring. Let us prove that it is a simple $\mathbb{R}$-algebra.

Pick k different real numbers $a_1, \dots, a_k$ and let $f \in \mathcal{S}(M)$ be the function defined by $f(p_i) = a_i$ for $i = 1, \dots, k$. Let us check that $\mathcal{S}(M) = \mathbb{R}[f]$. Indeed, each function

$$f_i := \frac{\prod_{j \neq i}(f - a_j)}{\prod_{j \neq i}(a_i - a_j)} \in \mathbb{R}[f]$$

satisfies $f_i(p_i) = 1$ and $f_i(p_j) = 0$ if $i \neq j$. Thus, each function $g \in \mathcal{S}(M)$ satisfies

$$g = \sum_{i=1}^{k} g(p_i) f_i \in \mathbb{R}[f],$$

and so $\mathcal{S}(M) = \mathbb{R}[f]$ is a simple $\mathbb{R}$-algebra. This proves (1) $\Longrightarrow$ (2).

To prove (2) $\Longrightarrow$ (1) recall that the Krull dimension of $\mathcal{S}(M)$ coincides with $\dim(M)$, see e.g. [17, Prop. 1.4] or [14, Thm. 1.1]. As $\mathcal{S}(M)$ is artinian then $\dim(M) = \dim(\mathcal{S}(M)) = 0$, and M is finite. □

Corollary 3.2. *Let $\pi : M \to N$ be a semialgebraic map and suppose that the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is either finite, integral, finitely generated, simple. Then, the fibers of π are finite sets.*

Proof. Pick a point $p \in N$ and consider the fiber $P := \pi^{-1}(p)$, which is a closed semialgebraic subset of M. By Lemma 2.1 the homomorphism $\mathcal{S}(M) \to \mathcal{S}(P),\ f \mapsto f|_P$ is surjective. Consider the commutative square

$$\begin{array}{ccc} \mathcal{S}(N) & \xrightarrow{\varphi_\pi} & \mathcal{S}(M) \\ \downarrow & & \downarrow \\ \mathcal{S}(\{p\}) = \mathbb{R} & \xrightarrow{\mathtt{j}} & \mathcal{S}(P) \end{array}$$

where the vertical arrows are the restriction mappings and $\mathtt{j}$ transforms each real number r into the constant function that only attains the value r. Applying Remark 2.6 it follows that $\mathcal{S}(P)$ is a finite $\mathbb{R}$-algebra and, by Proposition 3.1, the fiber P is finite. □

Corollary 3.3. *Let $\pi : M \to N$ be a surjective semialgebraic map where N is finite, and let $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ be the induced homomorphism. Then, the following conditions are equivalent:*

(1) The set M is finite.

(2) The homomorphism φ_π is finite.

(3) The homomorphism φ_π is integral.

(4) The homomorphism φ_π is simple.

(5) The homomorphism φ_π is finitely generated.

Proof. Notice that $\mathbb{R} \hookrightarrow \mathcal{S}(N) \to \mathcal{S}(M)$. Thus, if M is finite it follows from Proposition 3.1 that $\mathcal{S}(M)$ is a finite, integral, simple and finitely generated $\mathbb{R}$-algebra, so it is a finite, integral, simple and finitely generated $\mathcal{S}(N)$-algebra. This proves that condition (1) implies conditions (2), (3), (4) and (5).

Conversely, suppose that one among conditions (2), (3), (4) and (5) holds. Then, by Corollary 3.2, the fibers of π are finite. Hence, N being finite, the same holds for M. □

Proposition 3.4. *Let $\pi : M \to N$ be a closed and surjective semialgebraic map. Let Y be a closed semialgebraic subset of N and let us denote $X := \pi^{-1}(Y)$. Suppose that the map*

$$\pi|_{M \setminus X} : M \setminus X \to N \setminus Y$$

is injective. Then, the homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is finite (resp. integral, finitely generated or simple) if and only if $\widetilde{\varphi}_\pi : \mathcal{S}(Y) \to \mathcal{S}(X),\ g \mapsto g \circ (\pi|_X)$ is finite (integral, finitely generated or simple).

Proof. The only if part follows at once from Remark 2.6 because we have a commutative square

$$\begin{array}{ccc} \mathcal{S}(N) & \xrightarrow{\varphi_\pi} & \mathcal{S}(M) \\ \downarrow & & \downarrow \\ \mathcal{S}(Y) & \xrightarrow{\widetilde{\varphi}_\pi} & \mathcal{S}(X) \end{array}$$

where the vertical arrows are the restriction homomorphisms, which are surjective by Lemma 2.1 because X and Y are, respectively, closed subsets of M and N. For the converse, denote

$$\mathcal{J}(X) := \{f \in \mathcal{S}(M) : X \subset \mathcal{Z}_M(f)\} \quad \& \quad \mathcal{J}(Y) := \{g \in \mathcal{S}(N) : Y \subset \mathcal{Z}_N(g)\}.$$

Claim. The equality $\varphi_\pi(\mathcal{J}(Y)) = \mathcal{J}(X)$ holds.

Proof of the claim. For $g \in \mathcal{J}(Y)$ and $x \in X$ we have $\pi(x) \in Y$ and $(\varphi_\pi(g))(x) = (g \circ \pi)(x) = g(\pi(x)) = 0$, which proves the inclusion $\varphi_\pi(\mathcal{J}(Y)) \subset \mathcal{J}(X)$. Conversely, let $f \in \mathcal{J}(X)$ and define the function

$$g : N \to \mathbb{R},\ v \mapsto f(u),\ \text{ where } u \in \pi^{-1}(v) \text{ is arbitrary.}$$

Let us prove that $g \in \mathcal{S}(N)$. To check that g is a well defined function it suffices to see that for every point $v \in N$ the function f is constant on the (nonempty) fiber $\pi^{-1}(v)$. In case $v \in N \setminus Y$ there is nothing to check because $\pi^{-1}(v)$ is a singleton. On the other hand, if $v \in Y$ and $u \in \pi^{-1}(v) \subset \pi^{-1}(Y) = X$ we have $f(u) = 0$. To prove that g is continuous note that the triangle

$$\begin{array}{ccc} M & \xrightarrow{\pi} & N \\ & \searrow^{f} & \downarrow g \\ & & \mathbb{R} \end{array}$$

is commutative. Thus, the preimage $g^{-1}(C) = \pi(f^{-1}(C))$ of a closed semialgebraic subset C of $\mathbb{R}$ is a closed subset of N because f is continuous and π is a closed semialgebraic map. To see that g is a semialgebraic function, and therefore $g \in \mathcal{S}(N)$, it suffices to check that its graph $\Gamma(g)$ is a semialgebraic subset of $N \times \mathbb{R}$. Indeed, the mapping

$$\rho : M \to N \times \mathbb{R},\ x \mapsto (\pi(x), f(x))$$

is semialgebraic and, since $\pi : M \to N$ is surjective, $\Gamma(g) = \rho(M)$ is a semialgebraic set.

Thus $g \in \mathcal{S}(N)$ and, in fact, $g \in \mathcal{J}(Y)$ because given $y \in Y$ there exists, since π is surjective, a point $x \in X$ such that $y = \pi(x)$, so $g(y) = g(\pi(x)) = f(x) = 0$. Moreover, $\varphi_\pi(g) = g \circ \pi = f$ and the equality $\varphi_\pi(\mathcal{J}(Y)) = \mathcal{J}(X)$ is proved.

$\square_{\textit{Claim}}$

Suppose now that $\widetilde{\varphi}_\pi$ is a finite homomorphism and let us prove that φ_π is finite too. Let $g_1, \dots, g_r \in \mathcal{S}(X)$ be such that

$$\mathcal{S}(X) = \mathcal{S}(Y)g_1 + \cdots + \mathcal{S}(Y)g_r.$$

The constant function $G_{r+1} : M \to \mathbb{R},\ x \mapsto 1$ belongs to $\mathcal{S}(M)$ and, by Lemma 2.1 (2), there exist $G_1, \dots, G_r \in \mathcal{S}(M)$ with $G_i|_X = g_i$ for $1 \le i \le r$. Let us check the equality

$$\mathcal{S}(M) = \mathcal{S}(N)G_1 + \cdots + \mathcal{S}(N)G_r + \mathcal{S}(N)G_{r+1}.$$

Given $F \in \mathcal{S}(M)$ its restriction $f := F|_X \in \mathcal{S}(X)$. Thus, there exist $h_1, \dots, h_r \in \mathcal{S}(Y)$ such that

$$f = \widetilde{\varphi}_\pi(h_1)g_1 + \cdots + \widetilde{\varphi}_\pi(h_r)g_r = (h_1 \circ \pi|_X)g_1 + \cdots + (h_r \circ \pi|_X)g_r. \quad (3.3)$$

For $1 \le i \le r$ let $H_i \in \mathcal{S}(N)$ with $H_i|_Y = h_i$. By (3.3) we have

$$F - \sum_{i=1}^{r}(H_i \circ \pi)G_i \in \mathcal{J}(X).$$

Since we have proved that $\varphi_\pi(\mathcal{J}(Y)) = \mathcal{J}(X)$ there exists a function $H_{r+1} \in \mathcal{J}(Y)$ with

$$F - \sum_{i=1}^{r}(H_i \circ \pi)G_i = \varphi_\pi(H_{r+1}) = H_{r+1} \circ \pi = (H_{r+1} \circ \pi)G_{r+1}.$$

In other words,

$$F = \sum_{i=1}^{r+1}(H_i \circ \pi)G_i = \sum_{i=1}^{r+1}\varphi_\pi(H_i)G_i \in \mathcal{S}(N)G_1 + \cdots + \mathcal{S}(N)G_r + \mathcal{S}(N)G_{r+1}.$$

Next, suppose that $\widetilde{\varphi}_\pi$ is a simple homomorphism and let us prove that φ_π enjoys this property too.

Let $f \in \mathcal{S}(X)$ such that $\mathcal{S}(X) = \mathcal{S}(Y)[f]$ and let $F \in \mathcal{S}(M)$ with $F|_X = f$. We will see that $\mathcal{S}(M) = \mathcal{S}(N)[F]$. For each $H \in \mathcal{S}(M)$ its restriction $h := H|_X \in \mathcal{S}(X)$. Hence, there exists a polynomial $\mathtt{p} \in \mathcal{S}(Y)[\mathtt{t}]$ such that $h = \mathtt{p}(f)$. Write

$$\mathtt{p}(\mathtt{t}) := \sum_{i=0}^{n} g_i \mathtt{t}^{n-i}, \text{ where } g_i \in \mathcal{S}(Y)$$

and, for $0 \le i \le n$, let $G_i \in \mathcal{S}(N)$ with $G_i|_Y = g_i$. As $\sum_{i=0}^{n} \widetilde{\varphi}_\pi(g_i) f^{n-i} = \mathtt{p}(f) = h$ we have

$$H - \sum_{i=0}^{n}\varphi_\pi(G_i)F^{n-i} = H - \sum_{i=0}^{n}(G_i \circ \pi)F^{n-i} \in \mathcal{J}(X) = \varphi_\pi(\mathcal{J}(Y)).$$

Therefore there exists $G \in \mathcal{J}(Y) \subset \mathcal{S}(N)$ such that

$$H - \sum_{i=0}^{n} \varphi_\pi(G_i) F^{n-i} = \varphi_\pi(G).$$

Consequently, the polynomial $\mathtt{P}(\mathtt{t}) := (G_n + G) + \sum_{i=0}^{n-1} G_i \mathtt{t}^{n-i} \in \mathcal{S}(N)[\mathtt{t}]$ and $H = \mathtt{P}(F)$.

Arguing analogously one proves that if $\widetilde{\varphi}_\pi$ is a finitely generated homomorphism the same holds true for φ_π. So to finish it suffices to see that φ_π is an integral homomorphism whenever $\widetilde{\varphi}_\pi$ is so. Given $F \in \mathcal{S}(M)$ its restriction $f := F|_X$ is integral over $\mathcal{S}(Y)$, that is, there exist $g_1, \dots, g_n \in \mathcal{S}(Y)$ such that

$$f^n + \sum_{i=1}^{n} \widetilde{\varphi}_\pi(g_i) f^{n-i} = 0.$$

For $1 \leq i \leq n$ let $G_i \in \mathcal{S}(N)$ such that $G_i|_Y = g_i$. Then $F^n + \sum_{i=1}^{n} \varphi_\pi(G_i) F^{n-i} \in \mathcal{J}(X)$. Since $\mathcal{J}(X) = \varphi_\pi(\mathcal{J}(Y))$ there exists $G \in \mathcal{J}(Y) \subset \mathcal{S}(N)$ such that

$$F^n + \sum_{i=1}^{n} \varphi_\pi(G_i) F^{n-i} = \varphi_\pi(G).$$

Thus $\mathtt{P}(F) = 0$, where

$$\mathtt{P}(\mathtt{t}) := \mathtt{t}^n + \sum_{i=1}^{n-1} G_i \mathtt{t}^{n-i} + (G_n - G) \in \mathcal{S}(N)[\mathtt{t}]$$

is a monic polynomial, so F is integral over $\mathcal{S}(N)$. □

Corollary 3.5. *Let $\pi : M \to N$ be a surjective semialgebraic map such that $Y := \{y \in N : \operatorname{Card}(\pi^{-1}(y)) > 1\}$ is a finite set. Then, the following statements are equivalent:*

(1) The set $X := \pi^{-1}(Y)$ is finite.

(2) The homomorphism φ_π is finite.

(3) The homomorphism φ_π is integral.

(4) The homomorphism φ_π is simple.

(5) The homomorphism φ_π is finitely generated.

Proof. Notice that Y being finite it is a closed semialgebraic subset of N and, by its definition, the restriction

$$\pi|_{M \setminus X} : M \setminus X \to N \setminus Y$$

is injective. It follows from Proposition 3.4 that the ring homomorphism φ_π is finite (resp. integral, simple or finitely generated) if and only if the ring homomorphism

$$\widetilde{\varphi}_\pi : \mathcal{S}(Y) \to \mathcal{S}(X),\ g \mapsto g \circ (\pi|_X)$$

is finite (resp. integral, simple or finitely generated). Each one of these conditions is equivalent, by Corollary 3.3, to the finiteness of X, as wanted. □

Next we apply the results above to semialgebraic compactifications of locally compact semialgebraic sets.

Definition 3.6. (1) A *semialgebraic compactification* of a semialgebraic set M is a pair $(X, \mathtt{j})$ where X is a compact semialgebraic set and $\mathtt{j} : M \to X$ is a semialgebraic homeomorphism onto its image $\mathtt{j}(M)$, which is a dense subset of X. We denote $\partial^* X := X \setminus \mathtt{j}(M)$ the *remainder* of the compactification $(X, \mathtt{j})$ of M.

(2) Given two semialgebraic compactifications $X_1, \mathtt{j}_1)$ and $(X_2, \mathtt{j}_2)$ of a semialgebraic set M, we say that $(X_2, \mathtt{j}_2)$ *dominates* $(X_1, \mathtt{j}_1)$, and we write $(X_1, \mathtt{j}_1) \preccurlyeq (X_2, \mathtt{j}_2)$, if there exists a semialgebraic map $\pi : X_2 \to X_1$ such that $\pi \circ \mathtt{j}_2 = \mathtt{j}_1$. Notice that π is surjective since its image is a compact subset of X_1 that contains its dense subset $\pi(\mathtt{j}_2(M)) = \mathtt{j}_1(M)$. Moreover, the equality $\pi \circ \mathtt{j}_2 = \mathtt{j}_1$ determines π because for every point $p \in X_2$ there exists a sequence $\{x_n\} \subset M$ such that $p = \lim_{n\to\infty}\{\mathtt{j}_2(x_n)\}$, and

$$\pi(p) = \lim_{n\to\infty}\{\pi(\mathtt{j}_2(x_n))\} = \lim_{n\to\infty}\{\mathtt{j}_1(x_n)\}.$$

Then $\mathcal{S}(X_2)$ is an $\mathcal{S}(X_1)$-algebra with the structure endowed by the homomorphism

$$\varphi_\pi : \mathcal{S}(X_1) \to \mathcal{S}(X_2),\ f \mapsto f \circ \pi.$$

As $\pi^{-1}(\partial^* X_1) = \partial^* X_2$ by [12, Lemma 4.3], the semialgebraic mapping $\pi|_{\partial^* X_2} : \partial^* X_2 \to \partial^* X_1$ is surjective. Therefore, $\mathcal{S}(\partial^* X_2)$ is an $\mathcal{S}(\partial^* X_1)$-algebra with the structure endowed by the homomorphism

$$\partial^*\varphi_\pi : \mathcal{S}(\partial^* X_1) \to \mathcal{S}(\partial^* X_2),\ f \mapsto f \circ (\pi|_{\partial^* X_2}).$$

It was proved in [12, Cor. 4.14] that if M is a locally compact semialgebraic set and $(X, \mathtt{j})$ is a semialgebraic compactification of M then the residue $\partial^* X$ is a closed semialgebraic subset of X. Thus, the next Corollary 3.7 applies in this case.

Corollary 3.7. *Let $(X_1, \mathtt{j}_1)$ and $(X_2, \mathtt{j}_2)$ be two semialgebraic compactifications of a non compact semialgebraic set M such that $(X_2, \mathtt{j}_2)$ dominates $(X_1, \mathtt{j}_1)$ and $\partial^* X_1$ is a closed subset of X_1. Then, $\mathcal{S}(X_2)$ is a finite (resp. integral, simple or finitely generated) $\mathcal{S}(X_1)$-algebra if and only if $\mathcal{S}(\partial^* X_2)$ is a finite (resp. integral, simple or finitely generated) $\mathcal{S}(\partial^* X_1)$- algebra.*

Proof. Let $\pi : X_2 \to X_1$ be the unique semialgebraic map satisfying $\pi \circ \mathtt{j}_2 = \mathtt{j}_1$. As we have just remarked, $\pi^{-1}(\partial^* X_1) = \partial^* X_2$. In addition, the restriction

$$\pi|_{X_2 \setminus \pi^{-1}(\partial^* X_1)} : X_2 \setminus \pi^{-1}(\partial^* X_1) = \mathtt{j}_2(M) \to X_1 \setminus \partial^* X_1 = \mathtt{j}_1(M)$$

is injective because both $\mathtt{j}_1$ and $\mathtt{j}_2$ are injective mappings. Hence the result follows from Proposition 3.4 □

Corollary 3.8. *Let $(X_1, \mathtt{j}_1)$ and $(X_2, \mathtt{j}_2)$ be two semialgebraic compactifications of a semialgebraic set M such that $(X_2, \mathtt{j}_2)$ dominates $(X_1, \mathtt{j}_1)$.*
(1) If $\mathcal{S}(X_2)$ is a finite, integral, simple or finitely generated $\mathcal{S}(X_1)$-algebra, then the fibers of the unique semialgebraic map $\pi : X_2 \to X_1$ such that $\pi \circ \mathtt{j}_2 = \mathtt{j}_1$ are finite.
(2) Suppose that the union of the fibers of π with more than a point is a finite set. Then $\mathcal{S}(X_2)$ is a finite and a simple $\mathcal{S}(X_1)$- algebra.

Proof. (1) This is a particular case of Corollary 3.2.
(2) Denote $Y := \{y \in X_1 : \operatorname{Card}(\pi^{-1}(y)) > 1\}$. By hypothesis $\pi^{-1}(Y)$ is finite and, since π is surjective, Y is finite too. Then, the conclusion follows at once from Corollary 3.5. □

Corollary 3.9. *Let $(X_1, \mathtt{j}_1)$ and $(X_2, \mathtt{j}_2)$ be two semialgebraic compactifications of a semialgebraic set M such that $(X_2, \mathtt{j}_2)$ dominates $(X_1, \mathtt{j}_1)$ and the remainder $\partial^* X_1$ is a finite set. Then, the following conditions are equivalent:*
(1) The remainder $\partial^* X_2$ is finite.
(2) $\mathcal{S}(X_2)$ is a finite $\mathcal{S}(X_1)$-algebra.
(3) $\mathcal{S}(X_2)$ is integral over $\mathcal{S}(X_1)$.
(4) $\mathcal{S}(X_2)$ is a finitely generated $\mathcal{S}(X_1)$-algebra.
(5) $\mathcal{S}(X_2)$ is a simple $\mathcal{S}(X_1)$-algebra.

Proof. Let $\pi : X_2 \to X_1$ be the unique semialgebraic map such that $\pi \circ \mathtt{j}_2 = \mathtt{j}_1$. By [12, Lemma 4.3], $\pi^{-1}(\partial^* X_1) = \partial^* X_2$, so $\pi|_{\partial^* X_2} : \partial^* X_2 \to \partial^* X_1$ is a surjective semialgebraic mapping. Since $\partial^* X_1$ is finite, and $\mathcal{S}(\partial^* X_2)$ is an $\mathcal{S}(\partial^* X_1)$-algebra with the structure endowed by the homomorphism

$$\partial^* \varphi_\pi : \mathcal{S}(\partial^* X_1) \to \mathcal{S}(\partial^* X_2),\ f \mapsto f \circ (\pi|_{\partial^* X_2}),$$

the finiteness of $\partial^* X_2$ is equivalent, by Corollary 3.3 to any, and so all of the following conditions:

i) The homomorphism $\partial^* \varphi_\pi$ is finite; ii) $\partial^* \varphi_\pi$ is integral; iii) $\partial^* \varphi_\pi$ is simple and iv) $\partial^* \varphi_\pi$ is finitely generated. To finish it suffices to apply Corollary 3.7 because $\partial^* X_1$ being finite it is a closed subset of X_1. □

4. Locally injective semialgebraic mappings

Proposition 4.1. *Let $\pi : M \to N$ be a semialgebraic map such that the ring homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is integral. Then*
(1) The spectral map

$$\operatorname{Spec_s}(\pi) : \operatorname{Spec_s}(M) \to \operatorname{Spec_s}(N)$$

is proper and separated, that is, each pair of points in the same fiber admit disjoint open neighborhoods in $\operatorname{Spec_s}(M)$.

(2) $\operatorname{Spec_s}(\pi)$ *maps maximal ideals of* $\mathcal{S}(M)$ *into maximal ideals of* $\mathcal{S}(N)$. *Thus the restriction*

$$\beta_s\pi := \operatorname{Spec_s}(\pi)|_{\beta_s M} : \beta_s M \to \beta_s N$$

is a well defined proper map and it is unique making commutative the square

$$\begin{array}{ccc} M & \overset{\mathtt{j}_M}{\hookrightarrow} & \beta_s M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\beta_s\pi} \\ N & \overset{\mathtt{j}_N}{\hookrightarrow} & \beta_s N \end{array}$$

(3) The map π *is proper with finite fibers.*

Proof. (1) Since φ_π is integral the spectral map $\operatorname{Spec_s}(\pi)$ is closed, by [1, Thm. 5.10]. In addition, it is proved in [24, 5.23.3] (see also [11, Lemma 3.8]), that its fibers are compact.

Let us prove now that $\operatorname{Spec}_s(\pi)$ is a separated map. Let $\mathfrak{q}$ be a prime ideal in $\mathcal{S}(N)$ and let $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct points in the fiber $X := \operatorname{Spec_s}(\pi)^{-1}(\mathfrak{q})$ over $\mathfrak{q}$. We claim that it does not exist a prime ideal $\mathfrak{p} \in \operatorname{Spec}_s(M)$ contained in $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Otherwise, by [7, 2.4.2], the set of prime ideals in $\mathcal{S}(M)$ containing $\mathfrak{p}$ form a chain, which is false because, by [1, Cor. 5.9], $\mathfrak{p}_1 \not\subset \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subset \mathfrak{p}_1$. Hence by [24, 5.23.7], there exist open disjoint neighborhoods U_1 of $\mathfrak{p}_1$ and U_2 of $\mathfrak{p}_2$ in $\operatorname{Spec_s}(M)$, as claimed.

(2) Let $(\beta_s M, \mathtt{j}_M)$ and $(\beta_s N, \mathtt{j}_N)$ be, respectively, the semialgebraic Stone–Čech compactifications of M and N. The closed map $\operatorname{Spec_s}(\pi)$ transforms the closed points of $\operatorname{Spec_s}(M)$ into closed points of $\operatorname{Spec_s}(N)$. Thus, if $\mathtt{r}_N : \operatorname{Spec_s}(N) \to \beta_s N$ denotes the retraction that maps each prime ideal in $\mathcal{S}(N)$ to the unique maximal ideal containing it, we have

$$\beta_s\pi = \mathtt{r}_N \circ \operatorname{Spec_s}(\pi)|_{\beta_s M} = \operatorname{Spec_s}(\pi)|_{\beta_s M}.$$

Since $\beta_s M$ and $\beta_s N$ are compact and Hausdorff spaces $\beta_s\pi$ is a proper map. Its uniqueness making commutative the square in the statement follows at once from (2).

(3) Let us prove that $(\beta_s\pi)^{-1}(\mathtt{j}_N(N)) = \mathtt{j}_M(M)$. The inclusion $\mathtt{j}_M(M) \subset (\beta_s\pi)^{-1}(\mathtt{j}_N(N))$ is an straightforward consequence of the commutativity of the square above. Conversely, given $\mathfrak{m} \in (\beta_s\pi)^{-1}(\mathtt{j}_N(N))$ the maximal ideal $\mathfrak{n} := \beta_s\pi(\mathfrak{m})$ lies in $\mathtt{j}_N(N)$, i.e., there exists a point $y \in N$ such that $\mathfrak{n} = \mathfrak{m}_y$. Thus $\mathcal{S}(N)/\mathfrak{n} = \mathbb{R}$ and, since $\mathfrak{n} = \varphi_\pi^{-1}(\mathfrak{m})$, we have an algebraic field extension

$$\mathbb{R} = \mathcal{S}(N)/\mathfrak{n} \hookrightarrow \mathcal{S}(M)/\mathfrak{m},$$

because φ_π is an integral homomorphism. By [22, III.1] or [23, Thm. 5.12] $\mathcal{S}(M)$ is a real closed ring, so the quotient $\mathcal{S}(M)/\mathfrak{m}$ is a real closed field and the equality $\mathcal{S}(M)/\mathfrak{m} = \mathbb{R}$ holds. This implies, by [12, Cor. 3.11], that there exists a point $x \in M$ such that $\mathfrak{m} = \mathfrak{m}_x = \mathtt{j}_M(x) \in \mathtt{j}_M(M)$. Therefore,

$$\beta_s\pi|_{\mathtt{j}_M(M)} : \mathtt{j}_M(M) \to \mathtt{j}_N(N)$$

is a proper map too, and the same holds for

$$\pi = \jmath_N^{-1} \circ \beta_s\pi|_{\jmath_M(M)} \circ \jmath_M : M \to N.$$

Finally, the finiteness of the fibers of π follows at once from Corollary 3.2. □

Remark 4.2. By cell decomposition, for every semialgebraic map $\pi : M \to N$ whose fibers are finite –as in Proposition 4.1– there exists an integer $k > 0$ such that the fibers of π have, at most, k elements (see also [3, Thm. 9.3.2]).

The next examples show that local injectivity is somehow related to integral extensions.

Examples 4.3. (1) Let $\pi : M \to N$ be a closed semialgebraic map. Suppose that there exists a finite cover $\{M_i : 1 \le i \le k\}$ of M by closed semialgebraic subsets such that each restriction $\pi|_{M_i} : M_i \to N$ is injective. Then the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is integral.

Indeed, note that $N_i := \pi(M_i)$ is a closed semialgebraic subset of N, because π is a closed semialgebraic map, and $\pi|_{M_i} : M_i \to N_i$ is a closed and continuous semialgebraic bijection. Thus it is a semialgebraic homeomorphism. Consequently, for every $f \in \mathcal{S}(M)$ the function $g_i := f|_{M_i} \circ (\pi|_{M_i})^{-1} : N_i \to \mathbb{R}$ is semialgebraic and, since N_i is a closed semialgebraic subset of N, there exists, by Lemma 2.1 (2), a semialgebraic extension $G_i : N \to \mathbb{R}$ of g_i. Now, for each point $x \in M$ there exists an index $1 \le i \le k$ such that $x \in M_i$; thus $G_i(\pi(x)) = g_i(\pi(x)) = f(x)$, and henceforth the equality

$$\prod_{i=1}^{k}(f - G_i \circ \pi) = 0$$

shows that f is integral over $\mathcal{S}(N)$.

(2) Let $M := (a, b) \subset \mathbb{R}$ be an open interval, where $a \in \mathbb{R}\cup\{-\infty\}$ and $b \in \mathbb{R}\cup\{\infty\}$, and let $\pi : M \to \mathbb{R}$ be a non constant closed Nash function. Let $\pi' : M \to \mathbb{R}$ be its derivative, whose zeroset $\mathcal{Z}_M(\pi')$ is finite. Otherwise it would be an 1-dimensional semialgebraic set, and so it should contain an open subset V of M. By the Identity Principle $\pi' \equiv 0$, i.e., π would be constant, and this is false. Let $\mathcal{Z}_M(\pi') := \{a_1, \dots, a_\ell\}$ and consider the closed subsets of M

$$M_0 := (a, a_1], \quad M_1 := [a_1, a_2], \dots, M_{\ell-1} := [a_{\ell-1}, a_\ell] \quad \& \quad M_\ell := [a_\ell, b).$$

For $0 \le i \le \ell$ the restriction $\pi|_{M_i}$ is monotone, hence injective, and it follows from (1) that the homomorphism $\varphi_\pi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(M)$ is integral. This applies, for instance, to non constant polynomial maps $\pi : \mathbb{R} \to \mathbb{R}$, since they are closed Nash maps. Notice that φ_π is not necessarily finitely generated. To check this consider the Nash function $\pi : M := \mathbb{R} \to \mathbb{R}$, $t \mapsto t^2$. We have just proved that $\varphi_\pi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(M)$ is integral. If, in addition, it were finitely generated, then it would be finite and, by the next Theorem 4.5, the function π would be locally injective at $0 \in \mathbb{R}$, which is false.

(3) The Nash function $\pi : \mathbb{R} \to \mathbb{R}$, $t \mapsto \sqrt{1 + t^2} - t$ is not a closed map because $\pi(\mathbb{R}) = (0, +\infty)$ is not a closed subset of $\mathbb{R}$. Thus, by Proposition 4.1 (3), the induced homomorphism φ_π is not integral.

Before proving that finite homomorphisms between rings of semialgebraic functions are induced by locally injective semialgebraic maps we formulate Nakayama's Lemma as we will need for our purposes.

Lemma 4.4. (Nakayama) *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local rings and let $\psi : A \to B$ be a finite homomorphism of local rings such that the homomorphism $\overline{\psi} : A/\mathfrak{m}_A \to B/\psi(\mathfrak{m}_A)B$ defined by $\overline{\psi}(a+\mathfrak{m}_A) = \psi(a)+\psi(\mathfrak{m}_A)B$ is surjective. Then, ψ is surjective too.*

Proof. It suffices to prove that $B = \psi(\mathfrak{m}_A)B + \operatorname{im}\psi$ and apply [1, Cor. 2.7], because $\mathfrak{m}_A$ is the Jacobson radical of A. Given $b \in B$ there exists $a \in A$ such that $b + \psi(\mathfrak{m}_A)B = \psi(a) + \psi(\mathfrak{m}_A)B$, i.e., $b - \psi(a) \in \psi(\mathfrak{m}_A)B$, as wanted. □

We are ready to state and prove the main result of this work.

Theorem 4.5. *Let $\pi : M \to N$ be a semialgebraic map such that the ring homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is finite. Then the maps*

$$\operatorname{Spec_s}(\pi) : \operatorname{Spec_s}(M) \to \operatorname{Spec_s}(N), \quad \beta_s\pi : \beta_s M \to \beta_s N \quad \& \quad \pi : M \to N$$

are proper, separated, locally injective and their fibers are finite sets.

Proof. Since φ_π is finite it is integral, and it follows from Proposition 4.1 (1), that the map $\operatorname{Spec_s}(\pi) : \operatorname{Spec_s}(M) \to \operatorname{Spec_s}(N)$ is proper and separated. The properness of π and $\beta_s\pi$ has been proved in Proposition 4.1. In addition, since φ_π is finite it follows from [1, Ex. 4, Ch. 8] that the fibers of $\operatorname{Spec_s}(\pi)$ are finite. Since the fibers of π and $\beta_s\pi$ are subsets of the fibers of $\operatorname{Spec_s}(\pi)$, they are finite too. In addition they are Hausdorff spaces since M and $\beta_s M$ are so.

To prove that $\operatorname{Spec_s}(\pi)$, $\beta_s\pi$ and π are locally injective maps it suffices to demonstrate the following:

(4.a). For every $\mathfrak{m} \in \beta_s M$ there exists $f \in \mathcal{S}(M)$ such that $\mathfrak{m} \in \mathfrak{U} := \mathfrak{D}_{\operatorname{Spec_s}(M)}(f)$ and the restriction $\operatorname{Spec_s}(\pi)|_{\mathfrak{U}} : \mathfrak{U} \to \operatorname{Spec_s}(N)$ is injective. Once this is proved, every ideal $\mathfrak{p} \in \operatorname{Spec_s}(M)$ is contained in a maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$, so $f \notin \mathfrak{p}$, that is, $\mathfrak{U}$ is an open neighbourhood of $\mathfrak{p}$ in $\operatorname{Spec_s}(M)$, which proves that $\operatorname{Spec_s}(\pi)$ is locally injective. In addition, also the maps

$$\beta_s\pi|_{\mathfrak{U}\cap\beta_s M} = \operatorname{Spec_s}(\pi)|_{\mathfrak{U}\cap\beta_s M} : \mathfrak{U}\cap\beta_s M \to \beta_s N \quad \& \quad \pi|_{\mathfrak{U}\cap M} = \beta_s\pi|_{\mathfrak{U}\cap M} : \mathfrak{U}\cap M \to N$$

are injective, so both $\beta_s\pi$ and π are locally injective.

Let us prove (4.a). Denote $\mathfrak{n} := \operatorname{Spec_s}(\pi)(\mathfrak{m}) \in \beta_s N$, and let us show first that:

Claim. The induced homomorphism of local rings

$$\psi : A := \mathcal{S}(N)_{\mathfrak{n}} \to B := \mathcal{S}(M)_{\mathfrak{m}}, \ \frac{h}{v} \mapsto \frac{\varphi_\pi(h)}{\varphi_\pi(v)},$$

is finite, i.e., B is a finitely generated A-module via ψ.

Proof of the Claim. Note that the canonical homomorphism $\rho : \mathcal{S}(M) \to \mathcal{S}(M)_{\mathfrak{m}}$, $h \mapsto h/1$ is surjective. In fact, consider the ideal $\ker(\rho)$ of $\mathcal{S}(M)$, which is clearly contained in $\mathfrak{m}$. Let us show that $\mathfrak{m}$ is the unique maximal ideal in $\mathcal{S}(M)$ that contains $\ker(\rho)$. Indeed, if $\mathfrak{m}_1 \neq \mathfrak{m}$ is another maximal ideal in $\mathcal{S}(M)$ such that $\ker(\rho) \subset \mathfrak{m}_1$, and since $\beta_s M$ is a Hausdorff space, there exist $f, g \in \mathcal{S}(M)$ with $\mathfrak{m} \in \mathcal{D}_{\operatorname{Spec_s}(M)}(f)$ and $\mathfrak{m}_1 \in \mathcal{D}_{\operatorname{Spec_s}(M)}(f)$ such that $\mathcal{D}_{\operatorname{Spec_s}(M)}(f) \cap \mathcal{D}_{\operatorname{Spec_s}(M)}(f) = \varnothing$. Thus

$$\begin{aligned}\mathcal{Z}_{\operatorname{Spec_s}(M)}(fg) &= \mathcal{Z}_{\operatorname{Spec_s}(M)}(f) \cup \mathcal{Z}_{\operatorname{Spec_s}(M)}(g)\\ &= (\operatorname{Spec_s}(M) \setminus \mathcal{D}_{\operatorname{Spec_s}(M)}(f)) \cup (\operatorname{Spec_s}(M) \setminus \mathcal{D}_{\operatorname{Spec_s}(M)}(g))\\ &= \operatorname{Spec_s}(M) \setminus (\mathcal{D}_{\operatorname{Spec_s}(M)}(f) \cap \mathcal{D}_{\operatorname{Spec_s}(M)}(g)) = \operatorname{Spec_s}(M),\end{aligned}$$

which means that $fg = 0$ because $\mathcal{S}(M)$ is a reduced ring. Since $f \notin \mathfrak{m}$ this implies that $\rho(g) = 0$, which is a contradiction because $g \notin \mathfrak{m}_1$.

In particular, if $p : \mathcal{S}(M) \to \mathcal{S}(M)/\ker(\rho)$ denotes the natural projection, then $p(u)$ is a unit in $\mathcal{S}(M)/\ker(\rho)$ for every $u \in \mathcal{S}(M) \setminus \mathfrak{m}$. Therefore we can define the homomorphism

$$h : \mathcal{S}(M)_{\mathfrak{m}} \to \mathcal{S}(M)/\ker(\rho) : \tfrac{f}{u} \mapsto (f + \ker(\rho)(u + \ker(\rho))^{-1},$$

which is an isomorphism whose inverse is defined as $h^{-1}(f + \ker(\rho)) = \frac{f}{1}$. This shows that ρ is surjective and in fact $\mathcal{S}(M)/\ker(\rho) \simeq \mathcal{S}(M)_{\mathfrak{m}}$ is a local ring whose unique maximal ideal is $\mathfrak{m}/\ker(\rho)$.

We are ready to prove that the homomorphism $\psi : A \to B$ is finite. Consider the multiplicatively closed set $T := \mathcal{S}(N) \setminus \mathfrak{n}$. By [1, Cor. 3.4] the finiteness of φ_π implies that the homomorphism

$$\mathcal{S}(N)_{\mathfrak{n}} = T^{-1}\mathcal{S}(N) \to \varphi_\pi(T)^{-1}\mathcal{S}(M),\ \frac{h}{t} \mapsto \frac{\varphi_\pi(h)}{\varphi_\pi(t)}$$

is finite too. On the other hand, since $\rho : \mathcal{S}(M) \to \mathcal{S}(M)_{\mathfrak{m}}$ is surjective, the map

$$\varphi_\pi(T)^{-1}\mathcal{S}(M) \to \mathcal{S}(M)_{\mathfrak{m}},\ \frac{g}{v} \mapsto \frac{\varphi_\pi(g)}{\varphi_\pi(v)}$$

is a well defined and surjective homomorphism. Thus the homomorphism $\psi : \mathcal{S}(N)_{\mathfrak{n}} \to \mathcal{S}(M)_{\mathfrak{m}}$ is finite since it is the composition of the finite homomorphism $\mathcal{S}(N)_{\mathfrak{n}} \to \varphi_\pi(T)^{-1}\mathcal{S}(M)$ and the surjective homomorphism $\varphi_\pi(T)^{-1}\mathcal{S}(M) \to \mathcal{S}(M)_{\mathfrak{m}}$.

□*Claim*

Next we are going to prove that ψ is surjective. Let $\mathfrak{m}_A := \mathfrak{n}A$ and $\mathfrak{m}_B := \mathfrak{m}B$ be, respectively, the unique maximal ideals in the local rings A and B. Then $\mathfrak{m}_A B \subset \mathfrak{m}_B$, and we will prove right now that $\sqrt{\mathfrak{m}_A B} = \mathfrak{m}_B$. To that end it suffices to see that $\mathfrak{m}_B$ is the unique prime ideal of B that contains $\mathfrak{m}_A B$. Let $\mathfrak{p}$ be a prime ideal in B that contains $\mathfrak{m}_A B$. Then $\mathfrak{p} \subset \mathfrak{m}_B$ and, using [1, Cor. 5.9], to prove that $\mathfrak{p} = \mathfrak{m}_B$ it is enough to see that both prime ideals lie over $\mathfrak{m}_A$ via the integral homomorphism ψ. But $\mathfrak{m}_A \subset \psi^{-1}(\psi(\mathfrak{m}_A)B) \subset \psi^{-1}(\mathfrak{p})$ and, since $\mathfrak{m}_A$ is maximal, we have $\mathfrak{m}_A = \psi^{-1}(\mathfrak{p})$, whereas $\mathfrak{m}_A = \psi^{-1}(\mathfrak{m}_B)$ because $\mathfrak{n} = \varphi_\pi^{-1}(\mathfrak{m})$. Consequently,

$$\mathfrak{m} = \mathfrak{m}_B \cap \mathcal{S}(M) = \sqrt{\mathfrak{m}_A B} \cap \mathcal{S}(M) \subset \sqrt{\mathfrak{m}_A B \cap \mathcal{S}(M)} = \sqrt{\mathfrak{n}B \cap \mathcal{S}(M)}. \tag{4.4}$$

This implies $\mathfrak{m} = \sqrt{\mathfrak{n}B \cap \mathcal{S}(M)}$ because $\mathfrak{m}$ is maximal. We will show now that

$$\mathfrak{m} = \mathfrak{n}B \cap \mathcal{S}(M). \tag{4.5}$$

To prove it note that, by the finiteness of φ_π, there exist $f_1, \dots, f_r \in \mathcal{S}(M)$ such that

$$\mathcal{S}(M) = f_1\mathcal{S}(N) + \cdots + f_r\mathcal{S}(N). \tag{4.6}$$

On the other hand the field homomorphism

$$\phi : \kappa(\mathfrak{n}) := \mathcal{S}(N)/\mathfrak{n} \to \kappa(\mathfrak{m}) := \mathcal{S}(M)/\mathfrak{m},\ g + \mathfrak{n} \mapsto (g \circ \pi) + \mathfrak{m}$$

is finite because φ_π is so. Thus $\kappa(\mathfrak{m})$ is an algebraic extension of $\kappa(\mathfrak{n})$ and both are real closed fields, which implies that ϕ is surjective. Hence, for every $1 \le j \le r$ there exist $g_j \in \mathcal{S}(N)$ such that $f_j + \mathfrak{m} = (g_j \circ \pi) + \mathfrak{m}$, so $h_j := f_j - (g_j \circ \pi) \in \mathfrak{m}$.

By equality (4.4) there exists a positive integer $n \in \mathbb{Z}$ such that $h_j^n \in \mathfrak{n}B \cap \mathcal{S}(M)$ for $1 \le j \le r$. On the other hand, for each function $f \in \mathcal{S}(M)$ there exist $\ell_1, \dots, \ell_r \in \mathcal{S}(N)$ such that

$$f = \sum_{j=1}^r f_j \cdot (\ell_j \circ \pi) = \sum_{j=1}^r (h_j + (g_j \circ \pi)) \cdot (\ell_j \circ \pi) = \sum_{j=1}^r h_j \cdot (\ell_j \circ \pi) + \sum_{j=1}^r g_j \ell_j \circ \pi.$$

In other words,

$$\mathcal{S}(M) = h_1\mathcal{S}(N) + \cdots + h_r\mathcal{S}(N) + h_{r+1}\mathcal{S}(N), \tag{4.7}$$

where $h_{r+1} = \mathbf{1}_M$.

Let $m > n(r+1)$ be an odd positive integer. We claim that $h^m \in \mathfrak{n}B \cap \mathcal{S}(M)$ for every $h \in \mathfrak{m}$. To see this observe that there exist $q_1, \dots, q_{r+1} \in \mathcal{S}(N)$ such that

$$h = \sum_{j=1}^r h_j \cdot (q_j \circ \pi) + (q_{r+1} \circ \pi),$$

hence

$$\varphi_\pi(q_{r+1}) = q_{r+1} \circ \pi = h - \sum_{j=1}^r h_j \cdot (q_j \circ \pi) \in \mathfrak{m},$$

that is, $q_{r+1} \in \varphi_\pi^{-1}(\mathfrak{m}) = \mathfrak{n}$. Thus,

$$h^m = \Big(\sum_{j=1}^r h_j \cdot (q_j \circ \pi) + (q_{r+1} \circ \pi)\Big)^m \in \mathfrak{n}B \cap \mathcal{S}(M)$$

because it is a sum whose summands contain either a factor $h_j^{r_j}$ with $r_j \geq n$ or a power of $q_{r+1} \circ \pi$.

We are ready to check equality (4.5), where the inclusion $\mathfrak{n}B \cap \mathcal{S}(M) \subset \mathfrak{m}$ is evident. Conversely, let $f \in \mathfrak{m}$. Since m is odd there exists $h \in \mathcal{S}(M)$ such that $h^m = f \in \mathfrak{m}$. Thus $h \in \mathfrak{m}$ and, as we have just proved, this implies that $f = h^m \in \mathfrak{n}B \cap \mathcal{S}(M)$. Consequently $\mathfrak{m}_A B = \mathfrak{m}_B$ because

$$\mathfrak{m}_B = \mathfrak{m}B = (\mathfrak{n}B \cap \mathcal{S}(M))B \subset \mathfrak{n}B = \mathfrak{m}_A B \subset \mathfrak{m}_B.$$

Thus, the homomorphism $\psi : A \to B$ is surjective since it satisfies the hypothesis in Nakayama's Lemma 4.4; ψ induces a surjective homomorphism

$$\phi := \overline{\psi} : A/\mathfrak{m}_A = \kappa(\mathfrak{n}) \to B/\mathfrak{m}_A B = B/\mathfrak{m}B = \kappa(\mathfrak{m}).$$

In this way we have a commutative square

$$\begin{array}{ccc} \mathcal{S}(N) & \xrightarrow{\varphi_\pi} & \mathcal{S}(M) \\ \downarrow & & \downarrow \\ \mathcal{S}(N)_{\mathfrak{n}} & \xrightarrow{\psi} & \mathcal{S}(M)_{\mathfrak{m}} \end{array}$$

where φ_π is finite and ψ is surjective. Let $f_1, \dots, f_r \in \mathcal{S}(M)$ satisfying equality (4.6). As ψ is surjective there exist $h_1, \dots, h_r \in \mathcal{S}(N)$ and $g_1, \dots, g_r \in \mathcal{S}(N) \setminus \mathfrak{n}$ such that, for $1 \leq i \leq r$,

$$f_i = \frac{h_i \circ \pi}{g_i \circ \pi}.$$

Denote $p_i := h_i \cdot \prod_{j \neq i} g_j \in \mathcal{S}(N)$. Then

$$g := \prod_{i=1}^r g_i \in \mathcal{S}(N) \setminus \mathfrak{n} \quad \& \quad f_i = \frac{p_i \circ \pi}{g \circ \pi}.$$

Thus $f := g \circ \pi \in \mathcal{S}(M) \setminus \mathfrak{m}$ and we claim that the homomorphism

$$\widetilde{\varphi}_\pi : \mathcal{S}(N)_g \to \mathcal{S}(M)_f, \ \frac{u}{g^n} \mapsto \frac{u \circ \pi}{f^n}$$

is surjective. Indeed, given $\zeta \in \mathcal{S}(M)_f$ there exist $v \in \mathcal{S}(M)$ and a positive integer k such that $\zeta := \frac{v}{f^k}$. Write

$$v = f_1(u_1 \circ \pi) + \cdots + f_r(u_r \circ \pi),$$

for some $u_1, \dots, u_r \in \mathcal{S}(N)$. Then

$$\zeta = \frac{v}{f^k} = \frac{\sum_{i=1}^r f_i(u_i \circ \pi)}{f^k} = \frac{\sum_{i=1}^r (p_i \circ \pi)(u_i \circ \pi)}{(g \circ \pi)^{k+1}}.$$

Therefore,

$$\zeta' := \frac{\sum_{i=1}^r p_i u_i}{g^{k+1}} \in \mathcal{S}(N)_g \quad \text{and} \quad \widetilde{\varphi}_\pi(\zeta') = \zeta.$$

Finally, $\mathcal{U} := \mathcal{D}_{\mathrm{Spec_s}(M)}(f) = \mathrm{Spec}(\mathcal{S}(M)_f)$ is a neighbourhood of $\mathfrak{m}$ in $\mathrm{Spec_s}(M)$ and the map

$$\mathrm{Spec_s}(\pi)|_{\mathcal{U}} = \mathrm{Spec}(\widetilde{\varphi}_\pi) : \mathcal{U} \to \mathrm{Spec}(\mathcal{S}(N)_g) \hookrightarrow \mathrm{Spec_s}(N)$$

is injective, as claimed. □

Remark 4.6. It follows from the proof of the previous Theorem 4.5 that given a semialgebraic map $\pi : M \to N$ such that the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is finite, there exists a subset $\{f_i : i \in I\} \subset \mathcal{S}(M)$ such that

$$\beta_s M = \bigcup_{i\in I} \mathcal{D}_{\beta_s M}(f_i) \quad \& \quad \beta_s\pi|_{\mathcal{D}_{\beta_s M}(f_i)} : \mathcal{D}_{\beta_s M}(f_i) \to \beta_s N \ \text{ is injective for each } \ i \in I.$$

In addition we may assume that I is finite because $\beta_s M$ is compact. Thus $\{\mathcal{D}_M(f_i) : i \in I\}$ is a finite cover by open semialgebraic subsets of M and each restriction $\pi|_{\mathcal{D}_M(f_i)} : \mathcal{D}_M(f_i) \to N$ is injective.

Corollary 4.7. *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be compact semialgebraic sets. Then, a semialgebraic map $\pi : M \to N$ is locally injective if and only if the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is finite.*

Proof. Suppose first that π is locally injective. For each point $x \in M$ there exists an open ball $\mathcal{B}_x \subset \mathbb{R}^m$ centered at x such that the restriction $\pi|_{\mathrm{Cl}_M(\mathcal{B}_x)} : \mathrm{Cl}_M(\mathcal{B}_x) \to N$ is injective. As $\mathrm{Cl}_M(\mathcal{B}_x)$ is compact this implies that $\pi|_{\mathrm{Cl}_M(\mathcal{B}_x)}$ is a semialgebraic homeomorphism onto its image $\pi(\mathrm{Cl}_M(\mathcal{B}_x))$. Hence, the ring homomorphism

$$\mathcal{S}(\pi(\mathrm{Cl}_M(\mathcal{B}_x))) \to \mathcal{S}(\mathrm{Cl}_M(\mathcal{B}_x)),\ f \mapsto f \circ \pi$$

is an isomorphism. In addition, the restriction homomorphism $\mathcal{S}(N) \to \mathcal{S}(\pi(\mathrm{Cl}_M(\mathcal{B}_x)))$ is surjective by Lemma 2.1 (2). Therefore, the map

$$\mathcal{S}(N) \to \mathcal{S}(\mathrm{Cl}_M(\mathcal{B}_x)),\ f \mapsto f|_{\pi(\mathrm{Cl}_M(\mathcal{B}_x))} \circ \pi|_{\mathrm{Cl}_M(\mathcal{B}_x)}$$

is surjective too. Since M is compact there exist finitely many points $x_1, \dots, x_r \in M$ such that $M \subset \bigcup_{i=1}^r \mathcal{B}_{x_i}$. For $1 \le i \le r$ the difference $M \setminus \mathcal{B}_{x_i}$ is a closed semialgebraic subset of M and, by Lemma 2.1 (1), there exists $f_i \in \mathcal{S}(M)$ with $M \setminus \mathcal{B}_{x_i} = \mathcal{Z}_M(f_i)$. In fact, changing f_i by f_i^2 we may assume that $f_i(x) \ge 0$ for every point $x \in M$. This together with the inclusion $M \subset \bigcup_{i=1}^r \mathcal{B}_{x_i}$ implies that $\mathcal{Z}_M(f) = \varnothing$, where $f := \sum_{i=1}^r f_i$. Consequently each quotient $g_i := \frac{f_i}{f} \in \mathcal{S}(M)$ and all reduces to prove the equality

$$\mathcal{S}(M) = g_1\mathcal{S}(N) + \cdots + g_r\mathcal{S}(N).$$

Given $h \in \mathcal{S}(M)$ and $1 \le i \le r$, its restriction $h|_{\mathrm{Cl}_M(\mathcal{B}_{x_i})} \in \mathcal{S}(\mathrm{Cl}_M(\mathcal{B}_{x_i}))$ and there exists $u_i \in \mathcal{S}(N)$ such that $u_i|_{\pi(\mathrm{Cl}_M(\mathcal{B}_{x_i}))} \circ \pi|_{\mathrm{Cl}_M(\mathcal{B}_{x_i})} = h|_{\mathrm{Cl}_M(\mathcal{B}_{x_i})}$. Therefore,

$$h \cdot f_i = f_i \cdot (u_i \circ \pi) \ \text{ for } \ 1 \le i \le r,$$

which implies

$$h \cdot f = h \cdot \sum_{i=1}^{r} f_i = \sum_{i=1}^{r} h \cdot f_i = \sum_{i=1}^{r} f_i \cdot (u_i \circ \pi).$$

Dividing both members by f, that is a unit in $\mathcal{S}(M)$, and since $\frac{f_i}{f} = g_i$ we get

$$h = \sum_{i=1}^{r} \left(\frac{f_i}{f}\right) \cdot (u_i \circ \pi) = \sum_{i=1}^{r} g_i \cdot (u_i \circ \pi) \in g_1\mathcal{S}(N) + \cdots + g_r\mathcal{S}(N).$$

The converse follows straightforwardly from Theorem 4.5. □

Examples 4.8. (1) Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere and let $\mathbb{P}^n(\mathbb{R})$ be the n-dimensional real projective space. Both are compact real algebraic sets, hence semialgebraic; for the case of the projective space see [3, Thm. 3.4.4], where it is proved that $\mathbb{P}^n(\mathbb{R})$ can be understood as a real algebraic subset of $\mathbb{R}^{n^2}$. As the canonical projection

$$\pi : \mathbb{S}^n \to \mathbb{P}^n(\mathbb{R}),\ x := (x_0, \dots, x_n) \mapsto [x_0 : \cdots : x_n]$$

is a locally injective semialgebraic map the homomorphism $\varphi_\pi : \mathcal{S}(\mathbb{P}^n(\mathbb{R})) \to \mathcal{S}(\mathbb{S}^n)$ is, by Corollary 4.7, finite. In particular it is integral and finitely generated. However it is *not simple*. Otherwise there would exist $h \in \mathcal{S}(\mathbb{S}^n)$ such that $\mathcal{S}(\mathbb{S}^n) = \mathcal{S}(\mathbb{P}^n(\mathbb{R}))[h]$. By Borsuk-Ulam theorem, [20, Thm. 57.3] there exists a point $p \in \mathbb{S}^n$ such that $h(p) = h(-p)$. Consider the semialgebraic function

$$g : \mathbb{S}^n \to \mathbb{R},\ x \mapsto \|x - p\|.$$

Then there exist a positive integer d and functions $f_0, \dots, f_d \in \mathcal{S}(\mathbb{P}^n(\mathbb{R}))$ such that

$$g = (f_0 \circ \pi) + (f_1 \circ \pi)h + \cdots + (f_d \circ \pi)h^d.$$

Since $\pi(-p) = \pi(p)$ and $g(-p) = 2\|p\| \neq 0$ whereas $g(p) = 0$, we get a contradiction:

$$0 \neq g(-p) = \sum_{i=0}^{d} (f_i \circ \pi)(-p)h^i(-p) = \sum_{i=0}^{d} (f_i \circ \pi)(p)h^i(p) = g(p) = 0.$$

(2) Let $\mathtt{i} := \sqrt{-1}$ and $\tau : \mathbb{C} \to \mathbb{R}^2,\ z := x + \mathtt{i}y \mapsto (x, y)$. Fix a positive integer n and consider the analytic map $\phi : \mathbb{C} \to \mathbb{C},\ z \mapsto z^n$. The semialgebraic map

$$\pi := \tau \circ \phi \circ \tau^{-1} : M := \mathbb{R}^2 \to N := \mathbb{R}^2$$

is not locally injective at the origin, and it follows from Theorem 4.5 that the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is not finite. However φ_π is an integral homomorphism. To prove this, let $f \in \mathcal{S}(M)$ and consider the elementary symmetric forms in n variables

$$\sigma_j(\mathtt{x}_1, \dots, \mathtt{x}_n) := (-1)^j \sum_{1 \le i_1 < \dots < i_j \le n} \mathtt{x}_{i_1} \cdots \mathtt{x}_{i_j} \in \mathbb{Z}[\mathtt{x}_1, \dots, \mathtt{x}_n].$$

For every point $z \in N \setminus \{(0,0)\}$ let $\pi^{-1}(z) := \{z_1, \dots, z_n\} \subset M$. For $1 \le j \le n$ define the function

$$S_j(f) : N \setminus \{(0,0)\} \to \mathbb{R},\ z \mapsto \sigma_j(f(z_{i_1}), \dots, f(z_{i_n})).$$

Since $(S_j(f) \circ \tau)|_{\mathbb{C}\setminus\{0\}} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a locally bounded analytic function, $S_j(f)$ admits, by [18, Cor. 12.3], a continuous extension that we also denote $S_j(f) : N \to \mathbb{R}$. This function is semialgebraic, see the Appendix in [4]. Hence,

$$\mathtt{P}(\mathtt{t}) := \mathtt{t}^n + \sum_{j=1}^{n} S_j(f)\mathtt{t}^{n-j} \in \mathcal{S}(N)[\mathtt{t}]$$

is a monic polynomial and $\mathtt{P}(f) = 0$, or, more precisely,

$$f^n + \sum_{j=1}^{n} (S_j(f) \circ \pi) f^{n-j} = 0,$$

which proves that f is integral over $\mathcal{S}(N)$.

The previous Example 4.8 (2) leads us to look for a criterion that characterizes that the homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is integral, which we will do in Corollary 4.13. But first we need to do some work.

Lemma 4.9. *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and let $f_1, \dots, f_n \in \mathcal{S}(M)$. Consider the polynomial*

$$\mathtt{p}(\mathtt{t}) := \mathtt{t}^n + \sum_{j=1}^{n} f_j \mathtt{t}^{n-j} \in \mathcal{S}(M)[\mathtt{t}],$$

and suppose that for each point $x \in M$ the polynomial

$$\mathtt{p}_x(\mathtt{t}) := \mathtt{t}^n + \sum_{j=1}^{n} f_j(x)\mathtt{t}^{n-j} \in \mathbb{R}[\mathtt{t}]$$

splits in $\mathbb{R}[\mathtt{t}]$ as a product of (non necessarily distinct) factors of degree 1. Then, there exist $g_1, \dots, g_n \in \mathcal{S}(M)$ such that

$$\mathtt{p}(\mathtt{t}) = \prod_{j=1}^{n} (\mathtt{t} - g_j).$$

Proof. For every point $u := (u_1, \dots, u_n) \in \mathbb{R}^n$ let $\zeta_1(u), \dots, \zeta_n(u)$ be the real parts of the complex zeros of the polynomial

$$\mathtt{q}_u(\mathtt{t}) := \mathtt{t}^n + \sum_{j=1}^{n} u_j \mathtt{t}^{n-j} \in \mathbb{R}[\mathtt{t}]$$

listing each according to its multiplicity and indexed so that $\zeta_1(u) \leq \cdots \leq \zeta_n(u)$. It was proved in the Appendix of [4] that each $\zeta_j \in \mathcal{S}(\mathbb{R}^n)$.

Now notice that, for every point $x \in M$, the roots in $\mathbb{C}$ of the polynomial $\mathtt{p}_x(\mathtt{t})$ are the real numbers $\zeta_j(f_j(x))$. Since $\mathtt{p}_x$ is a monic polynomial this means that

$$\mathtt{p}_x(\mathtt{t}) = \prod_{j=1}^{n} (\mathtt{t} - \zeta_j(f_j(x))) \text{ for every point } x \in M.$$

Hence $g_j := \zeta_j \circ f_j \in \mathcal{S}(M)$ for $1 \leq j \leq n$ and $\mathtt{p}(\mathtt{t}) = \prod_{j=1}^{n}(\mathtt{t} - g_j)$. □

Corollary 4.10. *Let $\pi : M \to N$ be a semialgebraic map and let $g \in \mathcal{S}(M)$ be integral over $\mathcal{S}(N)$ via the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$. Then, there exist an integer $k > 0$ and $h_1, \dots, h_k \in \mathcal{S}(N)$ such that $\prod_{j=1}^{k}(g - h_j \circ \pi) = 0$.*

Proof. Since g is integral over $\mathcal{S}(N)$ there exist an integer $k > 0$ and $f_1, \dots, f_k \in \mathcal{S}(N)$ such that

$$g^k + (f_1 \circ \pi)g^{k-1} + \cdots + (f_{k-1} \circ \pi)g + (f_k \circ \pi) = 0.$$

Consider the semialgebraic map

$$\phi : N \to \mathbb{R}^k,\ y \mapsto (f_1(y), \dots, f_k(y)).$$

Then, with the notations in Lemma 4.9, $g(x) \in \{\zeta_j(\phi(\pi(x))) : 1 \leq j \leq k\}$ for each point $x \in M$ or, equivalently, the functions $h_j := \zeta_j \circ \phi \in \mathcal{S}(N)$ satisfy the equality in the statement. □

Definition 4.11. Let $M \subset \mathbb{R}^m$ be a semialgebraic set. A subset $\mathcal{F} \subset \mathcal{S}(M)$ *separates points* of M if for each pair of distinct points $p, q \in M$ there exists a function $f \in \mathcal{F}$ such that $f(p) \neq f(q)$.

Proposition 4.12. *Let $\pi : M \to N$ be a closed semialgebraic map. Suppose that there exist $f_1, \dots, f_k \in \mathcal{S}(M)$ such that each f_i is integral over $\mathcal{S}(N)$ via φ_π and $\mathcal{S}(N)[f_1, \dots, f_k]$ separates points of M. Then, there exists a finite cover $\{M_1, \dots, M_r\}$ by closed semialgebraic subsets of M such that each restriction $\pi|_{M_i} : M_i \to N$ is injective. In particular, φ_π is an integral homomorphism.*

Proof. Notice that the last part is the immediate consequence of the first one and Example 4.3 (1). To prove the first one notice that, by the previous Corollary 4.10, for $1 \leq i \leq k$ there exist $g_{i1}, \dots, g_{i\ell_i} \in \mathcal{S}(N)$ such that

$$\prod_{j=1}^{\ell_i} (f_i - g_{ij} \circ \pi) = 0.$$

Denote $h_{ij} := f_i - g_{ij} \circ \pi \in \mathcal{S}(M)$ and $Z_{ij} := \mathcal{Z}_M(h_{ij})$ for $1 \leq i \leq k$ and $1 \leq j \leq \ell_i$. Each Z_{ij} is a closed semialgebraic subset of M and, for fixed i, we have $M = \bigcup_{j=1}^{\ell_i} Z_{ij}$.

Let Σ be the set of all k-tuples $J := (j_1, \dots, j_k)$ with $1 \leq j_i \leq \ell_i$ and let

$$h_J := \sum_{i=1}^{k} h_{ij_i}^2 \in \mathcal{S}(M) \quad \& \quad M_J := \mathcal{Z}_M(h_J) = \bigcap_{i=1}^{k} \mathcal{Z}_M(h_{ij_i}).$$

Observe that for every point $x \in M$ and each index i with $1 \leq i \leq k$ there exists an index j_i with $1 \leq j_i \leq \ell_i$ such that $x \in Z_{ij_i}$. Thus $x \in M_J$, where $J := (j_1, \dots, j_k)$. In other words, the family $\{M_J : J \in \Sigma\}$ is a finite cover by closed semialgebraic subsets of M, and all reduces to check that each restriction $\pi|_{M_J} : M_J \to N$ is injective. Suppose, by the way of contradiction, that there exist two distinct points $p, q \in M_J$ with $\pi(p) = \pi(q)$. Since $\mathcal{S}(N)[f_1, \dots, f_k]$ separates points there exists a polynomial

$$\mathtt{p}(\mathtt{x}_1, \dots, \mathtt{x}_k) := \sum u_{i_1,\dots,i_k} \mathtt{x}_{i_1}^{\nu_{i_1}} \cdots \mathtt{x}_{i_k}^{\nu_{i_k}} \in \mathcal{S}(N)[\mathtt{x}_1, \dots, \mathtt{x}_k]$$

such that the function

$$f := \mathtt{p}(f_1, \dots, f_k) = \sum (u_{i_1,\dots,i_k} \circ \pi) f_{i_1}^{\nu_{i_1}} \cdots f_{i_k}^{\nu_{i_k}} \in \mathcal{S}(M)$$

satisfies $f(p) \neq f(q)$. But for each multiindex $(i_1, \dots, i_k)$ one has

$$(u_{i_1,\dots,i_k} \circ \pi)(p) = u_{i_1,\dots,i_k}(\pi(p)) = u_{i_1,\dots,i_k}(\pi(q)) = (u_{i_1,\dots,i_k} \circ \pi)(q),$$

so there exists some index i with $1 \leq i \leq k$ such that $f_i(p) \neq f_i(q)$. Since $p, q \in M_J$ we know that $h_{ij_i}(p) = h_{ij_i}(q) = 0$, that is,

$$f_i(p) = g_{ij_i}(\pi(p)) = g_{ij_i}(\pi(q)) = f_i(q),$$

a contradiction. □

Corollary 4.13. *Let $\pi : M \to N$ be a closed semialgebraic map. Then, the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is integral if and only if there exists a finite cover $\{M_i : 1 \leq i \leq k\}$ by closed semialgebraic subsets of M such that each restriction $\pi|_{M_i} : M_i \to N$ is injective.*

Proof. The "if part" follows straightforwardly from Example 4.3 (1). For the "only if" part suppose that $M \subset \mathbb{R}^m$ and choose

$$f_i : M \to \mathbb{R},\ x := (x_1, \dots, x_m) \mapsto x_i, \quad \text{for } i = 1, \dots, m.$$

As $f_1, \dots, f_m \in \mathcal{S}(M)$ and $\mathcal{S}(N)[f_1, \dots, f_k]$ separates points of M, the conclusion follows from Proposition 4.12. □

We finish the paper with an application of Corollary 4.10. Namely, it will allow us to determine the local injectivity of a semialgebraic map $\pi : M \to N$ between compact semialgebraic sets under the weakest hypothesis that φ_π is simple (instead of finite), provided that π is also surjective.

Theorem 4.14. *Let $\pi : M \to N$ be a surjective semialgebraic map between the compact semialgebraic sets M and N such that the induced homomorphism $\varphi_\pi : \mathcal{S}(N) \to \mathcal{S}(M)$ is simple. Then π is locally injective.*

Proof. By Corollary 4.7 it is enough to prove that φ_π is finite. By the hypothesis there exists a function $g \in \mathcal{S}(M)$ such that $\mathcal{S}(M) = \mathcal{S}(N)[g]$ and, using [1, Prop. 5.1], it suffices to see that g is integral over $\mathcal{S}(N)$. We may assume that $g(x) > 0$ for every $x \in M$. Indeed, since M is compact there exists $r := \min\{g(x) : x \in M\}$, and the function $g - r + 1$ is positive on M and $\mathcal{S}(N)[g] = \mathcal{S}(N)[g - r + 1]$. In addition, using again the compactness of M, there exists $\varepsilon > 0$ such that $g(x) < \varepsilon$ for each $x \in M$. Since $\mathcal{Z}_M(g) = \varnothing$ also the quotient $\frac{1}{g} \in \mathcal{S}(M) = \mathcal{S}(N)[g]$, and there exist an integer $n \geq 0$ and $f_0, \ldots, f_n \in \mathcal{S}(N)$ with

$$\frac{1}{g} = (f_0 \circ \pi) + (f_1 \circ \pi)g + (f_2 \circ \pi)g^2 + \cdots + (f_{n-1} \circ \pi)g^{n-1} + (f_n \circ \pi)g^n.$$

Dividing both members by g^n we get

$$\frac{1}{g^{n+1}} = (f_0 \circ \pi)\left(\frac{1}{g}\right)^n + (f_1 \circ \pi)\left(\frac{1}{g}\right)^{n-1} + (f_2 \circ \pi)\left(\frac{1}{g}\right)^{n-2} + \cdots + (f_{n-1} \circ \pi)\left(\frac{1}{g}\right) + (f_n \circ \pi).$$

Therefore $\frac{1}{g}$ is integral over $\mathcal{S}(N)$ and, by Corollary 4.10, there exist $h_0, \ldots, h_n \in \mathcal{S}(N)$ satisfying

$$\prod_{j=0}^{n}\left(\frac{1}{g} - h_j \circ \pi\right) = 0.$$

Hence, for each $x \in M$ there exists j with $0 \leq j \leq n$ such that $h_j(\pi(x)) = \frac{1}{g(x)}$. The function $\ell_j := \max\{h_j, \frac{1}{\varepsilon}\} \in \mathcal{S}(N)$ has empty zeroset and $\frac{1}{h_j(\pi(x))} = g(x) < \varepsilon$. Thus $h_j(\pi(x)) > \frac{1}{\varepsilon}$, that is, $\ell_j(\pi(x)) = \frac{1}{g(x)}$ or, equivalently, $g(x) = \frac{1}{\ell_j(\pi(x))}$. Consequently, g is integral over $\mathcal{S}(N)$, because

$$\prod_{j=0}^{n}\left(g - \frac{1}{\ell_j \circ \pi}\right) = 0.$$

□

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Declarations

Data Available. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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