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On convex polyhedra as regular images of $\mathbb{R}^n$

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Abstract

We show that convex polyhedra in $\mathbb{R}^n$ and their interiors are images of regular maps $\mathbb{R}^n \to \mathbb{R}^n$. As a main ingredient in the proof, given an n-dimensional, bounded, convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n \setminus \mathcal{K}$, we construct a semialgebraic partition $\{\mathcal{A}, \mathcal{B}, \mathcal{T}\}$ of the boundary $\partial\mathcal{K}$ of $\mathcal{K}$ determined by p, and compatible with the interiors of the faces of $\mathcal{K}$, such that $\mathcal{A}$ and $\mathcal{B}$ are semialgebraically homeomorphic to an $(n-1)$-dimensional open ball and $\mathcal{T}$ is semialgebraically homeomorphic to an $(n-2)$-dimensional sphere. Finally, we also prove that closed balls in $\mathbb{R}^n$ and their interiors are images of regular maps $\mathbb{R}^n \to \mathbb{R}^n$.

1. *Introduction*

This work generalizes to the n-dimensional setting the results concerning (real) regular images of the Euclidean plane developed by the third author in [**8**]. In fact, those results find their origin in the pioneer work concerning regular images of Euclidean spaces initiated by the first two authors in [**5**, **6**]. Before entering into further detail, we recall some terminology. Given a set $X \subset \mathbb{R}^n$, a *regular function on* X is a quotient $f = F_1/F_2$ of polynomials $F_1, F_2 \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ such that $F_2(x) \neq 0$ for every $x \in X$; and a map $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ is a *regular map* on X if each component f_i of f is a regular function on X. As one can expect, we say that a subset S of $\mathbb{R}^m$ is a *regular image* of $\mathbb{R}^n$ if there exists a regular map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $S = f(\mathbb{R}^n)$.

For every affine hyperplane $H \subset \mathbb{R}^n$ there exists a polynomial $\ell \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ of degree 1 such that $H = \{x \in \mathbb{R}^n : \ \ell(x) = 0\} \equiv \{\ell = 0\}$, and the sets

$$H^+ = \{x \in \mathbb{R}^n : \ \ell(x) \geqslant 0\} \equiv \{\ell \geqslant 0\} \quad \text{and} \quad H^- = \{x \in \mathbb{R}^n : \ \ell(x) \leqslant 0\} \equiv \{\ell \leqslant 0\}$$

are called the *closed half-spaces* defined by H. Observe that H^+ and H^- are the closures in $\mathbb{R}^n$ of the connected components of $\mathbb{R}^n \setminus H$; hence, they are completely determined by H. However, assigning H^+ and H^- to these half-spaces depends on the choice of the equation ℓ; of course, they are easily interchanged just considering $-\ell$ instead of ℓ to define H.

A *convex polyhedron in* $\mathbb{R}^n$ is a subset $\mathcal{K} \subset \mathbb{R}^n$ that can be written as a finite intersection $\mathcal{K} = \bigcap_{i=1}^r H_i^+$, where each H_i^+ is a closed half-space. We use the notation $\mathcal{K} = \langle H_1^+, \ldots, H_r^+ \rangle$. For convenience we allow this family of hyperplanes to be empty, and in such a case $\mathcal{K} = \mathbb{R}^n$. The dimension $\dim(\mathcal{K})$ of a convex polyhedron $\mathcal{K}$ corresponds to its dimension as a topological manifold with boundary.

In [**8**], the author proved the following statement.

Theorem 1.1. *Each* 2*-dimensional convex polygon in* $\mathbb{R}^2$ *and its interior are regular images of* $\mathbb{R}^2$.

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The purpose of this article is to prove that the previous statement can be generalized for n-dimensional convex polyhedra in $\mathbb{R}^n$ for $n \geqslant 2$; namely, we have the following theorem. In what follows, we assume $n \geqslant 2$.

THEOREM 1.2. *Each n-dimensional convex polyhedron in $\mathbb{R}^n$ and its interior are regular images of $\mathbb{R}^n$.*

Of course, if $\mathcal{K} \subset \mathbb{R}^n$ is a d-dimensional polyhedron for some $0 \leqslant d < n$, then $\mathcal{K}$ is contained in some d-dimensional affine subspace of $\mathbb{R}^n$ that can be identified with $\mathbb{R}^d \times \{0\}$ after an affine change of coordinates. Thus, it follows from Theorem 1.2 that $\mathcal{K}$ is a regular image of $\mathbb{R}^d$. This is why we are mostly concerned along this paper with n-dimensional polyhedra of $\mathbb{R}^n$.

To prove that the interior $\operatorname{Int}(\mathcal{K})$ of an n-dimensional bounded convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is the image of $\mathbb{R}^n$ under a regular map $\mathbb{R}^n \to \mathbb{R}^n$, it becomes crucial the partition of its boundary $\partial\mathcal{K} = \mathcal{K} \setminus \operatorname{Int}(\mathcal{K})$ (see Theorem 3.1) determined by an exterior point $p \in \mathbb{R}^n \setminus \mathcal{K}$, a construction that has interest by its own. Roughly speaking it works as follows in the generic case. Fix a point p that belongs neither to $\mathcal{K}$ nor to any of the hyperplanes containing the facets of $\mathcal{K}$. Each ray R from p intersects $\mathcal{K}$ in either the empty set or in a compact segment $I_R = [a_R, b_R]$, which is a singleton in case $a_R = b_R$. Next, we define the sets $\mathcal{A} = \{a_R : \ R \cap \operatorname{Int}(\mathcal{K}) \neq \varnothing\}$, $\mathcal{B} = \{b_R : R \cap \operatorname{Int}(\mathcal{K}) \neq \varnothing\}$ and $\mathcal{T} = \partial\mathcal{K} \setminus (\mathcal{A} \sqcup \mathcal{B})$, which constitute a partition of the boundary $\partial\mathcal{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ are open subsets of $\partial\mathcal{K}$ homeomorphic to the n-dimensional open ball and $\mathcal{T}$ is a closed subset of $\partial\mathcal{K}$ homeomorphic to the $(n-1)$-dimensional sphere. Moreover, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$ are compatible with the faces of $\mathcal{K}$. We use the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$ to prove in Proposition 4.4 and Corollary 4.5 the part of Theorem 1.2 concerning interiors of n-dimensional convex polyhedra. Moreover, as we see at the end of Section 3, the previous partition can be generalized, with some extra care, by choosing as p an arbitrary point $p \in \mathbb{R}^n \setminus \mathcal{K}$ and eliminating the boundedness hypothesis on $\mathcal{K}$; see Remark 3.4.

Once we know that the interiors of convex polyhedra are regular images of Euclidean spaces, the next step is to prove that also convex polyhedra themselves share the same property. This requires us to generalize the techniques about *scaffolds* (see Section 5) already introduced in [**8**, 4.7] in the 2-dimensional case. However, such generalization is not straightforward and needs a careful and subtle analysis of the behaviour of the restriction to $\mathcal{K}$ of suitable central projections $\pi : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ (see Lemma 5.1 and Corollary 5.8).

The interest of deciding whether a semialgebraic set is a regular image of $\mathbb{R}^n$ is out of any doubt, and it lies in the fact that the study of certain classical problems in Real Geometry concerning this kind of sets is reduced to the analysis of those problems on $\mathbb{R}^n$, for which many more tools have been developed. Let us recall some of them. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a regular map and let $S = f(\mathbb{R}^n)$. Then the optimization of a given regular function $g : S \to \mathbb{R}$ is equivalent to the optimization of the composition $g \circ f$ on $\mathbb{R}^n$, and in this way one can forget about contour conditions. Another classical problem is the characterization of those regular functions $g : \mathbb{R}^n \to \mathbb{R}$ that are either strictly positive or positive semidefinite on S. In case S is a basic closed semialgebraic set, these problems have been solved in [**7**]; see also [**3**, 4.4.3]. Note that g is strictly positive or positive semidefinite on S if and only if $g \circ f$ is strictly positive or positive semidefinite on $\mathbb{R}^n$, respectively, and both last questions are decidable, using, for instance, [**7**]. For more details about these applications and others, see [**5**, Section 1; **6**, Section 1].

In [**6**], the first two authors introduced the invariant $\mathrm{r}(S)$ for a semialgebraic set $S \subset \mathbb{R}^n$, as the least integer among those $m \geqslant 1$ such that $S = f(\mathbb{R}^m)$ for some regular map $f : \mathbb{R}^m \to \mathbb{R}^n$, or $\mathrm{r}(S) = +\infty$ if such an integer does not exist. It is proved there that $\mathrm{r}(S) \geqslant \dim S$. Hence, Theorem 1.2 says that if S is either a convex polyhedron or its interior as a topological manifold with boundary, then $\mathrm{r}(S) = \dim S$.

The article is organized as follows. In Section 2, we present the basic definitions and some relevant results about the geometry of convex polyhedra of $\mathbb{R}^n$. Section 3 is devoted to construct the aforementioned partition of the boundary $\partial\mathcal{K}$ of an n-dimensional bounded convex polyhedron determined by an exterior point. In fact, we also sketch how this construction can be also extended to unbounded convex polyhedra (see Remark 3.4). Next, in Section 4 we prove the second part of Theorem 1.2, namely, the interior of an n-dimensional convex polyhedron $\mathcal{K}$ is a regular image of $\mathbb{R}^n$. We study first the bounded case (see Proposition 4.4) whose proof runs by induction on the number of vertices of $\mathcal{K}$. We start this proof by showing the statement for an n-simplex in Lemma 4.1. However, the general case of bounded convex polyhedra is much more involved, and it requires the already mentioned partition of the boundary of the polyhedron, which is in the core of the proof of Proposition 4.4. At the end of this section, we achieve the unbounded case in Corollary 4.5 using Proposition 4.4 for bounded polyhedra and the reduction to the bounded case (Proposition 2.7). In Section 5, we prove the first part of Theorem 1.2. By means of Lemma 2.3, the problem is focused on polyhedra having at least one vertex, and this case is solved in Proposition 5.2. To approach Proposition 5.2, one requires the notion of *d-scaffold* of a d-dimensional face E of a polyhedron $\mathcal{K} \subset \mathbb{R}^n$ which, as already mentioned, extends to the n-dimensional setting the notion introduced by the third author in [**8**, 4.7] and plays a crucial role in the proof of Proposition 5.2. Such a d-scaffold is a semialgebraic topological manifold Γ semialgebraically homeomorphic to E satisfying $\operatorname{Int}(\Gamma) \subset \operatorname{Int}(\mathcal{K})$ and $\partial\Gamma = \partial E$. Finally, observe that the closed ball and its interior can be, respectively, seen as 'limits' of bounded convex regular polyhedra and their interiors, when the number of faces tends to infinity. Thus, it seems natural to ask whether they are regular images of $\mathbb{R}^n$ or not. We answer both questions in the affirmative in Section 6 and so $\mathrm{r}(\mathcal{B}_n) = \mathrm{r}(\overline{\mathcal{B}_n}) = n$, where $\mathcal{B}_n$ and $\overline{\mathcal{B}_n}$ denote the open and the closed n-dimensional ball, respectively.

2. *Preliminaries on convex polyhedra*

We begin this section by recalling certain terminology and properties concerning convex polyhedra. The references we have used concerning polyhedra and convex sets are [**1**, **2**].

2.1. *Convex polyhedra and their faces*

Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron. By Berger [**2**, 12.1.5] there exists a unique family $\{H_1, \dots, H_m\}$ of affine hyperplanes of $\mathbb{R}^n$ (which is empty if $\mathcal{K} = \mathbb{R}^n$) whose cardinality is minimal among those satisfying the equality $\mathcal{K} = \bigcap_{i=1}^m H_i^+ = \langle H_1^+, \dots, H_m^+\rangle$. This family $\{H_1, \dots, H_m\}$ is, in particular, irredundant and will be called the *minimal presentation of* $\mathcal{K}$. The *facets* of $\mathcal{K}$ are the intersections $F_i = H_i \cap \mathcal{K}$ (if any) for $i = 1, \dots, m$. Of course, $\mathbb{R}^n$ is the unique polyhedron of $\mathbb{R}^n$ without facets. Note that each $F_i = \langle H_i^-, H_1^+, \dots, H_m^+\rangle$ is a polyhedron contained in H_i. We also say that $F_1, \dots, F_m$ are the $(n-1)$-*faces* of $\mathcal{K}$. For $0 \leqslant j \leqslant n-2$, a subset of $\mathcal{K}$ is a j-face of $\mathcal{K}$ if it is a facet of some $(j+1)$-face of $\mathcal{K}$. In particular, the 0-faces are the *vertices* of $\mathcal{K}$ and the 1-faces are the edges of $\mathcal{K}$; note that if $\mathcal{K}$ has a vertex, then $m \geqslant n$ (see [**2**, 12.1.8–9]). In general, a face of $\mathcal{K}$ (which is not 'registered' as a facet) will be denoted by E to distinguish it from the facets $F_1, \dots, F_m$, and the affine subspace generated by E will be denoted by W to distinguish it from the hyperplanes $H_1, \dots, H_m$ containing the facets $F_1, \dots, F_m$.

2.1.1. Observe that, for each $i = 1, \dots, m$, the polyhedron $\mathcal{K}_i = \bigcap_{j \neq i} H_j^+$ contains $\mathcal{K}$ properly and it is called *the polyhedron obtained from* $\mathcal{K}$ *by eliminating the facet* F_i. Note that the number of facets of $\mathcal{K}$ exceeds in one unit the number of facets of $\mathcal{K}_i$. Of course, not all polyhedra are bounded, but every bounded polyhedron $\mathcal{K}$ is the convex hull of its set of vertices $\{v_1, \dots, v_r\}$, and we write $\mathcal{K} = [v_1, \dots, v_r]$ (see [**1**, 11.1.8]).

Next, given any set $T \subset \mathbb{R}^n$, we denote by $\operatorname{Int}_{\mathbb{R}^n}(T)$ the *relative interior* of T in $\mathbb{R}^n$ and by $\operatorname{Cl}_{\mathbb{R}^n}(T)$ its *relative closure* in $\mathbb{R}^n$. Next, let $X \subset \mathbb{R}^n$ denote either the polyhedron $\mathcal{K}$ or one of its faces. Note that X is a topological manifold with boundary, and denote by ∂X its boundary and by $\operatorname{Int}(X) = X \setminus \partial X$ its *interior*, that is, the largest topological manifold (without boundary) contained in X. In case $X = \{v\}$ is a singleton, we use the usual convention and write $\operatorname{Int}(X) = X$ and $\partial X = \varnothing$. The dimension $\dim X = \dim(\operatorname{Int}(X))$ of X is its dimension as a topological manifold with boundary. Observe that $\operatorname{Int}(X)$ coincides with the relative interior of X in the affine subspace of $\mathbb{R}^n$ generated by X, and that $X = \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Int}(X))$.

Note that each affine hyperplane $H \subset \mathbb{R}^n$ coincides with the boundary $\partial H^+ = \partial H^-$ of the closed half-spaces defined by H. On the other hand, observe that affine transformations are polynomial mappings and so all our statements do not depend on affine changes of coordinates. Thus, all through this work, we will freely use (affine) changes of coordinates. We denote by $B_n(p,r)$ the open ball of $\mathbb{R}^n$ centred at the point $p \in \mathbb{R}^n$ with radius $r > 0$, and by $\overline{B}_n(p,r)$ its closure.

In the following result, we represent the boundary and the interior of a polyhedron in terms of its minimal presentation; namely, we have the following lemma.

LEMMA 2.1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron, let $\{H_1, \dots, H_m\}$ be the minimal presentation of $\mathcal{K}$ and let $\{F_1, \dots, F_m\}$ be the facets of $\mathcal{K}$. Then $\partial\mathcal{K} = \bigcup_{i=1}^m F_i$ and $\operatorname{Int}(\mathcal{K}) = \bigcap_{i=1}^m (H_i^+ \setminus H_i)$.*

Proof. By Berger [2, 12.1.5], $\partial\mathcal{K} = \bigcup_{i=1}^m F_i = \bigcup_{i=1}^m (\mathcal{K} \cap H_i)$, and consequently,

$$\operatorname{Int}(\mathcal{K}) = \mathcal{K} \setminus \partial\mathcal{K} = \mathcal{K} \Big\backslash \bigcup_{i=1}^m (\mathcal{K} \cap H_i) = \bigcap_{j=1}^m H_j^+ \cap \bigcap_{i=1}^m (\mathbb{R}^n \setminus H_i) = \bigcap_{i=1}^m (H_i^+ \setminus H_i),$$

as required. $\square$

2.2. *Degenerate and nondegenerate polyhedra*

A convex polyhedron in $\mathbb{R}^n$ is *nondegenerate* if it has at least one vertex. Otherwise, we say that the polyhedron is *degenerate*. Let us present now some properties concerning degenerate convex polyhedra.

LEMMA 2.2. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron containing a line L. Then $\mathcal{K}$ is degenerate and each face E of $\mathcal{K}$ is a degenerate convex polyhedron that contains a line L_E parallel to L. In particular, the edges of $\mathcal{K}$, if any, are lines parallel to L.*

Proof. We may assume that $\mathcal{K} \subsetneq \mathbb{R}^n$ and let $\{H_1, \dots, H_m\}$ be the minimal presentation of $\mathcal{K}$. We claim that each H_i is parallel to L. Otherwise $H_i \cap L$ is a unique point, and so $L \not\subset H_i^+$. Therefore, $L \not\subset \mathcal{K}$, which is a contradiction.

Next, we prove the result for the facets of $\mathcal{K}$. Fix a facet $F_i = \mathcal{K} \cap H_i$ of $\mathcal{K}$ and a point $p_i \in F_i$. Let us prove that F_i contains the line L_i parallel to L and passing through p_i. Indeed, for $j = 1, \dots, m$ the hyperplane H_j is parallel to L, and so either $L_i \subset H_j$ or L_i is parallel to H_j. In particular, $L_i \subset H_i$ because $p_i \in L_i \cap H_i$. Observe also that $p_i \in \mathcal{K} \cap L_i \subset H_j^+ \cap L_i$. Therefore, $L_i \subset H_j^+$ and this implies

$$L_i \subset H_i \cap \bigcap_{j=1}^m H_j^+ = H_i \cap \mathcal{K} = F_i.$$

Now, given an arbitrary face E of $\mathcal{K}$, there exist, by Berger [**2**, 12.1.9], some facets $F_1, \dots, F_s$ of $\mathcal{K}$ such that $E = \bigcap_{j=1}^{s} F_j$. Pick a point $p \in E$ and note that, since the line L_E parallel to L and passing through p is contained in each facet F_j for $1 \leqslant j \leqslant s$, it is also contained in E. To complete the proof, it suffices to see that $\mathcal{K}$ has no vertex. Indeed, suppose that there exists a vertex E of $\mathcal{K}$. Since E is a face of $\mathcal{K}$, it should contain the line L_E, which is impossible. $\square$

LEMMA 2.3. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron. The following assertions are equivalent.*

(i) *The polyhedron $\mathcal{K}$ is degenerate.*

(ii) *Either $\mathcal{K} = \mathbb{R}^n$ or there exist $1 \leqslant k \leqslant n-1$ and a nondegenerate convex polyhedron $\mathcal{P} \subset \mathbb{R}^{n-k}$ such that, after a change of coordinates, $\mathcal{K} = \mathbb{R}^k \times \mathcal{P}$.*

Proof. The implication (ii) $\Longrightarrow$ (i) is clear, by Lemma 2.2, because $\mathcal{K}$ contains a line. Thus, let us prove the converse and suppose $\mathcal{K} \neq \mathbb{R}^n$. Let E be a face of $\mathcal{K}$ of minimal dimension. Since $\mathcal{K}$ is degenerate, it has no vertices and so $1 \leqslant \dim E = k < n$. Observe that since the facets of E (if any) are also faces of $\mathcal{K}$ whose dimension is strictly smaller than the one of E, it follows that E has no facets, and so it is affinely equivalent to $\mathbb{R}^k$ for some $1 \leqslant k \leqslant n-1$. Hence, after a change of coordinates, we may assume that

$$E = \{x \in \mathbb{R}^n : \ x_{k+1} = 0, \dots, x_n = 0\} = \mathbb{R}^k \times \{0\}.$$

Let $\{H_1, \dots, H_m\}$ be the minimal presentation of $\mathcal{K}$ and let $\ell_i = a_{i1}\mathtt{x}_1 + \dots + a_{in}\mathtt{x}_n + a_{i0}$ be a polynomial of degree 1 such that $H_i^+ = \{\ell_i \geqslant 0\}$. Since $E \subset \mathcal{K}$,

$$a_{i1}y_1 + \dots + a_{ik}y_k + a_{i0} = \ell_i(y, 0) \geqslant 0,$$

for all $y \in \mathbb{R}^k$. Thus, $a_{i1} = \dots = a_{ik} = 0$ for $1 \leqslant i \leqslant m$, that is, each $\ell_i = a_{i,k+1}\mathtt{x}_{k+1} + \dots + a_{in}\mathtt{x}_n + a_{i0}$. Hence, $\mathcal{K} = \mathbb{R}^k \times \mathcal{P}$ where

$$\mathcal{P} = \{z = (z_{k+1}, \dots, z_n) \in \mathbb{R}^{n-k} : \ \ell_1(0, z) \geqslant 0, \dots, \ell_m(0, z) \geqslant 0\}$$

is a convex polyhedron of $\mathbb{R}^{n-k}$. Note that there exists a face E' of $\mathcal{P}$ such that $E = \mathbb{R}^k \times E'$ and, comparing dimensions, $k = \dim E = k + \dim E'$. Therefore, $\dim E' = 0$, that is, E' is a vertex of $\mathcal{P}$, and so $\mathcal{P}$ is nondegenerate. $\square$

2.3. *Polyhedra facing upwards*

When one tries to represent a polyhedron $\mathcal{K} \subset \mathbb{R}^n$ and its interior as regular images of $\mathbb{R}^n$, it is a great advantage to place $\mathcal{K}$ in a suitable way. We say that an n-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ with minimal presentation $\mathfrak{H} = \{H_1, \dots, H_m\}$ is *facing upwards* if there exists a subfamily $\{H_{i_1}, \dots, H_{i_n}\}$ of $\mathfrak{H}$ whose common intersection is a vertex $v = (v_1, \dots, v_n)$ of $\mathcal{K}$ such that $\bigcap_{j=1}^{n} H_{i_j}^+ \setminus \{v\} \subset \{x_n > v_n\}$. Observe that v is the unique point of $\mathcal{K}$ with minimum x_n-coordinate and it will be called *the minimum vertex* of $\mathcal{K}$. First of all, let us check that, after a change of coordinates, every n-dimensional nondegenerate convex polyhedron is facing upwards.

LEMMA 2.4. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional nondegenerate convex polyhedron. Then we may assume, after a change of coordinates, that $\mathcal{K}$ is facing upwards and it does not intersect the hyperplane $\{x_n = 0\}$.*

Proof. Let $\{H_1, \dots, H_m\}$ be the minimal presentation of $\mathcal{K}$. Recall that since $\mathcal{K}$ is nondegenerate, $m \geqslant n$. We may assume, after a change of coordinates and up to reordering the indices $i=1,\dots,m$, that $\bigcap_{i=1}^n H_i=\{0\}$ is a vertex of $\mathcal{K}$ and $H_i^+=\{x_i\geqslant 0\}$ for $1\leqslant i\leqslant n$. Consequently,

$$\mathcal{K}\subset\{x_1\geqslant 0,\dots,x_n\geqslant 0\}\subset\{x_1+\dots+x_n>0\}\cup\{0\}.$$

Observe that after a new change of coordinates that transforms $\{x_1+\dots+x_n\geqslant 0\}$ onto $\{x_n\geqslant 1\}$, we are done. □

Let us see now several properties concerning polyhedra that are facing upwards.

Lemma 2.5. *Let $\{u_1,\dots,u_n\}\subset\mathbb{R}^n$ and let $\{\ell_1,\dots,\ell_n\}\subset\mathbb{R}^{n,*}$ be linear forms such that $\ell_k(u_k)>0$ and $\ell_j(u_k)=0$ if $j\neq k$. Then the sets $\mathcal{K}_1=\{\ell_1\geqslant 0,\dots,\ell_n\geqslant 0\}$ and $\mathcal{K}_2=\{\sum_{k=1}^n\lambda_k u_k:\ \lambda_1\geqslant 0,\dots,\lambda_n\geqslant 0\}$ coincide.*

Proof. First, observe that the condition $\ell_k(u_k)>0$ and $\ell_j(u_k)=0$ if $j\neq k$ guarantees that $\{u_1,\dots,u_n\}$ and $\{\ell_1,\dots,\ell_n\}$ are, respectively, a basis of $\mathbb{R}^n$ and its dual space $\mathbb{R}^{n,*}$. Consequently, the linear map $\Phi:\mathbb{R}^n\to\mathbb{R}^n,\ x\mapsto(\ell_1(x),\dots,\ell_n(x))$ is an isomorphism and $\Phi(\mathcal{K}_1)=\{y\in\mathbb{R}^n:\ y_1\geqslant 0,\dots,y_n\geqslant 0\}$. Note that, for each $k=1,\dots,n$, there exists a real positive number $t_k=\ell_k(u_k)>0$ such that $w_k=\Phi(u_k)=t_ke_k$, where e_k is the vector whose coordinates are all zero except the kth, which equals 1. Therefore,

$$\Phi(\mathcal{K}_2)=\left\{\sum_{k=1}^n\lambda_k w_k:\ \lambda_1\geqslant 0,\dots,\lambda_n\geqslant 0\right\}=\{(t_1\lambda_1,\dots,t_n\lambda_n):\lambda_1\geqslant 0,\dots,\lambda_n\geqslant 0\},$$

that is, $\Phi(\mathcal{K}_1)=\Phi(\mathcal{K}_2)$. Hence, Φ being injective, we get $\mathcal{K}_1=\mathcal{K}_2$. □

Lemma 2.6. *Let $\mathcal{K}\subset\mathbb{R}^n$ be an unbounded convex polyhedron facing upwards that does not intersect the hyperplane $\{x_n=0\}$. Consider the rational map*

$$f:\mathbb{R}^n\dashrightarrow\mathbb{R}^n,\ (x_1,\dots,x_n)\longmapsto\left(\frac{x_1}{x_n},\dots,\frac{x_{n-1}}{x_n},\frac{1}{x_n}\right).$$

Then $\mathrm{Cl}_{\mathbb{R}^n}(f(\mathcal{K}))\subset\mathbb{R}^n$ is a bounded convex polyhedron.

Proof. Since f can be interpreted as a transition map between two charts of the real projective space $\mathbb{RP}^n$, it preserves affine subspaces and the convexity of those subsets that do not intersect the hyperplane $\{x_n=0\}$. Hence, $f(\mathcal{K})$ is a convex subset of $\mathbb{R}^n$ and so, by Berger [**1**, 11.2.1], $\mathrm{Cl}_{\mathbb{R}^n}(f(\mathcal{K}))$ is a convex polyhedron of $\mathbb{R}^n$. Now, all reduces to check that $f(\mathcal{K})$ is a bounded set.

Indeed, let $\mathfrak{H}=\{H_1,\dots,H_m\}$ be the minimal presentation of $\mathcal{K}$. Since $\mathcal{K}$ is facing upwards, we may assume, after reordering the indices $i=1,\dots,m$ and applying a translation, that the common intersection of the family $\{H_1,\dots,H_n\}\subset\mathfrak{H}$ is the vertex $v=(0,\dots,0,1)$ of $\mathcal{K}$ and $\bigcap_{j=1}^n H_j^+\setminus\{v\}\subset\{x_n>1\}$. Moreover, since $\bigcap_{j=1}^n H_j=\{v\}$, there exists a basis $\mathcal{B}^*=\{\ell_1,\dots,\ell_n\}$ of $\mathbb{R}^{n,*}$ such that $H_i^+=\{\ell_i-\ell_i(v)\geqslant 0\}$ for $1\leqslant i\leqslant n$. Denote $\mathcal{Q}=\{\ell_1\geqslant 0,\dots,\ell_n\geqslant 0\}$. Hence,

$$\mathcal{K}\subset\bigcap_{j=1}^n H_j^+=v+\mathcal{Q}\quad\text{and}\quad\mathcal{Q}\setminus\{0\}\subset\{x_n>0\}.\tag{$\diamond$}$$

Let $\mathcal{B}=\{u_1,\dots,u_n\}\subset\mathbb{R}^n$ be the dual basis of $\mathcal{B}^*$. From Lemma 2.5, we deduce that

$$\mathcal{Q}=\{\lambda_1u_1+\dots+\lambda_nu_n:\ \lambda_1\geqslant 0,\dots,\lambda_n\geqslant 0\}.$$

Write $u_k = (u_{1k}, \dots, u_{nk})$ for $k = 1, \dots, n$ and observe that, by ($\diamond$), each $u_{nk} > 0$. We also define $u'_k = (u_{1k}, \dots, u_{n-1,k}) \in \mathbb{R}^{n-1}$. Let $M > 0$ be a positive real number such that $\|u'_k\| \leqslant M u_{nk}$.

Observe that, for each point $y \in \mathcal{Q}$, there exist nonnegative real numbers $\lambda_k \geqslant 0$ such that $y = \lambda_1 u_1 + \dots + \lambda_n u_n$. Hence, $(y_1, \dots, y_{n-1}) = \lambda_1 u'_1 + \dots + \lambda_n u'_n$, and so

$$\sqrt{y_1^2 + \dots + y_{n-1}^2} = \|(y_1, \dots, y_{n-1})\| \leqslant \sum_{k=1}^{n-1} \lambda_k \|u'_k\| \leqslant M \left(\sum_{k=1}^{n-1} \lambda_k u_{nk} \right) \leqslant M y_n.$$

Therefore, v being the minimum vertex of $\mathcal{K}$, it follows that

$$\mathcal{K} \subset v + \mathcal{Q} \subset v + \{x \in \mathbb{R}^n : \ x_1^2 + \dots + x_{n-1}^2 \leqslant M^2 x_n^2, \ x_n \geqslant 0\}.$$

Now, a straightforward computation shows that also

$$\mathcal{K} \subset \{x \in \mathbb{R}^n : \ x_1^2 + \dots + x_{n-1}^2 \leqslant M^2 x_n^2, \ x_n \geqslant 1\}.$$

Finally, given a point $z \in f(\mathcal{K})$, there exists $x \in \mathcal{K}$ such that $f(x) = z$. Hence,

$$\|z\|^2 = \|f(x)\|^2 = \left(\frac{1}{x_n} \right)^2 + \sum_{k=1}^{n-1} \left(\frac{x_k}{x_n} \right)^2 = \frac{1}{x_n^2} + \frac{x_1^2 + \dots + x_{n-1}^2}{x_n^2} < 1 + M^2,$$

which proves that $f(\mathcal{K})$ is a bounded set. □

The next result will allow us to reduce the proof of certain statements concerning convex polyhedra to the case of bounded convex polyhedra.

Proposition 2.7 (Reduction to bounded convex polyhedra). *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional, nondegenerate, unbounded, convex polyhedron in $\mathbb{R}^n$, which in addition is facing upwards and does not intersect the hyperplane $\{x_n = 0\}$. Consider the rational map*

$$f : \mathbb{R}^n \dashrightarrow \mathbb{R}^n, \ (x_1, \dots, x_n) \longmapsto \left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n} \right).$$

Then there exist an n-dimensional, bounded, convex polyhedron $\mathcal{K}' \subset \mathbb{R}^n$ and a face E' of $\mathcal{K}'$ such that f is regular on $\mathcal{K}' \setminus E'$ and satisfies the equality $f(\mathcal{K}' \setminus E') = \mathcal{K}$. Moreover, the restriction $f|_{\mathcal{K}' \setminus E'} : \mathcal{K}' \setminus E' \to \mathcal{K}$ is a biregular homeomorphism and $f(\operatorname{Int}(\mathcal{K}')) = \operatorname{Int}(\mathcal{K})$.

Before proving Proposition 2.7, we need the following preliminary result.

Lemma 2.8. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and let $H \subset \mathbb{R}^n$ be a hyperplane such that $\mathcal{K} \subset H^+$. Then $H \cap \mathcal{K}$ is either empty or a face of $\mathcal{K}$.*

Proof. We proceed by induction on the dimension of $\mathcal{K}$. If $n = \dim \mathcal{K} = 1$, then we may assume that $H^+ = \{x \geqslant 0\} \subset \mathbb{R}$. Observe that either $\mathcal{K} \subset \{x > 0\}$, and so $\mathcal{K} \cap H = \varnothing$, or $\mathcal{K} \cap H = \{0\}$, which is a face of $\mathcal{K}$. Assume now the result true for polyhedra whose dimension is smaller than n, and let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron. Since $\mathcal{K} \subset H^+$, we have

$$\operatorname{Int}(\mathcal{K}) = \operatorname{Int}_{\mathbb{R}^n}(\mathcal{K}) \subset \operatorname{Int}_{\mathbb{R}^n}(H^+) = \operatorname{Int}(H^+),$$

and $\mathcal{K} \cap H \subset \partial\mathcal{K}$. Let $F_1, \dots, F_m$ be the facets of $\mathcal{K}$. By Lemma 2.1, $\partial\mathcal{K} = \bigcup_{i=1}^m F_i$, and so

$$\mathcal{K} \cap H = \partial\mathcal{K} \cap H = \bigcup_{i=1}^m (F_i \cap H).$$

After reordering the indices $i=1,\dots,m$, we may assume that $\dim(F_1\cap H)\geqslant\dim(F_j\cap H)$ for $j=2,\dots,m$; hence, $\dim(\mathcal{K}\cap H)=\dim(F_1\cap H)=d\leqslant n-1$.

Next, let us check that $\mathcal{K}\cap H=F_1\cap H$. Indeed, let $H_1\subset\mathbb{R}^n$ be the hyperplane of $\mathbb{R}^n$ generated by F_1 and suppose, by way of contradiction, that there exists a point $p\in(\mathcal{K}\setminus F_1)\cap H$. Then $p\notin H_1$, because $F_1=H_1\cap\mathcal{K}$. Since $d=\dim(F_1\cap H)$, there exist affinely independent points $\{p_0,p_1,\dots,p_d\}\subset F_1\cap H\subset H_1$ and observe that also the points $\{p_0,p_1,\dots,p_d,p_{d+1}=p\}\subset\mathcal{K}\cap H$ are affinely independent because $p\notin H_1$. Therefore, their convex hull T has dimension $d+1$. But, $\mathcal{K}\cap H$ being convex, it contains T; hence,

$$d+1=\dim T\leqslant\dim(\mathcal{K}\cap H)=d,$$

which is a contradiction. Thus, $\mathcal{K}\cap H=F_1\cap H$. Since $F_1\subset H^+\cap H_1$ and $\dim F_1=n-1$, we deduce that $F_1\subset H\cap H_1$ or, by the induction hypothesis, either $F_1\cap H=F_1\cap H\cap H_1=\varnothing$ or $E=F_1\cap H=F_1\cap H\cap H_1$ is a face of F_1, and hence of $\mathcal{K}$. In the first case, $\mathcal{K}\cap H=F_1$ is a face of $\mathcal{K}$; in the second one, either $\mathcal{K}\cap H=\varnothing$ or $\mathcal{K}\cap H=F_1\cap H=E$ is a face of $\mathcal{K}$, as wanted. □

Now, we are ready to prove Proposition 2.7.

Proof of Proposition 2.7. Let $c_0>0$ denote the last coordinate of the minimum vertex of $\mathcal{K}$ and let us consider the rational map

$$f:\mathbb{R}^n\dashrightarrow\mathbb{R}^n,\quad(x_1,\dots,x_n)\longmapsto\left(\frac{x_1}{x_n},\dots,\frac{x_{n-1}}{x_n},\frac{1}{x_n}\right).$$

By Lemma 2.6, $\mathcal{K}'=\operatorname{Cl}_{\mathbb{R}^n}(f(\mathcal{K}))$ is a bounded convex polygon. Since $\mathcal{K}\cap\{x_n=0\}=\varnothing$ and $f|_{\mathbb{R}^n\setminus\{x_n=0\}}:\mathbb{R}^n\setminus\{x_n=0\}\to\mathbb{R}^n\setminus\{x_n=0\}$ is a regular involution, it follows that

$$\operatorname{Int}(\mathcal{K}')=\operatorname{Int}_{\mathbb{R}^n}(\mathcal{K}')=\operatorname{Int}_{\mathbb{R}^n}(f(\mathcal{K}))=f(\operatorname{Int}_{\mathbb{R}^n}(\mathcal{K}))\subset f(\mathcal{K})\subset\mathbb{R}^n\setminus\{x_n=0\}.$$

Thus, $\mathcal{K}'\cap\{x_n=0\}\subset\partial\mathcal{K}'$. Observe also that $\mathcal{K}'\subset\{x_n\geqslant0\}$, because the last coordinate of each point in $\mathcal{K}$ is $\geqslant c_0>0$. Moreover, $E'=\mathcal{K}'\cap\{x_n=0\}\neq\varnothing$ because $\mathcal{K}$ is unbounded, and we deduce from Lemma 2.8 that E' is a face of $\mathcal{K}'$. Note also that

$$\mathcal{K}'\setminus E'=\mathcal{K}'\setminus\{x_n=0\}=\operatorname{Cl}_{\mathbb{R}^n}(f(\mathcal{K}))\setminus\{x_n=0\}=f(\mathcal{K}),$$

and consequently $\mathcal{K}=f(f(\mathcal{K}))=f(\mathcal{K}'\setminus E')$, which proves the first part. Next, observe that the restriction $f|_{\mathcal{K}'\setminus E'}:\mathcal{K}'\setminus E'\to\mathcal{K}$ is a biregular homeomorphism whose inverse is the restriction $f|_{\mathcal{K}}:\mathcal{K}\to\mathcal{K}'\setminus E'$. To complete the proof, and since we have already seen that $\operatorname{Int}(\mathcal{K}')=f(\operatorname{Int}_{\mathbb{R}^n}(\mathcal{K}))=f(\operatorname{Int}(\mathcal{K}))$, we get $f(\operatorname{Int}(\mathcal{K}'))=f(f(\operatorname{Int}(\mathcal{K})))=\operatorname{Int}(\mathcal{K})$. □

Remark 2.9. Observe that if $H\subset\mathbb{R}^n$ is an affine subspace that does not intersect the hyperplane $\{x_n=0\}$, then so is $f(H)$. Moreover, if $\mathcal{P}\subset\mathbb{R}^n$ is a bounded convex polyhedron that does not intersect $\{x_n=0\}$, so is $f(\mathcal{P})$. Furthermore, if $\mathcal{P}\subset\mathbb{R}^n$ is a bounded convex polyhedron such that $\mathcal{P}\cap\{x_n=0\}$ is a face of $\mathcal{P}$, then $f(\mathcal{P})$ is an unbounded convex polyhedron. Conversely, an unbounded convex polyhedron $\mathcal{P}\subset\mathbb{R}^n$ that does not intersect $\{x_n=0\}$ is transformed by f onto a bounded convex polyhedron $\mathcal{P}'=\operatorname{Cl}_{\mathbb{R}^n}(f(\mathcal{P}))$ such that $\mathcal{P}'\cap\{x_n=0\}$ is a face of $\mathcal{P}'$. To prove the previous facts, which are well known, recall that f can be understood as a transition map between two charts of the real projective space $\mathbb{RP}^n$.

3. *Partitions of the boundary of a convex polyhedron*

The purpose of this section is to prove Theorem 3.1, which is the clue to demonstrate the second part of Theorem 1.2. This result, which has its own interest, provides, for each point

$p \in \mathbb{R}^n \setminus \mathcal{K}$, a natural partition determined by p of the boundary $\partial\mathcal{K}$ of the bounded convex polyhedron $\mathcal{K}$; namely, we have the following theorem.

THEOREM 3.1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional, bounded, convex polyhedron and let $p \in \mathbb{R}^n \setminus \mathcal{K}$ be an exterior point. Let $\mathfrak{R}$ be the collection of all rays R from p intersecting* $\operatorname{Int}(\mathcal{K})$ *and, for each $R \in \mathfrak{R}$, let a_R be the point in $\mathcal{K} \cap R$ closest to p. Let $\mathcal{A} = \{a_R : R \in \mathfrak{R}\}$, $\mathcal{T} = \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \mathcal{A}$ and $\mathcal{B} = \partial\mathcal{K} \setminus \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A})$. Then the following properties hold:*

(i) *The sets $\mathcal{A}, \mathcal{B}$ and $\mathcal{T}$ are pairwise disjoint subsets of $\partial\mathcal{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ are open in $\partial\mathcal{K}$ and connected, $\mathcal{T}$ is closed in $\partial\mathcal{K}$ and $\partial\mathcal{K} = \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{T}$.*

(ii) *The boundary $\partial\mathcal{K}$ is homeomorphic to the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, and there exist homeomorphisms $\varphi_1 : \overline{B}_{n-1}(0,1) \to \mathcal{A} \sqcup \mathcal{T}$ and $\varphi_2 : \overline{B}_{n-1}(0,1) \to \mathcal{B} \sqcup \mathcal{T}$ such that $\varphi_1(B_{n-1}(0,1)) = \mathcal{A}$, $\varphi_2(B_{n-1}(0,1)) = \mathcal{B}$ and $\varphi_i(\partial\overline{B}_{n-1}(0,1)) = \mathcal{T}$ for $i = 1, 2$.*

(iii) *If $F_1, \ldots, F_m$ are the facets of $\mathcal{K}$, then there exists $1 \leqslant k < m$ such that, after reordering the indices if necessary, $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = \bigcup_{i=1}^k F_i$, $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}) = \bigcup_{j=k+1}^m F_j$ and $\mathcal{T} = \bigcup_{i=1}^k \bigcup_{j=k+1}^m F_i \cap F_j$.*

(iv) *If E is a face of $\mathcal{K}$ and $\{F_{i_1}, \ldots, F_{i_e}\}$ is the collection of all the facets of $\mathcal{K}$ containing E, then $\operatorname{Int}(E) \subset \mathcal{A}$ if and only if $\operatorname{Int}(F_{i_r}) \subset \mathcal{A}$ and $\operatorname{Int}(E) \subset \mathcal{B}$ if and only if $\operatorname{Int}(F_{ir}) \subset B$ for $r = 1, \ldots, e$.*

We say that $\partial\mathcal{K} = \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{T}$ is the *partition of $\partial\mathcal{K}$ determined by the point p*. We approach the proof of Theorem 3.1 in two steps. First, we prove the result for a point p not contained in any of the hyperplanes of $\mathbb{R}^n$ generated by the facets of $\mathcal{K}$. Next, we proceed to the general case using the already proved situation. Before this though, we state the following technical result, whose proof is straightforward and is not included here.

LEMMA 3.2. *Let $p, q \in \mathbb{R}^n$ and let $0 < \delta < \operatorname{dist}(p,q)$. Let $H \subset \mathbb{R}^n$ be the hyperplane passing through q and perpendicular to the line joining p and q, and let R be the open ray with origin at p and passing through q. Consider the semialgebraic sets*

$$D = H \cap B_n(q,\delta) \quad \text{and} \quad C = \{p + t(y-p) : \ t \geqslant 0, \ y \in D\}.$$

Then $C \setminus (\overline{B}_n(q,\delta) \cup \{p\})$ is an open neighbourhood in $\mathbb{R}^n$ of $R \setminus (\overline{B}_n(q,\delta) \cup \{p\})$ which is contained in the open subset $\mathbb{R}^n \setminus H$ of $\mathbb{R}^n$.

3.1. *Proof of Theorem* 3.1 *with restrictions on the exterior point*

First recall that if $\mathcal{K} \subset \mathbb{R}^n$ is an n-dimensional, bounded, convex polyhedron, then, by [1, 11.3.4], $\mathcal{K}$ is homeomorphic to the closed ball $\overline{B}_n(0,1)$ via a homeomorphism $\varphi : \mathcal{K} \to \overline{B}_n(0,1)$. From the invariance of domain theorem, it follows that $\varphi(\partial\mathcal{K}) = \mathbb{S}^{n-1}$ and $\varphi(\operatorname{Int}(\mathcal{K})) = B_n(0,1)$.

Denote by H_i the hyperplane of $\mathbb{R}^n$ generated by the facet F_i, for $i = 1, \ldots, m$. Let H_i^+ be the closed half-space of $\mathbb{R}^n$ determined by H_i containing $\mathcal{K}$ and let $H_j^- = \mathbb{R}^n \setminus (H_j^+ \setminus H_j)$. Recall that, by Berger [2, 12.1.5], $\mathcal{K} = \bigcap_{i=1}^m H_i^+$ and $\mathcal{K} \subsetneq \bigcap_{j\neq i} H_j^+$ for each $i = 1, \ldots, m$. Moreover, $\partial\mathcal{K} = \bigcup_{i=1}^m F_i$ (see Lemma 2.1).

3.1.1. In what follows in this proof, we fix a point $p \notin \bigcup_{i=1}^m H_i \cup \mathcal{K}$ and denote by $\mathfrak{F}$ the family of all rays from p intersecting $\mathcal{K}$. Observe that, since $p \notin H_i$, the intersection $F_i \cap R$ is either empty or a singleton for each $R \in \mathfrak{F}$. Moreover, the intersection $\mathcal{K} \cap R$ is either a singleton or a compact segment $I_R \subset R$.

The distance to the point p defines a natural order relation in the segment I_R; namely, the smallest element $a_R\in I_R$ is the nearest point to p and the largest one is the furthest point $b_R\in I_R$ to p. Given two points $x,y\in I_R$, we say that $x\leqslant y$ if $\operatorname{dist}(x,p)\leqslant\operatorname{dist}(y,p)$. We set $I_R=[a_R,b_R]=\{x\in R:\ a_R\leqslant x\leqslant b_R\}$ and $(a_R,b_R)=\{x\in R:\ a_R<x<b_R\}$. Observe that $I_R=\{(1-\lambda)a_R+\lambda b_R:\ \lambda\in[0,1]\}$ and, given two points $x=(1-\lambda)a_R+\lambda b_R$ and $y=(1-\mu)a_R+\mu b_R$ in I_R, we have $x\leqslant y$ if and only if $\lambda\leqslant\mu$.

In the extremal case in which $\mathcal{K}\cap R$ is a singleton, we have $\mathcal{K}\cap R=[a_R,b_R]$ with $a_R=b_R$, and $(a_R,b_R)=\varnothing$. We keep the above notation along the rest of this proof. Next, we prove several facts about the intervals I_R and the points a_R,b_R.

3.1.2. *Let $R\in\mathfrak{F}$ and $I_R=[a_R,b_R]=\mathcal{K}\cap R$. Then $a_R,b_R\in\partial\mathcal{K}$ and $(a_R,b_R)\subset\operatorname{Int}(\mathcal{K})$.* Indeed, the statement is obvious if $a_R=b_R$; hence, we assume that $a_R\neq b_R$ and define $d=\operatorname{dist}(a_R,b_R)$. Indeed, suppose, by way of contradiction, that $a_R\in\operatorname{Int}(\mathcal{K})$; then, there exists $\varepsilon>0$ such that $B_n(a_R,\varepsilon)\subset\mathcal{K}$. Hence, the point $a_R-(\varepsilon/2d)(b_R-a_R)\in\mathcal{K}\cap R$ and it is closer to p than a_R, which is a contradiction. Hence, $a_R\in\partial\mathcal{K}$ and, analogously, $b_R\in\partial\mathcal{K}$.

Next, suppose that there exists a point $x\in(a_R,b_R)\cap\partial\mathcal{K}$. Since R intersects each facet of $\mathcal{K}$ in at most one point, the points a_R,x,b_R belong to different facets of $\mathcal{K}$, for example, $a_R\in F_1$, $b_R\in F_2$ and $x\in F_3$. Let H_3^+ be the half-space of $\mathbb{R}^n$ containing $\mathcal{K}$ whose boundary is H_3. Observe that $x\in H_3\cap(a_R,b_R)$ and, consequently, either $a_R\notin H_3^+$ or $b_R\notin H_3^+$, which is a contradiction.

3.1.3. *Let $\mathfrak{G}=\{R\in\mathfrak{F}:\ \#(R\cap\partial\mathcal{K})=2\}$, $A=\{a_R:\ R\in\mathfrak{G}\}$, $B=\{b_R:R\in\mathfrak{G}\}$ and $T=\{a_R:\ R\in\mathfrak{F}\setminus\mathfrak{G}\}$. Then $\partial\mathcal{K}=A\sqcup B\sqcup T$ and both A and B are open subsets of $\partial\mathcal{K}$. In particular, T is a closed subset of $\partial\mathcal{K}$.* Indeed, the equality $\partial\mathcal{K}=A\sqcup B\sqcup T$ is evident, and so $T=\partial\mathcal{K}\setminus(A\sqcup B)$. Hence, all reduces to prove that A and B are open subsets of $\partial\mathcal{K}$. To show this, it suffices to see that fixed a ray $R\in\mathfrak{G}$ with $I_R=[a_R,b_R]$, the points a_R and b_R are interior points of the sets A and B, respectively. To prove this, we fix a point $q\in(a_R,b_R)\subset\operatorname{Int}(\mathcal{K})$ and take $\delta>0$ such that $\overline{B}_n(q,\delta)\subset\operatorname{Int}(\mathcal{K})$. Let H be the hyperplane of $\mathbb{R}^n$ passing through q and perpendicular to the line joining p and q. Let $D_R=B_n(q,\delta)\cap H$ and consider the semialgebraic set $C=\{p+t(y-p):\ t\geqslant0,\ y\in D_R\}$. By Lemma 3.2, $C\setminus(\overline{B}_n(q,\delta)\cup\{p\})$ is an open neighbourhood in $\mathbb{R}^n$ of $R\setminus(\overline{B}_n(q,\delta)\cup\{p\})$ contained in the open subset $\mathbb{R}^n\setminus H$ of $\mathbb{R}^n$. This implies, in particular, that $a_R,b_R\in C\setminus(\overline{B}_n(q,\delta)\cup\{p\})$. Denote by H^- the closed half-space defined by H containing p, and let $H^+=(\mathbb{R}^n\setminus H^-)\cup H$. Note that $\mathbb{R}^n\setminus H=\operatorname{Int}(H^+)\cup\operatorname{Int}(H^-)$. Let $\mathfrak{F}_C\subset\mathfrak{F}$ be the family of rays from p passing through a point of D_R; by the conic structure of C, the equality $C=\bigcup_{S\in\mathfrak{F}_C}S$ holds.

Observe that if $S\in\mathfrak{F}_C$, then $S\cap\operatorname{Int}(\mathcal{K})\neq\varnothing$ and so $S\in\mathfrak{G}$. Thus, $T\cap C=\varnothing$ because $C=\bigcup_{S\in\mathfrak{F}_C}S$. Equivalently, $C\cap\partial\mathcal{K}\subset A\sqcup B$. Consider the open subsets of $\partial\mathcal{K}$:

$$\begin{aligned}U_1&=(C\setminus(\overline{B}_n(q,\delta)\cup\{p\}))\cap\operatorname{Int}(H^-)\cap\partial\mathcal{K}\quad\text{and}\\U_2&=(C\setminus(\overline{B}_n(q,\delta)\cup\{p\}))\cap\operatorname{Int}(H^+)\cap\partial\mathcal{K},\end{aligned}$$

which satisfy the equality $U_1\cup U_2=C\setminus(\overline{B}_n(q,\delta)\cup\{p\})\cap\partial\mathcal{K}$. For each $S\in\mathfrak{F}_C$, let m_S be the intersection point of H and S. Observe that $\{m_S\}=S\cap D_S$ and that there exist points $c_S,d_S\in\partial\overline{B}_n(q,\delta)\subset\operatorname{Int}(\mathcal{K})$ such that $m_S\in(c_S,d_S)$ and $S\cap\overline{B}_n(q,\delta)=[c_S,d_S]$. Therefore,

$$S\cap(C\setminus(\overline{B}_n(q,\delta)\cup\{p\}))\cap\mathcal{K}=[a_S,c_S)\cup(d_S,b_S].$$

Thus, since $C\setminus(\overline{B}_n(q,\delta)\cup\{p\})\subset\mathbb{R}^n\setminus H$, it follows

$$\begin{aligned}[a_S,c_S)&=S\cap(C\setminus(\overline{B}_n(q,\delta)\cup\{p\}))\cap\mathcal{K}\cap\operatorname{Int}(H^-)\quad\text{and}\\(d_S,b_S]&=S\cap(C\setminus(\overline{B}_n(q,\delta)\cup\{p\}))\cap\mathcal{K}\cap\operatorname{Int}(H^+).\end{aligned}$$

Hence, for each ray $S \in \mathfrak{F}_C$, we have

$$\{a_S\} = S \cap (C \setminus (\overline{B}_n(q,\delta) \cup \{p\})) \cap \operatorname{Int}(H^-) \cap \partial\mathcal{K} = S \cap U_1 \quad \text{and}$$
$$\{b_S\} = S \cap (C \setminus (\overline{B}_n(q,\delta) \cup \{P\})) \cap \operatorname{Int}(H^+) \cap \partial\mathcal{K} = S \cap U_2.$$

Consequently, $a_R \in U_1 = \bigcup_{S\in\mathfrak{F}_C} S \cap U_1 \subset A$ and $b_R \in U_2 = \bigcup_{S\in\mathfrak{F}_C} S \cap U_2 \subset B$, and this shows that a_R is an interior point of A and b_R is an interior point of B, as wanted.

3.1.4. *Both A and B are connected*: $\operatorname{Cl}_{\mathbb{R}^n}(A) = A \sqcup T$ *and* $\operatorname{Cl}_{\mathbb{R}^n}(B) = B \sqcup T$. *Moreover, $A = \mathcal{A}$, $B = \mathcal{B}$ and $T = \mathcal{T}$* (*see Theorem* 3.1 *for the definition of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$*). Indeed, since $p \notin \mathcal{K}$, there exists a polynomial $\ell \in \mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_n]$ of degree 1 such that $\ell(p) < 0$ and $\mathcal{K} \subset \{\ell > 0\}$. Let H' be the hyperplane of $\mathbb{R}^n$ passing through p and parallel to the hyperplane $H = \{\ell = 0\}$. Consider the central projection $\pi : \mathbb{R}^n \setminus H' \to H$ onto H with centre p. For every point $q \in \mathcal{K}$ denote by R_q the ray from p passing through q. Since $\ell(p)\ell(q) < 0$, it follows that $\pi(q) = R_q \cap H$.

Since $\mathcal{K}$ is a bounded convex polyhedron of $\mathbb{R}^n$ and π is a central projection, the image $\mathcal{P} = \pi(\mathcal{K}) \subset H$ is a bounded convex polyhedron contained in the hyperplane H. Note that π is an open map and $\operatorname{Int}(\mathcal{K})$ is an open subset of $\mathbb{R}^n \setminus H'$. Therefore, $\pi(\operatorname{Int}(\mathcal{K}))$ is an open subset of H and so of $\mathcal{P}$. Moreover, $\operatorname{Int}(\mathcal{K})$ being convex (see [**1**, 11.2.5]), its image $\pi(\operatorname{Int}(\mathcal{K}))$ is convex too.

The continuous map $\pi|_{\mathcal{K}} : \mathcal{K} \to H$ is proper, because $\mathcal{K}$ is compact and, consequently,

$$\operatorname{Cl}_H(\pi(\operatorname{Int}(\mathcal{K}))) = \pi(\operatorname{Cl}_{\mathcal{K}}(\operatorname{Int}(\mathcal{K}))) = \pi(\mathcal{K}) = \mathcal{P}.$$

By Berger [**1**, 11.2.5], and $\pi(\operatorname{Int}(\mathcal{K}))$ being convex, we have

$$\pi(\operatorname{Int}(\mathcal{K})) = \operatorname{Int}_H(\pi(\operatorname{Int}(\mathcal{K}))) = \operatorname{Int}_H(\operatorname{Cl}_H(\pi(\operatorname{Int}(\mathcal{K})))) = \operatorname{Int}_H(\mathcal{P}).$$

Observe that, by the very definition of π, A and B, we also have

$$\pi(A) = \pi(B) = \pi(\operatorname{Int}(\mathcal{K})) = \operatorname{Int}_H(\mathcal{P}),$$

and the restrictions $\pi|_T, \pi|_A$ and $\pi|_B$ are injective maps. Moreover, a point of $A \sqcup B$ and a point of T are not collinear with p and so the restrictions $\pi|_{A\sqcup T}$ and $\pi|_{B\sqcup T}$ are injective as well. Observe that $\pi(T) = \partial\mathcal{P}$, because

$$\begin{aligned}\partial\mathcal{P} \sqcup \operatorname{Int}_H(\mathcal{P}) = \mathcal{P} = \pi(\mathcal{K}) &= \pi(\partial\mathcal{K} \sqcup \operatorname{Int}(\mathcal{K})) = \pi(A \sqcup B \sqcup T \sqcup \operatorname{Int}(\mathcal{K}))\\ &= \pi(A) \cup \pi(B) \cup \pi(T) \cup \pi(\operatorname{Int}(\mathcal{K})) = \pi(T) \sqcup \operatorname{Int}_H(\mathcal{P}).\end{aligned}$$

From Paragraph 3.1.3 we know that A and B are open subsets of $\partial\mathcal{K}$, and therefore $\partial\mathcal{K} \setminus B = A \sqcup T$ and $\partial\mathcal{K} \setminus A = B \sqcup T$ are compact sets, and so the bijective maps $\pi|_{A\sqcup T} : A \sqcup T \to \mathcal{P}$ and $\pi|_{B\sqcup T} : B \sqcup T \to \mathcal{P}$ are in fact homeomorphisms. In particular, A and B are homeomorphic to $\pi(A) = \pi(B) = \operatorname{Int}_H(\mathcal{P})$, which is connected.

Let us check now the equalities $\operatorname{Cl}_{\mathbb{R}^n}(A) = A \sqcup T$ and $\operatorname{Cl}_{\mathbb{R}^n}(B) = B \sqcup T$. The inclusion $\operatorname{Cl}_{\mathbb{R}^n}(A) \subset A \sqcup T$ follows because $A \subset A \sqcup T = \partial\mathcal{K} \setminus B$ is a closed subset in $\mathbb{R}^n$. On the other hand, the map $\pi|_{\mathcal{K}} : \mathcal{K} \to H$ being proper,

$$\pi(\operatorname{Cl}_{\mathbb{R}^n}(A)) = \pi(\operatorname{Cl}_{\mathcal{K}}(A)) = \operatorname{Cl}_H(\pi(A)) = \operatorname{Cl}_H(\pi(\operatorname{Int}(\mathcal{K}))) = \mathcal{P} = \pi(A \sqcup T),$$

which implies the equality $\operatorname{Cl}_{\mathbb{R}^n}(A) = A \sqcup T$ because the restriction $\pi|_{A\sqcup T}$ is injective. Analogously one proves that $\operatorname{Cl}_{\mathbb{R}^n}(B) = B \sqcup T$.

Next, note that a ray $R \in \mathfrak{G}$ (see Paragraph 3.1.3) if and only if $R \cap \operatorname{Int}(\mathcal{K}) \neq \varnothing$, that is, $R \in \mathfrak{R}$ (see Theorem 3.1 for the definition of $\mathfrak{R}$). Hence, $\mathfrak{G} = \mathfrak{R}$ and so $A = \mathcal{A}$. Therefore, $\mathcal{T} = \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \mathcal{A} = \operatorname{Cl}_{\mathbb{R}^n}(A) \setminus A = (A \sqcup T) \setminus A = T$ and

$$\mathcal{B} = \partial\mathcal{K} \setminus \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = (A \sqcup T) \sqcup B \setminus \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = (A \sqcup T) \sqcup B \setminus (A \sqcup T) = B,$$

as wanted.

3.1.5. *There exist homeomorphisms* $\varphi_1 : \overline{B}_{n-1}(0,1) \to \mathcal{A} \sqcup \mathcal{T}$ *and* $\varphi_2 : \overline{B}_{n-1}(0,1) \to \mathcal{B} \sqcup \mathcal{T}$ *such that* $\varphi_1(B_{n-1}(0,1)) = \mathcal{A}$, $\varphi_2(B_{n-1}(0,1)) = \mathcal{B}$ *and* $\varphi_i(\partial\overline{B}_{n-1}(0,1)) = \mathcal{T}$ *for* $i = 1, 2$. We use all the notation introduced in the proof of Paragraph 3.1.4. By Berger [**1**, 11.3.4] there exists a homeomorphism $\varphi : \mathcal{P} \to \overline{B}_{n-1}(0,1)$. By the invariance of domain theorem, we deduce that $\varphi(\partial\mathcal{P}) = \partial\overline{B}_{n-1}(0,1)$ and $\varphi(\operatorname{Int}_H(\mathcal{P})) = B_{n-1}(0,1)$. Now, the homeomorphisms φ_1 and φ_2 we are looking for are, respectively, the compositions

$$\varphi_1 = (\pi|_{\mathcal{A}\sqcup\mathcal{T}})^{-1} \circ \varphi^{-1} : \overline{B}_{n-1}(0,1) \longrightarrow \mathcal{A} \sqcup \mathcal{T} \text{ and } \varphi_2 = (\pi|_{B\sqcup\mathcal{T}})^{-1} \circ \varphi^{-1} : \overline{B}_{n-1}(0,1) \longrightarrow \mathcal{B} \sqcup \mathcal{T}.$$

Let us check that they satisfy the required conditions. First,

$$\begin{aligned} &\varphi_1(B_{n-1}(0,1)) = (\pi|_{\mathcal{A}\sqcup\mathcal{T}})^{-1}(\operatorname{Int}_H(\mathcal{P})) = \mathcal{A} \\ \text{and} \quad &\varphi_2(B_{n-1}(0,1)) = (\pi|_{\mathcal{B}\sqcup\mathcal{T}})^{-1}(\operatorname{Int}_H(\mathcal{P})) = \mathcal{B}. \end{aligned}$$

Second,

$$\varphi_1(\partial B_{n-1}(0,1)) = (\pi|_{\mathcal{A}\sqcup\mathcal{T}})^{-1}(\partial\mathcal{P}) = \mathcal{T} = (\pi|_{\mathcal{B}\sqcup\mathcal{T}})^{-1}(\partial\mathcal{P}) = \varphi_2(\partial B_{n-1}(0,1)),$$

and we are done.

3.1.6. Recall that $F_1, \dots, F_m$ denote the facets of the polyhedron $\mathcal{K} \subset \mathbb{R}^n$ and $H_1, \dots, H_m$ the hyperplanes of $\mathbb{R}^n$ generated by them. *Then we have the following conditions*:

(1) $\operatorname{Int}_{\partial\mathcal{K}}(F_i) = \operatorname{Int}(F_i)$ *for* $i = 1, \dots, m$;

(2) for *each index* $i = 1, \dots, m$, *either* $\operatorname{Int}(F_i) \subset \mathcal{A}$ *or* $\operatorname{Int}(F_i) \subset \mathcal{B}$.

We begin by proving (1). By Berger [**2**, 12.1.5–7], $\operatorname{Int}(F_i) = \operatorname{Int}_{H_i}(F_i) = F_i \setminus \bigcup_{j\neq i} F_j$, and since each facet F_j is a closed subset of $\mathbb{R}^n$, we get

$$\begin{aligned} \operatorname{Int}_{\partial\mathcal{K}}(F_i) &= F_i \setminus \operatorname{Cl}_{\partial\mathcal{K}}(\partial\mathcal{K} \setminus F_i) = F_i \Bigg\backslash \operatorname{Cl}_{\partial\mathcal{K}}\Bigg(\bigcup_{j\neq i} F_j \setminus F_i\Bigg) \\ &= F_i \Bigg\backslash \bigcup_{j\neq i} \operatorname{Cl}_{\partial\mathcal{K}}(F_j \setminus F_i) = F_i \Bigg\backslash \bigcup_{j\neq i} F_j = \operatorname{Int}_{H_i}(F_i) = \operatorname{Int}(F_i). \end{aligned}$$

Next, we proceed with (2). Since $\mathcal{A}$ and $\mathcal{B}$ are, by Paragraph 3.1.3, open subsets of $\partial\mathcal{K}$ and $\operatorname{Int}(F_i)$ is connected, to prove our claim, it is enough to check that $\operatorname{Int}(F_i) \subset \mathcal{A} \sqcup \mathcal{B}$. Indeed, let $x \in \operatorname{Int}(F_i)$. We must prove that the ray R from p passing through x intersects $\operatorname{Int}(\mathcal{K})$ and that x is one of the extremes of $R \cap \mathcal{K}$.

Observe first that $\operatorname{dist}(x,p) > 0$, because $p \notin \mathcal{K}$. Also $\operatorname{dist}(x, H_j) > 0$ for $j \neq i$, because $x \in \operatorname{Int}(F_i) = F_i \setminus \bigcup_{j\neq i} F_j = F_i \setminus \bigcup_{j\neq i} H_j$. Thus, $\varepsilon = \min\{\operatorname{dist}(x,p),\ \operatorname{dist}(x,H_j) :\ j \neq i\}$ is a positive real number. Let us check that $B_n(x,\varepsilon) \cap (H_i^+ \setminus H_i) \subset \operatorname{Int}(\mathcal{K})$. Suppose, by way of contradiction, that there exists a point

$$y \in (B_n(x,\varepsilon) \cap (H_i^+ \setminus H_i)) \cap (\mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K})).$$

By Lemma 2.1, $y \in \mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K}) = \bigcup_{j=1}^m (\mathbb{R}^n \setminus (H_j^+ \setminus H_j))$. Consequently, there exists $j \neq i$ such that $y \in \mathbb{R}^n \setminus (H_j^+ \setminus H_j) = H_j^-$ and so

$$\operatorname{dist}(x, H_j) = \operatorname{dist}(x, H_j^-) \leqslant \operatorname{dist}(x,y) < \varepsilon \leqslant \operatorname{dist}(x, H_j),$$

which is a contradiction. Thus, $B_n(x,\varepsilon) \cap (H_i^+ \setminus H_i) \subset \operatorname{Int}(\mathcal{K})$.

Observe that, since p is an exterior point with respect to the open ball $B_n(x,\varepsilon)$ and $x \in H_i$ but $p \notin H_i$, the ray R from p passing through x intersects $B_n(x,\varepsilon) \cap (H_i^+ \setminus H_i)$ and so $R \cap \operatorname{Int}(\mathcal{K}) \neq \varnothing$. Hence, $R \cap \mathcal{K}$ is a closed (nontrivial) interval having x as one of its extremes because $x \in F_i \subset \partial\mathcal{K}$. Thus, $x \in \mathcal{A} \sqcup \mathcal{B}$, as wanted.

3.1.7. Therefore, by Paragraph 3.1.6(2), we may assume the existence of $k < m$ such that $\operatorname{Int}(F_i) \subset \mathcal{A}$ for $i = 1, \dots, k$ and $\operatorname{Int}(F_i) \subset \mathcal{B}$ for $i = k+1, \dots, m$. To ensure that $k < m$, just recall that $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = \mathcal{A} \cup \mathcal{T} \subsetneq \partial\mathcal{K} = \bigcup_{i=1}^m F_i = \mathcal{A} \cup \mathcal{T} \cup \mathcal{B}$ and $\mathcal{B}$ is a nonempty open subset of $\partial\mathcal{K}$.

3.1.8. *Moreover, with these notation we have the following properties*:

(1) given *indices* i, j *with* $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant m$, *the intersection* $F_i \cap F_j \subset \mathcal{T}$;

(2) $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = \bigcup_{i=1}^k F_i$, $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}) = \bigcup_{j=k+1}^m F_j$ *and* $\mathcal{T} = \bigcup_{i=1}^k \bigcup_{j=k+1}^m F_i \cap F_j$.

We first prove (1). Since each facet is a convex set, it coincides, by Berger [**1**, 11.2.5], with the closure of its interior. Thus, using Paragraph 3.1.4,

$$\begin{aligned} F_i \cap F_j &= \operatorname{Cl}_{H_i}(\operatorname{Int}_{H_i}(F_i)) \cap \operatorname{Cl}_{H_j}(\operatorname{Int}_{H_j}(F_j)) \\ &= \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Int}_{H_i}(F_i)) \cap \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Int}_{H_j}(F_j)) \\ &\subset \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \cap \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}) = (\mathcal{A} \sqcup \mathcal{T}) \cap (\mathcal{B} \sqcup \mathcal{T}) = \mathcal{T}. \end{aligned}$$

Next, we proceed with (2). Recall that $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \cap \mathcal{B} = (\mathcal{A} \sqcup \mathcal{T}) \cap \mathcal{B} = \varnothing$ and $\operatorname{Int}(F_i) \subset \mathcal{B}$ for $i = k+1, \dots, m$. Hence,

$$\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \bigcup_{i=1}^k \operatorname{Int}(F_i) = \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \bigcup_{i=1}^m \operatorname{Int}(F_i) \subset \partial\mathcal{K} \setminus \bigcup_{i=1}^m \operatorname{Int}(F_i) = \bigcup_{i=1}^m F_i \setminus \bigcup_{i=1}^m \operatorname{Int}(F_i).$$

Now, since $F_i \cap \operatorname{Int}_{H_j}(F_j) = \varnothing$ if $i \neq j$, we infer that

$$\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \bigcup_{i=1}^k \operatorname{Int}(F_i) \subset \bigcup_{i=1}^m (F_i \setminus \operatorname{Int}(F_i)).$$

Consequently, by Bochnak, Coste and Roy [**3**, 2.8.13],

$$\dim\left(\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \bigcup_{i=1}^k \operatorname{Int}(F_i)\right) \leqslant \dim\left(\bigcup_{i=1}^m (F_i \setminus \operatorname{Int}(F_i))\right) = n-2.$$

This implies, since $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A})$ is pure dimensional of dimension $n-1$, that

$$\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = \operatorname{Cl}_{\mathbb{R}^n}\left(\bigcup_{i=1}^k \operatorname{Int}(F_i)\right) = \bigcup_{i=1}^k F_i.$$

Analogously, $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}) = \bigcup_{j=k+1}^m F_j$ and so, using again Paragraph 3.1.5,

$$\mathcal{T} = (\mathcal{A} \sqcup \mathcal{T}) \cap (\mathcal{B} \sqcup \mathcal{T}) = \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \cap \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}) = \bigcup_{i=1}^k \bigcup_{j=k+1}^m F_i \cap F_j,$$

as wanted.

3.1.9. *Let E be a face of $\mathcal{K}$ and let $\mathfrak{E} = \{F_{i_1}, \dots, F_{i_e}\}$ be the collection of all the facets of $\mathcal{K}$ containing E. Then* $\operatorname{Int}(E) \subset \mathcal{A}$ *if and only if* $\operatorname{Int}(F_{i_r}) \subset \mathcal{A}$ *for all* $r = 1, \dots, e$. Indeed, we may assume, after reordering the indices $1 \leqslant i \leqslant k$, that $\mathfrak{E} = \{F_1, \dots, F_e\}$ and $\operatorname{Int}(F_i) \subset \mathcal{A}$ for $i = 1, \dots, e$. Then by Paragraph 3.1.8,

$$\operatorname{Int}(E) \subset E = \bigcap_{i=1}^e F_i \subset \bigcap_{i=1}^e \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Int}(F_i)) \subset \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) = \mathcal{A} \sqcup \mathcal{T},$$

and so all reduces to see that $\operatorname{Int}(E)\cap\mathcal{T}=\varnothing$. Suppose, by way of contradiction, the existence of a point $x\in\operatorname{Int}(E)\cap\mathcal{T}$. Since $x\in\mathcal{T}$ there exists, by Paragraph 3.1.8, a facet F_s of $\mathcal{K}$, with $s\geqslant k+1$, such that $x\in F_s$. Since $x\in\operatorname{Int}(E)\cap F_s$, we deduce that $E=\operatorname{Cl}_{\mathbb{R}^n}(E)\subset F_s$. But $F_s\in\mathfrak{E}$ and $\operatorname{Int}(F_s)\subset\mathcal{B}$, which is a contradiction.

Conversely, suppose that $\operatorname{Int}(E)\subset\mathcal{A}$, but $\operatorname{Int}(F_{i_j})\not\subset\mathcal{A}$ for some index $1\leqslant j\leqslant e$. By Paragraph 3.1.6, $\operatorname{Int}(F_{i_j})\subset\mathcal{B}$, and let us check that we may choose some index $1\leqslant s\leqslant e$ such that $\operatorname{Int}(F_{i_s})\subset\mathcal{A}$. Otherwise, all $F_{i_j}\in\mathfrak{E}$ satisfies $\operatorname{Int}(F_{i_j})\subset\mathcal{B}$ and proceeding as in the previous implication but swapping $\mathcal{A}$ for $\mathcal{B}$, we deduce that $\operatorname{Int}(E)\subset\mathcal{B}$, which is a contradiction. Hence, by Paragraph 3.1.8(1), $\operatorname{Int}(E)\subset E\subset F_{i_j}\cap F_{i_s}\subset\mathcal{T}$, which is false.

3.1.10. Observe that under our assumptions, that is, $\mathcal{K}$ bounded and $p\notin\bigcup_{i=1}^m H_i$, we have already proved Theorem 3.1; namely, (i) follows from Paragraphs 3.1.3 and 3.1.4; Paragraph 3.1.5 proves (ii); Paragraph 3.1.8(2) implies (iii), and (iv) is proved in Paragraph 3.1.9 (because Paragraph 3.1.9 also works if we substitute $\mathcal{A}$ by $\mathcal{B}$).

Next, we proceed to prove Theorem 3.1 in case p is an arbitrary point outside $\mathcal{K}$. Before that, we need a preliminary lemma.

Lemma 3.3. *Let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional, convex polyhedron and let $\{H_1,\dots,H_m\}$ be the minimal presentation of $\mathcal{K}$. Let $\ell_i\in\mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_n]$ be a polynomial of degree 1 such that $H_i^+=\{\ell_i\geqslant 0\}$ for $i=1,\dots,m$. Let $p\in\mathbb{R}^n\setminus\mathcal{K}$ such that*

$$\ell_1(p)\geqslant 0,\dots,\ell_s(p)\geqslant 0\quad\text{and}\quad \ell_{s+1}(p)<0,\dots,\ell_m(p)<0\ \text{ for some } 0\leqslant s<m.$$

Then, for each $\varepsilon>0$, there exists a point $q\in B_n(p,\varepsilon)$, such that $\ell_1(q)>0,\dots,\ell_s(q)>0$ and $\ell_{s+1}(q)<0,\dots,\ell_m(q)<0$.

Proof. Observe first that if $\ell_1(p)>0,\dots,\ell_s(p)>0$, then it suffices to choose $q=p$. Thus, after reordering the indices $1,\dots,s$, we may assume that there exists $1\leqslant k\leqslant s$ such that $\ell_1(p)=0,\dots,\ell_k(p)=0$ and $\ell_{k+1}(p)>0,\dots,\ell_s(p)>0$. Consider the n-dimensional convex polyhedron $\mathcal{K}'=\bigcap_{i=1}^k H_i^+$, which contains $\mathcal{K}=\bigcap_{i=1}^m H_i^+$. Note that $\{H_1,\dots,H_k\}$ is the minimal presentation of $\mathcal{K}'$, because $\{H_1,\dots,H_m\}$ is the minimal presentation of $\mathcal{K}$. Observe that $p\in\mathcal{K}'$ and, by Lemma 2.1, $p\notin\mathcal{K}'\setminus\bigcup_{i=1}^k H_i=\operatorname{Int}(\mathcal{K}')$, that is, $p\in\partial\mathcal{K}'$. Let $\delta=\min\{\varepsilon,\ \operatorname{dist}(p,H_i):\ i=k+1,\dots,m\}$, which is positive because $p\notin\bigcup_{i=k+1}^m H_i$. Note that, for each point $y\in B_n(p,\delta)$, we have

$$\ell_{k+1}(y)>0,\dots,\ell_s(y)>0\quad\text{and}\quad\ell_{s+1}(y)<0,\dots,\ell_m(y)<0.$$

On the other hand, since $p\in\mathcal{K}'=\operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Int}(\mathcal{K}'))$, there is a point $q\in\operatorname{Int}(\mathcal{K}')\cap B_n(p,\delta)$. Hence, $q\in B_n(p,\varepsilon)$, and it satisfies $\ell_1(q)>0,\dots,\ell_s(q)>0$ and $\ell_{s+1}(q)<0,\dots,\ell_m(q)<0$, as wanted. $\square$

3.2. *Proof of Theorem* 3.1 *with no restrictions on the exterior point*

Recall that $H_1,\dots,H_m$ denote the hyperplanes of $\mathbb{R}^n$ generated by the facets $F_1,\dots,F_m$ of $\mathcal{K}$. Since we have already proved Theorem 3.1 when $p\notin(\mathcal{K}\cup\bigcup_{i=1}^m H_i)$, it only remains to consider the case in which $p\in(\bigcup_{i=1}^m H_i)\setminus\mathcal{K}$. Thus, let p be such a point and, after reordering the indices if necessary, let $1\leqslant r_1\leqslant r_2<m$ be such that

$$p\in\bigcap_{i=1}^{r_1}H_i\cap\bigcap_{i=r_1+1}^{r_2}(H_i^+\setminus H_i)\cap\bigcap_{i=r_2+1}^{m}(\mathbb{R}^n\setminus H_i^+).$$

3.2.1. We repeat for p the construction we did in Paragraph 3.1.1 for a point in $\mathbb{R}^n \setminus (\bigcup_{i=1}^m H_i \cup \mathcal{K})$. Denote by $\mathfrak{F}$ the family of all rays R from p intersecting $\mathcal{K}$. Fix $R \in \mathfrak{F}$ and observe that the intersection $\mathcal{K} \cap R$ is either a singleton or a compact interval $I_R = [a_R, b_R]$, where a_R is the point in I_R closest to p and b_R is the furthest one. We define in I_R the same order relation we constructed in Paragraph 3.1.1. In the extremal case in which $\mathcal{K} \cap R$ is a singleton, we write $\mathcal{K} \cap R = [a_R, b_R]$, with $a_R = b_R$ and so $(a_R, b_R) = \varnothing$. Recall also that $\mathfrak{R} = \{R \in \mathfrak{F} : \ I_R \cap \operatorname{Int}(\mathcal{K}) \neq \varnothing\}$ and $\mathcal{A} = \{a_R : \ R \in \mathfrak{R}\}$. We also set $\mathcal{T} = \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}) \setminus \mathcal{A}$ and $\mathcal{B} = \partial\mathcal{K} \setminus \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A})$. The same proof of Paragraph 3.1.2 provides us the following.

3.2.2. *Let $R \in \mathfrak{F}$ and $I_R = [a_R, b_R] = \mathcal{K} \cap R$. Then $a_R, b_R \in \partial\mathcal{K}$. Moreover, if $R \in \mathfrak{R}$, then $(a_R, b_R) \subset \operatorname{Int}(\mathcal{K})$.* Next, by Lemma 3.3, there exists a point $q \in \bigcap_{i=1}^{r_2}(H_i^+ \setminus H_i) \cap \bigcap_{i=r_2+1}^m(\mathbb{R}^n \setminus H_i^+)$. Observe that $q \notin (\mathcal{K} \cup \bigcup_{i=1}^m H_i)$, and our next goal is to compare the sets $\mathcal{A}, \mathcal{B}$ and $\mathcal{T}$ determined by the point p, with those $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{T}'$ determined by the point q, whose properties were carefully studied in the first part of the proof of Theorem 3.1. In fact, we obtain the best possible answer.

3.2.3. *With the notation introduced above, $\mathcal{A} = \mathcal{A}'$ and so $\mathcal{B} = \mathcal{B}'$ and $\mathcal{T} = \mathcal{T}'$.* Indeed, let $a_R \in \mathcal{A}$ for some ray $R \in \mathfrak{R}$ from p. The strategy will be the following. We will prove first that the ray R_1 from q passing through a_R intersects $\operatorname{Int}(\mathcal{K})$. Consequently, $a_R \in \mathcal{K} \cap R_1 = [a_{R_1}, b_{R_1}]$ and, by Paragraphs 3.1.2 and 3.2.2, $(a_{R_1}, b_{R_1}) \subset \operatorname{Int}(\mathcal{K})$ and $a_R \in \partial\mathcal{K}$. Thus, $a_R \in \{a_{R_1}, b_{R_1}\}$. We shall see later that in fact $a_R \neq b_{R_1}$, and so $a_R = a_{R_1} \in \mathcal{A}'$, which proves the inclusion $\mathcal{A} \subset \mathcal{A}'$. The converse inclusion $\mathcal{A}' \subset \mathcal{A}$ follows analogously, but interchanging the roles of p and q, R and R_1, a_R and a_{R_1}, and b_R and b_{R_1}, and we do not include the details.

Hence, let us begin by proving that the ray R_1 from q passing by a_R intersects $\operatorname{Int}(\mathcal{K})$. To that end, let $\ell_1, \ldots, \ell_m \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ be polynomials of degree 1 such that each closed half-space $H_i^+ = \{\ell_i \geqslant 0\}$. Let us check first that $\ell_i(a_R) > 0$ for $i = 1, \ldots, r_2$. Indeed, since $a_R \in \mathcal{A}$, there exists a point $x \in \operatorname{Int}(\mathcal{K}) \cap R$ with $x > a_R$. Hence, there exists $\rho > 1$ such that $x = \rho a_R + (1-\rho)p$, and since $x \in \operatorname{Int}(\mathcal{K})$,

$$0 < \ell_i(x) = \rho(\ell_i(a_R)) + (1-\rho)\ell_i(p).$$

But $\ell_i(p) \geqslant 0$, and so $\ell_i(a_R) > 0$.

Next, recall that, by Lemma 2.1, $\operatorname{Int}(\mathcal{K}) = \bigcap_{i=1}^m (H_i^+ \setminus H_i) = \bigcap_{i=1}^m \{\ell_i > 0\}$. Therefore, we must check that $R_1 \cap \bigcap_{i=1}^m \{\ell_i > 0\} \neq \varnothing$. If $r_2 + 1 \leqslant i \leqslant m$, then we have $\ell_i(q) < 0$ and, for each $\rho > 1$, the point $z = \rho a_R + (1-\rho)q \in R_1 \cap \{\ell_i > 0\}$; namely,

$$\ell_i(z) = \ell_i(\rho a_R + (1-\rho)q) = \rho\ell_i(a_R) + (1-\rho)\ell_i(q) > 0.$$

Now let $i = 1, \ldots, r_2$ and recall that $\ell_i(q) > 0$. If $\ell_i(a_R) - \ell_i(q) \geqslant 0$, then for each positive real number $\rho > 0$ the point $z = \rho a_R + (1-\rho)q \in R_1 \cap \{\ell_i > 0\}$; namely,

$$\ell_i(z) = \rho\ell_i(a_R) + (1-\rho)\ell_i(q) = \ell_i(q) + \rho(\ell_i(a_R) - \ell_i(q)) > 0.$$

On the other hand, if $\ell_i(a_R) - \ell_i(q) < 0$, then the quotient $\lambda_i = \ell_i(q)/(\ell_i(q) - \ell_i(a_R)) > 1$, because both $\ell_i(q)$ and $\ell_i(a_R)$ are positive, since $1 \leqslant i \leqslant r_2$. Observe that if $1 < \rho < \lambda_i$, then the point $z = \rho a_R + (1-\rho)q \in R_1$ satisfies

$$\ell_i(z) = \ell_i(\rho a_R + (1-\rho)q) = \rho\ell_i(a_R) + (1-\rho)\ell_i(q) = \ell_i(q) + \rho(\ell_i(a_R) - \ell_i(q)) > 0.$$

Thus, if we choose $1 < \rho < \lambda_i$ for all $1 \leqslant i \leqslant r_2$ such that $\ell_i(a_R) - \ell_i(q) < 0$, then we find a point $z = \rho a_R + (1-\rho)q \in R_1 \cap \bigcap_{i=1}^m \{\ell_i > 0\} = R_1 \cap \operatorname{Int}(\mathcal{K})$.

Finally, all reduces to check that $a_R \neq b_{R_1}$. Assume, by way of contradiction, that $a_R = b_{R_1}$. Since $a_R \in \mathcal{A}$, it follows that $I_R = \mathcal{K} \cap R = [a_R, b_R]$ with $a_R < b_R$. Moreover, $a_R \in \partial\mathcal{K}$ and $\ell_i(a_R) > 0$ for $i = 1, \ldots, r_2$, which implies the existence of $r_2 + 1 \leqslant j \leqslant m$ such that $\ell_j(a_R) = 0$. On the other hand, there exists $\rho > 1$ such that $a_R = b_{R_1} = \rho a_{R_1} + (1-\rho)q$ or, equivalently,

$q = (1/(1-\rho))a_R + (-\rho/(1-\rho))b_{R_1}$. Therefore,

$$0 > \ell_j(q) = \ell_j\left(\frac{1}{1-\rho}a_R + \frac{-\rho}{1-\rho}a_{R_1}\right) = \frac{1}{1-\rho}\ell_j(a_R) + \frac{\rho}{\rho-1}\ell_j(a_{R_1}) = \frac{\rho}{\rho-1}\ell_j(a_{R_1}) \geqslant 0,$$

which is a contradiction. We are done.

To conclude the proof, observe that we have already seen in Paragraph 3.1.10 that $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}' = \mathcal{B}$ and $\mathcal{T}'$ satisfy (i)–(iv) in the Theorem 3.1, as wanted.

Remark 3.4. Theorem 3.1 can be generalized to an n-dimensional unbounded convex polyhedron by means of Lemma 2.4 and Proposition 2.7. In such a case, $\partial\mathcal{K}$ is homeomorphic to $\mathbb{R}^{n-1}$ (see [**1**, 11.3.8]) and, while $\mathcal{A}$ is always homeomorphic to the open ball $B_n(0,1)$, there are several possibilities concerning the topology of the sets $\mathcal{B}$ and $\mathcal{T}$; namely, depending upon the position of the point p, we may have:

(1) either $\mathcal{B} = \mathcal{T} = \varnothing$; or
(2) there exist homeomorphisms $\varphi_1 : \{x_n \geqslant 0\} \to \mathcal{A} \sqcup \mathcal{T}$ and $\varphi_2 : \{x_n \geqslant 0\} \to \mathcal{B} \sqcup \mathcal{T}$ such that $\varphi_1(\{x_n > 0\}) = \mathcal{A}$, $\varphi_2(\{x_n > 0\}) = \mathcal{B}$ and $\varphi_i(\{x_n = 0\}) = \mathcal{T}$ for $i = 1, 2$; or
(3) there exist homeomorphisms $\varphi_1 : \overline{B}_{n-1}(0,1) \to \mathcal{A} \sqcup \mathcal{T}$ and $\varphi_2 : \overline{B}_{n-1}(0,1) \setminus \{0\} \to \mathcal{B} \sqcup \mathcal{T}$ such that $\varphi_1(B_{n-1}(0,1)) = \mathcal{A}$, $\varphi_2(B_{n-1}(0,1) \setminus \{0\}) = \mathcal{B}$ and $\varphi_i(\partial\overline{B}_{n-1}(0,1)) = \mathcal{T}$ for $i = 1, 2$.

To prove all these facts, one can use Lemma 2.4, Proposition 2.7, Theorem 3.1 and the classical Schoenflies' Theorem (see [**4**]). Since this generalization of Theorem 3.1 is not necessary for our purposes and its proof is quite cumbersome, we do not include the details here.

4. *Interior of convex polyhedra as regular images of* $\mathbb{R}^n$

The goal of this section is to prove that the interior of a convex polyhedron of $\mathbb{R}^n$ is a regular image of $\mathbb{R}^n$. We begin by dealing with the most elementary example of a convex polyhedron.

Lemma 4.1. *The interior of an n-simplex Δ is a regular image of $\mathbb{R}^n$.*

Proof. First, observe that after a change of coordinates, we may assume that Δ is the n-simplex of vertices $(1, \overset{(k)}{\ldots}, 1, 0, \overset{(n-k)}{\ldots}, 0)$, where $k = 0, \ldots, n$. A straightforward computation shows that

$$\Delta = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_n \geqslant 0,\ 1 - x_1 \geqslant 0,\ x_{k-1} - x_k \geqslant 0,\ 2 \leqslant k \leqslant n\} \quad \text{and}$$
$$\operatorname{Int}(\Delta) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_n > 0,\ 1 - x_1 > 0,\ x_{k-1} - x_k > 0,\ 2 \leqslant k \leqslant n\}.$$

By Fernando and Gamboa [**5**, 1.6], there exists a polynomial map $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ whose image is the n-dimensional open orthant $Q_0 = \{x_1 > 0, \ldots, x_n > 0\}$. Now, if we compose f_1 with the rational map

$$f_2 : \mathbb{R}^n \dashrightarrow \mathbb{R}^n,\ x = (x_1, \ldots, x_n) \longmapsto \left(\frac{1}{x_1+1}, \ldots, \frac{1}{x_n+1}\right),$$

we obtain a regular map $f_2 \circ f_1 : \mathbb{R}^n \to \mathbb{R}^n$ whose image is the interior $\mathcal{C} = (0,1)^n$ of the closed cube $[0,1]^n$. Next, consider the polynomial map

$$f_3 : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ x = (x_1, \ldots, x_n) \longmapsto \left(\prod_{j=1}^{k} x_j\right)_{k=1,\ldots,n},$$

and let us check the equality $f_3(\mathcal{C}) = \operatorname{Int}(\Delta)$. Indeed, given a point $x \in \mathcal{C}$, let us denote $f_3(x) = y = (y_1, \dots, y_n)$. Observe that

$$y_n = \prod_{j=1}^{n} x_j > 0, \quad 1 - y_1 = 1 - x_1 > 0,$$

$$\text{and} \quad y_{k-1} - y_k = \prod_{j=1}^{k-1} x_j - \prod_{j=1}^{k} x_j = (1 - x_k) \prod_{j=1}^{k-1} x_j > 0 \quad \text{for } 2 \leqslant k \leqslant n.$$

Hence, $y = f_3(x) \in \operatorname{Int}(\Delta)$. Conversely, let $y = (y_1, \dots, y_n) \in \operatorname{Int}(\Delta)$ and consider the point $x = (y_1, y_2/y_1, \dots, y_k/y_{k-1}, \dots, y_n/y_{n-1}) \in \mathbb{R}^n$, which satisfies $f_3(x) = y$. Moreover, since $0 < y_n \leqslant y_k < y_{k-1} \leqslant y_1 < 1$ for $k = 2, \dots, n$, we get, $0 < y_k/y_{k-1} < 1$ for $k = 2, \dots, n$ and $0 < y_1 < 1$. Therefore, $x \in \mathcal{C}$ and, consequently, $f_3(\mathcal{C}) = \operatorname{Int}(\Delta)$.

Finally, we conclude that the image of the regular map $f_3 \circ f_2 \circ f_1 : \mathbb{R}^n \to \mathbb{R}^n$ is the interior of the n-simplex Δ. □

LEMMA 4.2. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional, bounded, convex polyhedron whose facets are $F_1, \dots, F_m$ and let $\mathsf{A} = \bigcup_{i=1}^{k} F_i$ for some $1 \leqslant k < m$ and $\mathsf{B} = \bigcup_{j=k+1}^{m} F_j$. Let H_i be the hyperplane of $\mathbb{R}^n$ generated by F_i for $i = 1, \dots, m$. Then there exists a rational function $h : \mathbb{R}^n \dashrightarrow \mathbb{R}$, which is regular on $\mathbb{R}^n \setminus (\bigcup_{i=1}^{k} H_i \cap \bigcup_{j=k+1}^{m} H_j)$, such that:*

(i) *h takes value 0 on $\mathsf{A} \setminus \mathsf{B}$ and 1 on $\mathsf{B} \setminus \mathsf{A}$;*
(ii) *$0 < h(p) < 1$ for each point $p \in \operatorname{Int}(\mathcal{K})$.*

Proof. For each index $i = 1, \dots, m$, let $\ell_i \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$ be a polynomial of degree 1 such that $H_i = \{\ell_i = 0\}$. The rational function defined by

$$h : \mathbb{R}^n \dashrightarrow \mathbb{R}, \ x \longmapsto \frac{\prod_{i=1}^{k} \ell_i^2(x)}{\prod_{i=1}^{k} \ell_i^2(x) + \prod_{j=k+1}^{m} \ell_j^2(x)}$$

satisfies the conditions in the statement. □

REMARK 4.3. Observe, moreover, that $\mathcal{K} \setminus (\mathsf{A} \cap \mathsf{B}) \subset \mathbb{R}^n \setminus (\bigcup_{i=1}^{k} H_i \cap \bigcup_{j=k+1}^{m} H_i)$.

We are ready to prove the second part of Theorem 1.2 in case $\mathcal{K}$ is bounded; namely, we have the following proposition.

PROPOSITION 4.4. *The interior of an n-dimensional, bounded, convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a regular image of $\mathbb{R}^n$.*

Proof. Since $\dim \mathcal{K} = n$ and $\mathcal{K}$ is, by Berger [**1**, 11.6.8] and Berger [**2**, 12.1.9], the convex hull of the set $\mathfrak{V}$ of its vertices, $\mathfrak{V}$ has at least $n+1$ elements, and $n+1$ of them are affinely independent. We proceed by induction on the cardinality of $\mathfrak{V}$. Observe that if $\#\mathfrak{V} = n+1$, then $\mathcal{K}$ is an n-simplex and, by Lemma 4.1, $\operatorname{Int}(\mathcal{K})$ is a regular image of $\mathbb{R}^n$.

Let us consider an n-dimensional, bounded, convex polyhedron $\mathcal{K}$ whose set of vertices is $\mathfrak{V} = \{v_1, \dots, v_s\}$ and $s > n+1$. We may assume that its subset $\mathfrak{V}' = \{v_2, \dots, v_s\}$ is not contained in a hyperplane of $\mathbb{R}^n$. After a change of coordinates, we may also assume that v_1 is the origin of $\mathbb{R}^n$. Consider the n-dimensional, bounded, convex polyhedron

$\mathcal{K}'$ whose set of vertices is $\mathfrak{V}'$, and observe that $v_1 \in \mathbb{R}^n \setminus \mathcal{K}'$. Since $\mathcal{K}$ and $\mathcal{K}'$ are, respectively, the convex hulls of $\mathfrak{V}$ and $\mathfrak{V}'$, we have, by Berger [1, 11.1.8.6], the equalities

$$\mathcal{K} = \left\{\sum_{i=1}^{s} \lambda_i v_i :\ \lambda_i \geqslant 0,\ \sum_{i=1}^{s} \lambda_i = 1\right\} \quad \text{and} \quad \mathcal{K}' = \left\{\sum_{i=2}^{s} \mu_i v_i :\ \mu_i \geqslant 0,\ \sum_{i=2}^{s} \mu_i = 1\right\}.$$

Observe that $\mathcal{K} = \{\lambda p :\ p \in \mathcal{K}' \ \&\ 0 \leqslant \lambda \leqslant 1\}$ because v_1 is the origin. Moreover, one can check that

$$\operatorname{Int}(\mathcal{K}) = \left\{\sum_{i=1}^{s} \lambda_i v_i :\ \lambda_i > 0,\ \sum_{i=1}^{s} \lambda_i = 1\right\}$$

$$\text{and} \quad \operatorname{Int}(\mathcal{K}') = \left\{\sum_{i=2}^{s} \mu_i v_i :\ \mu_i > 0,\ \sum_{i=2}^{s} \mu_i = 1\right\}.$$

In particular, $\operatorname{Int}(\mathcal{K}) = \{\lambda p :\ p \in \operatorname{Int}(\mathcal{K}') \ \&\ 0 < \lambda < 1\}$. In what follows, we use the notation already introduced in Paragraph 3.2.1. Let $\mathfrak{R}$ be the family of all rays from v_1 that intersect $\operatorname{Int}(\mathcal{K}')$ and let $I_R = \mathcal{K}' \cap R = [a_R, b_R]$, where a_R is the nearest point of I_R to v_1 and b_R is the furthest one. By Paragraph 3.2.2, $a_R, b_R \in \partial\mathcal{K}'$ and $(a_R, b_R) \subset \operatorname{Int}(\mathcal{K}')$ for all $R \in \mathfrak{R}$. Observe that $\operatorname{Int}(\mathcal{K}') = \bigcup_{R\in\mathfrak{R}} (a_R, b_R)$, and so $\operatorname{Int}(\mathcal{K}) = \bigcup_{R\in\mathfrak{R}} (0, b_R)$, where $(0, b_R) = \{\lambda b_R :\ \lambda \in (0,1)\}$.

Let $\mathcal{A}'$,$\mathcal{B}'$ and $\mathcal{T}'$ be the sets constructed in Theorem 3.1 for the point v_1 and the polyhedron $\mathcal{K}'$, and let $F_1', \dots, F_m'$ be the facets of $\mathcal{K}'$. By Theorem 3.1(iii), we may assume that there exists an index $1 \leqslant k < m$ such that $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}') = \bigcup_{i=1}^{k} F_i'$, $\operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}') = \bigcup_{j=k+1}^{m} F_j'$ and

$$\mathcal{T}' = \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{A}') \cap \operatorname{Cl}_{\mathbb{R}^n}(\mathcal{B}') = \bigcup_{i=1}^{k} \bigcup_{j=k+1}^{m} F_i' \cap F_j'.$$

By Lemma 4.2 and Remark 4.3, there exists a rational function $h : \mathbb{R}^n \dashrightarrow \mathbb{R}$, which is regular on $\mathcal{K}' \setminus \mathcal{T}'$, such that $h|_{\mathcal{A}'} \equiv 0$, $h|_{\mathcal{B}'} \equiv 1$ and $0 < h(p) < 1$ for any point $p \in \operatorname{Int}(\mathcal{K}')$.

We claim now that the rational map

$$f_1 : \mathbb{R}^n \dashrightarrow \mathbb{R}^n,\ x = (x_1, \dots, x_n) \longmapsto (x_1 h(x), \dots, x_n h(x)),$$

which is regular on $\mathcal{K}' \setminus \mathcal{T}'$, maps $\operatorname{Int}(\mathcal{K}') \subset \mathcal{K}' \setminus \mathcal{T}'$ onto $\operatorname{Int}(\mathcal{K})$. To prove this, let us consider a ray $R \in \mathfrak{R}$ and recall that $\mathcal{K}' \cap R = [a_R, b_R]$, where $a_R \in \mathcal{A}'$ and $b_R \in \mathcal{B}'$ (see Paragraphs 3.1.3 and 3.2.3). We have $f_1(a_R) = v_1$, $f_1(b_R) = b_R$ and, since $h(\operatorname{Int}(\mathcal{K}')) \subset (0,1)$, it follows that $f_1((a_R, b_R)) = (0, b_R)$. Thus,

$$f_1(\operatorname{Int}(\mathcal{K}')) = f_1\left(\bigcup_{R\in\mathfrak{R}} (a_R, b_R)\right) = \bigcup_{R\in\mathfrak{R}} f_1((a_R, b_R)) = \bigcup_{R\in\mathfrak{R}} (0, b_R) = \operatorname{Int}(\mathcal{K}).$$

By induction hypothesis, $\operatorname{Int}(\mathcal{K}') = f_2(\mathbb{R}^n)$ for a regular map $f_2 : \mathbb{R}^n \to \mathbb{R}^n$, and so $f = f_1 \circ f_2 : \mathbb{R}^n \to \mathbb{R}^n$ is a regular map satisfying $f(\mathbb{R}^n) = \operatorname{Int}(\mathcal{K})$. We are done. □

As announced, Proposition 2.7 together with Proposition 4.4, allows us to prove the second part of Theorem 1.2 eliminating the boundedness hypothesis; namely, we have the following corollary.

Corollary 4.5. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron. Then* $\operatorname{Int}(\mathcal{K})$ *is a regular image of* $\mathbb{R}^n$.

Proof. Suppose first that $\mathcal{K}$ is nondegenerate. Then, by Proposition 2.7, there exist a nondegenerate, bounded, convex polyhedron $\mathcal{K}'$ and a rational map $h : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ that is regular on $\operatorname{Int}(\mathcal{K}')$ such that $h(\operatorname{Int}(\mathcal{K}')) = \operatorname{Int}(\mathcal{K})$. By Proposition 4.4, there exists a regular map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(\mathbb{R}^n) = \operatorname{Int}(\mathcal{K}')$ and so $f = h \circ g : \mathbb{R}^n \to \mathbb{R}^n$ is a regular map whose image is $\operatorname{Int}(\mathcal{K})$.

Next, assume that $\mathcal{K}$ is degenerate. Thus, by Lemma 2.3, either $\mathcal{K} = \mathbb{R}^n$ (and so $\mathcal{K}$ is trivially a regular image of $\mathbb{R}^n$) or, after a change of coordinates, there exist an index $1 \leqslant k \leqslant n-1$ and a nondegenerate convex polyhedron $\mathcal{P} \subset \mathbb{R}^{n-k}$ such that $\mathcal{K} = \mathbb{R}^k \times \mathcal{P}$. Observe that $n = \dim \mathcal{K} = \dim \mathbb{R}^k + \dim \mathcal{P}$, that is, $\dim \mathcal{P} = n - k$. Note also that $\operatorname{Int}(\mathcal{K}) = \mathbb{R}^k \times \operatorname{Int}(\mathcal{P})$. We apply now what we have just proved to the $(n-k)$-dimensional nondegenerate convex polyhedron $\mathcal{P} \subset \mathbb{R}^{n-k}$. Hence, there exists a regular map $h_1 : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ whose image is $\operatorname{Int}(\mathcal{P})$. Therefore, the regular map

$$f_1 : \mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k}, \ (y, z) \longrightarrow (y, h_1(z))$$

satisfies $f_1(\mathbb{R}^n) = \mathbb{R}^k \times \operatorname{Int}(\mathcal{P}) = \operatorname{Int}(\mathcal{K})$, and we are done. □

5. *Convex polyhedra as regular images of* $\mathbb{R}^n$

The purpose of this section is to prove the remaining part of Theorem 1.2, that is, *each n-dimensional convex polyhedron in $\mathbb{R}^n$ is a regular image of $\mathbb{R}^n$*. The key results to show this are the following lemma, together with Corollary 4.5.

LEMMA 5.1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional, bounded, convex polyhedron and let E be a face of $\mathcal{K}$. Let $Y \subset \partial\mathcal{K}$ be such that $E \cap Y = \varnothing$. Then there exist a rational map $f : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ and an algebraic subset $Z \subset \mathbb{R}^n$ such that $Z \cap \mathcal{K} = \partial E$, which is empty if* $\dim E = 0$, *and satisfy the following conditions*:

(i) *f is regular on $\mathbb{R}^n \setminus Z$*;
(ii) *$f(\operatorname{Int}(\mathcal{K}) \cup Y) = \operatorname{Int}(\mathcal{K}) \cup Y \cup \operatorname{Int} E$.*

Assume for a while we have already proved Lemma 5.1 and let us demonstrate the following proposition.

PROPOSITION 5.2. *Each n-dimensional, nondegenerate, convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a regular image of $\mathbb{R}^n$.*

Proof. First, suppose that $\mathcal{K}$ is unbounded. After a change of coordinates, we may assume that $\mathcal{K}$ is facing upwards (use Lemma 2.4). By Proposition 2.7, there exist an n-dimensional bounded convex polyhedron $\mathcal{K}' \subset \mathbb{R}^n$, a face E' of $\mathcal{K}'$ and a rational map $h : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ that is regular on $\mathcal{K}' \setminus E'$ and satisfies $h(\mathcal{K}' \setminus E') = \mathcal{K}$.

5.1.

Thus, to prove the statement, it is enough to prove the following condition: *If $\mathcal{K} \subset \mathbb{R}^n$ is an n-dimensional, bounded, convex polyhedron and E_0 is either the empty set or a face of $\mathcal{K}$, then there exists a regular map $f : \mathbb{R}^n \to \mathbb{R}^n$ whose image is $\mathcal{K} \setminus E_0$.*

Indeed, for each $0 \leqslant d \leqslant n-1$, let $\mathfrak{C}_d$ be the family of those faces of $\mathcal{K}$ of dimension at most d not contained in E_0 and $\mathfrak{C}_{-1} = \varnothing$. Recall that if E is a face of $\mathcal{K}$, then either $E \subset E_0$ or $E_0 \cap \operatorname{Int}(E) = \varnothing$.

Let us define, for $0 \leqslant d \leqslant n-1$, the semialgebraic set

$$\begin{aligned}\mathcal{K}_{(d)} &= (\mathcal{K} \setminus E_0) \Bigg\backslash \left(\bigcup_{E \in \mathfrak{E}_{d-1}} E \right) = \operatorname{Int}(\mathcal{K}) \cup \bigcup_{E \in \mathfrak{E}_{n-1} \setminus \mathfrak{E}_{d-1}} \operatorname{Int}(E) \\ &= \operatorname{Int}(\mathcal{K}) \cup \bigcup_{d \leqslant k \leqslant n-1} \bigcup_{E \in \mathfrak{E}_k \setminus \mathfrak{E}_{k-1}} \operatorname{Int}(E),\end{aligned}$$

and note that $\mathcal{K}_{(0)} = \mathcal{K} \setminus E_0$. Recall that, by Proposition 4.4, there exists a regular map $f_n : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_n(\mathbb{R}^n) = \operatorname{Int}(\mathcal{K}) = \mathcal{K}_{(n)}$. Let us check that, for each $d = 0, \ldots, n-1$, there exists a rational map $f_d : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ that is regular on $\mathcal{K}_{(d+1)}$ such that $f_d(\mathcal{K}_{(d+1)}) = \mathcal{K}_{(d)}$. Once this is proved, the image of the regular map $f = f_0 \circ \ldots \circ f_n : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{K} \setminus E_0$, and we will be done.

Thus, we fix $0 \leqslant d \leqslant n-1$ and observe that $\mathcal{K}_{(d)} \setminus \mathcal{K}_{(d+1)} = \bigcup_{E \in \mathfrak{E}_d \setminus \mathfrak{E}_{d-1}} \operatorname{Int}(E)$. We write $\mathfrak{E}_d \setminus \mathfrak{E}_{d-1} = \{E_1, \ldots, E_r\}$ and note that $\operatorname{Int}(E_j) \cap E_i = \varnothing$ if $i \neq j$. Moreover, $\mathcal{K}_{(d+1)} \cap E_i = Y_i \cap E_i = \varnothing$ for $i = 1, \ldots, r$, where

$$Y_i = (\mathcal{K}_{(d+1)} \setminus \operatorname{Int}(\mathcal{K})) \cup \bigcup_{j=1}^{i-1} \operatorname{Int}(E_j) \subset \partial \mathcal{K}.$$

Now, for each $i = 1, \ldots, r$, there exist, by Lemma 5.1, an algebraic set $Z_i \subset \mathbb{R}^n$, such that $Z_i \cap \mathcal{K} = \partial E_i \subset \mathcal{K} \setminus \mathcal{K}_{(d)}$, and a rational map $g_i : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ that is regular on $\mathbb{R}^n \setminus Z_i$ and satisfies $g_i(\operatorname{Int}(\mathcal{K}) \cup Y_i) = \operatorname{Int}(\mathcal{K}) \cup Y_i \cup \operatorname{Int}(E_i)$. Hence, the composition $f_d = g_r \circ \ldots \circ g_1 : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ is a rational map that is regular on $\mathcal{K}_{(d+1)}$ such that

$$f_d(\mathcal{K}_{(d+1)}) = \operatorname{Int}(\mathcal{K}) \cup Y_r \cup \operatorname{Int}(E_r) = \mathcal{K}_{(d+1)} \cup \bigcup_{i=1}^{r} \operatorname{Int}(E_i) = \mathcal{K}_{(d)},$$

as wanted. □

As a straightforward consequence of Proposition 5.2, we prove the remaining part of Theorem 1.2; namely, we have the following corollary.

COROLLARY 5.3. *Every n-dimensional, convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a regular image of $\mathbb{R}^n$.*

Proof. In view of Proposition 5.2, we may assume that $\mathcal{K}$ is degenerate. Thus, by Lemma 2.3, either $\mathcal{K} = \mathbb{R}^n$ (and so $\mathcal{K}$ is trivially a regular image of $\mathbb{R}^n$) or, after a change of coordinates, there exist an index $1 \leqslant k \leqslant n-1$ and a nondegenerate convex polyhedron $\mathcal{P} \subset \mathbb{R}^{n-k}$ such that $\mathcal{K} = \mathbb{R}^k \times \mathcal{P}$. By Proposition 5.2, there exists a regular map $g : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ whose image is $\mathcal{P}$. Hence, the image of the regular map

$$f : \mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k}, (y, z) \longrightarrow (y, g(z))$$

is $\mathbb{R}^k \times \mathcal{P} = \mathcal{K}$, and we are done. □

Therefore, 'it only remains' to prove Lemma 5.1 and, in order to prove it, we need to introduce some terminology and technical results. A *d-scaffold of a d-face E* of an n-dimensional, bounded, convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a semialgebraic topological manifold Γ semialgebraically homeomorphic to E such that $\operatorname{Int}(\Gamma) \subset \operatorname{Int}(\mathcal{K})$ and $\partial\Gamma = \partial E$ (see also [8, 4.7] for the 2-dimensional case).

LEMMA 5.4. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional, bounded, convex polyhedron, and let E be one of its d-faces. Define $\mathtt{y} = (\mathtt{x}_1, \ldots, \mathtt{x}_d)$, $\mathtt{z} = (\mathtt{x}_{d+1}, \ldots, \mathtt{x}_n)$, and suppose that:*

(1) *the polyhedron $\mathcal{K}$ is contained in the half-space $\{x_n \geqslant 0\}$, and the hyperplane $\{x_n = 0\}$ contains a facet of $\mathcal{K}$;*

(2) *$W = \{x_{d+1} = 0, \ldots, x_n = 0\}$ is the affine subspace of $\mathbb{R}^n$ generated by E.*

Let $q = (q_1, \ldots, q_n) \in \operatorname{Int}(\mathcal{K})$ and $\alpha_i = q_i/q_n$ for $i = d+1, \ldots, n$. Then there exist a rational function $f : \mathbb{R}^n \dashrightarrow \mathbb{R}$ and a polynomial $P \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_d] = \mathbb{R}[\mathtt{y}]$ positive on $\operatorname{Int}(E)$ and identically zero on ∂E, such that the following properties hold:

(i) *The semialgebraic set $\Gamma = \{(y, \alpha_{d+1}P(y), \ldots, \alpha_{n-1}P(y), P(y)) \in \mathbb{R}^n : \ (y, 0) \in E\}$ is a d-scaffold of the d-face E contained in the affine subspace generated by $E \cup \{q\}$.*

(ii) *The restriction to Γ of the projection*

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ x = (x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_d, 0, \ldots, 0) = (y, 0)$$

induces a semialgebraic homeomorphism between Γ and E.

(iii) *There exists an algebraic set $Z \subset \mathbb{R}^n$ such that $\mathcal{K} \cap Z = \partial E$ and the function f is regular on $\mathbb{R}^n \setminus Z$.*

(iv) *The function f satisfies the equalities $f|_{\partial\mathcal{K}\setminus\partial E} \equiv 1$ and $f|_{\operatorname{Int}(\Gamma)} \equiv 0$.*

(v) *For every point $p \in \operatorname{Int}(\mathcal{K}) \setminus \operatorname{Int}(\Gamma)$ we have $0 < f(p) < 1$.*

(vi) *If $\dim E = 0$, then $Z = \varnothing$.*

Proof. Observe first that since $q \in \operatorname{Int}(\mathcal{K}) \subset \{x_n > 0\}$, the quotients $\alpha_i = q_i/q_n$ are well defined for $i = d+1, \ldots, n$. Observe also that W can be written as

$$W = \{x_{d+1} - \alpha_{d+1}x_n = 0, \ldots, x_{n-1} - \alpha_{n-1}x_n = 0, \ x_n = 0\}.$$

Let $\mathfrak{H} = \{H_1, \ldots, H_m\}$ be the minimal presentation of $\mathcal{K}$ and let $\ell_i \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$, with $1 \leqslant i \leqslant m$, be polynomials of degree 1 such that $H_i^+ = \{\ell_i \geqslant 0\}$. Observe that

$$E = \mathcal{K} \cap W = \{(y, 0) \in \mathbb{R}^n : \ \ell_i(y, 0) \geqslant 0, \ i = 1, \ldots, m\}.$$

After reordering the indices $\{1, \ldots, m\}$ if necessary, we may assume the existence of an index $1 \leqslant r \leqslant m$ such that the polynomials $a_k(\mathtt{x}_1, \ldots, \mathtt{x}_d) = \ell_k(\mathtt{x}_1, \ldots, \mathtt{x}_d, 0, \ldots, 0)$ are not identically zero exactly for $k = 1, \ldots, r$. Moreover, since $\mathcal{K}$ is bounded, 0 does not belong to all the facets of $\mathcal{K}$ and so there exists at least one index $j = 1, \ldots, m$ such that $\ell_j(0) > 0$. Note that

$$E = \mathcal{K} \cap W = \{(y, 0) \in \mathbb{R}^n : \ a_k(y) \geqslant 0, \ k = 1, \ldots, r\}.$$

Define $\alpha = (\alpha_{d+1}, \ldots, \alpha_n)$ and, for each integer $M > 0$, consider the polynomial

$$P_M(\mathtt{x}_1, \ldots, \mathtt{x}_d) = \frac{\prod_{k=1}^r a_k(\mathtt{x}_1, \ldots, \mathtt{x}_d)}{M},$$

and the semialgebraic set

$$\begin{aligned}\Gamma_M = \{x = (y, z) \in \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} : \ a_k(y) \geqslant 0, \ z_i = \alpha_i P_M(y), \\ 1 \leqslant k \leqslant r, \ d+1 \leqslant i \leqslant n\} = \{(y, \alpha P_M(y)) \in \mathbb{R}^n : \ (y, 0) \in E\}.\end{aligned}$$

5.1.1. *We claim that $\Gamma = \Gamma_M$ is, for large enough M, the d-scaffold of E we are looking for.* Indeed, note that the restriction to Γ_M of the projection

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ x = (x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_d, 0, \ldots, 0) = (y, 0)$$

induces, for each $M>0$, a semialgebraic homeomorphism between Γ_M and E. Hence, $\partial\Gamma_M=\{(y,\alpha P_M(y))\in\mathbb{R}^n:\ (y,0)\in\partial E\}$, and since $\partial E=E\cap\bigcup_{k=1}^r\{a_k=0\}$, the restriction $P_M|_{\partial E}\equiv 0$. Therefore, since $E\subset\{x_{d+1}=\ldots=x_n=0\}$, it follows that $\partial\Gamma_M=\partial E$.

5.1.1.1. Let us check now that

$$\operatorname{Int}(\Gamma_M)\subset\operatorname{Int}(\mathcal{K})\quad\text{for } M \text{ large enough.}$$

Observe first that

$$\operatorname{Int}(\Gamma_M)=\Gamma_M\setminus\partial\Gamma_M=\Gamma_M\setminus\partial E=\{(y,\alpha P_M(y))\in\mathbb{R}^n:\ (y,0)\in\operatorname{Int}(E)\}\subset\{x_n>0\}.$$

The last inclusion is due to the fact that, for each point $(y,0)\in\operatorname{Int}(E)$, the product $a_1(y)\ldots a_r(y)$ is positive, and so the nth coordinate x_n of $x\in\operatorname{Int}(\Gamma_M)$ is positive too. For $i=1,\ldots,m$ define

$$A_i(\mathtt{x}_1,\ldots,\mathtt{x}_d,\mathtt{x}_n)=\ell_i(\mathtt{x}_1,\ldots,\mathtt{x}_d,\alpha_{d+1}\mathtt{x}_n,\ldots,\alpha_{n-1}\mathtt{x}_n,\mathtt{x}_n)\in\mathbb{R}[\mathtt{x}_1,\ldots,\mathtt{x}_d,\mathtt{x}_n],$$

and note that, for $k=1,\ldots,r$, there exists $b_{kn}\in\mathbb{R}$ such that $A_k(\mathtt{y},\mathtt{x}_n)=a_k(\mathtt{y})+b_{kn}\mathtt{x}_n$. On the other hand, $\ell_i(\mathtt{x}_1,\ldots,\mathtt{x}_d,0,\ldots,0)\equiv 0$ for $i=r+1,\ldots,m$, and so $A_i(\mathtt{y},\mathtt{x}_n)=b_{in}\mathtt{x}_n$ for some $b_{in}\in\mathbb{R}$. In fact, $b_{in}>0$ for $i=r+1,\ldots,m$. To check this, note that $q_n>0$ because $q\in\operatorname{Int}(\mathcal{K})$, and also

$$b_{in}q_n=A_i(q_1,\ldots,q_d,q_n)=\ell_i(q)>0.$$

Next, consider the affine subspace

$$V=\{x\in\mathbb{R}^n:\ x_i=\alpha_i x_n,\ d+1\leqslant i\leqslant n-1\},$$

generated by $E\cup\{q\}$. Since the hyperplane $\{x_n=0\}$ contains a facet of $\mathcal{K}\subset\{x_n\geqslant 0\}$, we deduce that

$$\begin{aligned}V\cap\mathcal{K}=\{x\in\mathbb{R}^n:\ x_n\geqslant 0,\ A_k(y,x_n)=a_k(y)+b_{kn}x_n\geqslant 0,\\ x_i=\alpha_i x_n,\ 1\leqslant k\leqslant r,\ d+1\leqslant i\leqslant n-1\}.\end{aligned}$$

Moreover, a straightforward computation shows that

$$\begin{aligned}V\cap\operatorname{Int}(\mathcal{K})=\{x\in\mathbb{R}^n:\ x_n>0,\ A_k(x)=a_k(x)+b_{kn}x_n>0,\\ x_i=\alpha_i x_n,\ 1\leqslant k\leqslant r,\ d+1\leqslant i\leqslant n-1\}.\end{aligned}$$

Observe that if $x=(y,z)\in\{x_n=P_M(y)\}$, then

$$A_k(y,x_n)=a_k(y)+b_{kn}x_n=a_k(y)+\frac{b_{kn}\prod_{i=1}^r a_i(y)}{M}=a_k(y)\left(1+\frac{b_{kn}\prod_{i\neq k}a_i(y)}{M}\right).$$

Since $\mathcal{K}$ is bounded, it is compact, and so there exists $M_0>0$ such that if $M\geqslant M_0$, then $1+b_{kn}\prod_{i\neq k}a_i(y)/M>0$ for each point $x=(y,z)\in\mathcal{K}$ and each $k=1,\ldots,r$. Fix $M\geqslant M_0$ and observe that if $x=(y,z)\in\mathcal{K}\cap\{x_n=P_M(y)\}$, then

$$A_k(y,x_n)\geqslant 0\iff a_k(y)\geqslant 0\quad\text{and}\quad A_k(y,x_n)>0\iff a_k(y)>0.$$

Consequently,

$$\begin{aligned} V\cap\mathcal{K}\cap\{x_n=P_M(y)\}&=\{x\in\mathbb{R}^n:\ P_M(y)\geqslant 0,\ A_k(y,x_n)=a_k(y)+b_{kn}P_M(y)\geqslant 0,\\ &\qquad x_i=\alpha_iP_M(y),\ 1\leqslant k\leqslant r,\ d+1\leqslant i\leqslant n\}\\ &=\{x\in\mathbb{R}^n:\ a_k(y)\geqslant 0,\ x_i=\alpha_iP_M(y),\ 1\leqslant k\leqslant r,\ d+1\leqslant i\leqslant n\}\\ &=\Gamma_M, \end{aligned}$$

which in particular implies that $\Gamma_M\subset V$. Moreover,

$$\begin{aligned} V\cap\operatorname{Int}(\mathcal{K})\cap\{x_n=P_M(y)\}&=\{x\in\mathbb{R}^n:\ P_M(y)>0,A_k(y,x_n)=a_k(y)+b_{kn}P_M(y)>0,\\ &\qquad x_i=\alpha_iP_M(x),\ 1\leqslant k\leqslant r,\ d+1\leqslant i\leqslant n\}\\ &=\{x\in\mathbb{R}^n: a_k(y)>0,x_i=\alpha_iP_M(x),1\leqslant k\leqslant r,d+1\leqslant i\leqslant n\}\\ &=\operatorname{Int}(\Gamma_M), \end{aligned}$$

and so $\operatorname{Int}(\Gamma_M)\subset\operatorname{Int}(\mathcal{K})$. Recall also that $\partial\Gamma_M=\partial E\subset\partial\mathcal{K}$ and so $\Gamma_M\cap\partial\mathcal{K}=\partial E$.

5.1.1.2. Finally, consider the rational function $f:\mathbb{R}^n\dashrightarrow\mathbb{R}$ given by the formula

$$f(\mathtt{x})=\frac{\sum_{j=d+1}^n(\mathtt{x}_j-\alpha_jP_M(\mathtt{y}))^2}{\prod_{i=1}^m\ell_i^2(\mathtt{x})+\sum_{j=d+1}^n(\mathtt{x}_j-\alpha_jP_M(\mathtt{y}))^2}.$$

The function f is regular outside the zero set Z of the polynomial in the denominator $\prod_{i=1}^m\ell_i^2(\mathtt{x})+\sum_{j=d+1}^n(\mathtt{x}_j-\alpha_jP_M(\mathtt{y}))^2$. Recall that $\partial\mathcal{K}=\mathcal{K}\cap\{x\in\mathbb{R}^n:\ \prod_{i=1}^m\ell_i(x)=0\}$. Thus, since $\operatorname{Int}(\Gamma_M)\subset\operatorname{Int}(\mathcal{K})$ and $\partial\Gamma_M=\partial E\subset\partial\mathcal{K}$, we have

$$\begin{aligned} \mathcal{K}\cap Z&=\mathcal{K}\cap\left\{x\in\mathbb{R}^n:\ \prod_{i=1}^m\ell_i(x)=0\right\}\cap\{x\in\mathbb{R}^n:\ x_j=\alpha_jP_M(y),\ j=d+1,\dots,n\}\\ &=\partial\mathcal{K}\cap V\cap\{x_n=P_M(y)\}=\partial\mathcal{K}\cap\Gamma_M=\partial E. \end{aligned}$$

Now, a straightforward computation shows that this function f and the algebraic set Z satisfy the conditions in the statement.

Notice finally that if $\dim E=0$, then $E=\{0\}$ and $\Gamma_M=\operatorname{Int}(\Gamma_M)=\{p\}$ is a singleton contained in $\operatorname{Int}(\mathcal{K})$. In fact, $Z=\{p\}\cap\bigcap_{i=1}^m H_i\subset\operatorname{Int}(\mathcal{K})\cap\bigcap_{i=1}^m H_i=\varnothing$, as wanted. □

LEMMA 5.5. *Let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional, nondegenerate, convex polyhedron, and let v be a vertex of $\mathcal{K}$. Then, after a change of coordinates, we may assume that*:

(i) *the vertex v is the origin*;
(ii) *the intersections $F_i=\mathcal{K}\cap\{x_i=0\}$ are facets of $\mathcal{K}$ for $i=1,\dots,n$*;
(iii) *for each $k=1,\dots,n-1$, the intersection $E_k=\mathcal{K}\cap\{x_{k+1}=0,\dots,x_n=0\}$ is a face of $\mathcal{K}$, and the intersections $\mathcal{K}\cap\{x_j=0,x_{k+1}=0,\dots,x_n=0\}$ are facets of E_k for $j=1,\dots,k$*;
(iv) *The polyhedron $\mathcal{K}$ satisfies $\mathcal{K}\subset\bigcap_{i=1}^n\{x_i\geqslant 0\}$.*

Proof. We proceed by induction on n. If $n=1$, then we may assume, after a change of coordinates, that $v=0$ and $\mathcal{K}$ is either $[0,1]$ or $[0,+\infty)$, and the statement follows. Assume the result to be true for $n-1$ and let us check that it holds for n. Let F be a facet of $\mathcal{K}$ that contains v. After a change of coordinates in $\mathbb{R}^n$, we may assume that the hyperplane of $\mathbb{R}^n$ generated by F is $H=\{x_n=0\}$. Note that $\mathcal{P}=\mathcal{K}\cap\{x_n=0\}$ is an $(n-1)$-dimensional, nondegenerate, convex polyhedron contained in $H\equiv\mathbb{R}^{n-1}\times\{0\}$ and having v as one of its vertices. By induction hypothesis, there exists a change of coordinates in $\mathbb{R}^{n-1}\times\{0\}$ such that

(1) the vertex v is the origin;
(2) the intersections $G_i=\mathcal{P}\cap\{x_i=0\}$ are facets of $\mathcal{P}$ for $i=1,\dots,n-1$;

(3) for each $k=1,\dots,n-2$, the intersection $E'_k=\mathcal{P}\cap\{x_{k+1}=0,\dots,x_{n-1}=0\}$ is a face of $\mathcal{P}$, and the intersections $\{x_j=0,x_{k+1}=0,\dots,x_{n-1}=0\}\cap\mathcal{P}$ are facets of E'_k for $j=1,\dots,k$;

(4) The polyhedron $\mathcal{P}$ satisfies $\mathcal{P}\subset\bigcap_{i=1}^{n-1}\{x_i\geqslant 0\}$.

By Berger [**2**, 12.1.5], the facets of $\mathcal{P}$ are intersections with the hyperplane H of those facets of $\mathcal{K}$ that intersect H. Thus, there exist hyperplanes H_i of $\mathbb{R}^n$ generated by facets F_i of $\mathcal{K}$ such that each facet G_i of $\mathcal{P}$ has the form $\mathcal{P}\cap H_i$, for $i=1,\dots,n-1$. Hence,

$$\mathcal{K}\cap\{x_n=0\}\cap H_i=\mathcal{P}\cap H_i=G_i=\mathcal{P}\cap\{x_i=0\}=\mathcal{K}\cap\{x_i=0,\ x_n=0\}.$$

Recall that $\mathcal{K}\cap\{x_i=0,x_n=0\}$ is a facet of $\mathcal{P}$ and so its dimension equals $n-2$. Hence, $H_i\cap\{x_n=0\}=\{x_i=0,x_n=0\}$ and, consequently, there exist real numbers $a_i\in\mathbb{R}$ such that $H_i=\{x_i-a_ix_n=0\}$ for $i=1,\dots,n-1$.

Therefore, $\bigcap_{i=1}^{n-1}H_i=\{t(a_1,\dots,a_{n-1},1):\ t\in\mathbb{R}\}$. After a change of coordinates that fixes the hyperplane $\{x_n=0\}$ and transforms the vector $(a_1,\dots,a_{n-1},1)$ into the vector $(0,\dots,0,1)$, we may assume that $H_i=\{x_i=0\}$ for $i=1,\dots,n-1$. Moreover, after changing the sign of the variable x_n if necessary, we may assume also that $\mathcal{K}\subset\{x_n\geqslant 0\}$. Observe that in this way the four conditions of the statement are satisfied in a straightforward manner, as wanted. □

LEMMA 5.6. *Let $\varepsilon>0$ and let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional, nondegenerate, convex polyhedron such that:*

(1) *the origin is a vertex v of $\mathcal{K}$;*

(2) *the intersections $F_i=\mathcal{K}\cap\{x_i=0\}$ are facets of $\mathcal{K}$ for $i=1,\dots,n$;*

(3) *for each $k=1,\dots,n-1$, the intersection $E_k=\mathcal{K}\cap\{x_{k+1}=0,\dots,x_n=0\}$ is a face of $\mathcal{K}$ and the intersections $\{x_j=0,x_{k+1}=0,\dots,x_n=0\}\cap\mathcal{K}$ are facets of E_k for $j=1,\dots,k$;*

(4) *The polyhedron $\mathcal{K}$ satisfies $\mathcal{K}\subset\{x_1\geqslant 0,\dots,x_n\geqslant 0\}$.*

For each $i=1,\dots,n$, let $\mathcal{K}_i$ be the polyhedron obtained from $\mathcal{K}$ by eliminating (See Paragraph 2.1.1 for a precise definition of the polyhedron $\mathcal{K}_i$ obtained from $\mathcal{K}$ by eliminating the facet F_i.) the facet F_i. Then, for each $i=1,\dots,n$, there exists a point

$$p_i\in(\mathcal{K}_i\cap\{x_{i+1}=0,\dots,x_n=0\}\cap B_n(0,\varepsilon))\setminus\mathcal{K}$$

such that the affine subspace of $\mathbb{R}^n$ generated by $\{p_1,\dots,p_n\}$ has dimension $n-1$ and does not intersect the n-dimensional closed orthant $\{x_1\geqslant 0,\dots,x_n\geqslant 0\}$.

Proof. We proceed by induction on n. If $n=1$, then either $\mathcal{K}=[0,a]$ where $a>0$, or $\mathcal{K}=[0,+\infty)$. Then the point $p_1=-\varepsilon/2$ satisfies our requirements. Assume that the result is true for $n-1$ and let us check that it is also true for n. Consider the polyhedron $\mathcal{P}=\mathcal{K}\cap\{x_n=0\}$, which satisfies analogous conditions to (1)–(4) in the $(n-1)$-dimensional setting, and define $F'_i=\mathcal{P}\cap\{x_i=0\}=\mathcal{K}\cap\{x_i=0,\ x_n=0\}$ for $i=1,\dots,n-1$. Let $\mathcal{P}_i$ be the polyhedron obtained from $\mathcal{P}$ by eliminating the facet F'_i. By the induction hypothesis, for each $i=1,\dots,n-1$ there exists a point

$$p_i\in(\mathcal{P}_i\cap\{x_{i+1}=0,\dots,x_{n-1}=0\}\cap(B_n(0,\varepsilon)\cap\{x_n=0\}))\setminus\mathcal{P},$$

such that the affine subspace L_{n-1} of $\{x_n=0\}$ generated by $\{p_1,\dots,p_{n-1}\}$ has dimension $n-2$, and L_{n-1} does not intersect the semialgebraic set $\{x_1\geqslant 0,\dots,x_{n-1}\geqslant 0,\ x_n=0\}$. Observe that

$$\mathcal{P}_i\cap\{x_{i+1}=0,\dots,x_{n-1}=0\}=\mathcal{K}_i\cap\{x_{i+1}=0,\dots,x_{n-1}=0,\ x_n=0\},$$

and, consequently,

$$p_1,\dots,p_{n-1}\in(\mathcal{K}_i\cap\{x_{i+1}=0,\dots,x_{n-1}=0,x_n=0\}\cap B_n(0,\varepsilon))\setminus\mathcal{K}.$$

Since $\{x_n = 0\} \cap \mathcal{K}$ is a facet of $\mathcal{K}$ that contains the vertex $v = 0$ and $\mathcal{K} \subset \{x_n \geqslant 0\}$, there exists a point $p_n \in \mathcal{K}_n \cap B_n(0, \varepsilon) \setminus \mathcal{K} = \mathcal{K}_n \cap B_n(0, \varepsilon) \cap \{x_n < 0\}$. The coordinates of the point $p_n = (p_{1n}, \dots, p_{nn})$ satisfy $p_{1n} \geqslant 0, \dots, p_{n-1,n} \geqslant 0$ and $p_{nn} < 0$, because $\mathcal{K}_n \cap \{x_n < 0\} \subset \bigcap_{i=1}^{n-1} \{x_i \geqslant 0\} \cap \{x_n < 0\}$.

Observe that since the points $\{p_1, \dots, p_{n-1}\} \subset \{x_n = 0\}$ are affinely independent, the affine subspace L_n of $\mathbb{R}^n$ generated by $\{p_1, \dots, p_n\}$ has dimension $n-1$. To conclude the proof, it only remains to check that $L_n \cap \{x_1 \geqslant 0, \dots, x_n \geqslant 0\} = \varnothing$.

Indeed, note that $L_n = \{(1-\lambda)q + \lambda p_n : \ q \in L_{n-1}, \ \ \lambda \in \mathbb{R}\}$ and suppose, by way of contradiction, that there exists a point $z \in L_n \cap \{x_1 \geqslant 0, \dots, x_n \geqslant 0\}$. In particular, $z = (z_1, \dots, z_n) = (1-\lambda)q + \lambda p_n$ for some point $q = (q_1, \dots, q_{n-1}, 0) \in L_{n-1}$ and $\lambda \in \mathbb{R}$. Since $0 \leqslant z_n = \lambda p_{nn}$ and $p_{nn} < 0$, it follows that $\lambda \leqslant 0$. On the other hand, since $q \in L_{n-1}$ and L_{n-1} does not intersect the set $\{x_1 \geqslant 0, \dots, x_{n-1} \geqslant 0, \ x_n = 0\}$, there exists an index $i = 1, \dots, n-1$ such that $q_i < 0$. Observe that $z_i = (1-\lambda)q_i + \lambda p_{in} < 0$, because $\lambda \leqslant 0$, $p_{in} \geqslant 0$ and $q_i < 0$. Thus, $z \notin \{x_1 \geqslant 0, \dots, x_n \geqslant 0\}$, which is a contradiction. Hence, $L_n \cap \{x_1 \geqslant 0, \dots, x_n \geqslant 0\} = \varnothing$, and we are done. □

From now on we denote by $\overrightarrow{W}$ the *direction* of the affine subspace $W \subset \mathbb{R}^n$, that is, the vector subspace of $\mathbb{R}^n$ parallel to W.

LEMMA 5.7. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and let E be a d-dimensional face of $\mathcal{K}$ for some $0 \leqslant d \leqslant n-1$. Denote by W the affine subspace of $\mathbb{R}^n$ generated by E. Then there exist $n-d$ affinely independent points $p_1, \dots, p_{n-d} \in \mathbb{R}^n$ such that*:

(i) *the affine subspace L of $\mathbb{R}^n$ generated by $\{p_1, \dots, p_{n-d}\}$ satisfies $\overrightarrow{W} \cap \overrightarrow{L} = \{0\}$, $W \cap L = \varnothing$ and $(L + \overrightarrow{W}) \cap \mathcal{K} = \varnothing$;*
(ii) *for each point $p \in \mathcal{K}$, the $(n-d)$-simplex $[p, p_1, \dots, p_{n-d}]$ intersects E exactly at one point*;
(iii) *$[p, p_1, \dots, p_{n-d}] \cap E \subset \operatorname{Int}(E)$ if and only if $p \in \mathcal{K} \setminus \partial E$.*

Proof. First note that, applying [**2**, 12.1.5] recursively, there exist facets $F_1', \dots, F_s'$ of $\mathcal{K}$ such that $\partial E = E \cap \bigcup_{i=1}^s F_i'$.

5.1.2. For each $i = 1, \dots, s$, we denote by H_i' the hyperplane of $\mathbb{R}^n$ generated by F_i', and by $H_i'^{+}$ the closed half-space of $\mathbb{R}^n$ containing $\mathcal{K}$ determined by H_i'. Let $q_0 \in \operatorname{Int}(E)$ and let $\varepsilon = \min\{\operatorname{dist}(q_0, H_i') : \ i = 1, \dots, s\}$. Observe that $\varepsilon > 0$, because $q_0 \notin \bigcup_{i=1}^s H_i'$. Since $\dim \mathcal{K} = n$, there exist $q_1, \dots, q_{n-d} \in \operatorname{Int}(\mathcal{K})$ such that the affine subspace V of $\mathbb{R}^n$ generated by $q_0, q_1, \dots, q_{n-d}$ has dimension $r = n-d$ and $E \cap V = \{q_0\}$. After a change of coordinates, we may assume that $V = \{x_{r+1} = 0, \dots, x_n = 0\}$. Define $\mathcal{P} = \mathcal{K} \cap V$ and note that $\operatorname{Int}([q_0, q_1, \dots, q_{n-d}]) \subset \operatorname{Int}(\mathcal{P})$ is an open subset of V. Thus, V is the affine subspace of $\mathbb{R}^n$ generated by $\mathcal{P}$ and so $\operatorname{Int}(\mathcal{P}) = \operatorname{Int}_V(\mathcal{P})$. Moreover, $E \cap V = \{q_0\}$ is a face of $\mathcal{P}$, that is, q_0 is a vertex of $\mathcal{P}$ and

$$\operatorname{Int}(\mathcal{P}) = \mathcal{P} \setminus \partial\mathcal{P} = \mathcal{P} \setminus \partial\mathcal{K} \subset \operatorname{Int}(\mathcal{K}).$$

To check these last facts, it suffices to observe that $\mathcal{P} = \mathcal{K} \cap V$, and to apply [**2**, 12.1.5–7] recursively.

5.1.3. To simplify notation, we identify in what follows $V = \mathbb{R}^r \times \{0\} \equiv \mathbb{R}^r$. By Lemma 5.5, after a change of coordinates in $\mathbb{R}^r \times \{0\}$ we may assume that:

(1) The point q_0 is the origin;
(2) the intersections $G_i = \mathcal{P} \cap \{x_i = 0\}$ are facets of $\mathcal{P}$ for $i = 1, \dots, r$;

(3) for each $k = 1, \dots, r-1$, the intersection $E_k = \mathcal{P} \cap \{x_{k+1} = 0, \dots, x_r = 0\}$ is a face of $\mathcal{P}$ and the intersections $R_j = \{x_j = 0\} \cap E_k \cap \mathcal{P}$ are facets of E_k for $j = 1, \dots, k$;
(4) The polyhedron $\mathcal{P}$ satisfies $\mathcal{P} \subset \bigcap_{i=1}^{r} \{x_i \geqslant 0\}$.

Observe that, by Berger [**2**, 12.1.5], the facets of $\mathcal{P}$ are intersections of the facets of $\mathcal{K}$ with V. Moreover, all facets of $\mathcal{K}$ containing the point $q_0 \in \operatorname{Int}(E)$ contain also E, because E is a face of $\mathcal{K}$. Thus, the facets G_i of $\mathcal{P}$ chosen above are intersections of $\mathcal{P}$ with facets F_i of $\mathcal{K}$ containing E. Moreover, the hyperplane of V generated by G_i is $\{x_i = 0\} \cap V$.

We may assume, after a change of coordinates fixing V, that the d-dimensional affine subspace $W = \{x_1 = 0, \dots, x_r = 0\}$. Therefore, since for $i = 1, \dots, r$ the hyperplane H_i of $\mathbb{R}^n$ generated by F_i contains the union $W \cup (\{x_i = 0\} \cap V)$, it also contains the sum $W + (\{x_i = 0\} \cap V)$, which implies $H_i = \{x_i = 0\}$ for $i = 1, \dots, r$. Moreover, $\mathcal{K} \subset \{x_1 \geqslant 0, \dots, x_r \geqslant 0\}$ because $\mathcal{K}$ is contained either in $\{x_i \geqslant 0\}$ or $\{x_i \leqslant 0\}$ for $1 \leqslant i \leqslant r$ and $\varnothing \neq \mathcal{P} \subset \mathcal{K} \cap \{x_1 \geqslant 0, \dots, x_r \geqslant 0\}$.

5.1.4. Recall that $\mathcal{P}_i$ and $\mathcal{K}_i$ denote the polyhedron obtained from $\mathcal{P}$ and $\mathcal{K}$ by eliminating the facets G_i and F_i, respectively. By Lemma 5.6, there exists, for each $i = 1, \dots, r$, a point (See Paragraph 5.1.2 for the definition of $\varepsilon > 0$ and H'_i for $i = 1, \dots, s$.)

$$p_i \in (\mathcal{P}_i \cap \{x_{i+1} = 0, \dots, x_r = 0\} \cap (B_n(0, \varepsilon) \cap V)) \setminus \mathcal{P} \subset V$$

such that the affine subspace $L \subset V$ generated by $p_1, \dots, p_r$ has dimension $r - 1 = n - d - 1$ and it does not intersect the semialgebraic set $\{x_1 \geqslant 0, \dots, x_r \geqslant 0\} \cap V$. Observe that $\mathcal{P}_i = \mathcal{K}_i \cap V$ and so $p_i \in (\mathcal{K}_i \cap V \cap \{x_{i+1} = 0, \dots, x_r = 0\}) \setminus (\mathcal{K} \cap V)$ for $i = 1, \dots, r$. Besides, since $\mathcal{P} = \mathcal{K} \cap V \subset \{x_1 \geqslant 0, \dots, x_r \geqslant 0\} \cap V$ and $L \subset V$ does not intersect $\mathcal{P}$, we deduce that $L \cap \mathcal{K} = \varnothing$. Moreover, since $\operatorname{dist}(p_i, q_0) < \varepsilon$ for $i = 1, \dots, r$, the point $q_0 \in \bigcap_{i=1}^{s} (H_i'^{+} \setminus H'_i)$ and

$$\varepsilon = \min\{\operatorname{dist}(q_0, H'_i) : \ i = 1, \dots, s\} = \min\{\operatorname{dist}(q_0, (\mathbb{R}^n \setminus H_i'^{+}) \cup H'_i) : \ i = 1, \dots, s\},$$

it follows that $\{p_1, \dots, p_r\} \subset \bigcap_{i=1}^{s} (H_i'^{+} \setminus H'_i)$.

Moreover, note that since

$$V = \{x_{r+1} = 0, \dots, x_n = 0\} \quad \text{and} \quad p_i \in (\mathcal{P}_i \cap \{x_{i+1} = 0, \dots, x_r = 0\}) \setminus \mathcal{P} \subset V,$$

we have $p_i = (p_{1i}, \dots, p_{i-1,i}, -p_{ii}, 0, \dots, 0)$, where $p_{ji} \geqslant 0$ for $j = 1, \dots, i-1$ and $p_{ii} > 0$. Observe that L does not intersect $W = \{x_1 = 0, \dots, x_r = 0\}$ and $\overrightarrow{L} \cap \overrightarrow{W} = \{0\}$ because $\overrightarrow{L} \subset \overrightarrow{V}$ and $\overrightarrow{V} \cap \overrightarrow{W} = \{0\}$. In addition, $L + \overrightarrow{W}$ does not intersect $\mathcal{K}$ because

$$\mathcal{K} \subset \{x_1 \geqslant 0, \dots, x_r \geqslant 0\},\ W = \{x_1 = 0, \dots, x_r = 0\} \text{ and } L \cap \{x_1 \geqslant 0, \dots, x_r \geqslant 0\} = \varnothing.$$

Thus, the affine subspace L satisfies condition (i) in the statement.

5.1.5. Let us check that, for each point $p = (y_1, \dots, y_n) \in \mathcal{K}$, the simplex $[p, p_1, \dots, p_r]$ intersects E in just one point. Indeed, consider the equation

$$\begin{aligned} -(y_1, \dots, y_n) = -p &= \sum_{i=1}^{r} \lambda_i p_i - q \\ &= \sum_{i=1}^{r} \lambda_i (p_{1i}, \dots, p_{i-1,i}, -p_{ii}, 0, \dots, 0) + (0, \overset{(r)}{\dots}, 0, \beta_{r+1}, \dots, \beta_n), \end{aligned} \tag{$\diamond$}$$

where $q = (0, \overset{(r)}{\dots}, 0, -\beta_{r+1}, \dots, -\beta_n)$ is a generic point of W. The previous equation is equivalent to a triangular system of linear equations, which has a unique solution, that we denote as $(\lambda_1, \dots, \lambda_r, \beta_{r+1}, \dots, \beta_n)$, because its matrix of coefficients has maximal rank n. Note also that since $p_{ii} > 0$ and each $p_{ji} \geqslant 0$ for $j = 1, \dots, i-1$, it follows that $\lambda_1 \geqslant 0, \dots, \lambda_r \geqslant 0$. Hence, $\mu = 1 + \sum_{i=1}^{r} \lambda_i > 0$. If we write $\mu_0 = 1/\mu$ and $\mu_i = \lambda_i/\mu$ for $i = 1, \dots, r$, we obtain $q' = q/\mu = \mu_0 p + \sum_{i=1}^{r} \mu_i p_i$, where $\sum_{i=0}^{r} \mu_i = 1$, and each $\mu_i \geqslant 0$. Thus, $q' \in [p, p_1, \dots, p_r] \cap W$.

Let $\{H_1,\dots,H_m\}$ be the minimal presentation of $\mathcal{K}$. Since $p\in\mathcal{K}$ and $p_i\in\bigcap_{j=r+1}^m H_j^+$ for $i=1,\dots,r$, we deduce that $q'\in[p,p_1,\dots,p_r]\subset\bigcap_{j=r+1}^m H_j^+$. Hence,

$$q'\in W\cap\bigcap_{j=r+1}^m H_j^+\subset\bigcap_{i=1}^r\{x_i\geqslant 0\}\cap\bigcap_{j=r+1}^m H_j^+=\mathcal{K},$$

and so $q'\in W\cap\mathcal{K}=E$. Thus, $[p,p_1,\dots,p_r]\cap E\neq\varnothing$; and in fact, this intersection is a unique point because the system $(\diamond)$ has a unique solution, as we have already observed. All this proves part (ii) in the statement.

5.1.6. To complete our discussion, we will prove that $[p,p_1,\dots,p_r]\cap E\subset\operatorname{Int}(E)$ if and only if $p\in\mathcal{K}\setminus\partial E$. It is clear that if $p\in\partial E$, then $p\in[p,p_1,\dots,p_r]\cap E\setminus\operatorname{Int}(E)$. Suppose now that $p\in\mathcal{K}\setminus\partial E$ and let us check that $\{q'\}=[p,p_1,\dots,p_r]\cap E\subset\operatorname{Int}(E)$.

We distinguish two cases. Assume first that $p=(y_1,\dots,y_n)\in\mathcal{K}\setminus E$. Then there exists an index $j\in\{1,\dots,r\}$ such that $y_j>0$. Thus, the solution $(\lambda_1,\dots,\lambda_r,\beta_{r+1},\dots,\beta_n)$ of equation $(\diamond)$ satisfies $\lambda_1,\dots,\lambda_r\geqslant 0$ and $\lambda_j>0$; hence, $\mu_1,\dots,\mu_r\geqslant 0$ and $\mu_j>0$. Recall also that $\mu_0=1/\mu>0$ (see Paragraph 5.1.5) and $p_i\in\bigcap_{k=1}^s(H_k'^+\setminus H_k')$ for $i=1,\dots,r$ (see Paragraph 5.1.4). Using this information, we next prove that

$$q'=\frac{q}{\mu}=\mu_0p+\sum_{i=1}^r\mu_ip_i\in\bigcap_{k=1}^s(H_k'^+\setminus H_k').$$

Indeed, let ℓ_k be a polynomial of degree 1 such that $H_k^+=\{\ell_k\geqslant 0\}$; hence, $H_k^+\setminus H_k=\{\ell_k>0\}$. Since $\sum_{j=0}^r\mu_j=1$ and $p\in\mathcal{K}\subset\bigcap_{k=1}^s(H_k'^+\setminus H_k')$, one deduces that

$$\ell_k(q')=\ell_k\left(\mu_0p+\sum_{i=1}^r\mu_ip_i\right)=\mu_0\ell_k(p)+\sum_{i=1}^r\mu_i\ell_k(p_i)\geqslant\mu_j\ell_k(p_j)>0,$$

for $k=1,\dots,s$. Thus, $q'\in E\setminus\bigcup_{k=1}^s H_k=\operatorname{Int}(E)$.

Next, if $p\in\operatorname{Int}(E)$, then the uniqueness of the solution of $(\diamond)$ implies that the intersection $[p,p_1,\dots,p_{n-d}]\cap E=\{p\}\subset\operatorname{Int}(E)$. This proves part (iii), and we are done. $\square$

Corollary 5.8. *Let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional, convex polyhedron and let E be a d-dimensional face of $\mathcal{K}$ for some $0\leqslant d\leqslant n-1$. Let W be the affine subspace of $\mathbb{R}^n$ generated by E. Then there exists an affine subspace $L\subset\mathbb{R}^n$ of dimension $n-d-1$ such that $L\cap W=\varnothing$, $\overrightarrow{L}\cap\overrightarrow{W}=\{0\}$ and $\mathcal{K}\cap(L+\overrightarrow{W})=\varnothing$, and the projection $\pi:\mathbb{R}^n\setminus(L+\overrightarrow{W})\to W$ of centre L and basis W satisfies the following conditions*:

(i) *$\pi|_E=\mathrm{id}_E$ and $\pi(\mathcal{K}\setminus\partial E)=\operatorname{Int}(E)$;*
(ii) *for each $p\in\mathcal{K}$, there exist $q\in L$ and $\lambda\in[0,1]$ such that $\pi(p)=\lambda p+(1-\lambda)q$.*

Proof. First, by Lemma 5.7, there exist $n-d$ affinely independent points $p_1,\dots,p_{n-d}\in\mathbb{R}^n$ such that:

(1) the affine subspace L generated by $p_1,\dots,p_{n-d}$ satisfies $\overrightarrow{L}\cap\overrightarrow{W}=\{0\}$, $L\cap W=\varnothing$ and $\mathcal{K}\cap(L+\overrightarrow{W})=\varnothing$;
(2) for each point $p\in\mathcal{K}$, the $(n-d)$-simplex $[p,p_1,\dots,p_{n-d}]$ intersects E in exactly one point;
(3) $[p,p_1,\dots,p_{n-d}]\cap E\subset\operatorname{Int}(E)$ if and only if $p\in\mathcal{K}\setminus\partial E$.

By its very definition, $\pi(x)=(\{x\}+L)\cap W$, where $\{x\}+L$ denotes the affine subspace of $\mathbb{R}^n$ generated by x and L. Note that $\{\pi(p)\}=[p,p_1,\dots,p_{n-d}]\cap E$ for each point $p\in\mathcal{K}$, and so there exist $\lambda_0,\lambda_1,\dots,\lambda_{n-d}\geqslant 0$ such that $\sum_{i=0}^{n-d}\lambda_i=1$ and $\pi(p)=\lambda_0p+\sum_{i=1}^{n-d}\lambda_ip_i$.

Now we distinguish two possibilities: On the one hand, if $\lambda_0 = 1$, then $\pi(p) = p$. On the other hand, if $\lambda_0 \neq 1$, then set $0 < \mu = \sum_{i=1}^{n-d} \lambda_i = 1 - \lambda_0 \leqslant 1$, $\mu_i = \lambda_i/\mu$ and $q = \sum_{i=1}^{n-d} \mu_i p_i \in L$. In any case, we have $\pi(p) = \lambda_0 p + (1-\lambda_0)q$ for some $q \in L$, where $0 \leqslant \lambda_0 \leqslant 1$.

A straightforward computation shows that the central projection $\pi : \mathbb{R}^n \setminus (L + \overrightarrow{W}) \to W$ satisfies $\pi|_E = \mathrm{id}_E$ and $\pi(\mathcal{K} \setminus \partial E) = \mathrm{Int}(E)$, as wanted. $\square$

Finally, we are ready to prove Lemma 5.1.

5.2. *Proof of Lemma* 5.1

Let us define $d = \dim E$.

5.2.1. We study first the case $d = 0$, that is, $E = \{v\} = \mathrm{Int}(E)$ is a vertex of $\mathcal{K}$. Note that $\partial E = \varnothing$, and we choose Z as the empty set. Consider the constant map $\pi : \mathbb{R}^n \to \mathbb{R}^n,\ p \mapsto v$. By Lemma 5.4, there exist a regular function $g : \mathbb{R}^n \to \mathbb{R}$ and a point $q \in \mathrm{Int}(\mathcal{K})$ such that $g|_{\partial\mathcal{K}} \equiv 1$, $g(q) = 0$ and $0 < g(p) < 1$ for all $p \in \mathrm{Int}(\mathcal{K}) \setminus \{q\}$. The regular map

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ p \longmapsto g(p)p + (1 - g(p))v$$

satisfies the conditions of the lemma. Indeed, observe that $f|_{\partial\mathcal{K}} \equiv \mathrm{id}_{\partial\mathcal{K}}$ and $f(q) = v$. Moreover, the equality $f(\mathrm{Int}(\mathcal{K}) \setminus \{q\}) = \mathrm{Int}(\mathcal{K})$ holds. The inclusion $f(\mathrm{Int}(\mathcal{K}) \setminus \{q\}) \subset \mathrm{Int}(\mathcal{K})$ follows at once from [**1**, 11.2.4]. Conversely, let $a \in \mathrm{Int}(\mathcal{K})$ and consider the line L passing through the points v and a. Let $b \in \partial\mathcal{K}$ be the point such that $\mathcal{K} \cap L$ is the segment $[v, b]$ joining the points v and b. Suppose first that this segment does not contain q. By Berger [**1**, 11.2.4], the open interval $(v, b) \subset \mathrm{Int}(\mathcal{K})$. Moreover, since $f(v) = v$ and $f(b) = b$, and the segment $[v, b]$ is convex, the image of the restriction $f|_{[v,b]} : [v, b] \to \mathcal{K}$ is $[v, b]$. Thus, there exists a point $a' \in (v, b) \subset \mathrm{Int}(\mathcal{K})$ such that $f(a') = a$, and so $\mathrm{Int}(\mathcal{K}) \subset f(\mathrm{Int}(\mathcal{K}) \setminus \{q\})$. On the other hand, if $q \in [v, b]$, then we use a similar argument substituting the segment $[v, b]$ by $[q, b]$. Therefore, for each subset $Y \subset \partial\mathcal{K}$,

$$f(\mathrm{Int}(\mathcal{K}) \cup Y) = \mathrm{Int}(\mathcal{K}) \cup Y \cup \{v\} = \mathrm{Int}(\mathcal{K}) \cup Y \cup \mathrm{Int}(E),$$

which solves this case.

5.2.2. Hence, in what follows we assume that $1 \leqslant d \leqslant n - 1$. We denote by W the affine subspace of $\mathbb{R}^n$ generated by E. By Corollary 5.8, there exists an $(n - d - 1)$-dimensional affine subspace L of $\mathbb{R}^n$ such that $L \cap W = \varnothing$, $\overrightarrow{L} \cap \overrightarrow{W} = \{0\}$, $\mathcal{K} \cap (L + \overrightarrow{W}) = \varnothing$, and the projection $\pi : \mathbb{R}^n \setminus (L + \overrightarrow{W}) \to W$ of centre L and basis W satisfies the following conditions:

(1) $\pi|_E = \mathrm{id}_E$ and $\pi(\mathcal{K} \setminus \partial E) = \mathrm{Int}(E)$;

(2) for all $p \in \mathcal{K}$ there exist $q \in L$ and $\lambda \in [0, 1]$ such that $\pi(p) = \lambda p + (1 - \lambda)q$.

5.2.2.1. Let us check that, after a change of coordinates, we may assume that:

(a) $W = \{x_{d+1} = 0, \ldots, x_n = 0\}$ and the origin is a vertex of $\mathcal{K}$;

(b) $L = \{x_1 = 0, \ldots, x_d = 0,\ x_{d+1} = -1\}$;

(c) $\mathcal{K} \subset \{x_n \geqslant 0\}$ and the hyperplane $\{x_n = 0\}$ contains a facet of $\mathcal{K}$.

Indeed, if $d = n - 1$, then we may assume that $W = \{x_n = 0\}$, the origin is a vertex of $\mathcal{K}$ and $\mathcal{K} \subset \{x_n \geqslant 0\}$. Observe that in this case $E \subset \{x_n = 0\}$ is a facet of $\mathcal{K}$. From (2) above, we deduce that L is a point contained in $\{x_n < 0\}$. Thus, after a change of coordinates that keeps fixed the closed half-space $\{x_n \geqslant 0\}$, we may assume that $L = \{(0, \ldots, 0, -1)\}$.

Next, consider the case $1 \leqslant d \leqslant n - 2$. Let $p_0, \ldots, p_d \in W$ be affinely independent points such that p_0 is a vertex of $\mathcal{K}$ (recall that $\mathcal{K}$ is bounded) and let $p_{d+1}, \ldots, p_n \in L$ be affinely independent points. Observe that $\{p_0, p_1, \ldots, p_d, p_{d+1}, \ldots, p_n\}$ is an affine reference of $\mathbb{R}^n$.

Thus, after a change of coordinates, we may assume that

$$\begin{aligned}
p_0 &= 0,\\
p_i &= (0,\dots,0,\overset{(i)}{1},0,\dots,0), \quad \text{for } i=1,\dots,d,\\
p_{d+1} &= (0,\dots,0,\overset{(d+1)}{-1},0,\dots,0), \text{ and}\\
p_j &= (0,\dots,0,\overset{(d+1)}{-1},0,\dots,0,\overset{(j)}{1},0,\dots,0), \quad \text{for } j=d+2,\dots,n.
\end{aligned}$$

After this change of coordinates,

$$W=\{x_{d+1}=0,\dots,x_n=0\} \quad \text{and} \quad L=\{x_1=0,\dots,x_d=0,x_{d+1}=-1\}.$$

Note that the equations of the facets of $\mathcal{K}$ containing E have the form $\alpha_{d+1}\mathtt{x}_{d+1}+\ldots+\alpha_n\mathtt{x}_n=0$. After a change of coordinates that keeps L invariant and W fixed, we may assume that $\{x_n=0\}$ contains a facet of $\mathcal{K}$ and $\mathcal{K}\subset\{x_n\geqslant 0\}$; here we are using the fact that $d\leqslant n-2$. With these coordinates, $V=L+\overrightarrow{W}=\{x_{d+1}=-1\}$ and

$$\pi:\mathbb{R}^n\setminus V\longrightarrow H,\ x\longmapsto\left(\frac{x_1}{x_{d+1}+1},\dots,\frac{x_d}{x_{d+1}+1},0,\dots,0\right).$$

5.2.2.2. Next, we claim the following: *There exist a d-scaffold Γ of the d-face E of $\mathcal{K}$, an algebraic subset $Z_0\subset\mathbb{R}^n$ and a rational function $g:\mathbb{R}^n \dashrightarrow \mathbb{R}$, such that*:

(i) $\pi(\operatorname{Int}(\Gamma))=\operatorname{Int}(E)$;
(ii) $\mathcal{K}\cap Z_0=\partial E$;
(iii) *g is regular on $\mathbb{R}^n\setminus Z_0$;*
(iv) $g|_{\partial\mathcal{K}\setminus\partial E}\equiv 1$, $g|_{\operatorname{Int}(\Gamma)}\equiv 0$ *and*
(v) $0<g(p)<1$ *for every point* $p\in\operatorname{Int}(\mathcal{K})\setminus\operatorname{Int}(\Gamma)$.

Indeed, recall that $V=L+\overrightarrow{W}=\{x_{d+1}=-1\}$ and consider the rational map

$$h:\mathbb{R}^n \dashrightarrow \mathbb{R}^n,\ x\longmapsto\frac{1}{1+x_{d+1}}x,$$

which is regular on $\mathbb{R}^n\setminus V$ and can be interpreted as the restriction to suitable charts of the homography

$$\Psi:\mathbb{RP}^n\longrightarrow\mathbb{RP}^n,\ (x_0:x_1:\ldots:x_n)\longmapsto(x_0+x_{d+1}:x_1:\ldots:x_n).$$

Thus, h preserves convexity and affine subspaces not contained in V. Observe also that

$$h^{-1}:\mathbb{R}^n \dashrightarrow \mathbb{R}^n,\ x\longmapsto\frac{1}{1-x_{d+1}}x,$$

and the projection $\pi:\mathbb{R}^n\setminus V\to W\subset\mathbb{R}^n\setminus V$ of centre L and basis W is the composition $\pi=\psi^{-1}\circ\rho\circ\psi$, where

$$\rho:\mathbb{R}^n\longrightarrow\mathbb{R}^n,\ (x_1,\dots,x_n)\longmapsto(x_1,\dots,x_d,0,\dots,0).$$

Since $h|_{\mathbb{R}^n\setminus V}:\mathbb{R}^n\setminus V\to\mathbb{R}^n\setminus V$ is a biregular diffeomorphism and $\mathcal{K}\subset\mathbb{R}^n\setminus V$, to prove claim 5.2.2.2, it is enough to show the following.

5.2.2.3. *There exist a d-scaffold Γ_0 of the d-dimensional face $h(E)$ of $h(\mathcal{K})$, an algebraic subset $Z_0'\subset\mathbb{R}^n$ and a rational function $g_0:\mathbb{R}^n \dashrightarrow \mathbb{R}$ such that*:

(i) $\rho(\operatorname{Int}(\Gamma_0))=\operatorname{Int}(h(E))$;
(ii) $h(\mathcal{K})\cap Z_0'=\partial h(E)$;
(iii) *g_0 is regular on $\mathbb{R}^n\setminus Z_0'$;*
(iv) $g_0|_{\partial h(\mathcal{K})\setminus\partial h(E)}\equiv 1$, $g_0|_{\operatorname{Int}(\Gamma_0)}\equiv 0$ *and*
(v) $0<g_0(p)<1$ *for every* $p\in\operatorname{Int}(h(\mathcal{K}))\setminus\operatorname{Int}(\Gamma_0)$.

Indeed, observe that h preserves the hyperplanes $\{x_i = 0\}$ for $i = 1, \dots, n$ and, changing the sign of the variable $\mathtt{x}_n$ if necessary, we may assume that $h(\mathcal{K}) \subset \{x_n \geqslant 0\}$. Thus, we are under the hypotheses of Lemma 5.4, and a straightforward computation shows that 5.2.2.3 holds. Thus, also 5.2.2.2 holds.

5.2.2.4. Now we are ready to prove Lemma 5.1 in case $1 \leqslant d \leqslant n-1$. With the notation of 5.2.2.2, consider the rational map

$$f : \mathbb{R}^n \dashrightarrow \mathbb{R}^n,\ p \longmapsto g(p)p + (1 - g(p))\pi(p),$$

which is regular on $\mathbb{R}^n \setminus Z$, where $Z = Z_0 \cup (L + \overrightarrow{W})$. Observe that $Z \cap \mathcal{K} = Z_0 \cap \mathcal{K} = \partial E$. Now let us check that

$$f(\operatorname{Int}(\mathcal{K}) \cup Y) = \operatorname{Int}(\mathcal{K}) \cup Y \cup \operatorname{Int}(E).$$

Since $g|_{\partial\mathcal{K}\setminus\partial E} \equiv 1$ and $Y \cap E = \varnothing$, it follows that f is regular on Y and $f|_Y = \mathrm{id}_Y$. Thus, $f(Y) = Y$. Moreover, $f(\operatorname{Int}(\Gamma)) = \operatorname{Int}(E)$ because $g|_{\operatorname{Int}(\Gamma)} \equiv 0$ and $\pi(\operatorname{Int}(\Gamma)) = \operatorname{Int}(E)$ (see 5.2.2.2). Hence, it only remains to check that $f(\operatorname{Int}(\mathcal{K}) \setminus \partial E) = \operatorname{Int}(\mathcal{K})$.

Indeed, let $p \in \operatorname{Int}(\mathcal{K}) \setminus \partial E$, and observe that $0 < g(p) < 1$. Thus, since $p \in \operatorname{Int}(\mathcal{K})$ and $\pi(p) \in \operatorname{Int}(E) \subset \mathcal{K}$, we deduce from [**1**, 11.2.4] that $f(p) \in \operatorname{Int}(\mathcal{K})$. Conversely, let $a \in \operatorname{Int}(\mathcal{K})$ and let $\pi(a) \in \operatorname{Int}(E)$ (see Paragraph 5.2.2(1)). Next, consider the line T that contains the points a and $\pi(a)$. Let b be the point of $\partial\mathcal{K}$ such that $\mathcal{K} \cap T$ is the segment $[\pi(a), b]$ joining the points $\pi(a)$ and b. Note that, again by Berger [**1**, 11.2.4], the open interval $(\pi(a), b) \subset \operatorname{Int}(\mathcal{K})$. Moreover, observe that, since $f(\pi(a)) = \pi(a)$, $f(b) = b$, and the segment $[\pi(a), b]$ is convex, the image of the restriction $f|_{[\pi(a),b]} : [\pi(a), b] \to \mathcal{K}$ is the segment $[\pi(a), b]$. Thus, there exists a point $a' \in (\pi(a), b) \subset \operatorname{Int}(\mathcal{K})$ such that $f(a') = a$, which shows that $\operatorname{Int}(\mathcal{K}) \subset f(\operatorname{Int}(\mathcal{K}))$. But in fact, since $f(\operatorname{Int}(\Gamma)) = \operatorname{Int}(E) \subset \partial\mathcal{K}$, it follows that $a' \in \operatorname{Int}(\mathcal{K}) \setminus \operatorname{Int}(\Gamma)$, and we are done.

6. *The open and the closed ball as regular images of* $\mathbb{R}^n$

As commented in Section 1, a closed ball and its interior can be constructed as 'limits' of bounded, convex, regular polyhedra and their interiors when the number of facets tends to infinity. In this section, we show that both are regular images of $\mathbb{R}^n$ and so their invariant r coincides with their common dimension n. Of course, the centre and radius of the ball are irrelevant, and so we just deal with the open ball $\mathcal{B}_n \subset \mathbb{R}^n$ of centre the origin and radius 1, and its closure. We begin with the open ball. The finding of the regular map realizing $\mathcal{B}_n$ as a regular image of $\mathbb{R}^n$ is strongly inspired by the solution for $n = 2$ previously obtained in [**6**, 6.3.a].

Lemma 6.1. *The open ball $\mathcal{B}_n \subset \mathbb{R}^n$ of centre the origin and radius 1 is a regular image of $\mathbb{R}^n$.*

Proof. Consider the inverse h of the stereographic projection

$$\pi_N : \mathbb{S}^n \setminus \{p_N\} \longrightarrow \mathbb{R}^n,\ y = (y_1, \dots, y_{n+1}) \longrightarrow \left(\frac{y_1}{1 - y_{n+1}}, \dots, \frac{y_n}{1 - y_{n+1}}\right)$$

from the north pole $p_N = (0, \dots, 0, 1)$ of the sphere

$$\mathbb{S}^n = \{y = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} :\ y_1^2 + \dots + y_{n+1}^2 = 1\}.$$

Recall that h is given by

$$h : \mathbb{R}^n \longrightarrow \mathbb{S}^n \setminus \{p_N\},\ x = (x_1, \dots, x_n) \longrightarrow \left(\frac{2x_1}{\|x\|^2 + 1}, \dots, \frac{2x_n}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1}\right).$$

Define $\mathcal{H}_n = \{x \in \mathbb{R}^n : x_1 > 0\}$ and observe that $h(\mathcal{H}_n) = \mathbb{S}^n \cap \mathcal{H}_{n+1}$. Consider the orthogonal projection $f : \mathbb{R}^{n+1} \to \mathbb{R}^n,\ y = (y_1, \dots, y_{n+1}) \mapsto (y_2, \dots, y_{n+1})$, which satisfies $(f \circ h)(\mathcal{H}_n) = f(\mathbb{S}^n \cap \mathcal{H}_{n+1}) = \mathcal{B}_n$. Note that $\mathcal{H}_n$ is the interior of a convex polyhedron of $\mathbb{R}^n$ and so there exists, by Corollary 4.5, a regular map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(\mathbb{R}^n) = \mathcal{H}_n$. Hence, $\mathcal{B}_n = (f \circ h \circ g)(\mathbb{R}^n)$ is a regular image of $\mathbb{R}^n$. □

We finally show that also the closed ball is a regular image of $\mathbb{R}^n$.

Lemma 6.2. *The closed ball $\overline{\mathcal{B}}_n \subset \mathbb{R}^n$ of centre the origin and radius 1 is a regular image of $\mathbb{R}^n$.*

Proof. Consider first the univariate polynomial $g = \frac{16}{9}\mathtt{t}^4 - \frac{44}{9}\mathtt{t}^2 + \frac{28}{9} \in \mathbb{R}[\mathtt{t}^2]$ and the product $h(\mathtt{t}) = \mathtt{t}g(\mathtt{t})$, which satisfies the following properties:

$$h(0) = h(1) = 0, \quad h\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad h'\left(\frac{1}{2}\right) = 0.$$

Moreover, the derivative of h is $h'(\mathtt{t}) = \frac{4}{9}(2\mathtt{t} - 1)(2\mathtt{t} + 1)(5\mathtt{t}^2 - 7)$ and so $h'|_{[0,1/2)} > 0$ and $h'|_{(1/2,1]} < 0$. This implies in particular that $h([0,1)) = [0,1]$.

Next, since $g \in \mathbb{R}[\mathtt{t}^2]$, the map

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ (x_1, \dots, x_n) \longmapsto g(\sqrt{x_1^2 + \dots + x_n^2})(x_1, \dots, x_n)$$

is polynomial. If we restrict f to any closed segment $S_v = \{tv \in \mathbb{R}^n : t \in [0,1)\}$ from the origin, where $v \in \mathbb{R}^n$ is a unitary vector, then we obtain $f(tv) = tg(t)v = h(t)v$, and consequently

$$f(S_v) = \{h(t)v \in \mathbb{R}^n : t \in [0,1)\} = \{sv \in \mathbb{R}^n : s \in [0,1]\} = \mathrm{Cl}_{\mathbb{R}^n}(S_v).$$

Observe that $\mathcal{B}_n = \bigcup_{v \in \mathbb{S}^n} S_v$ and $\overline{\mathcal{B}}_n = \bigcup_{v \in \mathbb{S}^n} \mathrm{Cl}_{\mathbb{R}^n}(S_v)$. Therefore,

$$f(\mathcal{B}_n) = f\left(\bigcup_{v \in \mathbb{S}^n} S_v\right) = \bigcup_{v \in \mathbb{S}^n} f(S_v) = \bigcup_{v \in \mathbb{S}^n} \mathrm{Cl}_{\mathbb{R}^n}(S_v) = \overline{\mathcal{B}}_n.$$

Using now Lemma 6.1, we conclude that $\overline{\mathcal{B}}_n$ is a regular image of $\mathbb{R}^n$. □

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