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ON CHAINS OF PRIME IDEALS IN RINGS OF SEMIALGEBRAIC FUNCTIONS

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Abstract

In this work, we study the structure of non-refinable chains of prime ideals in the (real closed) rings $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ of semialgebraic and bounded semialgebraic functions on a semialgebraic set $M \subset \mathbb{R}^m$. We pay special attention to the prime z-ideals of $\mathcal{S}(M)$ and the minimal prime ideals of both rings. For the last, a decomposition of each semialgebraic set as an irredundant finite union of closed pure dimensional semialgebraic subsets plays a crucial role. We prove moreover the existence of maximal ideals in the ring $\mathcal{S}(M)$ of prefixed height whenever M is non-compact.

1. Introduction

A subset $M \subset \mathbb{R}^m$ is said to be *basic semialgebraic* if it can be written as

$$M = \{x \in \mathbb{R}^m : f_1(x) > 0, \ldots, f_r(x) > 0, g(x) = 0\} := \{f_1 > 0, \ldots, f_r > 0, g = 0\}$$

for some polynomials $f_1, \ldots, f_r, g \in \mathbb{R}[\mathrm{x}] := \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_m]$. The finite unions of basic semialgebraic sets are called *semialgebraic sets*. *Pure dimensional semialgebraic sets*, that is, those semialgebraic sets, for which the *local dimension function* $M \to \mathbb{R},\ p \mapsto \dim_p(M)$ is a constant function, will play a crucial role in this work. A continuous function $f : M \to \mathbb{R}$ is said to be *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{m+1}$. Usually, a semialgebraic function means a function that is not necessarily continuous and whose graph is semialgebraic. However, since all semialgebraic functions occurring in this article are continuous, we assume the continuity condition whenever we refer to them. Similarly, a continuous semialgebraic map $\varphi : M \to N$ between semialgebraic sets will be called a semialgebraic map.

The sum and product of functions, defined pointwise, endow the set $\mathcal{S}(M)$ of semialgebraic functions on M with a natural structure of a commutative ring (whose unity is the function with constant value 1). It is obvious that the subset $\mathcal{S}^*(M)$ of bounded semialgebraic functions on M is a real subring of $\mathcal{S}(M)$. In the following, we denote with $\mathcal{S}^\diamond(M)$ either $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$ if the involved statements or arguments are valid for both rings.

As is well-known, the rings $\mathcal{S}^\diamond(M)$ are particular cases of the so-called *real closed rings* introduced by Schwartz in the 1980s (see [20]). The theory of real closed rings has been deeply developed till now in a fruitful attempt to establish new foundations for semi-algebraic geometry with relevant interconnections to model theory, see [5, 6, 19–25, 28–30]. Moreover, this theory, which generalizes the classical techniques concerning the semi-algebraic spaces of Delfs–Knebusch (see [8]), provides

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a powerful machinery to approach problems concerning certain rings of real-valued functions and contributes to achieve a better understanding of the algebraic and topological properties of such rings. We highlight some of them:

(1) Rings of real-valued continuous functions on Tychonoff spaces;
(2) rings of semi-algebraic functions on semi-algebraic sets of an arbitrary real closed field and, more generally,
(3) rings of definable continuous functions on definable sets in o-minimal expansions of fields.

For the sake of completeness, we recall the characterization of real closed rings that adapts better to the content of this work; see [21] for a ring theoretic analysis of the concept of real closed rings. As such a characterization is quite involving, we quote only some of the involved objects without entering into further details.

DEFINITION 1.1 A ring A is *real closed* if it satisfies the following conditions:

(i) A is a reduced ring.
(ii) The support map supp : $\operatorname{Spec}_r(A) \to \operatorname{Spec}(A),\ \alpha \mapsto \mathfrak{p}_\alpha = \alpha \cap (-\alpha)$ is *identifying*, that is, it is a homeomorphism, which induces a bijection between the constructible subsets of $\operatorname{Spec}_r(A)$ and those of $\operatorname{Spec}(A)$.
(iii) For each $\mathfrak{p} \in \operatorname{Spec}(A)$ we have:
  (a) The quotient field $R := \operatorname{qf}(A/\mathfrak{p})$ is a real closed field and $A/\mathfrak{p}$ is integrally closed in R, and
  (b) Each $\mathfrak{Q} \in \operatorname{Spec}(A/\mathfrak{p})$ is convex with respect to the unique ordering of $A/\mathfrak{p}$.
(iv) A finite sum of radical ideals of A is a radical ideal of A.

Of course, $\operatorname{Spec}(A)$ denotes the Zariski spectrum of A endowed with the Zariski topology while $\operatorname{Spec}_r(A)$ denotes the real spectrum of A endowed with the spectral topology. We refer the reader to [2, Section 7.1] for further details concerning the real spectrum of a ring A and its constructible subsets.

1.1. *Main results*

The main purpose of this work is to understand the structure of non-refinable chains of prime ideals of the ring $\mathcal{S}^*(M)$ for an arbitrary semialgebraic set $M \subset \mathbb{R}^m$ (not necessarily locally closed). This somehow completes the work already began in [12], in which we studied some algebraic, topological and functorial properties of the Zariski and maximal spectra of the rings $\mathcal{S}^\diamond(M)$ for an arbitrary semialgebraic set $M \subset \mathbb{R}^m$. Moreover, our results generalize some similar already known ones for the o-minimal context in the exponentially bounded and polynomially bounded cases that are developed under the assumption of local closedness (see [28] for further details). We also recall here that $\mathcal{S}^*(M)$ can be understood as the *ring of holomorphy* of the real closed ring $\mathcal{S}(M)$ in the sense of [29, p. 40]. This provides some valuable information in relation with the chains of prime ideals containing an ideal $\mathfrak{b}$ of $\mathcal{S}^*(M)$. To that end, one can use Gelfand–Kolmogorov's Theorem for rings with normal spectrum and the related results concerning rings of holomorphy (see [29, §10]) applied to the pair of rings $\mathcal{S}^*(M) \subset \mathcal{S}(M)$.

Of course, many of the results we obtain in this work are still valid for an arbitrary real closed field but to ease the exposition we only focus on the field $\mathbb{R}$. Next, we point out our main results in more detail.

1.1.1 Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{S}(M)$ and $\mathfrak{m}^*$ the unique maximal ideal of $\mathcal{S}^*(M)$ containing the prime ideal $\mathfrak{m} \cap \mathcal{S}^*(M)$. Then all non-refinable chains of prime ideals in $\mathcal{S}^*(M)$ whose last member is $\mathfrak{m}^*$ contain $\mathfrak{m} \cap \mathcal{S}^*(M)$ and they share the members of the chain that are between both ideals (see Proposition 5.1). If M is moreover locally compact, the immediate successor of $\mathfrak{m} \cap \mathcal{S}^*(M)$ in such a chain is characterized topologically in Theorem 6.1; in fact, we show in Theorem 6.8 how far the previous description works if M is not locally compact. Such a characterization is inspired by the analogous results for classical rings of continuous functions (see [**15**, 14.27; **17**]). The proof of Theorem 6.1 is a paradigm of how the algebraic and topological arguments mix in a subtle way.

1.1.2 In addition, we present several examples (see Examples 5.5 and 5.6), which illustrate different types of chains of prime ideals. In the same manner, we approach the computation of the height of each maximal ideal $\mathfrak{m}^*$ in $\mathcal{S}^*(M)$ by using suitable semialgebraic compactifications of M. In fact, we prove the existence of a semialgebraic compactification X of M and a non-refinable chain of prime ideals in $\mathcal{S}(X)$ whose length equals $\operatorname{ht}(\mathfrak{m}^*)$ and which has $\mathfrak{m}^* \cap \mathcal{S}(X)$ as its last member (see Corollaries 5.8 and 5.9). However, such a chain need not to be a chain of maximal length among those ending with $\mathfrak{m}^* \cap \mathcal{S}(X)$ (see Remark 5.10).

1.1.3 The pure dimensional semialgebraic sets enjoy a particularly useful property: the complements of subsets of smaller dimension are dense. With the aim of using this fact, we present a finite decomposition $\mathcal{B}_M = \{\mathcal{B}_i(M)\}_{i=1}^r$ of any semialgebraic set M into closed pure dimensional semialgebraic subsets of different dimensions such that $\dim(\mathcal{B}_i(M) \cap \mathcal{B}_j(M)) < \min\{\dim(\mathcal{B}_i(M)), \dim(\mathcal{B}_j(M))\}$ if $i \neq j$, see Proposition and Definition 3.2. This decomposition is unique, up to reordering, and its members $\mathcal{B}_i(M)$ are called the *bricks* of M.

1.1.4 It is worthwhile to mention that the map $\mathfrak{q} \mapsto \mathfrak{q} \cap \mathcal{S}^*(M)$ establishes a bijection between the sets of minimal prime ideals of $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$. In fact, the decomposition $\mathcal{B}_M$ of M plays a crucial role in the characterization of minimal prime ideals of both rings (see Theorem 4.1). Moreover, in our context, a radical ideal of $\mathcal{S}^\diamond(M)$ is prime if and only if it contains a minimal prime ideal of $\mathcal{S}^\diamond(M)$ (see Lemma 5.3 and Corollary 5.4). We refer the reader to [**25**] for a careful study of the main properties of the minimal points (that is, minimal prime ideals) of the Zariski spectrum of a ring.

Furthermore, we prove that each minimal prime ideal of $\mathcal{S}(M)$ is a prime z-ideal (see Corollary 4.7). Recall that an ideal $\mathfrak{a}$ of $\mathcal{S}(M)$ is a *z-ideal* if it contains all functions whose zeroset contains the zeroset of a function in $\mathfrak{a}$ (see [**21**, Section 2]). Taking advantage of the study of minimal prime ideals of $\mathcal{S}(M)$, we analyse the main properties of the *semialgebraic depth* (see Definition and Proposition 2.8) on the set of prime z-ideals of $\mathcal{S}(M)$.

1.1.5 Finally, we study some properties of the maximal ideals of both rings $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$. In Theorem 7.1, we prove that for each non-compact pure dimensional semialgebraic set M of dimension d and for each $0 \leq r < d$, there exists a free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$ such that $\operatorname{ht}(\mathfrak{m}) = r$ but $\operatorname{ht}(\mathfrak{m}^*) = d$, where $\mathfrak{m}^*$ is the unique maximal ideal of $\mathcal{S}^*(M)$ that contains $\mathfrak{m} \cap \mathcal{S}^*(M)$. This result guarantees the existence of maximal ideals in $\mathcal{S}(M)$, which are also minimal prime ideals. If M does not have isolated points, no prime ideal of $\mathcal{S}^*(M)$ is simultaneously a maximal and minimal prime ideal (see Corollary 7.2). This kind of results are in some sense related to Bröcker's ultrafilter theorem (see [4, Section 4]).

1.2. *Structure of the article*

This work is organized as follows. In Section 2, we provide preliminary terminology and results concerning Zariski and maximal spectra of rings of semialgebraic and bounded semialgebraic functions we use along with the rest of the work. Most results in Section 2 arise from the general theory of real closed rings commented in Section 1 as well as from [10–13]; we include them (without proofs) for the sake of completeness. In Section 3, we present the decomposition of a semialgebraic set into bricks and recall the main properties of the set $M_{\rm lc}$ of points in M, which admit a compact neighbourhood in M (see [8, 9.14–9.21]). Section 4 is devoted to characterize the minimal prime ideals of $\mathcal{S}^\diamond(M)$ and to study the behaviour of the semialgebraic depth on the set of prime z-ideals. In Section 5, we obtain our main results concerning the study of chains of prime ideals of $\mathcal{S}^*(M)$. Section 6 is dedicated to analyse the immediate successor of the prime ideal $\mathfrak{m}\cap\mathcal{S}^*(M)$ in any non-refinable chain of prime ideals in $\mathcal{S}^*(M)$ ending with a free maximal ideal $\mathfrak{m}^*$ of the ring $\mathcal{S}^*(M)$. Finally, we approach the construction of maximal ideals in $\mathcal{S}(M)$ of prefixed height in Section 7.

2. Preliminaries on spectra of rings of semialgebraic functions

2.1. *Generalities about semialgebraic sets and semialgebraic functions*

Let M be a semialgebraic set. For each $f\in\mathcal{S}^\diamond(M)$ and each semialgebraic subset $N\subset M$, we denote $Z_N(f):=\{x\in N : f(x)=0\}$ and $D_N(f):=N\setminus Z_N(f)$. For $N=M$, we say that $Z_M(f)$ is the *zeroset* of f. We begin by writing each closed semialgebraic subset of M as the zeroset of a single semialgebraic function on M that can be chosen bounded.

LEMMA 2.1 *Let Z be a closed semialgebraic subset of the semialgebraic set $M\subset\mathbb{R}^m$. Then there exists a bounded semialgebraic function $h\in\mathcal{S}^*(M)$ such that $Z=Z_M(h)$.*

Proof. Take for instance $h=\min\{1,\operatorname{dist}(\cdot,Z)\}$. □

Sometimes it will be useful to assume that the semialgebraic set M we are working with is bounded. Such an assumption can be done without loss of generality, which we see in the following remark. In what follows, we, respectively, denote the open and closed balls of $\mathbb{R}^m$ of centre x and radius $\varepsilon>0$ with $\mathbb{B}_m(x,\varepsilon)$ and $\bar{\mathbb{B}}_m(x,\varepsilon)$; their common boundary is denoted with $\mathbb{S}^{m-1}(x,\varepsilon)$.

REMARK 2.2 Let $M\subset\mathbb{R}^m$ be a semialgebraic set. The semialgebraic homeomorphism

$$\varphi:\mathbb{B}_m(0,1)\to\mathbb{R}^m,\quad x\mapsto\frac{x}{\sqrt{1-\|x\|^2}}$$

induces a ring isomorphism $\mathcal{S}(M)\to\mathcal{S}(N),\ f\mapsto f\circ\varphi$ that maps $\mathcal{S}^*(M)$ onto $\mathcal{S}^*(N)$, where $N:=\varphi^{-1}(M)$. So, if necessary, we may always assume that M is bounded.

A key ingredient in the development of some results is the following semialgebraic version of the *Tietze–Urysohn extension lemma* due to Delfs and Knebusch (see [7]).

THEOREM 2.3 *Let $N \subset M \subset \mathbb{R}^m$ be a semialgebraic set such that N is closed in M. Then the homomorphism $\mathcal{S}^\diamond(M) \to \mathcal{S}^\diamond(N),\ F \mapsto F|_N$ is surjective.*

2.2. *The z-ideals of the ring $\mathcal{S}(M)$*

We recall the notion of a z-ideal of the ring $\mathcal{S}(M)$ and some remarkable properties (for a more detailed analysis of this concept see [**21**, Section 2; **11**, Section 3]). Whenever we consider an ideal of $\mathcal{S}^\diamond(M)$, we refer to a proper ideal of $\mathcal{S}^\diamond(M)$.

DEFINITION 2.4 Let $M \subset \mathbb{R}^m$ be a semialgebraic set. An ideal $\mathfrak{a}$ of $\mathcal{S}(M)$ is a *z-ideal* if we have $g \in \mathfrak{a}$ whenever there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}(M)$ satisfying $Z_M(f) \subset Z_M(g)$.

REMARK 2.5 Let $\varphi : N \to M$ be a semialgebraic map between the semialgebraic sets $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ and let $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$, $h \mapsto h \circ \varphi$. If $\mathfrak{a}$ is a z-ideal of $\mathcal{S}(N)$, then $\phi^{-1}(\mathfrak{a})$ is a z-ideal of $\mathcal{S}(M)$.

We recall the Nullstellensatz for the ring $\mathcal{S}(M)$ (see for instance [**28**, Section 2; **11**, 3.4]).

THEOREM 2.6 (Nullstellensatz) *Let $M \subset \mathbb{R}^m$ be a locally compact semialgebraic set and $\mathfrak{a}$ an ideal of $\mathcal{S}(M)$. Then $\mathfrak{a}$ is a z-ideal if and only if $\mathfrak{a}$ is radical. In particular, if $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}$ is a z-ideal.*

2.3. *Coheight and semialgebraic depth*

We present the notion and main properties of the *semialgebraic depth* of a prime ideal of $\mathcal{S}(M)$. As far as we know, this invariant was first introduced in [**10**, 4.4] where it has been used to approach the computation of the Krull dimension of rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set. In this work, we will provide further applications of this invariant, and in fact, we extend the results concerning the semialgebraic depth obtained in [**10**, 4.6] for the prime z-ideals of $\mathcal{S}(M)$, where M is an arbitrary semialgebraic set (not necessarily locally compact); see Lemma 4.10–Corollary 4.14. Before introducing this notion, we recall the concept of coheight. Namely:

DEFINITIONS 2.7 (Coheight) Let $\mathfrak{p} \subset \mathfrak{q}$ be two prime ideals of a commutative ring A with unity such that its Krull dimension is finite. The *coheight of* $\mathfrak{p}$ *in* $\mathfrak{q}$ is the maximum of the integers $r \geq 0$ such that there exists a chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{q}$.

If A is a Gelfand ring (for example, the rings of semialgebraic functions, see [**22**]), we define the *coheight of a prime ideal* $\mathfrak{p} \subset A$ as the coheight of $\mathfrak{p}$ in the unique maximal ideal of A containing $\mathfrak{p}$. In particular, the height of a maximal ideal $\mathfrak{m}$ of A is the maximum of the coheights of the minimal prime ideals of A contained in $\mathfrak{m}$.

DEFINITION AND PROPOSITION 2.8 Semialgebraic depth Let $M \subset \mathbb{R}^m$ be a semialgebraic set. We define the *semialgebraic depth* of a prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$ as $\mathrm{d}_M(\mathfrak{p}) := \min\{\dim(Z_M(f)) : f \in \mathfrak{p}\}$.

One of the main properties of this invariant, as stated in [**10**, 4.4], is the following.

(2.8.1) *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime z-ideals of $\mathcal{S}(M)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$. Then $\mathrm{d}_M(\mathfrak{p}) < \mathrm{d}_M(\mathfrak{q})$.*

In fact, if $M \subset \mathbb{R}^m$ is a locally compact semialgebraic set, all prime ideals of $\mathcal{S}(M)$ are by Theorem 2.6 z-ideals (see [11, 3.5; 21, Section 2]), so Paragraph 2.8.1 applies to each pair of prime ideals $\mathfrak{q} \subsetneq \mathfrak{p}$ of $\mathcal{S}(M)$.

(2.8.2) In this locally compact semialgebraic setting, the following is proved in [10, 4.6]:

(i) *For every prime ideal* $\mathfrak{p}$ *of* $\mathcal{S}(M)$ *it holds* $\mathrm{d}_M(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) \leq \dim(M)$.
(ii) *Let* $\mathfrak{p} \subset \mathfrak{q}$ *be prime ideals of* $\mathcal{S}(M)$. *Then the coheight of* $\mathfrak{p}$ *in* $\mathfrak{q}$ *is* $\leq \mathrm{d}_M(\mathfrak{p}) - \mathrm{d}_M(\mathfrak{q})$.
(iii) *The height of a maximal ideal* $\mathfrak{m}$ *of* $\mathcal{S}(M)$ *is less than or equal to the maximum of* $\mathrm{d}_M(\mathfrak{p})$, *where* $\mathfrak{p}$ *runs over the minimal prime ideals of* $\mathcal{S}(M)$ *contained in* $\mathfrak{m}$.

2.4. *Zariski spectra of rings of semialgebraic functions*

We recall some remarkable properties concerning the Zariski spectra of rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set, which follow from the fact that both rings are real closed rings (see [20, 21, 23, 28, 29]). We also refer the reader to [2, Section 1, Section 7] for further details concerning real fields and the real spectrum of a commutative ring with unity.

The *Zariski spectrum of* $\mathcal{S}^\diamond(M)$ that we denote with $\mathrm{Spec}_s^\diamond(M) := \mathrm{Spec}(\mathcal{S}^\diamond(M))$ for notational simplicity is the set of all prime ideals of $\mathcal{S}^\diamond(M)$. This set $\mathrm{Spec}_s^\diamond(M)$ is usually endowed with the Zariski topology, which has the family of sets $\mathcal{D}_{\mathrm{Spec}_s^\diamond(M)}(f) := \{\mathfrak{p} \in \mathrm{Spec}_s^\diamond(M) : f \notin \mathfrak{p}\}$, where $f \in \mathcal{S}^\diamond(M)$, as a basis of open sets. We write $\mathcal{Z}_{\mathrm{Spec}_s^\diamond(M)}(f) := \mathrm{Spec}_s^\diamond(M) \setminus \mathcal{D}_{\mathrm{Spec}_s^\diamond(M)}(f)$.

The field $\mathrm{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})$ is real closed, see for instance [22] or [14], so it admits a unique ordering $\leq$, having the squares as its cone of non-negative elements. Therefore, the map $\mathfrak{p} \mapsto (\mathfrak{p}, \leq)$ is a bijection between the Zariski and the real spectrum of $\mathcal{S}^\diamond(M)$, and to simplify we identify $\mathrm{Spec}_s^\diamond(M)$ with the real spectrum of $\mathcal{S}^\diamond(M)$. Recall that the *real spectrum of a commutative ring* A is the collection of all pairs of the type $(\mathfrak{p}, \leq)$, where $\mathfrak{p}$ is a real prime ideal and $\leq$ is an ordering of the orderable field $\mathrm{qf}(A/\mathfrak{p})$. An ideal $\mathfrak{a}$ of A is said to be real if it holds $a_1, \dots, a_r \in \mathfrak{a}$ whenever $a_1^2 + \cdots + a_r^2 \in \mathfrak{a}$ for $a_1, \dots, a_r \in A$; we refer the reader to [2, §4,§7] for further details concerning the real spectrum. The usual topology for the real spectrum of a commutative ring A is the *spectral topology*: In our case the real spectrum of $A := \mathcal{S}^\diamond(M)$ has the family of sets

$$\mathcal{U}_{\mathrm{Spec}_s^\diamond(M)}(f_1, \dots, f_r) := \{\mathfrak{p} \in \mathrm{Spec}_s^\diamond(M) : f_1 + \mathfrak{p} > 0, \dots, f_r + \mathfrak{p} > 0 \text{ in } \mathrm{qf}(\mathcal{S}^\diamond(M)/\mathfrak{p})\},$$

as a basis of open sets, where $f_1, \dots, f_r \in \mathcal{S}^\diamond(M)$. This topology coincides with the Zariski one introduced before. Moreover, M (endowed with the Euclidean topology) can be embedded in $\mathrm{Spec}_s^\diamond(M)$ as a dense subspace, via the embedding $\phi : M \to \mathrm{Spec}_s^\diamond(M),\ \ p \mapsto \mathfrak{m}^\diamond{}_p$, where $\mathfrak{m}^\diamond{}_p$ denotes the maximal ideal of all functions in $\mathcal{S}^\diamond(M)$ vanishing at p.

2.4.1 The prime ideals of $\mathcal{S}^\diamond(M)$ satisfy a 'convexity condition', which is ubiquitous for real closed rings. Namely: *Given a prime ideal* $\mathfrak{p}$ *of* $\mathcal{S}^\diamond(M)$ *and* $f, g \in \mathcal{S}^\diamond(M)$ *with* $g \in \mathfrak{p}$ *and* $0 \leq f(x) \leq g(x)$ *for each point* $x \in M$, *then also* $f \in \mathfrak{p}$. This convexity condition can be translated to the ordering of the real closed ring $\mathcal{S}^\diamond(M)/\mathfrak{p}$: *If* $0 \leq f + \mathfrak{p} \leq g + \mathfrak{p}$ *in the ring* $\mathcal{S}^\diamond(M)/\mathfrak{p}$, *then we may assume* $0 \leq f(x) \leq g(x)$ *for all* $x \in M$.

2.4.2 An important consequence of the convexity that we use frequently in this work is the following: *The set of prime ideals of the ring* $\mathcal{S}^\diamond(M)$ *containing a given prime ideal* $\mathfrak{p}$ *form a chain.*

It is a well-known fact that the ring $\mathcal{S}(M)$ of semialgebraic functions on a semialgebraic set M can be understood as a suitable localization of the ring $\mathcal{S}^*(M)$ of bounded semialgebraic functions on M. More precisely, we have:

LEMMA 2.9 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathcal{W}(M) \subset \mathcal{S}^*(M)$ the multiplicative set of those functions $f \in \mathcal{S}^*(M)$ such that $Z_M(f) = \emptyset$. Then*

(i) *$\mathcal{S}(M) = \mathcal{S}^*(M)_{\mathcal{W}(M)}$ is the localization of $\mathcal{S}^*(M)$ at the multiplicative set $\mathcal{W}(M)$.*
(ii) *If $\mathfrak{S}(M) \subset \operatorname{Spec}_s^*(M)$ denotes the set of prime ideals of $\mathcal{S}^*(M)$, which do not meet $\mathcal{W}(M)$, then the Zariski spectrum of $\mathcal{S}(M)$ is in one-to-one correspondence with $\mathfrak{S}(M)$ via the maps*

$$\mathtt{j} : \operatorname{Spec}_s(M) \to \mathfrak{S}(M), \mathfrak{p} \mapsto \mathfrak{p} \cap \mathcal{S}^*(M) \quad \textit{and} \quad \mathtt{j}^{-1} : \mathfrak{S}(M) \to \operatorname{Spec}_s(M), \mathfrak{p} \mapsto \mathfrak{p}\mathcal{S}(M).$$

We compile some results concerning the Zariski spectra of the rings $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$; for further details concerning their proofs we refer the reader to [12, Section 4–5]. Recall that, given a semialgebraic map $\varphi : N \to M$, there exists a (unique continuous) spectral map $\operatorname{Spec}_s^\diamond(\varphi) : \operatorname{Spec}_s^\diamond(N) \to \operatorname{Spec}_s^\diamond(M)$, which extends φ. The next result provides a nice description of the closure of a semialgebraic subset $N \subset M \subset \mathbb{R}^m$ in $\operatorname{Spec}_s^\diamond(M)$. Namely,

LEMMA 2.10 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets and $\mathtt{j} : N \hookrightarrow M$ the inclusion map. Consider the homomorphism $\phi : \mathcal{S}^\diamond(M) \to \mathcal{S}^\diamond(N),\ f \to f|_N$ and a prime ideal $\mathfrak{p}$ of $\mathcal{S}^\diamond(M)$. Then*

(i) *$\mathfrak{p} \in \operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ if and only if $\ker\phi \subset \mathfrak{p}$.*
(ii) *If M is moreover locally compact, $\mathfrak{p} \in \operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$ if and only if there exists $h \in \mathfrak{p}$ such that $Z_M(h) \subset \operatorname{Cl}_M(N)$.*
(iii) *If N is moreover closed in M, then $\operatorname{Spec}_s^\diamond(N) \cong \operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(N)$ via $\operatorname{Spec}_s^\diamond(\mathtt{j})$.*

The next result can be seen as a first attempt to show the rigidity of the chains of prime ideals in $\mathcal{S}^*(M)$. We will come back to this item in Section 5.

COROLLARY 2.11 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets such that N is closed in M. Let $\mathfrak{p} \in \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N)$ be a prime ideal and $\mathfrak{m}^*$ the unique maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{p}$. Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{S}(M)$ such that $\mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$. Then $\mathfrak{m} \in \operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$ and $\mathfrak{m} \cap \mathcal{S}^*(M) \in \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N)$.*

Proof. Let $\mathtt{j} : N \hookrightarrow M$ be the inclusion map and denote the induced homomorphisms with $\phi_1 : \mathcal{S}(M) \to \mathcal{S}(N),\ f \mapsto f|_N$ and $\phi_2 : \mathcal{S}^*(M) \to \mathcal{S}^*(N),\ f \mapsto f|_N$ that are by Theorem 2.3 surjective. We obtain the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\phi_1} & \mathcal{S}(N) \\ \uparrow & & \downarrow \\ \mathcal{S}^*(M) & \xrightarrow{\phi_2} & \mathcal{S}^*(N) \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} \operatorname{Spec}_s^*(N) & \xrightarrow{\operatorname{Spec}_s^*(\mathtt{j})} & \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N) & \hookrightarrow & \operatorname{Spec}_s^*(M) \\ \uparrow & & \uparrow & & \uparrow \\ \operatorname{Spec}_s(N) & \xrightarrow{\operatorname{Spec}_s(\mathtt{j})} & \operatorname{Cl}_{\operatorname{Spec}_s(M)}(N) & \hookrightarrow & \operatorname{Spec}_s(M) \end{array}$$

in which the first maps in the rows of the second diagram are by Lemma 2.10(iii) homeomorphisms. Since $\mathfrak{p} \in \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N)$, also $\mathfrak{m}^* \in \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N)$; in fact, $\mathfrak{n}^* = \operatorname{Spec}_s^*(\mathtt{j})^{-1}(\mathfrak{m}^*)$ is a maximal ideal

$\mathcal{S}^*(N)$ because $\operatorname{Spec}_s^*(\mathtt{j})^{-1}$ preserves closed points. Let us consider the ideal $\mathfrak{q} := \operatorname{Spec}_s^*(\mathtt{j})^{-1}(\mathfrak{p}) \subset \mathfrak{n}^*$ and let $\mathfrak{n}$ be the unique maximal ideal of $\mathcal{S}(N)$ satisfying $\mathfrak{n} \cap \mathcal{S}^*(N) \subset \mathfrak{n}^*$. By the correspondence theorem, $\operatorname{Spec}_s(\mathtt{j})(\mathfrak{n}) \in \operatorname{Cl}_{\operatorname{Spec}_s(M)}(N)$ is a maximal ideal of $\mathcal{S}(M)$ and, in fact, it equals $\mathfrak{m}$ because $\operatorname{Spec}_s{}^*(\mathtt{j})(\mathfrak{n}^*) = \mathfrak{m}^*$. Thus, also $\mathfrak{m} \cap \mathcal{S}^*(M) \in \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(N)$, as required. □

Later, it will be useful to approach the study of $\operatorname{Spec}_s(M)$ by comparing it with other already known spectra (see [**12**, 4.8-14]). To that end, we first need to introduce the following concept:

DEFINITION AND PROPOSITION 2.12 Given semialgebraic sets $Y \subset M \subset \mathbb{R}^m$, we denote $\mathcal{E}(Y) := \{f \in \mathcal{S}(M) : \ Z_M(f) = Y\}$ and define the *spectral envelope of* Y as the union $\mathcal{L}(Y) := \bigcup_{f \in \mathcal{E}(Y)} \mathcal{Z}_{\operatorname{Spec}_s(M)}(f)$. This set $\mathcal{L}(Y)$ satisfies (*by Lemma* 2.10(*i*))

$$Y \subset \operatorname{Cl}_{\operatorname{Spec}_s(M)}(Y) \subset \bigcap_{f \in \mathcal{E}(Y)} \mathcal{Z}_{\operatorname{Spec}_s(M)}(f) \subset \mathcal{L}(Y).$$

In fact, if M is locally compact, we have the equality $\operatorname{Cl}_{\operatorname{Spec}_s(M)}(Y) = \mathcal{L}(Y)$.

The spectral envelope has been studied in detail in [**12**, 4.8-14]. We now state its essential properties without proofs that are used frequently in Sections 4 and 5.

THEOREM 2.13 *Let* $N \subset M \subset \mathbb{R}^m$ *be semialgebraic sets such that* N *is open in* M *and locally compact. Denote* $Y := M \setminus N$ *and let* $\mathtt{j} : N \hookrightarrow M$ *be the inclusion map. The following properties hold*:

(i) *Let* $\mathfrak{q} \in \operatorname{Spec}_s(N)$ *and* $\mathfrak{p} := \operatorname{Spec}_s(\mathtt{j})(\mathfrak{q})$. *Then* $\mathrm{d}_N(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p})$.
(ii) *Let* $\mathfrak{p} \in \operatorname{Spec}_s(M) \setminus \mathcal{L}(Y)$. *Then* $\mathfrak{p}$ *is a z-ideal of* $\mathcal{S}(M)$, $\mathfrak{q} := \mathfrak{p}\mathcal{S}(N)$ *is a prime z-ideal of* $\mathcal{S}(N)$ *and* $\operatorname{Spec}_s(\mathtt{j})^{-1}(\mathfrak{p}) = \{\mathfrak{q}\}$.
(iii) *The map* $\operatorname{Spec}_s(\mathtt{j}) : \operatorname{Spec}_s(N) \to \operatorname{Spec}_s(M)$ *is a homeomorphism onto its image* $\operatorname{Spec}_s(M) \setminus \mathcal{L}(Y)$.
(iv) *If* $N \subsetneq M$ *is moreover a dense subset of* M *and* $\mathfrak{p}$ *is a minimal prime ideal of* $\mathcal{S}(M)$, *then* $\mathfrak{p} \notin \mathcal{L}(Y)$.

We take advantage of the following straightforward lemma concerning some general properties of chains of prime ideals of an arbitrary ring. In its statement, the reader could observe the shadow of the spectral envelope introduced above.

LEMMA 2.14 *Let* A *and* B *be two commutative rings with unity and let* $X \subset \operatorname{Spec}(A)$ *and* $Y \subset \operatorname{Spec}(B)$ *be*, *respectively*, *arbitrary unions of closed subsets of* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(B)$. *Suppose that there exists a homeomorphism* $\gamma : \operatorname{Spec}(A) \setminus X \to \operatorname{Spec}(B) \setminus Y$. *The following properties hold*:

(i) *Let* $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ *be a chain of prime ideals in* A *such that* $\mathfrak{p}_r \notin X$. *Then* $\gamma(\mathfrak{p}_0) \subsetneq \cdots \subsetneq \gamma(\mathfrak{p}_r)$ *is a chain of prime ideals in* B *such that* $\gamma(\mathfrak{p}_r) \notin Y$.
(ii) $\mathfrak{p} \in \operatorname{Spec}(A) \setminus X$ *is a minimal prime ideal of* A *if and only if* $\gamma(\mathfrak{p}) \in \operatorname{Spec}(B) \setminus Y$ *is a minimal prime ideal of* B.

Next, we compare the chains of prime ideals of $\mathcal{S}^\diamond(M)$ and those of $\mathcal{S}(X) = \mathcal{S}^*(X)$, where X is a semialgebraic compactification of a locally compact semialgebraic set M. Recall that a compactification $(X, \mathtt{j})$ of M is said to be a *semialgebraic compactification* of M if $\mathtt{j} : M \to X$ is a semialgebraic map (see also [**13**]).

LEMMA 2.15 *Let $M \subset \mathbb{R}^m$ be a locally compact semialgebraic set and $(X, \mathtt{j})$ a semialgebraic compactification of M. Denote $Y := X \setminus \mathtt{j}(M)$ and $Z := \operatorname{Cl}_{\operatorname{Spec}_s^*(X)}(Y) = \operatorname{Cl}_{\operatorname{Spec}_s(X)}(Y)$. Then*

(i) *The map $\operatorname{Spec}_s(\mathtt{j}) : \operatorname{Spec}_s(M) \to \operatorname{Spec}_s(X)$ is a homeomorphism onto its image $\operatorname{Spec}_s(X) \setminus Z$.*
(ii) *The map $\operatorname{Spec}_s^*(\mathtt{j}) : \operatorname{Spec}_s^*(M) \to \operatorname{Spec}_s^*(X)$ is surjective and the restriction map $\operatorname{Spec}_s^*(\mathtt{j})| : \operatorname{Spec}_s^*(M) \setminus \operatorname{Spec}_s^*(\mathtt{j})^{-1}(Z) \to \operatorname{Spec}_s^*(X) \setminus Z$ is a homeomorphism.*
(iii) *For each $\mathfrak{p} \in \operatorname{Spec}_s^*(X) \setminus Z$ the fiber $\operatorname{Spec}_s^*(\mathtt{j})^{-1}(\mathfrak{p})$ is a singleton.*
(iv) *If $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_s$ is a chain of prime ideals in $\mathcal{S}(X)$ such that $Z_M(f) \neq \emptyset$ for all $f \in \mathfrak{p}_s$, then $\mathfrak{p}_0\mathcal{S}(M) \subsetneq \cdots \subsetneq \mathfrak{p}_s\mathcal{S}(M)$ is a chain of prime ideals in $\mathcal{S}(M)$ of the same length.*
(v) *Given a chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in $\mathcal{S}(X)$, there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ in $\mathcal{S}^*(M)$ such that $\operatorname{Spec}_s(\mathtt{j})(\mathfrak{q}_i) = \mathfrak{p}_i$ for $i = 0, \ldots, r$.*

Proof. Parts (ii), (iii) and (v) follow from [12, 5.1, 5.4, 5.11]. Moreover, as X is locally compact, $\mathcal{L}(Y) = \operatorname{Cl}_{\operatorname{Spec}_s(X)}(Y)$, so (i) and (iv) follow straightforwardly from Theorem 2.13 and Lemma 2.14(i). Observe that the condition $Z_M(f) \neq \emptyset$ for all $f \in \mathfrak{p}_s$ in part (iv) guarantees $\mathfrak{p}_i \notin \mathcal{L}(Y)$ for all $i = 0, \ldots, s$. □

2.5. *Maximal spectra*

In this section, we focus our attention on a relevant subspace of $\operatorname{Spec}_s^\diamond(M)$: its maximal spectrum. We expose some properties and results of this space that are useful later. For further details see [13, 29].

2.5.1 We denote the collection of all maximal ideals of $\mathcal{S}^\diamond(M)$ with $\beta_s^\diamond M$. As usual, we consider the topology induced by the Zariski topology of $\operatorname{Spec}_s^\diamond(M)$ in $\beta_s^\diamond M$. Given $f, f_1, \ldots, f_r \in \mathcal{S}^\diamond(M)$, we denote in the following:

$$\mathcal{D}_{\beta_s^\diamond M}(f) := \mathcal{D}_{\operatorname{Spec}_s^\diamond(M)}(f) \cap \beta_s^\diamond M,$$
$$\mathcal{U}_{\beta_s^\diamond M}(f_1, \ldots, f_r) := \mathcal{U}_{\operatorname{Spec}_s^\diamond(M)}(f_1, \ldots, f_r) \cap \beta_s^\diamond M,$$
$$\mathcal{Z}_{\beta_s^\diamond M}(f) := \beta_s^\diamond M \setminus \mathcal{D}_{\beta_s^\diamond M}(f) = \mathcal{Z}_{\operatorname{Spec}_s^\diamond(M)}(f) \cap \beta_s^\diamond M.$$

By [2, 7.1.25(ii)], $\beta_s^\diamond M$ is a compact and Hausdorff space and it contains M as a dense subspace, that is, $\beta_s^\diamond M$ is a Hausdorff compactification of M. Observe that if M is compact, then the injective continuous map $\phi : M \to \beta_s^\diamond M,\ p \mapsto \mathfrak{m}^\diamond{}_p$ is in fact bijective (because in this case M is dense and closed in $\beta_s^\diamond M$) and so $\beta_s^\diamond M \equiv M$. As it happens for rings of continuous functions (see [15, Section 7]), the respective maximal spectra $\beta_s M$ and $\beta_s^* M$ of $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ are homeomorphic (see [29, Section 10] or [13, 3.5] for full details). More precisely,

2.5.2 The map $\Phi : \beta_s M \to \beta_s^* M$, which maps each maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$ to the unique maximal ideal $\mathfrak{m}^*$ of $\mathcal{S}^*(M)$ that contains $\mathfrak{m} \cap \mathcal{S}^*(M)$, is a homeomorphism. Moreover, $\Phi(\mathfrak{m}_p) = \mathfrak{m}_p^*$ for all $p \in M$.

Thus, it is not an abuse of notation to denote every maximal ideal of $\mathcal{S}^*(M)$ with $\mathfrak{m}^*$. Moreover, $\mathfrak{m}$ will denote the unique maximal ideal of $\mathcal{S}(M)$ such that $\mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$.

2.5.3 Observe that the inclusion map $\mathbb{R} \hookrightarrow \mathcal{S}^*(M)/\mathfrak{m}^*$, $r \mapsto r + \mathfrak{m}^*$ is an (injective) homomorphism of ordered fields; in fact, it is an isomorphism because $\mathcal{S}^*(M)/\mathfrak{m}^*$ is an archimedean extension of $\mathbb{R}$. Thus, since $\mathbb{R}$ admits a unique automorphism, there is no ambiguity to refer to $f + \mathfrak{m}^* \in \mathcal{S}^*(M)/\mathfrak{m}^* \cong \mathbb{R}$ as a real number for every $f \in \mathcal{S}^*(M)$. In particular, if $\mathfrak{m}^* = \mathfrak{m}_p^*$ for some point $p \in M$, the isomorphism $\mathcal{S}^*(M)/\mathfrak{m}_p^* \to \mathbb{R}$ identifies $f + \mathfrak{m}_p^*$ with $f(p)$. Therefore, each bounded semialgebraic function $f : M \to \mathbb{R}$ defines a natural extension $\hat{f} : \beta_s^* M \to \mathbb{R}$, $\mathfrak{m}^* \to f + \mathfrak{m}^*$, which is continuous because given real numbers $a < b$, then $\hat{f}^{-1}((a, b)) = \mathcal{U}_{\beta_s^* M}(f - a, b - f)$. Of course, since M is dense in $\beta_s^* M$, $\hat{f}$ is the unique continuous extension of f to $\beta_s^* M$.

2.5.4 In contrast to what happens when dealing with ideals of polynomial rings, the zeroset of a prime ideal $\mathfrak{p}$ of $\mathcal{S}^\diamond(M)$ does not provide substantial information about $\mathfrak{p}$ because it is either a point or the emptyset. An ideal $\mathfrak{a}$ of $\mathcal{S}^\diamond(M)$ is said to be *fixed* if all functions in $\mathfrak{a}$ vanish simultaneously at some point of M. Otherwise, the ideal $\mathfrak{a}$ is said to be *free*. The fixed maximal ideals of the ring $\mathcal{S}^\diamond(M)$ are those of the form $\mathfrak{m}_p^\diamond$ where $p \in M$. Clearly, $\mathfrak{m}_p \cap \mathcal{S}^*(M) = \mathfrak{m}_p^*$ for each point $p \in M$. In fact, the equality $\mathfrak{m} \cap \mathcal{S}^*(M) = \mathfrak{m}^*$ characterizes the fixed maximal ideals of $\mathcal{S}^\diamond(M)$ (see [13, 3.7]). Namely,

$$\mathfrak{m}^* \text{ is a fixed ideal } \iff \mathfrak{m} \text{ is a fixed ideal } \iff \mathfrak{m} \cap \mathcal{S}^*(M) = \mathfrak{m}^* \iff \operatorname{ht}(\mathfrak{m}) = \operatorname{ht}(\mathfrak{m}^*).$$

3. Dismantling and local compactness of semialgebraic sets

In this section, we present a dismantling of a semialgebraic set $M \subset \mathbb{R}^m$ into pure dimensional semialgebraic subsets that we call *bricks*, which are useful to characterize the minimal prime ideals of $\mathcal{S}^\diamond(M)$. We recall the main properties of the set M_{lc} of points of a semialgebraic set $M \subset \mathbb{R}^m$, which have a compact neighbourhood in M (see [8, 9.14–9.21]). This requires some preparation.

Definition 3.1 Let $M \subset \mathbb{R}^m$ be a d-dimensional semialgebraic set. We denote the set of *regular points* of M with $\operatorname{Reg}(M)$, that is, those points $x \in M$, which have a neighbourhood V^x in M analytically diffeomorphic to $\mathbb{R}^d$. We denote $\delta(M) := M \setminus \operatorname{Reg}(M)$. By [27], $\operatorname{Reg}(M)$ is an open semialgebraic subset of M and, since it is an analytic manifold, we deduce from [26, I.3.9] that $\operatorname{Reg}(M)$ is a Nash manifold. In fact, the set $\operatorname{Reg}(M)$ of regular points of M is non-empty and $\delta(M)$ is a semialgebraic set of dimension $\leq d - 1$.

3.1. *Decomposition into pure dimensional semialgebraic bricks*

A main difference between real and complex algebraic geometry is the existence of real irreducible algebraic sets with pieces of arbitrary prescribed dimensions, see [1, 4.4]. This kind of sets have a wilder behaviour than the pure dimensional ones. So, our next aim is to decompose a semialgebraic set $M \subset \mathbb{R}^m$ into an irredundant finite union of closed pure dimensional semialgebraic subsets of M. Consider the following semialgebraic sets, defined recursively: $T_0 := M$, $M_i := \operatorname{Cl}_M(\operatorname{Reg}(T_{i-1}))$, $T_i := \delta(T_{i-1}) \setminus \bigcup_{k=1}^i M_k$ for $i \geq 1$.

By Definition 3.1, each M_i is either empty or a pure dimensional semialgebraic set. Moreover, it follows from Definition 3.1 that for $i \geq 2$

$$\dim(M_i) = \dim(\mathrm{Cl}_M(\mathrm{Reg}(T_{i-1}))) = \dim(T_{i-1}) = \dim\left(\delta(T_{i-2}) \Big\backslash \bigcup_{k=1}^{i-1} M_k\right)$$
$$\leq \dim(\delta(T_{i-2})) < \dim(T_{i-2}) = \dim(\mathrm{Cl}_M(\mathrm{Reg}(T_{i-2}))) = \dim(M_{i-1}).$$

In particular, there are only finitely many non-empty M_is, say the first r, and so the same happens to the T_is. Observe also

$$T_k = \mathrm{Reg}(T_k) \cup \delta(T_k) \subset M_{k+1} \cup (\delta(T_k) \setminus M_{k+1}) \subset \bigcup_{j=1}^{k+1} M_j \cup T_{k+1}.$$

Therefore, $M = T_0 = \bigcup_{i=1}^{r} M_i$, and we have written M as a union of pure dimensional closed semialgebraic subsets of M with $\dim(M_i) \neq \dim(M_j)$ for $i \neq j$ and

$$\dim(M_i \cap M_j) < \min\{\dim(M_i), \dim(M_j)\}.$$

Indeed, since each M_i is pure dimensional, it is enough to check that if $i > j$, then $M_i \setminus M_j = M_i \setminus (M_i \cap M_j)$ is dense in M_i. Otherwise, there exists an open subset $V \subset \mathbb{R}^m$ such that $\emptyset \neq V \cap M_i \subset M_j$. Since $M_i = \mathrm{Cl}_M(\mathrm{Reg}(T_{i-1}))$, we have

$$\emptyset \neq V \cap \mathrm{Reg}(T_{i-1}) \subset M_j \cap T_{i-1} = M_j \cap \left(\delta(T_{i-2}) \Big\backslash \bigcup_{k=1}^{i-1} M_k\right) \subset M_j \setminus M_j = \emptyset,$$

which is a contradiction. Thus, $\dim(M_i \cap M_j) < \dim(M_i) < \dim(M_j)$.

3.1.1 Therefore, $M_i \setminus \bigcup_{j\neq i} M_j$ is dense in M_i. Moreover, $\mathrm{Cl}_M(M \setminus (M_i \cap M_j)) = M$ if $i \neq j$. Indeed, for each $k = 1, \ldots, r$ the dimension of $M_k \cap (M_i \cap M_j)$ is strictly smaller than $\dim(M_k)$ and so $\mathrm{Cl}_M(M_k \setminus (M_i \cap M_j)) = M_k$. Thus, $\mathrm{Cl}_M(M \setminus (M_i \cap M_j)) = M$.

3.1.2 Let $d_i := \dim(M_i)$ and S_i be the set of points $p \in M$ such that the germ M_p is pure dimensional and has dimension d_i. Notice that M_i is the closure of S_i in M.

Proposition and Definition 3.2 Let $M \subset \mathbb{R}^m$ be a semialgebraic set. Then the family $\{M_1, \ldots, M_r\}$ of semialgebraic subsets of M we have constructed is unique and satisfies the following properties:

(i) Each M_i is a pure dimensional and closed subset of M.
(ii) $M = \bigcup_{i=1}^{r} M_i$.
(iii) $M_i \setminus \bigcup_{j\neq i} M_j$ is dense in M_i.
(iv) $\dim(M_i) > \dim(M_{i+1})$ for $i = 1, \ldots, r-1$. In particular, $\dim(M_1) = \dim(M)$.

We call the sets M_i the *bricks* of M and denote the family of bricks of M with $\mathcal{B}_M := \{\mathcal{B}_i(M) := M_i : i = 1, \dots, r\}$.

Proof. We proceed by induction. If M has dimension 0, the result is trivially true. We assume that the result is true for semialgebraic sets of dimension $\leq d-1$ and have to prove that it also holds for $d := \dim(M)$.

Let $N_1, \dots, N_s$ be another family of semialgebraic subsets of M satisfying conditions (i) to (iv). Let us see first that $M_1 = N_1$. Note that $T := \bigcup_{j=2}^{s} N_j$ has dimension $\leq d-1$ and so, since M_1 is pure dimensional, $M_1 \setminus T$ is dense in M_1. Observe also

$$M_1 \setminus T \subset M \setminus T = \bigcup_{j=1}^{s} N_j \setminus T = N_1 \setminus T \subset N_1$$

and as N_1 is closed in M, we get $M_1 \subset N_1$. By symmetry $N_1 \subset M_1$ and so $N_1 = M_1$.

Next, let us see $S := \bigcup_{i=2}^{r} M_i = T$. Note that $M_1 \cup T = N_1 \cup T = M = M_1 \cup S$ and $S \setminus M_1$ is dense in S. We have

$$S \setminus M_1 \subset M \setminus M_1 = (M_1 \cup T) \setminus M_1 = T \setminus M_1 \subset T,$$

and since T is closed in M, we deduce $S \subset T$. Thus, again by symmetry, $S = T$. Now, the families $M_2, \dots, M_r$ and $N_2, \dots, N_s$ satisfy all properties (i) to (iv) for the set $T = S$, and we finish by applying the induction hypotheses to $S = T$. □

COROLLARY 3.3 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets such that N is dense in M. Then the families of bricks of N and M satisfy the following relations*:

(i) $\mathcal{B}_M = \{\mathcal{B}_i(M) = \operatorname{Cl}_M(\mathcal{B}_i(N))\}_i$.
(ii) $\mathcal{B}_N = \{\mathcal{B}_i(N) = \mathcal{B}_i(M) \cap N\}_i$.
(iii) $\operatorname{Spec}_s^*(\mathtt{j})(\operatorname{Cl}_{\operatorname{Spec}_s^*(N)}(\mathcal{B}_i(N))) = \operatorname{Cl}_{\operatorname{Spec}_s^*(M)}(\mathcal{B}_i(M))$, *where* $\mathtt{j} : N \hookrightarrow M$ *is the inclusion map for each index* i.

Proof. For the first equality it is enough to check that the family of semialgebraic sets $\{\operatorname{Cl}_M(\mathcal{B}_i(N))\}_i$ satisfies the properties (i)–(iv) in Proposition and Definition 3.2 relative to M. This is immediate for properties (i), (iii) and (iv) in Proposition and Definition 3.2 because the $\mathcal{B}_i(N)$s are the bricks of N. Let us see that they also satisfy property (ii) in Proposition and Definition 3.2. Indeed, since $\mathcal{B}_i(N) \subset N$, we have

$$\begin{aligned}\mathcal{B}_i(N) \Big\backslash \bigcup_{j \neq i} \operatorname{Cl}_M(\mathcal{B}_j(N)) &= \mathcal{B}_i(N) \Big\backslash \bigcup_{j \neq i} \operatorname{Cl}_M(\mathcal{B}_j(N)) \cap N \\ &= \mathcal{B}_i(N) \Big\backslash \bigcup_{j \neq i} \operatorname{Cl}_N(\mathcal{B}_j(N)) = \mathcal{B}_i(N) \Big\backslash \bigcup_{j \neq i} \mathcal{B}_j(N),\end{aligned}$$

which is a dense subset of $\mathcal{B}_i(N)$. Thus, the difference $\operatorname{Cl}_M(\mathcal{B}_i(N)) \setminus \bigcup_{j \neq i} \operatorname{Cl}_M(\mathcal{B}_j(N))$ is dense in $\operatorname{Cl}_M(\mathcal{B}_i(N))$ and we are done.

The second part is an immediate consequence of the first one. Since $\mathcal{B}_i(N)$ is closed in N, we get

$$\mathcal{B}_i(M) \cap N = \operatorname{Cl}_M(\mathcal{B}_i(N)) \cap N = \operatorname{Cl}_N(\mathcal{B}_i(N)) = \mathcal{B}_i(N),$$

as wanted.

Finally, statement (iii) is a particular case of [12, 5.15] because $C := \mathcal{B}_i(M)$ is by Proposition and Definition 3.2 a closed subset of M and $C_1 := C \cap N = \mathcal{B}_i(M) \cap N = \mathcal{B}_i(N)$ is dense in C. □

COROLLARY 3.4 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets such that N is open in M and let $\mathcal{B}_M := \{\mathcal{B}_i(M) : 1 \le i \le r\}$ be the family of bricks of M. Let $1 \le i_1 < \cdots < i_s \le r$ be those indices $1 \le i \le r$ such that the intersection $\mathcal{B}_i(M) \cap N$ is non-empty. Then $\mathcal{B}_N = \{\mathcal{B}_j(N) = \mathcal{B}_{i_j}(M) \cap N : 1 \le j \le s\}$ is the family of bricks of N.*

Proof. Note that each $\mathcal{B}_j(N) = \mathcal{B}_{i_j}(M) \cap N$ is an open pure dimensional semialgebraic subset of $\mathcal{B}_{i_j}(M)$ with $\dim(\mathcal{B}_j(N)) = \dim(\mathcal{B}_{i_j}(M))$. Moreover, $N = \bigcup_{j=1}^{s} \mathcal{B}_j(N)$ and

$$\dim(\mathcal{B}_j(N)) = \dim(\mathcal{B}_{i_j}(M)) > \dim(\mathcal{B}_{i_{j+1}}(M)) = \dim(\mathcal{B}_{j+1}(N)).$$

Finally, as N is open in M, it follows

$$\begin{aligned} &\operatorname{Cl}_{\mathcal{B}_j(N)}\Big(\mathcal{B}_j(N) \setminus \bigcup_{\ell \neq j} \mathcal{B}_\ell(N)\Big) \\ &\quad = N \cap \operatorname{Cl}_{\mathcal{B}_{i_j}(M)}\Big(\Big(\mathcal{B}_{i_j}(M) \setminus \bigcup_{\ell \neq j} \mathcal{B}_{i_\ell}(M)\Big) \cap N\Big) \\ &\quad = N \cap \operatorname{Cl}_{\mathcal{B}_{i_j}(M)}\Big(\mathcal{B}_{i_j}(M) \setminus \bigcup_{\ell \neq j} \mathcal{B}_{i_\ell}(M)\Big) = N \cap \mathcal{B}_{i_j}(M) = \mathcal{B}_j(N). \end{aligned}$$

This proves, using Proposition and Definition 3.2, that $\mathcal{B}_N$ is the family of bricks of N. □

REMARK 3.5 The bricks of a semialgebraic set are somehow related to the definable semialgebraic blocks introduced in [18, 3.2]. Recall that a *basic definable semialgebraic block* or *basic block* of dimension κ in $\mathbb{R}^n$ is a connected definable set $U \subset \mathbb{R}^n$ of dimension κ contained in some semialgebraic set A of dimension κ such that every point x of U is a $\mathcal{C}^1$-regular point of dimension κ in U and in A. Dimension zero is allowed: a point is a basic block. A *definable semialgebraic block* or *block* is the image of a basic block U under a semialgebraic map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ defined and continuous on a semialgebraic set containing U. For further details on the involved concepts we refer the reader to [18].

In [18, 3.3], it is presumed that the image of a basic block under a map $f : \mathbb{R}^n \to \mathbb{R}^m$ with semialgebraic graph is the union of finitely many basic blocks. We propose a short proof for this fact here.

Indeed, let $U \subset \mathbb{R}^n$ be a basic block of dimension d and A a connected semialgebraic set that contains U and is a $\mathcal{C}^1$ manifold of dimension d. Note that U is an open subset of A. Let $f : A \to \mathbb{R}^m$ be a map with semialgebraic graph Γ. By [2, 2.9.10], Γ is the disjoint union of finitely many semialgebraic smooth submanifolds Γ_i, each of them semialgebraically diffeomorphic to $(-1, 1)^{\dim(\Gamma_i)}$.

Let $A_i \subset A$ be the semialgebraic set such that Γ_i is the graph of $f|_{A_i}$. As Γ_i is a semialgebraic smooth manifold, so is A_i. Note that A is the pairwise disjoint union of the A_is; hence, $U = \bigsqcup_i U_i$ where $U_i = A_i \cap U$. As $f(U) = \bigcup_i f(U_i)$, it is enough to show that $f(U_i)$ is a finite union of basic blocks. Therefore, we assume from the beginning that we are in the following situation: *U is an open definable subset of a connected semialgebraic smooth manifold A of dimension d and $f : A_i \to \mathbb{R}^m$ is a semialgebraic smooth map*. By means of [2, 9.2.1-3] we can stratify A as a pairwise disjoint finite union of semialgebraic smooth submanifolds A_j such that $f|_{A_j}$ has constant rank. Thus, we assume in addition that *$f : A \to \mathbb{R}^m$ has constant rank*.

By the constant rank theorem [2, 9.6.1], we conclude that f is an open map and $f(A)$ is a semialgebraic smooth manifold of dimension $\kappa := d - \mathrm{rk}(f)$. Thus, $f(U)$ is an open definable subset of $f(A)$; hence, $f(U)$ is a definable $\mathcal{C}^1$-manifold of dimension κ contained in the semialgebraic smooth manifold $f(A)$ of dimension κ. To finish, we must show that $f(U)$ is a finite union of basic blocks, but this follows straightforwardly considering the (definable) connected components of $f(U)$ (use [9, 4.3]).

3.2. *Local compactness of semialgebraic sets*

Local closedness has been revealed in the semialgebraic setting as an important property for the validity of results, which are in the core of semialgebraic geometry. As is well-known, the locally closed subsets of a locally compact topological space coincide with the locally compact ones (see for instance [3, Section 9.7. Proposition 12-13]). If $M \subset \mathbb{R}^m$ is a semialgebraic set, then $\mathrm{Cl}_{\mathbb{R}^m}(M)$ and $U = \mathbb{R}^m \setminus (\mathrm{Cl}_{\mathbb{R}^m}(M) \setminus M)$ are also semialgebraic sets. If M is moreover locally compact, then U is open in $\mathbb{R}^m$ and so $M = \mathrm{Cl}_{\mathbb{R}^m}(M) \cap U$ can be written as the intersection of a closed and an open semialgebraic subset of $\mathbb{R}^m$.

To obtain certain results for an arbitrary semialgebraic set M, we compare the rings $\mathcal{S}^\diamond(M)$ and $\mathcal{S}^\diamond(M_{\mathrm{lc}})$ in the subsequent sections, where M_{lc} is the largest locally compact and dense subset of M. It turns out that M_{lc} is semialgebraic, and its construction is the main goal of the next result whose proof follows from [8, 9.14–9.21].

THEOREM 3.6 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set. Define*

$$\rho_0(M) := \mathrm{Cl}_{\mathbb{R}^m}(M) \setminus M \quad \textit{and} \quad \rho_1(M) := \rho_0(\rho_0(M)) = \mathrm{Cl}_{\mathbb{R}^m}(\rho_0(M)) \cap M.$$

The following properties hold:

(i) *Let $C \subset M$ be a closed semialgebraic subset of M. Then $\rho_1(C) \subset \rho_1(M)$.*
(ii) *The set $M_{\mathrm{lc}} := M \setminus \rho_1(M) = \mathrm{Cl}_{\mathbb{R}^m}(M) \setminus \mathrm{Cl}_{\mathbb{R}^m}(\rho_0(M))$, which is semialgebraic, is the largest locally compact and dense subset of M.*
(iii) *M_{lc} equals the set of points of M, which have a compact neighbourhood in M.*

4. Minimal prime ideals and semialgebraic depth

4.1. *Minimal prime ideals of rings of semialgebraic functions*

In the first part of this section, we characterize the minimal prime ideals of the ring $\mathcal{S}^\diamond(M)$ for an arbitrary semialgebraic set M and study some properties of this kind of ideals. For instance, all minimal prime ideals are z-ideals. This fact will be useful in the second part of this section to approach the study of the semialgebraic depth of prime z-ideals of the ring $\mathcal{S}(M)$ of an arbitrary semialgebraic set M. Our main result concerning minimal prime ideals of the ring $\mathcal{S}^\diamond(M)$ is the following.

THEOREM 4.1 (Minimal prime ideals) *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p}$ a prime ideal of $\mathcal{S}^\diamond(M)$. Let $\{\mathcal{B}_i(M)\}_{i=1}^r$ be the bricks of M. Then $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}^\diamond(M)$ if and only if there exists $i = 1, \dots, r$ such that for each $f \in \mathfrak{p}$ its zeroset $Z_M(f)$ contains a non-empty open subset of $\mathcal{B}_i(M)$.*

To approach the proof of Theorem 4.1, we recall an elementary but useful criterion of minimality for prime ideals that is used along this work (see also [16]).

LEMMA 4.2 *Let A be a reduced commutative ring with unity and $\mathfrak{p}$ a prime ideal of A. Then $\mathfrak{p}$ is a minimal prime ideal of A if and only if for every $f \in \mathfrak{p}$ there exists $g \in A \setminus \mathfrak{p}$ such that $fg = 0$.*

A useful consequence of the previous criterion is the following result, which reduces the proof of Theorem 4.1 to the case $\mathcal{S}^\diamond(M) = \mathcal{S}(M)$ (see also [29, 14.6; 23, Section 7; 25]).

LEMMA 4.3 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\operatorname{Min}(\mathcal{S}^\diamond(M))$ denote the family of minimal prime ideals of the ring $\mathcal{S}^\diamond(M)$. Then $\mathfrak{p} \cap \mathcal{W}(M) = \emptyset$ for every $\mathfrak{p} \in \operatorname{Min}(\mathcal{S}^*(M)$, and the map*

$$\psi : \operatorname{Min}(\mathcal{S}^*(M)) \to \operatorname{Min}(\mathcal{S}(M)), \quad \mathfrak{p} \mapsto \mathfrak{p}\mathcal{S}(M)$$

is a bijection whose inverse map is $\psi^{-1} : \operatorname{Min}(\mathcal{S}(M)) \to \operatorname{Min}(\mathcal{S}^*(M)),\ \mathfrak{q} \mapsto \mathfrak{q} \cap \mathcal{S}^*(M)$.

Proof. Let $\mathfrak{p} \in \operatorname{Min}(\mathcal{S}^*(M))$ and suppose by contradiction $\mathfrak{p} \cap \mathcal{W}(M) \neq \emptyset$. Let $f \in \mathfrak{p} \cap \mathcal{W}(M)$; since $\mathfrak{p}$ is a minimal prime ideal, there exists a function $g \in \mathcal{S}^*(M) \setminus \mathfrak{p}$ such that $fg = 0$. But as $Z_M(f)$ is empty, it follows that $Z_M(g) = M$, or equivalently $g = 0$, which is a contradiction.

This implies by Lemma 2.9 that $\mathfrak{p}\mathcal{S}(M)$ is a minimal prime ideal of $\mathcal{S}(M)$ for every minimal prime ideal $\mathfrak{p} \in \operatorname{Min}(\mathcal{S}^*(M))$. Thus, the map ψ is well defined and by Lemma 2.9 injective. To prove its surjectivity, we must show that for every $\mathfrak{q} \in \operatorname{Min}(\mathcal{S}(M))$ it holds $\mathfrak{q} \cap \mathcal{S}^*(M) \in \operatorname{Min}(\mathcal{S}^*(M))$. Otherwise there would exist $\mathfrak{q} \in \operatorname{Min}(\mathcal{S}(M))$ and $\mathfrak{p} \in \operatorname{Min}(\mathcal{S}^*(M))$ such that $\mathfrak{p} \subsetneq \mathfrak{q} \cap \mathcal{S}^*(M)$. But $\mathfrak{p} \cap \mathcal{W}(M) = \emptyset$ and so $\mathfrak{p}\mathcal{S}(M)$ is by Lemma 2.9 a prime ideal of $\mathcal{S}(M)$, which is strictly contained in $\mathfrak{q}$. This is against the minimality of $\mathfrak{q}$, as wanted. □

REMARK 4.4 If we consider in $\operatorname{Min}(\mathcal{S}^\diamond(M))$ the topology induced by the one of $\operatorname{Spec}_s^\diamond(M)$, it follows from Lemma 2.9 that the map ψ in Lemma 4.3 is a homeomorphism.

We are ready to prove Theorem 4.1. As commented above, it is by Lemma 4.3 sufficient to deal with the case $\mathcal{S}^\diamond(M) = \mathcal{S}(M)$. We assume first that M is locally compact.

Proof of Theorem 4.1 *for M locally compact* Let $\mathfrak{p}$ be a prime ideal of $\mathcal{S}(M)$. Since each brick $\mathcal{B}_j(M)$ is closed in M, there exist by Lemma 2.1 semialgebraic functions $g_j \in \mathcal{S}(M)$ such that $\mathcal{B}_j(M) = Z_M(g_j)$. Since $M = Z_M(g_1 \cdots g_r)$ and so the product $\prod_{i=1}^r g_i = 0 \in \mathfrak{p}$, there exists $k = 1, \dots, r$ such that $g_k \in \mathfrak{p}$.

(4.1.1) Let us see that if $\mathfrak{p}$ is a minimal prime ideal, such index k is unique.

Suppose by contradiction that there exists $k \neq \ell$ such that $g_k, g_\ell \in \mathfrak{p}$. Then $h = g_k^2 + g_\ell^2 \in \mathfrak{p}$ and $Z_M(h) = \mathcal{B}_k(M) \cap \mathcal{B}_\ell(M)$. By the minimality of $\mathfrak{p}$ there exists a function $b \in \mathcal{S}(M) \setminus \mathfrak{p}$ such that

$hb = 0$. Thus, the function b vanishes on the dense subset $M \setminus Z_M(h) = M \setminus (\mathcal{B}_k(M) \cap \mathcal{B}_\ell(M))$ of M (see Proposition and Definition 3.2) and so $b = 0$, which is a contradiction.

We proceed now by proving the equivalence claimed in Theorem 4.1.

($\Longrightarrow$) Let $1 \le k \le r$ be the unique index such that $g_k \in \mathfrak{p}$. Let us prove now that if $f \in \mathfrak{p}$, then $Z_M(f)$ contains a non-empty open subset of $\mathcal{B}_k(M) = Z_M(g_k)$. Since $\mathfrak{p}$ is a minimal prime ideal, there exists $g \in \mathcal{S}(M) \setminus \mathfrak{p}$ such that $fg = 0$.

If $Z_M(f)$ does not contain a non-empty open subset of the pure dimensional semialgebraic set $\mathcal{B}_k(M)$, then $\dim(Z_M(f) \cap \mathcal{B}_k(M)) < \dim(\mathcal{B}_k(M))$ and so $\mathcal{B}_k(M) \setminus (Z_M(f) \cap \mathcal{B}_k(M))$ is a dense subset of $\mathcal{B}_k(M)$ contained in $Z_M(g)$ because $\mathcal{B}_k(M)$ is pure dimensional. Consequently, $g|_{\mathcal{B}_k(M)} = 0$, that is, $Z_M(g_k) \subset Z_M(g)$. This implies by Theorem 2.6 that $g \in \mathfrak{p}$, which is a contradiction. Thus, f vanishes identically on a non-empty open subset of $\mathcal{B}_k(M)$, as claimed.

($\Longleftarrow$) Conversely, let $1 \le k \le r$ be such that $Z_M(f)$ contains a non-empty open subset of $\mathcal{B}_k(M)$ for all $f \in \mathfrak{p}$. Hence, the product $g := \prod_{j \neq k} g_j \notin \mathfrak{p}$ because $\mathcal{B}_k(M) \setminus Z_M(g) = \mathcal{B}_k(M) \setminus \bigcup_{j \neq k} \mathcal{B}_k(M)$ is a dense subset of $\mathcal{B}_k(M)$ and so $Z_M(g)$ does not contain a non-empty open subset of $\mathcal{B}_k(M)$. Note that $M \setminus \mathcal{B}_k(M) \subset Z_M(g)$. Thus, $g_k g = 0 \in \mathfrak{p}$, so $g_k \in \mathfrak{p}$.

Fix $f \in \mathfrak{p}$ and let us find a function in $\mathcal{S}(M) \setminus \mathfrak{p}$ whose product with f is zero. Note that $a = f^2 + g_k^2 \in \mathfrak{p}$ and $Y := Z_M(a) = Z_M(f) \cap \mathcal{B}_k(M) \subset \mathcal{B}_k(M)$ is a closed subset of M, which contains a non-empty open subset of $\mathcal{B}_k(M)$. Consider the semialgebraic set $Z := \delta(Y) \cup \delta(\mathcal{B}_k(M))$ whose dimension is by Definition 3.1 strictly smaller than $\dim(Y) = \dim(\mathcal{B}_k(M))$. Observe that since Y and $\mathcal{B}_k(M)$ are closed in M, also Z is by Definition 3.1 closed in M. Moreover, $Y \setminus Z$ is a Nash manifold contained in the Nash manifold $\mathcal{B}_k(M) \setminus Z$. As both manifolds have the same dimension, $Y \setminus Z$ is an open subset of $\mathcal{B}_k(M) \setminus Z$; hence, of $\mathcal{B}_k(M)$. Therefore,

$$C := \mathcal{B}_k(M) \setminus (Y \setminus Z) = (\mathcal{B}_k(M) \setminus Y) \cup Z$$

is a closed semialgebraic set in $\mathcal{B}_k(M)$; hence, in M. By Lemma 2.1, there exists a semialgebraic function $c \in \mathcal{S}(M)$ such that $C = Z_M(c)$. We claim that $c \notin \mathfrak{p}$. Otherwise, the sum $b = c^2 + a^2 \in \mathfrak{p}$ and its zeroset satisfies

$$Z_M(b) = Z_M(c) \cap Y = C \cap Y = Z \cap Y \subset Z.$$

Thus, $\dim(Z_M(b)) \le \dim(Z) < \dim(\mathcal{B}_k(M))$ and so $b \in \mathfrak{p}$ but $Z_M(b)$ does not contain any non-empty open subset of $\mathcal{B}_k(M)$, which is against the hypothesis; hence, $c \notin \mathfrak{p}$. Observe also

$$Z_M(cf) = Z_M(c) \cup Z_M(f) \supset (\mathcal{B}_k(M) \setminus Y) \cup Z \cup Z_M(a) = (\mathcal{B}_k(M) \setminus Y) \cup Z \cup Y = \mathcal{B}_k(M).$$

Hence, $cg \in \mathcal{S}(M) \setminus \mathfrak{p}$ and $fcg = 0$. Consequently, $\mathfrak{p}$ is a minimal prime ideal, as wanted. □

We proceed to prove Theorem 4.1 in the general setting. Namely,

Proof of Theorem 4.1 *for arbitrary* M Denote $Y := \rho_1(M)$, and $M_{\text{lc}} := M \setminus Y$, which is open and dense in M (see Theorem 3.6). This together with Corollary 3.3 implies that the bricks of M_{lc} are $\mathcal{B}_i(M_{\text{lc}}) = \mathcal{B}_i(M) \cap M_{\text{lc}}$ and $\mathcal{B}_i(M_{\text{lc}})$ is open and dense in $\mathcal{B}_i(M)$ for $i = 1, \dots, r$.

Consider the inclusion map $\mathtt{j} : M_{\text{lc}} \hookrightarrow M$ and recall that the induced map $\operatorname{Spec}_s(\mathtt{j}) : \operatorname{Spec}_s(M_{\text{lc}}) \to \operatorname{Spec}_s(M)$ is a homeomorphism onto its image $\operatorname{Spec}_s(M) \setminus \mathcal{L}(Y)$ by Theorem 2.13(iii). Moreover, since $M_{\text{lc}} \subsetneq M$ is dense in M, this image $\operatorname{Spec}_s(M) \setminus \mathcal{L}(Y)$ contains all minimal prime ideals of $\mathcal{S}(M)$.

Let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{S}(M)$. Then its preimage $\mathfrak{q} := \operatorname{Spec_s}(\mathtt{j})^{-1}(\mathfrak{p})$ is by Lemma 2.14(ii) a minimal prime ideal of $\mathcal{S}(M_{\rm lc})$. Thus, since $M_{\rm lc}$ is locally compact, there exists $k = 1, \dots, r$ such that for each $f \in \mathfrak{q}$ its zeroset $Z_{M_{\rm lc}}(f)$ contains a non-empty open subset of $\mathcal{B}_k(M_{\rm lc}) = \mathcal{B}_k(M) \cap M_{\rm lc}$. Hence, since $\mathfrak{p} = \mathfrak{q} \cap \mathcal{S}(M)$ and $\mathcal{B}_k(M) \cap M_{\rm lc}$ is open and dense in $\mathcal{B}_k(M)$, we deduce that for each $f \in \mathfrak{p}$ the zeroset $Z_M(f)$ contains a non-empty open subset of $\mathcal{B}_k(M)$.

Conversely, let $\mathfrak{p}$ be a prime ideal of $\mathcal{S}(M)$ and suppose the existence of $k = 1, \dots, r$ such that for all $f \in \mathfrak{p}$ the set $Z_M(f)$ contains a non-empty open subset of $\mathcal{B}_k(M)$. Hence, $\mathfrak{p} \notin \mathcal{L}(Y)$ because $M_{\rm lc}$ is dense in M. Thus, $\operatorname{Spec_s}(\mathtt{j})^{-1}(\mathfrak{p}) = \mathfrak{p}\mathcal{S}(M_{\rm lc})$ is a prime ideal by Theorem 2.13(ii). Again, by the density of $M_{\rm lc}$ in M, the set $Z_{M_{\rm lc}}(f)$ contains a non-empty open subset of $\mathcal{B}_k(M) \cap M_{\rm lc} = \mathcal{B}_k(M_{\rm lc})$ for all $f \in \mathfrak{p}\mathcal{S}(M_{\rm lc})$. Hence, $\mathfrak{p}\mathcal{S}(M_{\rm lc})$ is a minimal prime ideal of $\mathcal{S}(M_{\rm lc})$ and $\mathfrak{p} = \operatorname{Spec_s}(\mathtt{j})(\mathfrak{p}\mathcal{S}(M_{\rm lc}))$ is a minimal prime ideal of $\mathcal{S}(M)$ by Lemma 2.14(ii), as wanted. □

As a straightforward consequence of Theorem 4.1, we deduce the following: If the semialgebraic depth of a prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$ coincides with the dimension of M, then $\mathfrak{p}$ is a minimal prime ideal. Namely,

Corollary 4.5 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p}$ a prime ideal of $\mathcal{S}(M)$ such that $\mathrm{d}_M(\mathfrak{p}) = \dim(M)$. Then $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}(M)$.*

It is worthwhile to mention that minimal prime ideals enjoy a nice behaviour with respect to contraction.

Proposition 4.6 *Let $N \subset M \subset \mathbb{R}^m$ be semialgebraic sets, $\mathtt{j} : N \hookrightarrow M$ the inclusion and $\operatorname{Spec}^\diamond_{\rm s}(\mathtt{j}) : \operatorname{Spec}^\diamond_{\rm s}(N) \to \operatorname{Spec}^\diamond_{\rm s}(M)$ the induced map. Let $\mathfrak{q}$ be a minimal prime ideal of $\mathcal{S}^\diamond(N)$ and denote $\mathfrak{p} := \operatorname{Spec}^\diamond_{\rm s}(\mathtt{j})(\mathfrak{q})$. Then*:

(i) *If N is dense in M, then $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}^\diamond(M)$.*
(ii) *If N is open in M, then $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}^\diamond(M)$.*

Proof. (i) Let $\mathcal{B}_1(N), \dots, \mathcal{B}_r(N)$ be the bricks of N. The bricks of $M = \operatorname{Cl}_M(N)$ are $\operatorname{Cl}_M(\mathcal{B}_1(N)), \dots, \operatorname{Cl}_M(\mathcal{B}_r(N))$ by Corollary 3.3.

Note that the dimension of each non-empty open semialgebraic subset V of $\mathcal{B}_i(N)$ equals $\dim(\mathcal{B}_i(N))$ while by [2, 2.8.13] and Theorem 3.6

$$\dim(\rho_1(\mathcal{B}_i(N))) = \dim(\rho_0(\rho_0(\mathcal{B}_i(N)))) \le \dim(\mathcal{B}_i(N)) - 2.$$

On the other hand, observe that the set $(\mathcal{B}_i(N))_{\rm lc} = \mathcal{B}_i(N) \setminus \rho_1(\mathcal{B}_i(N))$ is open in $\operatorname{Cl}_{\mathbb{R}^m}(\mathcal{B}_i(N))$ by Theorem 3.6 and so it is also open in $\operatorname{Cl}_M(\mathcal{B}_i(N))$. Hence, $V \cap (\mathcal{B}_i(N))_{\rm lc} = V \setminus \rho_1(\mathcal{B}_i(N))$ is a non-empty open subset of $\operatorname{Cl}_M(\mathcal{B}_i(N))$. This together with Theorem 4.1 implies that $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}^\diamond(M)$.

(ii) Let $\mathfrak{B}_M := \{\mathcal{B}_i(M) : 1 \le i \le r\}$ be the family of bricks of M. Then $\mathfrak{B}_N := \{\mathcal{B}_j(N) = \mathcal{B}_{i_j}(M) \cap N : 1 \le j \le s\}$ is the family of bricks of N by Corollary 3.4, for those indices $1 \le i_1 < \cdots < i_s \le r$ such that $\mathcal{B}_{i_j}(M) \cap N \ne \emptyset$. Since $\mathfrak{q}$ is a minimal prime ideal of $\mathcal{S}^\diamond(N)$, there exists by Theorem 4.1 an index $1 \le j \le s$ such that the zeroset of each function in $\mathfrak{q}$ contains a non-empty subset of $\mathcal{B}_j(N)$. Now, for every $f \in \mathfrak{p}$ its restriction $f|_N \in \mathfrak{q}$ and so $Z_N(f)$ contains a non-empty open subset V of $\mathcal{B}_j(N)$. But since N is open in M, the set V is a non-empty subset of $\mathcal{B}_{i_j}(M)$ contained in $Z_M(f)$. Thus, $\mathfrak{p}$ is by Theorem 4.1 a minimal prime ideal of $\mathcal{S}^\diamond(M)$. □

As one can expect, the minimal prime ideals of $\mathcal{S}(M)$ are z-ideals.

COROLLARY 4.7 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p}$ a minimal prime ideal of $\mathcal{S}(M)$. Then $\mathfrak{p}$ is a z-ideal.*

Proof. Let $Y := M \setminus M_{\rm lc}$. Since $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}(M)$, we know by Theorem 2.13(iv) and the fact that $M_{\rm lc}$ is dense in M (see Theorem 3.6) that $\mathfrak{p} \notin \mathcal{L}(Y)$. Thus, by Theorem 2.13(iii) there exists a prime ideal $\mathfrak{q}$ of $\mathcal{S}(M_{\rm lc})$ such that $\mathfrak{p} = \mathfrak{q} \cap \mathcal{S}(M)$. But as $M_{\rm lc}$ is locally compact, $\mathfrak{q}$ is a z-ideal (see Theorem 2.6), so $\mathfrak{p}$ is by Remark 2.5 a z-ideal, too. □

4.2. *Further properties about prime z-ideals and semialgebraic depth*

The remaining part of this section is devoted to study the semialgebraic depth for prime z-ideals that has already been introduced in Definition and Proposition 2.8 (see also [10, 4.4]). In particular, we extend the properties in Paragraph 2.8.2 to the prime z-ideals of the ring $\mathcal{S}(M)$ of semialgebraic functions on an arbitrary semialgebraic set $M \subset \mathbb{R}^m$.

DEFINITION 4.8 Let $Y \subset M \subset \mathbb{R}^m$ be semialgebraic sets such that Y is closed in M. The inclusion map $\mathtt{j} : Y \hookrightarrow M$ induces by Lemma 2.10(iii) a homeomorphism

$$\operatorname{Spec}_s^\diamond(\mathtt{j}) : \operatorname{Spec}_s^\diamond(Y) \to \operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(Y) \subset \operatorname{Spec}_s^\diamond(M).$$

We say that a prime ideal $\mathfrak{p} \in \operatorname{Spec}_s^\diamond(M)$ is a *minimal prime ideal in* $\operatorname{Cl}_{\operatorname{Spec}_s^\diamond(M)}(Y)$ if $\operatorname{Spec}_s^\diamond(\mathtt{j})^{-1}(\mathfrak{p})$ is a minimal prime ideal of the ring $\mathcal{S}^\diamond(Y)$. Observe that if $\mathfrak{p}$ is a minimal prime ideal in $\operatorname{Cl}_{\operatorname{Spec}_s(M)}(Y)$, then $\mathfrak{p}$ is by Remarks 2.5 and Corollary 4.7 a z-ideal of $\mathcal{S}(M)$.

LEMMA 4.9 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p} \subset \mathcal{S}(M)$ a prime z-ideal. Fix a function $F \in \mathfrak{p}$ and write $N := Z_M(F)$. Then*:

(i) *There exists a prime z-ideal $\mathfrak{q}$ of $\mathcal{S}(N)$ such that $\mathcal{S}(M)/\mathfrak{p} \cong \mathcal{S}(N)/\mathfrak{q}$.*
(ii) *If $\mathrm{d}_M(\mathfrak{p}) = \dim(N)$, then the ideal $\mathfrak{q}$ is a minimal prime ideal of $\mathcal{S}(N)$ and $\mathrm{d}_N(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p})$.*

Proof. (i) The map $\phi : \mathcal{S}(M) \to \mathcal{S}(N),\ \ f \mapsto f|_N$ is by Theorem 2.3 an epimorphism. Moreover, $\ker\phi \subset \mathfrak{p}$ because $\mathfrak{p}$ is a z-ideal and $F \in \mathfrak{p}$. Therefore, $\mathfrak{q} \equiv \mathfrak{p}/\ker\phi$ is a prime ideal of $\mathcal{S}(N)$ and since ϕ is an epimorphism, $\mathcal{S}(M)/\mathfrak{p} \cong \mathcal{S}(N)/\mathfrak{q}$. Moreover, let us see that $\mathfrak{q}$ is a z-ideal. Indeed, let $g \in \mathcal{S}(N)$ and $h \in \mathfrak{q}$ such that $Z_N(h) \subset Z_N(g)$. There exist $G \in \mathcal{S}(M)$ and $H \in \mathfrak{p}$ such that $G|_N = g$ and $H|_N = h$. Let $G_1 := G^2 + F^2 \in \mathcal{S}(M)$ and $H_1 := H^2 + F^2 \in \mathfrak{p}$. Then $Z_M(H_1) = Z_N(h) \subset Z_N(g) = Z_M(G_1)$ and therefore $G_1 \in \mathfrak{p}$. Thus, $G \in \mathfrak{p}$ and so $g \in \mathfrak{q}$. Consequently, $\mathfrak{q}$ is a z-ideal.

(ii) Using Corollary 4.5, all is reduced to check the equality $\mathrm{d}_N(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p}) = \dim(N)$. For each $a \in \mathfrak{q}$ there exists $A \in \mathfrak{p}$ such that $a = A|_N$ and as $\mathrm{d}_M(\mathfrak{p}) = \dim(N)$, we obtain

$$\dim(N) = \mathrm{d}_M(\mathfrak{p}) \leq \dim(Z_M(F^2 + A^2)) = \dim(Z_N(a)) \leq \dim(N).$$

Hence, $\mathrm{d}_N(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p}) = \dim(N)$, as wanted. □

LEMMA 4.10 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p} \subsetneq \mathfrak{q}$ two prime z-ideals of $\mathcal{S}(M)$. Then the number of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$, when excluding $\mathfrak{p}$ and $\mathfrak{q}$, is bounded above by $\mathrm{d}_M(\mathfrak{p}) - \mathrm{d}_M(\mathfrak{q}) - 1$. In particular, if $\mathrm{d}_M(\mathfrak{p}) = \mathrm{d}_M(\mathfrak{q}) + 1$, there is no prime ideal between $\mathfrak{p}$ and $\mathfrak{q}$.*

Proof. Let us prove that we may assume that $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}(M)$ such that $d := \dim(M) = \mathrm{d}_M(\mathfrak{p})$. Let $F \in \mathfrak{p}$ be such that $\mathrm{d}_M(\mathfrak{p}) = \dim(N)$, where $N := Z_M(F)$. By Lemma 2.10(iii), the spectral map $\operatorname{Spec_s}(\mathtt{j}) : \operatorname{Spec_s}(N) \to \operatorname{Cl}_{\operatorname{Spec_s}(M)}(N) \subset \operatorname{Spec_s}(M)$ induced by the inclusion map $\mathtt{j} : N \hookrightarrow M$ is a homeomorphism and by 4.8 and 4.9 $\mathfrak{p} \in \operatorname{Cl}_{\operatorname{Spec_s}(M)}(N)$ is a minimal prime ideal in $\operatorname{Cl}_{\operatorname{Spec_s}(M)}(N)$. Thus, every chain of prime ideals of $\mathcal{S}(M)$ containing the prime ideal $\mathfrak{p}$ corresponds to a chain of prime ideals of $\mathcal{S}(N)$ of the same length via $\operatorname{Spec_s}(\mathtt{j})^{-1}$. Using this, we may assume from the beginning that $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{S}(M)$ such that $\mathrm{d}_M(\mathfrak{p}) = \dim(M)$.

Let $r := \mathrm{d}_M(\mathfrak{p}) - \mathrm{d}_M(\mathfrak{q}) - 1$ and suppose there exist $r+1$ prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_{r+1}$ of $\mathcal{S}(M)$ between $\mathfrak{p}$ and $\mathfrak{q}$, that is, $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_{r+1} \subsetneq \mathfrak{p}_{r+2} = \mathfrak{q}$. By [10, 4.11], there exists a semialgebraic compactification $(X, \mathtt{j}_1)$ of M and a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_{r+2}$ of $\mathcal{S}(X)$ such that $\mathfrak{p}_i \cap \mathcal{S}(X) = \mathfrak{q}_i$ for $0 \le i \le r+2$. Now we obtain by Paragraph 2.8.2

$$d \ge \mathrm{d}_X(\mathfrak{q}_0) > \overset{(r+2)}{\cdots} > \mathrm{d}_X(\mathfrak{q}_{r+2}) \ge \mathrm{d}_M(\mathfrak{p}_{r+2}) = \mathrm{d}_M(\mathfrak{q}) = d - r - 1,$$

which is a contradiction. Thus, there are at most r prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$. □

REMARK 4.11 The previous bound is not always sharp: for a counterexample see Example 5.5.

LEMMA 4.12 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p}$ a prime ideal of $\mathcal{S}(M)$. Then there exists a unique prime z-ideal $\mathfrak{q}$ of $\mathcal{S}(M)$ such that $\mathfrak{p} \subset \mathfrak{q}$ and $\mathrm{d}_M(\mathfrak{p}) = \mathrm{d}_M(\mathfrak{q})$. The prime ideal $\mathfrak{q}$ will be denoted with $\mathfrak{p}^z$.*

Proof. Let $f \in \mathfrak{p}$ such that $\mathrm{d}_M(\mathfrak{p}) = \dim(Z_M(f))$ and define $N := Z_M(f) \subset M$. The homomorphism $\phi : \mathcal{S}(M) \to \mathcal{S}(N),\ \ g \mapsto g|_N$ is surjective by Theorem 2.3. Consider the radical ideal $\mathfrak{q} := \sqrt{\mathfrak{p} + \ker \phi}$, which is a prime ideal by the ulterior result Corollary 5.4. Let us first see that $\mathrm{d}_M(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p})$. The inequality $\mathrm{d}_M(\mathfrak{q}) \le \mathrm{d}_M(\mathfrak{p})$ is evident because $\mathfrak{p} \subset \mathfrak{q}$. For the converse let $h \in \mathfrak{q}$. Then there exist $\ell \ge 1$, $a \in \mathfrak{p}$ and $b \in \ker \phi$ such that $h^\ell = a + b$. Therefore,

$$Z_M(h) = Z_M(h^\ell) = Z_M(a+b) \supset Z_M(a) \cap Z_M(b) \supset Z_M(a) \cap Z_M(f) = Z_M(a^2 + f^2)$$

and so $\dim(Z_M(h)) \ge \dim(Z_M(a^2 + f^2)) \ge \mathrm{d}_M(\mathfrak{p})$ because $a^2 + f^2 \in \mathfrak{p}$. Thus, $\mathrm{d}_M(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{p})$.

To prove that $\mathfrak{q}$ is a z-ideal, it is sufficient to see that $\phi(\mathfrak{q})$ is a prime z-ideal. Once this is done, $\mathfrak{q} = \phi^{-1}(\phi(\mathfrak{q}))$ is by Remark 2.5 a z-ideal. The primality of $\phi(\mathfrak{q}) \equiv \mathfrak{q}/\ker\phi$ is immediate because $\mathfrak{q}$ is a prime ideal that contains $\ker\phi$. Moreover, $\phi(\mathfrak{q})$ is by Corollary 4.7 a z-ideal because it is a minimal prime ideal of $\mathcal{S}(N)$. Indeed, the equality $\mathrm{d}_N(\phi(\mathfrak{q})) = \mathrm{d}_M(\mathfrak{q}) = \dim(N)$ implies the minimality of $\phi(\mathfrak{q})$ by Corollary 4.5.

Concerning the uniqueness, let $\mathfrak{q}_1$ be another prime z-ideal of $\mathcal{S}(M)$ such that $\mathfrak{p} \subset \mathfrak{q}_1$ and $\mathrm{d}_M(\mathfrak{q}_1) = \mathrm{d}_M(\mathfrak{p})$. The collection of all prime ideals of $\mathcal{S}(M)$ containing $\mathfrak{p}$ constitutes a chain by Paragraph 2.4.2. Hence, we may assume that $\mathfrak{q} \subset \mathfrak{q}_1$ and the equality $\mathrm{d}_M(\mathfrak{q}) = \mathrm{d}_M(\mathfrak{q}_1)$ implies by Paragraph 2.8.1 that both ideals coincide. □

Now it is obvious that all maximal ideals of $\mathcal{S}(M)$ are z-ideals.

COROLLARY 4.13 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set. Then all maximal ideals of $\mathcal{S}(M)$ are z-ideals.*

The next result extends Paragraph 2.8.2 for arbitrary semialgebraic sets.

COROLLARY 4.14 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set.*

(i) *Let $\mathfrak{p} \subset \mathfrak{q}$ be two prime z-ideals of $\mathcal{S}(M)$. Then the coheight of $\mathfrak{p}$ in $\mathfrak{q}$ is $\leq \mathrm{d}_M(\mathfrak{p}) - \mathrm{d}_M(\mathfrak{q})$. In particular, the coheight of a prime z-ideal is bounded above by $\mathrm{d}_M(\mathfrak{p})$.*

(ii) *For every prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$, the inequality $\mathrm{d}_M(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) \leq \dim(M)$ holds. In fact, if $\mathfrak{p}$ is not a z-ideal, then the inequality is strict.*

(iii) *The height of a maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$ is less than or equal to the maximum of $\mathrm{d}_M(\mathfrak{p})$, where $\mathfrak{p}$ runs over the minimal prime ideals of $\mathcal{S}(M)$ contained in $\mathfrak{m}$.*

Proof. Part (i) follows straightforwardly from Lemma 4.10.

(ii) Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r := \mathfrak{p}$ be a chain of prime ideals in $\mathcal{S}(M)$ such that $r = \mathrm{ht}(\mathfrak{p})$. Since $\mathfrak{p}_0$ is a minimal prime ideal, it is by Corollary 4.7 also a z-ideal. Let $\mathfrak{p}^z$ be the unique prime z-ideal of $\mathcal{S}(M)$ such that $\mathfrak{p} \subset \mathfrak{p}^z$ and $\mathrm{d}_M(\mathfrak{p}) = \mathrm{d}_M(\mathfrak{p}^z)$. By (i), we have

$$\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}^z) \leq \mathrm{d}_M(\mathfrak{p}_0) - \mathrm{d}_M(\mathfrak{p}^z) \leq \dim(M) - \mathrm{d}_M(\mathfrak{p}).$$

Note that if $\mathfrak{p}$ is not a prime z-ideal, then $\mathfrak{p} \subsetneq \mathfrak{p}^z$ and so $\mathrm{ht}(\mathfrak{p}) < \mathrm{ht}(\mathfrak{p}^z) \leq \dim(M) - \mathrm{d}_M(\mathfrak{p})$.

(iii) Let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{S}(M)$ contained in $\mathfrak{m}$. Both $\mathfrak{p}$ and $\mathfrak{m}$ are z-ideals. By (i), the coheight of $\mathfrak{p}$ is bounded above by $\mathrm{d}_M(\mathfrak{p})$. Since $\mathrm{ht}(\mathfrak{m})$ is the maximum of the lengths of the non-refinable chains of prime ideals contained in $\mathfrak{m}$, which coincides with the maximum of the coheights of the minimal prime ideals contained in $\mathfrak{m}$, we conclude that $\mathrm{ht}(\mathfrak{m})$ is bounded above by the maximum of $\mathrm{d}_M(\mathfrak{p})$, where $\mathfrak{p}$ runs over the minimal prime ideals of $\mathcal{S}(M)$ contained in $\mathfrak{m}$, as wanted. □

5. Chains of prime ideals

We analyse some properties of the chains of prime ideals in $\mathcal{S}^\diamond(M)$ for an arbitrary semialgebraic set M. In the second part, we compare the spectra $\operatorname{Spec}_s^\diamond(M)$ and $\operatorname{Spec}_s^\diamond(X)$, where X is a suitable semialgebraic compactification of a locally compact semialgebraic set M.

5.1. *Structure of non-refinable chains and criterions of primality*

We begin with the following result, which studies the structure of the non-refinable chains of prime ideals of $\mathcal{S}^*(M)$. We refer the reader to [28] for a similar approach to the chains of prime ideals for the o-minimal context in the exponentially bounded and polynomially bounded cases under the assumption of local closedness.

PROPOSITION 5.1 (Chains of prime ideals) *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ a non-refinable chain of prime ideals in $\mathcal{S}^*(M)$. Then*

(i) *$\mathfrak{p}_0$ is a minimal prime ideal of $\mathcal{S}^*(M)$ and there exists a minimal prime ideal of $\mathcal{S}(M)$ whose intersection with $\mathcal{S}^*(M)$ is $\mathfrak{p}_0$. Moreover, $\mathfrak{p}_r = \mathfrak{m}^*$ is a maximal ideal of $\mathcal{S}^*(M)$.*

(ii) *There exists* $0 \le k \le r$ *such that* $\mathfrak{p}_k = \mathfrak{m} \cap \mathcal{S}^*(M)$, *where* $\mathfrak{m}$ *is the unique maximal ideal of* $\mathcal{S}(M)$ *such that* $\mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$.

(iii) *If* M *is locally compact and* $\mathfrak{p}_r = \mathfrak{m}^*$ *is a free ideal, then* $k < r$ *and* $\mathfrak{p}_{k+1}$ *is the set of all functions* $f \in \mathcal{S}^*(M)$ *whose unique continuous extension* $\hat{f} : \beta_s^* M \to \mathbb{R}$ *vanishes on a neighbourhood of* $\mathfrak{m}^*$ *in* $\partial M := \beta_s^* M \setminus M$.

Proof. The first assertion is immediate (its second part follows from Lemmas 2.9 and 4.3). Let us have a look at the second one. By (i) and Lemma 4.3, $\mathfrak{q}_0 = \mathfrak{p}_0 \mathcal{S}(M)$ is a minimal prime ideal of $\mathcal{S}(M)$ and $\mathfrak{q}_0 \cap \mathcal{S}^*(M) = \mathfrak{p}_0$. By Paragraph 2.4.2 and [10, 4.1], the set of prime ideals of $\mathcal{S}(M)$ containing $\mathfrak{q}_0$ is a totally ordered set with respect to inclusion; denote such prime ideals with $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_k$. Of course, $\mathfrak{q}_k = \mathfrak{n}$ is a maximal ideal of the ring $\mathcal{S}(M)$.

By [22] (see also Paragraph 2.4.2 and [10, 4.1]), the set of prime ideals of $\mathcal{S}^*(M)$ containing $\mathfrak{p}_0$ is a finite totally ordered set with respect to inclusion, which is $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{m}^*$. Moreover, the intersection $\mathfrak{q}_\ell \cap \mathcal{S}^*(M)$ is a prime ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{p}_0$ for each $\ell = 0, \dots, k$. In particular, since $\mathfrak{p}_0$ is contained in $\mathfrak{p}_r = \mathfrak{m}^*$, we have $\mathfrak{n} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$ and so by Paragraph 2.5.2, $\mathfrak{n} = \mathfrak{m}$. Thus, by Lemma 2.9 our chain of prime ideals of $\mathcal{S}^*(M)$ looks as follows:

$$\mathfrak{p}_0 = \mathfrak{q}_0 \cap \mathcal{S}^*(M) \subsetneq \cdots \subsetneq \mathfrak{p}_{k-1} = \mathfrak{q}_{k-1} \cap \mathcal{S}^*(M) \subsetneq \mathfrak{p}_k = \mathfrak{m} \cap \mathcal{S}^*(M) \subsetneq \mathfrak{p}_{k+1} \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{m}^*.$$

Statement (iii) follows directly from Theorem 6.1. □

REMARKS 5.2 (i) It follows straightforwardly from Paragraph 2.4.2 that if $\mathfrak{m}^*$ is a free maximal ideal, the subchain

$$\mathfrak{p}_k = \mathfrak{m} \cap \mathcal{S}^*(M) \subsetneq \mathfrak{p}_{k+1} \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{m}^*$$

is the same for any non-refinable chain of prime ideals in $\mathcal{S}^*(M)$ ending at $\mathfrak{m}^*$. Indeed, $\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(\{\mathfrak{p}_k\}) = \{\mathfrak{p}_k, \dots, \mathfrak{p}_r = \mathfrak{m}^*\}$.

(ii) Let $\mathcal{W}(M) := \{f \in \mathcal{S}^*(M) : \ Z_M(f) = \emptyset\}$ and $\mathfrak{p} \in \mathrm{Spec}_s^*(M)$ be a prime ideal such that $\mathfrak{p} \cap \mathcal{W}(M) \neq \emptyset$. Let $\mathfrak{m}^*$ be the unique maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{p}$ and $\mathfrak{m}$ the unique maximal ideal of $\mathcal{S}(M)$ such that $\mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$. Then $\mathfrak{m} \cap \mathcal{S}^*(M) \subsetneq \mathfrak{p} \subset \mathfrak{m}^*$.

Indeed, consider a chain of prime ideals in $\mathcal{S}^*(M)$ admitting no refinement and having $\mathfrak{p}$ and $\mathfrak{m}^*$ as two of its members. By Proposition 5.1(ii), also $\mathfrak{p}_0 := \mathfrak{m} \cap \mathcal{S}^*(M)$ occurs in this chain. As $\mathfrak{p}_0 \cap \mathcal{W}(M) = \emptyset$, we deduce $\mathfrak{m} \cap \mathcal{S}(M) \subsetneq \mathfrak{p} \subset \mathfrak{m}^*$.

Next, we present a criterion to characterize prime ideals of $\mathcal{S}^*(M)$, which is strongly inspired by Gillman and Jerison [15, 2.9] and Tressl [28, 2.7], and which uses the Nullstellensatz [11, 3.11] for $\mathcal{S}^*(M)$ in a crucial way.

LEMMA 5.3 *Let* $M \subset \mathbb{R}^m$ *be a semialgebraic set and* $\mathfrak{a}$ *a radical ideal of* $\mathcal{S}^*(M)$. *The following conditions are equivalent*:

(i) *The ideal* $\mathfrak{a}$ *is prime.*

(ii) *The ideal* $\mathfrak{a}$ *contains a prime ideal* $\mathfrak{p}$ *of* $\mathcal{S}^*(M)$.

(iii) *For all* $f, g \in \mathcal{S}^*(M)$ *such that* $fg \equiv 0$ *it holds either* $f \in \mathfrak{a}$ *or* $g \in \mathfrak{a}$.

(iv) *For every function* $h \in \mathcal{S}^*(M)$ *there exists a function* $g \in \mathfrak{a}$ *such that the sign of the continuous extension* $\hat{h} : \beta_s^* M \to \mathbb{R}$ *of* h *is constant on* $\mathcal{Z}_{\beta_s^* M}(g)$.

Proof. It is enough to show (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). To prove (ii) $\Longrightarrow$ (iii), let $f, g \in \mathcal{S}^*(M)$ with $fg \equiv 0 \in \mathfrak{p}$. Hence, either $f \in \mathfrak{p} \subset \mathfrak{a}$ or $g \in \mathfrak{p} \subset \mathfrak{a}$.

For (iii) $\Longrightarrow$ (iv), define $f := \max\{h, 0\}$ and $g := \min\{h, 0\}$. Clearly, both $f, g \in \mathcal{S}^*(M)$ and $fg \equiv 0$. Thus, for instance, $g \in \mathfrak{a}$, and let us check that the sign of $\hat{h}$ is constant on $\mathcal{Z}_{\beta_s^* M}(g)$. Observe that $h = f + g$ and $|h| = f - g$. Hence, if $\mathfrak{m}^* \in \mathcal{Z}_{\beta_s^* M}(g)$, we have

$$\hat{h}(\mathfrak{m}^*) = h + \mathfrak{m}^* = f + g + \mathfrak{m}^* = f + \mathfrak{m}^* = f - g + \mathfrak{m}^* = |h| + \mathfrak{m}^* = (\sqrt{|h|} + \mathfrak{m}^*)^2.$$

Therefore, the sign of $\hat{h}$ is constant on $\mathcal{Z}_{\beta_s^* M}(g)$, as wanted.

Finally, we prove (iv) $\Longrightarrow$ (i). Let $f_1, f_2 \in \mathcal{S}^*(M)$ such that $f_1 f_2 \in \mathfrak{a}$. We may assume the existence of $g \in \mathfrak{a}$ such that the continuous extension $\hat{h}$ of $h = |f_1| - |f_2|$ to $\beta_s^* M$ is non-negative on $\mathcal{Z}_{\beta_s^* M}(g)$, and so $\mathcal{Z}_{\beta_s^* M}(g) \cap \mathcal{Z}_{\beta_s^* M}(f_1) \subset \mathcal{Z}_{\beta_s^* M}(f_2)$. Therefore,

$$\mathcal{Z}_{\beta_s^* M}(g^2 + f_1^2 f_2^2) \subset \mathcal{Z}_{\beta_s^* M}(g^2 + f_2^2) \subset \mathcal{Z}_{\beta_s^* M}(f_2).$$

By [**11**, 3.11], there exists an integer $k \geq 1$ and a semialgebraic function $a \in \mathcal{S}^*(M)$ such that $f_2^k = (g^2 + f_1^2 f_2^2)a \in \mathfrak{a}$ and because the last is a radical ideal, we obtain $f_2 \in \mathfrak{a}$. □

The following result is the counterpart of Lemma 5.3 for the ring $\mathcal{S}(M)$.

COROLLARY 5.4 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{a}$ a radical ideal of $\mathcal{S}(M)$. The following conditions are equivalent*:

(i) *The ideal $\mathfrak{a}$ is prime.*
(ii) *The ideal $\mathfrak{a}$ contains a prime ideal of $\mathcal{S}(M)$.*
(iii) *For all $f, g \in \mathcal{S}(M)$ such that $fg \equiv 0$ it holds either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are straightforward. For (iii) $\Longrightarrow$ (i) observe that $\mathfrak{p} = \mathfrak{a} \cap \mathcal{S}^*(M)$ is a radical ideal. Given $f, g \in \mathcal{S}^*(M)$ such that $fg = 0$, then either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$, that is, either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Thus, $\mathfrak{p}$ is by Lemma 5.3 a prime ideal of $\mathcal{S}^*(M)$ and therefore $\mathfrak{a} = \mathfrak{p}\mathcal{S}(M)$ is by Lemma 2.9 a prime ideal, too. □

We develop some examples to enlighten the behaviour of non-refinable chains of prime ideals in $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$.

EXAMPLE 5.5 Consider the prime ideal

$$\mathfrak{p} := \{f \in \mathcal{S}(\mathbb{R}^2) : \exists \varepsilon > 0 \mid f(t, e^t) = 0 \ \forall t \in [0, \varepsilon]\}$$

in $\mathcal{S}(\mathbb{R}^2)$, which is contained in the maximal ideal $\mathfrak{m}_p$ of semialgebraic functions vanishing at the point $p := (0, 1) \in \mathbb{R}^2$. Note first that $\mathrm{d}_{\mathbb{R}^2}(\mathfrak{p}) = 2$; to prove this fact, just observe $\overline{Z_{\mathbb{R}^2}(f)}^{\mathrm{zar}} = \mathbb{R}^2$ for all $f \in \mathfrak{p}$. Thus, $\mathfrak{p}$ is by Corollary 4.5 a minimal prime ideal of $\mathcal{S}(\mathbb{R}^2)$.

Let us sketch now the proof that there does not exist any prime ideal $\mathfrak{q}$ of $\mathcal{S}(\mathbb{R}^2)$ between $\mathfrak{p}$ and $\mathfrak{m}_p$. Since $\dim \mathcal{S}(\mathbb{R}^2) = 2$, such a prime ideal $\mathfrak{q}$ should have by Paragraph 2.8.2 height one. Moreover, since $\mathbb{R}^2$ is locally compact, $0 = \mathrm{d}_M(\mathfrak{m}_p) < \mathrm{d}_M(\mathfrak{q}) < \mathrm{d}_M(\mathfrak{p}) = 2$ (see Paragraph 2.8.1), and so $\mathrm{d}_M(\mathfrak{q}) = 1$. Let $g \in \mathfrak{q}$ be such that $Z_{\mathbb{R}^2}(g)$ is a semialgebraic curve through the point p. As the parametrization $t \mapsto (t, e^t)$ is not semialgebraic, we find $f \in \mathfrak{p} \subset \mathfrak{q}$ such that $Z_{\mathbb{R}^2}(f) \cap Z_{\mathbb{R}^2}(g) = \{p\}$. Thus, $h := f^2 + g^2 \in \mathfrak{q}$ and $Z_{\mathbb{R}^2}(h) = \{p\}$; hence, $\mathrm{d}_M(\mathfrak{q}) = 0$, which is a contradiction.

Consequently, although $\mathfrak{m}_p$ is a maximal ideal of the bidimensional ring $\mathcal{S}(\mathbb{R}^2)$, the chain of prime ideals $\mathfrak{p} \subsetneq \mathfrak{m}_p$ admits no refinement and $0 = \mathrm{d}_M(\mathfrak{m}_p) < \mathrm{d}_M(\mathfrak{p}) = 2$.

EXAMPLE 5.6 Let $N := \{(x, y) \in \mathbb{R}^2 : \ x > 0, \ y > 0\}$ and define

$$\mathfrak{n} := \{f \in \mathcal{S}(N) : \ \exists \varepsilon > 0 \text{ such that } \forall s \in (0, \varepsilon], \ \exists \delta > 0 \mid f(t, st) = 0 \ \forall t \in (0, \delta)\}.$$

(5.6.1) Let us check first that $\mathfrak{n}$ is a prime ideal of $\mathcal{S}(N)$ (we will show afterwards that it is in fact a maximal ideal) such that $\mathrm{d}_N(\mathfrak{n}) = 2$, and so $\mathrm{ht}(\mathfrak{n}) = 0$. Since $\mathfrak{n}$ is a radical ideal, it is by Corollary 5.4 sufficient to find a prime ideal of $\mathcal{S}(N)$ contained in $\mathfrak{n}$. Let $M := N \cup \{(x, 0) : \ x > 0\}$ and consider the inclusion map $\mathtt{j} : N \hookrightarrow M$ as well as the closed semialgebraic subset $Y := \{(x, 0) : \ x > 0\} = M \setminus N$ of M. Since M is locally compact, the map $\mathrm{Spec_s}(\mathtt{j}) : \mathrm{Spec_s}(N) \to \mathrm{Spec_s}(M) \setminus \mathrm{Cl}_{\mathrm{Spec_s}(M)}(Y)$ is by Definition and Proposition 2.12 and Theorem 2.13(iii) a homeomorphism. Consider the semialgebraic map $\varphi : M \to M, \ (s, t) \mapsto (t, st)$, whose restriction $\psi = \varphi|_N : N \to N$ is a homeomorphism. Consider the prime ideal

$$\mathfrak{p}_1 := \{f \in \mathcal{S}(M) : \exists \varepsilon > 0 \text{ such that } f(t, 0) = 0 \ \forall \ t \in (0, \varepsilon)\}$$

of $\mathcal{S}(M)$ and let $\mathfrak{p}_0$ be a minimal prime ideal of $\mathcal{S}(M)$ contained in $\mathfrak{p}_1$.

By Theorem 2.13(ii) and (iv), $\mathfrak{p}_0\mathcal{S}(N)$ is a prime ideal of $\mathcal{S}(N)$. Thus, also $\mathfrak{q}_0 = \phi(\mathfrak{p}_0\mathcal{S}(N))$ is a prime ideal of $\mathcal{S}(N)$ because $\phi : \mathcal{S}(N) \to \mathcal{S}(N), \ g \mapsto g \circ \psi^{-1}$ is an isomorphism. Now the reader can check that $\mathfrak{q}_0 \subset \mathfrak{n}$, which proves the primality of $\mathfrak{n}$.

Moreover, $\mathrm{d}_N(\mathfrak{n}) = 2$ since $\overline{Z_N(f)}^{\mathrm{zar}} = \mathbb{R}^2$ for all $f \in \mathfrak{n}$; hence, $\mathfrak{n}$ is a minimal prime ideal of $\mathcal{S}(N)$.

(5.6.2) We next sketch the proof of the maximality of $\mathfrak{n}$ in $\mathcal{S}(N)$. Thus, $\mathfrak{n}$ is simultaneously a maximal and a minimal prime ideal of $\mathcal{S}(N)$.

We must check that there exists no prime ideal $\mathfrak{q}$ of $\mathcal{S}(N)$ such that $\mathfrak{n} \subsetneq \mathfrak{q}$. Suppose by contradiction that such an ideal exists. Hence, $\mathrm{ht}(\mathfrak{q}) \geq 1$, and this implies that $\mathrm{d}_N(\mathfrak{q}) = 1$ since $\mathfrak{n}$ is free.

Let $f \in \mathfrak{q}$ such that $\dim(Z_N(f)) = 1$. Since $\mathfrak{q}$ is a prime ideal, we may assume that $Z_N(f)$ is a parametrizable Nash branch whose closure in $\mathbb{R}^2$ meets the line $\{y = 0\}$ at the point $p = (0, 0)$. Now the reader can construct a semialgebraic function $b \in \mathfrak{n}$ such that $Z_N(b) \cap Z_N(f) = \emptyset$; hence, $h := b^2 + f^2 \in \mathfrak{q}$ is a unit in $\mathcal{S}(N)$, which is a contradiction. This example will be generalized later in Theorem 7.1.

(5.6.3) Consider the set $\mathfrak{n}^*$ of all functions $f \in \mathcal{S}^*(N)$ such that

$$\lim_{s \to 0} F_f(s) = 0 \quad \text{where } F_f(s) = \lim_{t \to 0} f(t, st) \text{ for } s > 0, st > 0.$$

Let us check that $\mathfrak{n}^*$ is a maximal ideal of $\mathcal{S}^*(N)$. First, notice that $F_f : \mathbb{R} \to \mathbb{R}, \ s \mapsto F_f(s)$ is a well-defined function and its graph is semialgebraic. Thus, there exists the lateral limit of F_f at 0^+ (see [**11**, 2.6]), and so the function F_f is well-defined. Now, it follows straightforwardly that $\mathcal{S}^*(N)/\mathfrak{n}^* \cong \mathbb{R}$ and therefore $\mathfrak{n}^*$ is a maximal ideal.

To finish, let us see that $\mathrm{ht}(\mathfrak{n}^*) = 2$. Let $\mathfrak{q}$ be the subset of $\mathcal{S}^*(N)$ consisting of all bounded semialgebraic functions f on N such that there exists a real number $\varepsilon > 0$ such that $F_f(s) = 0$ for all $s \in (0, \varepsilon)$. Arguing as above, one checks that $\mathfrak{q}$ is a radical ideal. Observe also $\mathfrak{n} \cap \mathcal{S}^*(N) \subsetneq \mathfrak{q} \subsetneq \mathfrak{n}^*$, where $\mathfrak{n}$ is the maximal ideal of $\mathcal{S}(N)$ defined above. It follows from Lemma 5.3 that $\mathfrak{q}$ is a prime ideal of $\mathcal{S}^*(N)$ and so $\mathrm{ht}(\mathfrak{n}^*) = 2$. In particular, $\mathfrak{n}^*$ is not the immediate successor of $\mathfrak{n} \cap \mathcal{S}^*(N)$.

REMARKS 5.7

(i) The situation above will be generalized in Theorem 7.1 by proving that for each non-compact pure dimensional semialgebraic set N of dimension d and for each $0 \leq r < d$, there exists a free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(N)$ such that $\operatorname{ht}(\mathfrak{m}) = r$ and $\operatorname{ht}(\mathfrak{m}^*) = d$.

(ii) All of the above shows that not all non-refinable chains of prime ideals in $\mathcal{S}^\diamond(M)$ have the same length because there exist maximal ideals whose height is smaller than the dimension of M. This is a well-known fact for rings of polynomial functions on a semialgebraic set (see [2, 7.5.9 & 10.3.4]).

5.2. *Comparing chains of prime ideals*

The second goal in this section is to extract information about chains of prime ideals in $\mathcal{S}^*(M)$, where M is a locally compact semialgebraic set, from the one given by the chains of prime ideals in $\mathcal{S}(X)$, where X runs on the semialgebraic compactifications of M. Namely,

COROLLARY 5.8 *Let $M \subset \mathbb{R}^m$ be a locally compact semialgebraic set and X a semialgebraic compactification of M. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{S}(M)$ and $\mathfrak{m}^*$ the unique maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{m} \cap \mathcal{S}^*(M)$. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ be a chain of prime ideals in $\mathcal{S}(X)$ such that $\mathfrak{p}_k = \mathfrak{m} \cap \mathcal{S}(X)$ for some index $0 \leq k \leq r$. Then $\mathfrak{p}_r \subset \mathfrak{m}^* \cap \mathcal{S}(X)$ and $r \leq \operatorname{ht}(\mathfrak{m}^*)$.*

Proof. By Lemma 2.15(v), there exists a chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ in $\mathcal{S}^*(M)$ such that each intersection $\mathfrak{q}_i \cap \mathcal{S}(X) = \mathfrak{p}_i$. Next, we prove $\mathfrak{q}_k = \mathfrak{m} \cap \mathcal{S}^*(M)$ for the index k such that $\mathfrak{p}_k = \mathfrak{m} \cap \mathcal{S}(X)$. Then it follows from Paragraph 2.4.2 that each prime ideal occurring in a chain of prime ideals of $\mathcal{S}^*(M)$ containing $\mathfrak{m} \cap \mathcal{S}^*(M)$ is contained in $\mathfrak{m}^*$. Thus, $\mathfrak{q}_r \subset \mathfrak{m}^*$ and therefore $\mathfrak{p}_r = \mathfrak{q}_r \cap \mathcal{S}(X) \subset \mathfrak{m}^* \cap \mathcal{S}(X)$ and $r \leq \operatorname{ht}(\mathfrak{m}^*)$. So, we only have to prove $\mathfrak{q}_k = \mathfrak{m} \cap \mathcal{S}^*(M)$.

Indeed, as M is locally compact, M is an open and dense subset of X; hence, if $\mathtt{j} : M \hookrightarrow X$ denotes the inclusion map, then $\operatorname{Spec}_s(\mathtt{j}) : \operatorname{Spec}_s(M) \to \operatorname{Spec}_s(X)$ is by Theorem 2.13(ii) a homeomorphism onto its image $\operatorname{Spec}_s(X) \setminus \mathcal{L}(Y)$, where $Y := X \setminus M$. By Lemma 2.15(ii), the map $\operatorname{Spec}_s^*(j) : \operatorname{Spec}_s^*(M) \to \operatorname{Spec}_s^*(X)$ is surjective and its restriction

$$\operatorname{Spec}_s^*(\mathtt{j})\,|: \operatorname{Spec}_s^*(M) \setminus \operatorname{Spec}_s^*(\mathtt{j})^{-1}(\operatorname{Cl}_{\operatorname{Spec}_s^*(X)}(Y)) \to \operatorname{Spec}_s^*(X) \setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(X)}(Y)$$

is by Lemma 2.15(ii) a homeomorphism. Moreover, $\mathcal{S}(X) = \mathcal{S}^*(X)$, $\operatorname{Spec}_s(X) = \operatorname{Spec}_s^*(X)$ and $\mathcal{L}(Y) = \operatorname{Cl}_{\operatorname{Spec}_s(X)}(Y)$. Moreover,

$$\mathfrak{m} \cap \mathcal{S}(X) \in \operatorname{Spec}_s(X) \setminus \mathcal{L}(Y) = \operatorname{Spec}_s(X) \setminus \operatorname{Cl}_{\operatorname{Spec}_s(X)}(Y) = \operatorname{Spec}_s^*(X) \setminus \operatorname{Cl}_{\operatorname{Spec}_s^*(X)}(Y).$$

By Lemma 2.15(iii), there exists a unique prime ideal $\mathfrak{q} \in \operatorname{Spec}_s^*(M)$ such that $\mathfrak{q} \cap \mathcal{S}(X) = \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}_s(X) \setminus \operatorname{Cl}_{\operatorname{Spec}_s(X)}(Y)$. In particular, since the prime ideals $\mathfrak{m} \cap \mathcal{S}^*(M)$ and $\mathfrak{q}_k$ satisfy

$$(\mathfrak{m} \cap \mathcal{S}^*(M)) \cap \mathcal{S}(X) = \mathfrak{m} \cap \mathcal{S}(X) = \mathfrak{p}_k = \mathfrak{q}_k \cap \mathcal{S}(X),$$

we deduce the equality $\mathfrak{m} \cap \mathcal{S}^*(M) = \mathfrak{q}_k$ and we are done. □

The previous result provides the following characterization of the height of a maximal ideal $\mathfrak{m}^*$ of $\mathcal{S}^*(M)$ (if M is locally compact) in terms of the lengths of those chains of prime ideals in $\mathcal{S}(X)$

passing through $\mathfrak{m} \cap \mathcal{S}(X)$, where X runs along all semialgebraic compactifications of M. Its proof follows straightforwardly from the proof of [10, 4.11], Proposition 5.1 and Corollary 5.8 and we leave the details to the reader.

COROLLARY 5.9 *Let $M \subset \mathbb{R}^m$ be a locally compact semialgebraic set. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{S}(M)$ and $\mathfrak{m}^*$ the unique maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{m} \cap \mathcal{S}^*(M)$. For every semialgebraic compactification X of M denotes the maximum length of those chains of prime ideals in $\mathcal{S}(X)$ passing through $\mathfrak{m} \cap \mathcal{S}(X)$ with $\mathrm{h}_X(\mathfrak{m}^*)$. Then $\mathrm{ht}(\mathfrak{m}^*) = \max_X\{\mathrm{h}_X(\mathfrak{m}^*)\}$, where X runs along all the semialgebraic compactifications of M.*

REMARKS 5.10 (i) The previous result (Corollary 5.9) is false if we substitute $\mathrm{h}_X(\mathfrak{m}^*)$ by $\mathrm{ht}(\mathfrak{m}^* \cap \mathcal{S}(X))$. Thus, the 'important' ideal in the previous result is $\mathfrak{m}$ instead of $\mathfrak{m}^*$, contrary to what one would expect. To show this, we propose two examples:

(1) Consider the compact semialgebraic subset X of $\mathbb{R}^2$ given by

$$X := \{(x+1)^2 + y^2 \leq 1\} \cup \{0 \leq x \leq 1,\, y = 0\}$$

and the point $p := (0, 0)$. Clearly, X is a compactification of the locally compact semialgebraic set $M := X \setminus \{p\}$. Let $\mathfrak{m}^* := \{f \in \mathcal{S}^*(M) :\ \lim_{t\to 0^+} f(t, 0) = 0\}$, which is a maximal ideal of $\mathcal{S}^*(M)$. Then $\mathfrak{m}^* \cap \mathcal{S}(X) = \mathfrak{m}_p$, and so $\mathrm{ht}(\mathfrak{m}^* \cap \mathcal{S}(X)) = 2$ while $\mathrm{ht}(\mathfrak{m}^*) = 1$.

(2) Let X be a closed disc centred at the origin $p := (0, 0) \in \mathbb{R}^2$, which is a semialgebraic compactification of the locally compact semialgebraic set $M := X \setminus \{p\}$. Consider the analytic path $\alpha : (0, \varepsilon) \to M,\ t \mapsto (t, e^t - 1)$ and the maximal ideal

$$\mathfrak{m}^*_\alpha := \left\{ f \in \mathcal{S}^*(M) :\ \lim_{t\to 0}(f \circ \alpha)(t) = 0 \right\}$$

of $\mathcal{S}^*(M)$. The maximality of $\mathfrak{m}^*_\alpha$ is proved straightforwardly. In fact, it also holds that

$$\mathfrak{m}_\alpha := \{f \in \mathcal{S}(M) :\ \exists \varepsilon > 0 \text{ such that } (f \circ \alpha)|_{(0,\varepsilon]} = 0\}$$

is the unique maximal ideal of $\mathcal{S}(M)$ such that $\mathfrak{m}_\alpha \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*_\alpha$. Moreover, $\mathfrak{m}_\alpha$ is a minimal prime ideal of $\mathcal{S}(M)$ because $\mathrm{d}_M(\mathfrak{m}_\alpha) = 2$. As one can check, the chain $\mathfrak{m}_\alpha \cap \mathcal{S}^*(M) \subsetneq \mathfrak{m}^*_\alpha$ is non-refinable. Hence, $\mathrm{ht}(\mathfrak{m}^*_\alpha) = 1$. However, $\mathrm{ht}(\mathfrak{m}^*_\alpha \cap \mathcal{S}(X)) = 2$ because $\mathfrak{m}^*_\alpha \cap \mathcal{S}(X) = \mathfrak{m}_p$.

(ii) With the notation of Corollary 5.9, $\mathrm{h}_X(\mathfrak{m}^*) < \mathrm{ht}(\mathfrak{m}^*)$ for a suitable compactification X of M. Indeed, consider Paragraph 5.6.3. There $N := \{(x, y) \in \mathbb{R}^2 :\ x > 0, y > 0\}$ and $\mathfrak{n}^*$ is a maximal ideal of $\mathcal{S}^*(N)$ of height 2 such that the unique maximal ideal $\mathfrak{n}$ of $\mathcal{S}(N)$ with $\mathfrak{n} \cap \mathcal{S}^*(N) \subset \mathfrak{n}^*$ has height 0. Let X be a semialgebraic compactification of N by one point p, the inclusion map $\mathtt{j} : M \hookrightarrow X$ and let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{S}(X)$ consisting of those functions of $\mathcal{S}(X)$ vanishing at p. It follows from Lemma 2.15(i) that the induced map $\mathrm{Spec}_s(\mathtt{j}) : \mathrm{Spec}_s(N) \to \mathrm{Spec}_s(X) \setminus \{p \equiv \mathfrak{m}_p\}$ is a homeomorphism. Thus, since $\mathfrak{n}$ is a minimal and maximal ideal of $\mathcal{S}(N)$, the chain $\mathfrak{n} \cap \mathcal{S}(X) \subsetneq \mathfrak{m}_p$ is the unique one in the ring $\mathcal{S}(X)$, in which the prime ideal $\mathfrak{n} \cap \mathcal{S}(X)$ occurs. Thus, $\mathrm{h}_X(\mathfrak{n}^*) = 1$ while $\mathrm{ht}(\mathfrak{n}^*) = 2$.

6. Immediate successor

Given a free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$, the set of prime ideals of $\mathcal{S}^*(M)$ containing $\mathfrak{m} \cap \mathcal{S}^*(M)$ is a chain (see Paragraph 2.4.2), and our goal is to describe the immediate successor of $\mathfrak{m} \cap \mathcal{S}^*(M)$, that is, the smallest prime ideal of $\mathcal{S}^*(M)$ containing properly $\mathfrak{m} \cap \mathcal{S}^*(M)$. For technical reasons, the existence of a semialgebraic equation of the remainder $\partial M := \beta_{\rm s}^* M \setminus M$ of M in $\beta_{\rm s}^* M$ will be important, see Remark 6.4 and Lemma 6.5. This implies that ∂M is closed in $\beta_{\rm s}^* M$ or equivalently that M is locally compact. This is why we restrict ourselves to the locally compact case in the first part of this section. In the second part, we show how far the previous description extends in case M is an arbitrary semialgebraic set (see Theorem 6.8).

6.1. *Immediate successor in the locally compact setting*

The statement and proof of the next result is strongly inspired by Mandelker [17, 6] and by Gillman and Jerison [15, 14.25-27], and it corresponds to the unsettled Proposition 5.1(iii).

THEOREM 6.1 *Let $M \subset \mathbb{R}^m$ be a locally compact but not compact semialgebraic set. Let $\mathfrak{m}_0$ be a free maximal ideal of $\mathcal{S}(M)$ and $\mathfrak{m}_0^*$ the only maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{m}_0 \cap \mathcal{S}^*(M)$. Then*

(i) *The set $\mathfrak{q}_0$ of all semialgebraic functions $f \in \mathcal{S}^*(M)$, whose extension $\hat{f}$ to $\beta_{\rm s}^* M$ vanishes on a neighbourhood of $\mathfrak{m}_0^*$ in ∂M is a prime ideal of $\mathcal{S}^*(M)$, which strictly contains $\mathfrak{m}_0 \cap \mathcal{S}^*(M)$.*

(ii) *Let $\mathfrak{q}$ be a prime ideal of $\mathcal{S}^*(M)$, which strictly contains $\mathfrak{m}_0 \cap \mathcal{S}^*(M)$. Then $\mathfrak{q}_0 \subset \mathfrak{q}$.*

Before proving Theorem 6.1, we need to recall some results concerning compactifications of a semialgebraic set. These and other related results are studied in detail in [8, §9] and [13, §4] and we refer the reader to them for further details.

(6.2) Compactifications of a semialgebraic set. Given two compactifications $(X_1, \mathtt{j}_1)$ and $(X_2, \mathtt{j}_2)$ of a semialgebraic set $M \subset \mathbb{R}^m$, we say that $(X_2, \mathtt{j}_2)$ *dominates* $(X_1, \mathtt{j}_1)$, and we write $(X_1, \mathtt{j}_1) \preccurlyeq (X_2, \mathtt{j}_2)$ if there exists a continuous surjective map $\rho : X_2 \to X_1$ such that $\rho \circ \mathtt{j}_2 = \mathtt{j}_1$. Note that since $\mathtt{j}_i(M)$ is dense in X_i for $i = 1, 2$, the map ρ is unique with such property. The domination relation $\preccurlyeq$ is an order relation in the set of all compactifications of M that is up to a homeomorphism compatible with the embeddings. In fact, given two compactifications $(X_1, \mathtt{j}_1)$ and $(X_2, \mathtt{j}_2)$ of M such that $(X_1, \mathtt{j}_1) \preccurlyeq (X_2, \mathtt{j}_2)$ with X_2 Hausdorff, the continuous surjection $\rho : X_2 \to X_1$ such that $\rho \circ \mathtt{j}_2 = \mathtt{j}_1$ satisfies the equalities $\rho^{-1}(X_1 \setminus \mathtt{j}_1(M)) = X_2 \setminus \mathtt{j}_2(M)$ and $\rho(X_2 \setminus \mathtt{j}_2(M)) = X_1 \setminus \mathtt{j}_1(M)$. Recall that each semialgebraic compactification $(X, \mathtt{j})$ of M satisfies by [13, 4.6] that $(X, \mathtt{j}) \preccurlyeq (\beta_{\rm s}^* M, \phi)$, where $\phi : M \to \beta_{\rm s}^* M,\ \ p \mapsto \mathfrak{m}_p^*$. Moreover, given $f_1, \dots, f_r \in \mathcal{S}^*(M)$, there exist by [13, 4.1] a semialgebraic compactification $(X, \mathtt{j})$ of M and semialgebraic functions $F_i \in \mathcal{S}(X)$ such that $F_i \circ \mathtt{j} = f_i$, for $i = 1, \dots, r$.

We also need to prove that if M is locally compact, then the closed subset $\partial M := \beta_{\rm s}^* M \setminus M$ of $\beta_{\rm s}^* M$ has a semialgebraic equation. This will be very useful in order to prove Theorem 6.1. We verify first the following more general result.

LEMMA 6.3 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set. Then there exists a function $f \in \mathcal{S}^*(\mathbb{R}^m)$, which is positive at each point of $M_{\rm lc}$ such that $\partial M \cup \rho_1(M) = \mathcal{Z}_{\beta_{\rm s}^* M}(f|_M)$.*

Proof. If M is compact, we have $M = M_{\rm lc} = \beta_{\rm s}{}^* M$, and so the constant semialgebraic function $1 \in \mathcal{S}^*(M)$ does the work. In the following, we limit ourselves to a non-compact M. By Remark 2.2, we may assume that M is bounded. Moreover, $M_{\rm lc}$ is open and dense in the compact semialgebraic set $X := \operatorname{Cl}_{\mathbb{R}^m}(M) = \operatorname{Cl}_{\mathbb{R}^m}(M_{\rm lc})$. By Lemma 2.1, there exists $g \in \mathcal{S}^*(M)$ such that $X \setminus M_{\rm lc} = Z_X(g)$, and let $f := g^2$. Let us check that $\partial M \cup \rho_1(M) = \mathcal{Z}_{\beta_s^* M}(f|_M)$. Note $X \setminus M_{\rm lc} = Z_X(f)$. On the other hand, as we observed in 6.2, there exists a surjective continuous map $\rho : \beta_s^* M \to X$, which is the identity on M and satisfies $\partial M = \rho^{-1}(X \setminus M)$ and $\rho^{-1}(\rho_1(M)) = \rho_1(M)$. Thus, $\partial M \cup \rho_1(M)$ is the zeroset of the continuous function $f \circ \rho$. Note that the latter is the unique continuous extension $\widehat{f|_M}$ of $f|_M$ to $\beta_s^* M$, and so $\partial M \cup \rho_1(M) = \mathcal{Z}_{\beta_s^* M}(f|_M)$, as wanted. □

REMARK 6.4 In particular, if $M \subset \mathbb{R}^m$ is a locally compact but not compact semialgebraic set, there exists a function $f \in \mathcal{S}^*(\mathbb{R}^m)$, which is positive at each point of M and such that $\partial M = \mathcal{Z}_{\beta_s^* M}(f|_M)$.

LEMMA 6.5 *Let $M \subset \mathbb{R}^m$ be a locally compact but not compact semialgebraic set. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{S}(M)$ and $\mathfrak{p}$ a prime ideal of $\mathcal{S}^*(M)$ such that $\mathfrak{m} \cap \mathcal{S}^*(M) \subsetneq \mathfrak{p}$. Then there exists a function $b \in \mathfrak{p}$, which is positive at each point of M and satisfies $\mathcal{Z}_{\beta_s^* M}(b) = \partial M$.*

Proof. By Remark 6.4, there exists a function $a \in \mathcal{S}^*(M)$, which is positive at each point of M such that $\mathcal{Z}_{\beta_s^* M}(a) = \partial M$. Now it is enough to find a function $c \in \mathfrak{p}$ such that $\mathcal{Z}_{\beta_s^* M}(c) \subset \partial M$ and to choose $b := ac$. Pick a function $f \in \mathfrak{p} \setminus \mathfrak{m}$; hence, $f \in S(M) \setminus \mathfrak{m}$, and so there exists $g \in S(M)$ such that $h := 1 - fg \in \mathfrak{m}$. Then the function

$$c := \frac{1}{1 + g^2 + h^2} = \left(\frac{g}{1 + g^2 + h^2}\right) f + \frac{h}{1 + g^2 + h^2}$$

does the work because $Z_M(c)$ is empty and $\frac{h}{1+g^2+h^2} \in \mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{p}$. □

Another ingredient to prove Theorem 6.1 is the following elementary lemma whose proof is left to the reader.

LEMMA 6.6 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{m}$ a maximal ideal of $\mathcal{S}(M)$. Let $\mathfrak{m}^*$ be the only maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{p} := \mathfrak{m} \cap \mathcal{S}^*(M)$ and let $h \in \mathcal{S}^*(M)$. Then*

(i) *The canonical homomorphisms $\pi^* : \mathcal{S}^*(M)/\mathfrak{p} \to \mathcal{S}^*(M)/\mathfrak{m}^*,\ f + \mathfrak{p} \to f + \mathfrak{m}^*$ and $\pi : \mathcal{S}^*(M)/\mathfrak{p} \hookrightarrow \mathcal{S}(M)/\mathfrak{m},\ f + \mathfrak{p} \to f + \mathfrak{m}$ are order preserving.*
(ii) *If $h + \mathfrak{m}^* > 0$ in $\mathcal{S}^*(M)/\mathfrak{m}^*$, then $h + \mathfrak{m} > 0$ in $\mathcal{S}(M)/\mathfrak{m}$.*
(iii) *If $h + \mathfrak{m} > 0$ in $\mathcal{S}(M)/\mathfrak{m}$, then $h + \mathfrak{m}^* \geq 0$ in $\mathcal{S}^*(M)/\mathfrak{m}^*$.*

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1 (i) It is obvious that $\mathfrak{q}_0$ is a radical ideal. Thus, by Lemma 5.3 it is sufficient to check that $\mathfrak{q}_0$ strictly contains the prime ideal $\mathfrak{m}_0 \cap \mathcal{S}^*(M)$. Let $\Phi : \beta_s M \to \beta_s^* M$ be the homeomorphism that maps each maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$ to the unique maximal ideal $\mathfrak{m}^*$ of $\mathcal{S}^*(M)$ containing $\mathfrak{m} \cap \mathcal{S}^*(M)$ (see Paragraph 2.5.2). Since M is locally compact, there exists by Remark 6.4 a function $b \in \mathcal{S}^*(M)$ that is positive at each point of M and such that $\partial M = \beta_s^* M \setminus M = \mathcal{Z}_{\beta_s^* M}(b)$.

Let $g \in \mathfrak{m}_0 \cap \mathcal{S}^*(M)$ and let us prove that $g \in \mathfrak{q}_0$. Consider the open subset of ∂M defined by $W := \{\mathfrak{m}^* \in \partial M : (b - g^2) + \mathfrak{m} > 0\}$, where the positivity has its obvious meaning in the real closed field $\mathcal{S}(M)/\mathfrak{m}$. To show that $g \in \mathfrak{q}_0$, it is enough to prove that $\mathfrak{m}_0^* \in W$ and $\hat{g}$ vanishes on W, that is, $g \in \mathfrak{m}^*$ for each $\mathfrak{m}^* \in W$. Observe first that b is a positive unit in $\mathcal{S}(M)$ and so $b + \mathfrak{m}_0 > 0$ in $\mathcal{S}(M)/\mathfrak{m}_0$, while $g^2 + \mathfrak{m}_0 = 0$. Thus, $(b - g^2) + \mathfrak{m}_0 = b + \mathfrak{m}_0 > 0$ and therefore $\mathfrak{m}_0^* \in W$.

Next, take $\mathfrak{m}^* \in W$. The inequality $(b-g^2)+\mathfrak{m} > 0$ implies by Lemma 6.6 that $(b-g^2)+\mathfrak{m}^* \geq 0$. But $b \in \mathfrak{m}^*$ and so $-g^2+\mathfrak{m}^* \geq 0$, that is, $g \in \mathfrak{m}^*$. Hence, $\mathfrak{m}_0 \cap \mathcal{S}^*(M) \subset \mathfrak{q}_0$ and all is reduced to check that the inclusion is strict. But $b \in \mathfrak{q}_0 \setminus (\mathfrak{m}_0 \cap \mathcal{S}^*(M))$ because $\mathcal{Z}_{\beta_s^* M}(b) = \partial M$.

(ii) We may assume without loss of generality that M is bounded. Since $\mathfrak{m}_0 \cap \mathcal{S}^*(M) \subsetneq \mathfrak{q}_0$ and $\mathfrak{m}_0 \cap \mathcal{S}^*(M) \subsetneq \mathfrak{q}$, there exist by Lemma 6.5 functions $a_0 \in \mathfrak{q}_0$ and $a \in \mathfrak{q}$ such that $\mathcal{Z}_{\beta_s^* M}(a_0) = \partial M = \mathcal{Z}_{\beta_s^* M}(a)$. Hence, the function $c := a_0 a \in \mathfrak{q}_0 \cap \mathfrak{q}$ and $\mathcal{Z}_{\beta_s^* M}(c) = \partial M$. Let us see now that $\mathfrak{q}_0 \subset \mathfrak{q}$. Take a function $g \in \mathfrak{q}_0$. To prove that $g \in \mathfrak{q}$, it is enough to see $g_1 := g^2 + c^2 \in \mathfrak{q}$. Since $g_1 \in \mathfrak{q}_0$, its extension $\hat{g}_1 : \beta_s^* M \to \mathbb{R}$ vanishes on a neighbourhood of $\mathfrak{m}_0^*$ in ∂M. Therefore, there exists $f \in \mathcal{S}^*(M)$ such that $\mathfrak{m}_0^* \in \mathcal{D}_{\beta_s^* M}(f) \cap \partial M \subset \mathcal{Z}_{\beta_s^* M}(g_1)$. After replacing f by $f^2 + c^2$, we may assume $\mathcal{Z}_{\beta_s^* M}(f) \subset \partial M$. We claim:

(6.1.1) *There exists a function $h \in \mathfrak{m}_0 \cap \mathcal{S}^*(M)$ such that $h(x) \geq 0$ for each $x \in M$ and*

$$\mathcal{Z}_{\beta_s^* M}(h) \cap \partial M \subset \mathcal{D}_{\beta_s^* M}(f) \cap \partial M \subset \mathcal{Z}_{\beta_s^* M}(g_1).$$

Indeed, each maximal ideal $\mathfrak{m}^*$ of $\mathcal{S}^*(M)$ containing f is distinct from $\mathfrak{m}_0^*$, which is the unique one containing $\mathfrak{m}_0 \cap \mathcal{S}^*(M)$. Thus, for each $\mathfrak{m}^* \in \mathcal{Z}_{\beta_s^* M}(f)$, there exists a function $h_{\mathfrak{m}^*} \in \mathfrak{m}_0 \cap \mathcal{S}^*(M) \setminus \mathfrak{m}^*$, that is, $\mathcal{Z}_{\beta_s^* M}(f) \subset \bigcup_{\mathfrak{m}^* \in \mathcal{Z}_{\beta_s^* M}(f)} \mathcal{D}_{\beta_s^* M}(h_{\mathfrak{m}^*})$. By the compactness of $\mathcal{Z}_{\beta_s^* M}(f)$ there exist $h_1, \ldots, h_r \in \mathfrak{m}_0 \cap \mathcal{S}^*(M)$ such that

$$\mathcal{Z}_{\beta_s^* M}(f) \subset \bigcup_{i=1}^{r} \mathcal{D}_{\beta_s^* M}(h_i) = \mathcal{D}_{\beta_s^* M}(h),$$

where $h := \sum_{i=1}^r h_i^2 \in \mathfrak{m}_0 \cap \mathcal{S}^*(M)$. This function satisfies Paragraph 6.1.1.

By 6.2, there exist a semialgebraic compactification X of M and semialgebraic functions $H, G_1 \in \mathcal{S}(X)$ such that $H|_M = h$ and $G_1|_M = g_1$. Moreover there exists, also by 6.2, a surjective continuous map $\gamma : \beta_s^* M \to X$, which is the identity on M, and $\gamma(\partial M) = X \setminus M$. Recall that $\mathcal{S}(X) = \mathcal{S}^*(X)$ by the compactness of X and so all maximal ideals of this ring are fixed (see Subsection 2.5). Note that $\hat{h} = H \circ \gamma$ and $\hat{g}_1 = G_1 \circ \gamma$ are by Paragraph 2.5.3 the unique continuous extensions of h and g_1 to $\beta_s^* M$.

(6.1.2) *The closed semialgebraic subsets of X*

$$C_1 := Z_X(H) \cup Z_X(G_1) \quad \textit{and} \quad C_2 := \mathrm{Cl}_X(X \setminus (Z_X(G_1) \cup M))$$

satisfy the following inclusions: $C_1 \cap C_2 \subset Z_X(G_1)$ and

$$X \setminus M \subset Z_X(G_1) \cup \mathrm{Cl}_X((X \setminus M) \setminus Z_X(G_1)) = Z_X(G_1) \cup C_2 \subset C_1 \cup C_2.$$

The second one is not difficult. For the first one, suppose by contradiction the existence of a point $p \in (C_1 \cap C_2) \setminus Z_X(G_1)$; hence,

$$p \in (Z_X(H) \setminus Z_X(G_1)) \cap (X \setminus \mathrm{Int}_X(Z_X(G_1) \cup M)).$$

Recall that M is open in X and consequently

$$M = \mathrm{Int}_X(M) \subset \mathrm{Int}_X(Z_X(G_1) \cup M).$$

Therefore $p \in X \setminus M$, and so there exists a point $\mathfrak{m}^* \in \partial M$ such that $\gamma(\mathfrak{m}^*) = p$. Since $p \in (Z_X(H) \setminus Z_X(G_1)) \cap (X \setminus M)$, we deduce

$$\mathfrak{m}^* \in (\mathcal{Z}_{\beta_s^* M}(h) \setminus \mathcal{Z}_{\beta_s^* M}(g_1)) \cap \partial M \quad (*)$$

because the formulas

$$\hat{h}(\mathfrak{m}^*) = (H \circ \gamma)(\mathfrak{m}^*) = H(p) = 0 \quad \text{and} \quad \hat{g}_1(\mathfrak{m}^*) = (G_1 \circ \gamma)(\mathfrak{m}^*) = G_1(p) \neq 0$$

imply by 6.2 that $h \in \mathfrak{m}^*$ and $g_1 \notin \mathfrak{m}^*$. However, condition $(*)$ contradicts Paragraph 6.1.1, and so Paragraph 6.1.2 holds.

(6.1.3) Consider the following semialgebraic function on $C := C_1 \cup C_2$:

$$\eta : C \to \mathbb{R},\ x \mapsto \begin{cases} G_1(x) & \text{if } x \in C_2, \\ 0 & \text{if } x \in C_1. \end{cases}$$

By Theorem 2.3, there exists a semialgebraic function $\overline{\eta} : X \to \mathbb{R}$ such that $\overline{\eta}|_C = \eta$. Note that by Paragraph 6.1.2 $X \setminus M \subset Z_X(G_1) \cup C_2 \subset C$, that is, $\overline{\eta}|_{X\setminus M} = \eta|_{X\setminus M} = G_1|_{X\setminus M}$, and therefore $X \setminus M \subset Z_X(\overline{\eta} - G_1)$. Thus, $(\overline{\eta} - G_1) \circ \gamma$ vanishes identically on ∂M, that is,

$$\mathcal{Z}_{\beta_s^* M}(c) = \partial M \subset \mathcal{Z}_{\beta_s^* M}((\overline{\eta} - G_1) \circ (\gamma|_M)).$$

Applying the Nullstellensatz for $\mathcal{S}^*(M)$ (see [**11**, 3.11]) and because $c \in \mathfrak{q}$, we deduce $(\overline{\eta} - G_1) \circ (\gamma|_M) \in \mathfrak{q}$. On the other hand,

$$Z_M(H) = Z_X(H) \cap M \subset C_1 \cap M \subset Z_M(\overline{\eta}).$$

Hence, $Z_M(h) = Z_M(H \circ \gamma) \subset Z_M(\overline{\eta} \circ \gamma)$. Therefore, by Theorem 2.6, it holds $\overline{\eta} \circ (\gamma|_M) \in \mathfrak{m}_0 \cap \mathcal{S}^*(M) \subset \mathfrak{q}$ because $h \in \mathfrak{m}_0 \cap \mathcal{S}^*(M)$ and $\mathfrak{m}_0$ is a z-ideal. Finally, we obtain

$$g_1 = G_1 \circ (\gamma|_M) = \overline{\eta} \circ (\gamma|_M) - (\overline{\eta} - G_1) \circ (\gamma|_M) \in \mathfrak{q},$$

as wanted. □

Remark 6.7 A key point in the proof of Theorem 6.1 is the existence of a semialgebraic equation of ∂M in $\beta_s^* M$ (or equivalently that M is locally compact). Such hypothesis is also essential in the counterpart of Theorem 6.1 for rings of continuous functions (see [**15**, 14.25-27]).

6.2. *Immediate successor in the non-locally compact setting*

In this section, we study to what extent the description for the immediate successor of $\mathfrak{m} \cap \mathcal{S}^*(M)$ in $\operatorname{Spec}_s^*(M)$ proposed in Theorem 6.1, where $\mathfrak{m}$ is a free maximal ideal of $\mathcal{S}(M)$, still works if M is not locally compact.

Theorem 6.8 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{m}$ a free maximal ideal of $\mathcal{S}(M)$. Let $Y := \rho_1(M)$ and let $h \in \mathcal{S}^*(M)$ be an equation of $\partial M \cup Y$ (see Lemma* 6.3*). Let $\mathfrak{a}$ be the set of all bounded* *semialgebraic functions* $f \in \mathcal{S}^*(M)$ *whose extension* $\hat{f}$ *to* $\beta_s^* M$ *vanishes on a neighbourhood of* $\mathfrak{m}^*$ *in* ∂M. *The following properties hold*:

(i) *If* $\mathfrak{m}^* \in \partial M \setminus \mathrm{Cl}_{\beta_s^* M}(Y)$, *then* $\mathfrak{a}$ *is the immediate successor of* $\mathfrak{m} \cap \mathcal{S}^*(M)$ *in any non-refinable chain of prime ideals of* $\mathcal{S}^*(M)$ *ending at* $\mathfrak{m}^*$.

(ii) *If* $\mathfrak{m}^* \in \mathrm{Cl}_{\beta_s^* M}(Y) \setminus Y$, *then* $\mathfrak{a}$ *equals the intersection of all prime ideals* $\mathfrak{q}$ *of* $\mathcal{S}^*(M)$ *such that* $h \in \mathfrak{q} \subset \mathfrak{m}^*$. *Moreover, if* $\mathfrak{F}$ *denotes the collection of all prime ideals* $\mathfrak{p} \in \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)$ *contained in* $\mathfrak{m}^*$, *then* $\mathfrak{a} \subset \bigcap_{\mathfrak{p} \in \mathfrak{F}} \mathfrak{p} \subset \mathfrak{m} \cap \mathcal{S}^*(M)$.

Proof. Recall first that if $\mathtt{j} : M_{\mathrm{lc}} \hookrightarrow M$ denotes the inclusion map, the induced maps $\mathrm{Spec}_s^*(\mathtt{j}) : \mathrm{Spec}_s^*(M_{\mathrm{lc}}) \to \mathrm{Spec}_s^*(M)$ and $\beta_s^* \mathtt{j} : \beta_s^* M_{\mathrm{lc}} \to \beta_s^* M$ are surjective (see [**12**, 5.1 & 6.7]). Moreover, if $\mathtt{r}_M : \mathrm{Spec}_s^*(M) \to \beta_s^* M$ denotes the natural continuous retraction, which maps each prime ideal of $\mathcal{S}^*(M)$ onto the unique maximal ideal of $\mathcal{S}^*(M)$ containing it, the diagram

$$\begin{array}{ccc}
\mathrm{Spec}_s^*(M_{\mathrm{lc}}) \setminus \mathrm{Spec}_s^*(\mathtt{j})^{-1}(\mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)) & \xrightarrow{\cong} & \mathrm{Spec}_s^*(M) \setminus \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y) \\
{\scriptstyle \mathtt{j}_{M_{\mathrm{lc}}}} \uparrow \downarrow {\scriptstyle \mathtt{r}_{M_{\mathrm{lc}}}} & & {\scriptstyle \mathtt{j}_{M}} \uparrow \downarrow {\scriptstyle \mathtt{r}_{M}} \\
\beta_s^* M_{\mathrm{lc}} \setminus (\beta_s^* \mathtt{j})^{-1}(\mathrm{Cl}_{\beta_s^* M}(Y)) & \xrightarrow{\cong} & \beta_s^* M \setminus \mathrm{Cl}_{\beta_s^* M}(Y)
\end{array}$$

is commutative and the map in the upper row is a homeomorphism that extends the homeomorphism in the bottom row (see [**12**, 6.7]). By Corollary 2.11, $\mathfrak{m} \cap \mathcal{S}^*(M) \in \mathrm{Cl}_{\mathrm{Spec}_s^*(M)}(Y)$ if and only if $\mathfrak{m}^* \in \mathrm{Cl}_{\beta_s^* M}(Y)$. In this way, statement (i) follows from Theorem 6.1. Before entering into the proof of statement (ii), we need the following.

(6.8.1) $\mathrm{Cl}_{\beta_s^* M}(\partial M) = \beta_s^* M \setminus M_{\mathrm{lc}}$.

Indeed, observe that $U := \beta_s^* M \setminus \mathrm{Cl}_{\beta_s^* M}(\partial M) \subset \beta_s^* M \setminus \partial M = M$ is an open subset of $\beta_s^* M$, hence of M, and it is locally compact because $\beta_s^* M$ is so. Thus, each point in U admits a compact neighbourhood in M. Therefore, we have by Theorem 3.6(iii) $U = \beta_s^* M \setminus \mathrm{Cl}_{\beta_s^* M}(\partial M) \subset M_{\mathrm{lc}}$ or equivalently $\beta_s^* M \setminus M_{\mathrm{lc}} \subset \mathrm{Cl}_{\beta_s^* M}(\partial M)$. Conversely, M_{lc} is open in $\beta_s^* M = \mathrm{Cl}_{\beta_s^* M}(M_{\mathrm{lc}})$ and it does not meet ∂M; hence, $\mathrm{Cl}_{\beta_s^* M}(\partial M) \cap M_{\mathrm{lc}} = \varnothing$, and so $\mathrm{Cl}_{\beta_s^* M}(\partial M) = \beta_s^* M \setminus M_{\mathrm{lc}}$.

We proceed with (ii). Let $\mathfrak{m}^* \in \mathrm{Cl}_{\beta_s^* M}(Y) \setminus Y$ and $f \in \mathfrak{a}$. Then $\hat{f}$ vanishes on a neighbourhood V of $\mathfrak{m}^*$ in ∂M.

(6.8.2) We claim: $\hat{f}$ *also vanishes on a neighbourhood of* $\mathfrak{m}^*$ *in* $\beta_s^* M \setminus M_{\mathrm{lc}}$.

Indeed, let W be a neighbourhood of $\mathfrak{m}^*$ in $\beta_s^* M \setminus M_{\mathrm{lc}}$ such that $\partial M \cap W = V$ and let us check $W \subset \mathrm{Cl}_{\beta_s^* M}(V) \subset \mathcal{Z}_{\beta_s^* M}(\hat{f})$. Using Paragraph 6.8.1,

$$\begin{aligned}
W &= (\beta_s^* M \setminus M_{\mathrm{lc}}) \cap W = \mathrm{Cl}_{\beta_s^* M}(\partial M) \cap W \\
&= \mathrm{Cl}_{\beta_s^* M}(\partial M \cap W) \cap W \subset \mathrm{Cl}_{\beta_s^* M}(V) \subset \mathcal{Z}_{\beta_s^* M}(\hat{f}),
\end{aligned}$$

which proves our claim.

(6.8.3) Now let $\mathtt{i} : Y \hookrightarrow M$ denote the inclusion map and consider the ring homomorphisms $\phi_1 : \mathcal{S}(M) \to \mathcal{S}(Y),\ \ g \mapsto g|_Y$ and $\phi_2 : \mathcal{S}^*(M) \to \mathcal{S}^*(Y),\ \ g \mapsto g|_Y$ as well as the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Spec}_{\rm s}^*(Y) & \xrightarrow[\cong]{\operatorname{Spec}_{\rm s}^*(\mathtt{i})} & \operatorname{Cl}_{\operatorname{Spec}_{\rm s}^*(M)}(Y) & \hookrightarrow & \operatorname{Spec}_{\rm s}^*(M)\\
\uparrow & & \uparrow & & \uparrow\\
\operatorname{Spec}_{\rm s}(Y) & \xrightarrow[\cong]{\operatorname{Spec}_{\rm s}(\mathtt{i})} & \operatorname{Cl}_{\operatorname{Spec}_{\rm s}(M)}(Y) & \hookrightarrow & \operatorname{Spec}_{\rm s}(M)
\end{array}$$

Since Y is a closed semialgebraic subset of M, both ϕ_1 and ϕ_2 are by Theorem 2.3 surjective while the first maps in the rows of the diagram are by Lemma 2.10(iii) homeomorphisms.

(6.8.4) We claim: *For every prime ideal* $\mathfrak{p}\subset\mathfrak{m}^*$ *of* $\mathcal{S}^*(M)$ *that is minimal in* $\operatorname{Cl}_{\operatorname{Spec}_{\rm s}^*(M)}(Y)$ *we have* $\mathfrak{a}\subset\mathfrak{p}\subset\mathfrak{m}\cap\mathcal{S}^*(M)$.

We begin by proving the inclusion $\mathfrak{a}\subset\mathfrak{p}$. Given $f\in\mathfrak{a}$, there exists by Paragraph 6.8.2 a function $g\in\mathcal{S}^*(M)\setminus\mathfrak{m}^*$ such that $\mathcal{D}_{\beta_{\rm s}^* M}(g)\cap(\beta_{\rm s}^* M\setminus M_{\rm lc})\subset\mathcal{Z}_{\beta_{\rm s}^* M}(f)\cap(\beta_{\rm s}^* M\setminus M_{\rm lc})$; hence, $\partial M\cup Y=\beta_{\rm s}^* M\setminus M_{\rm lc}\subset\mathcal{Z}_{\beta_{\rm s}^* M}(fg)$. Therefore, $Z_Y(fg)=Y$, which implies $fg\in\mathfrak{p}$. Since $g\notin\mathfrak{m}^*$ and $\mathfrak{p}\subset\mathfrak{m}^*$, it follows $f\in\mathfrak{p}$.

Next, we prove the second inclusion $\mathfrak{p}\subset\mathfrak{m}\cap\mathcal{S}^*(M)$. Define $\mathfrak{g}:=\operatorname{Spec}_{\rm s}^*(\mathtt{i})^{-1}(\mathfrak{p})$ and $\mathfrak{b}:=\mathfrak{g}\mathcal{S}(Y)$; observe that as $\mathfrak{g}$ is a minimal prime ideal of $\mathcal{S}^*(Y)$, it follows from Lemma 4.3 that $\mathfrak{b}$ is a minimal prime ideal of $\mathcal{S}(Y)$. Let $\mathfrak{n}$ be the unique maximal ideal of $\mathcal{S}(Y)$ that contains $\mathfrak{b}$ and observe $\operatorname{Spec}_{\rm s}(\mathtt{i})(\mathfrak{b})=\phi_1^{-1}(\mathfrak{b})\subset\phi_1^{-1}(\mathfrak{n})=\operatorname{Spec}_{\rm s}(\mathtt{i})(\mathfrak{n})$; in fact, the last one is a maximal ideal of $\mathcal{S}(M)$ because ϕ_1 is surjective. Moreover,

$$\mathfrak{p}=\operatorname{Spec}_{\rm s}^*(\mathtt{i})(\mathfrak{g})=\operatorname{Spec}_{\rm s}^*(\mathtt{i})(\mathfrak{b}\cap\mathcal{S}^*(Y))=\operatorname{Spec}_{\rm s}(\mathtt{i})(\mathfrak{b})\cap\mathcal{S}^*(M)\subset\operatorname{Spec}_{\rm s}(\mathtt{i})(\mathfrak{n})\cap\mathcal{S}^*(M).$$

Since $\mathfrak{p}\subset\mathfrak{m}^*$, we deduce by Paragraph 2.4.2 that $\operatorname{Spec}(\mathtt{i})(\mathfrak{n})\cap\mathcal{S}^*(M)\subset\mathfrak{m}^*$. As $\operatorname{Spec}_{\rm s}(\mathtt{i})(\mathfrak{n})$ is a maximal ideal of $\mathcal{S}(M)$, we conclude by Paragraph 2.5.2 that $\operatorname{Spec}_{\rm s}(\mathtt{i})(\mathfrak{n})=\mathfrak{m}$. Therefore, $\mathfrak{p}\subset\mathfrak{m}\cap\mathcal{S}^*(M)$ and Paragraph 6.8.4 is proved. Observe that the last part of statement (ii) is an immediate consequence of Paragraph 6.8.4.

(6.8.5) To finish, let $\{\mathfrak{q}_\lambda\}_{\lambda\in\Lambda}$ be the collection of all prime ideals of $\mathcal{S}^*(M)$ satisfying $h\in\mathfrak{q}_\lambda\subset\mathfrak{m}^*$. We have to check $\mathfrak{a}=\bigcap_{\lambda\in\Lambda}\mathfrak{q}_\lambda$.

We prove first that $\mathfrak{a}\subset\bigcap_{\lambda\in\Lambda}\mathfrak{q}_\lambda$. Fix $\lambda\in\Lambda$ and let $f\in\mathfrak{a}$ and $g\in\mathcal{S}^*(M)\setminus\mathfrak{m}^*$ such that $\mathcal{D}_{\beta_{\rm s}^* M}(g)\cap(\partial M\cup Y)\subset\mathcal{Z}_{\beta_{\rm s}^* M}(f)\cap(\partial M\cup Y)$ (see Paragraph 6.8.2); hence, $\mathcal{Z}_{\beta_{\rm s}^* M}(h)=\partial M\cup Y\subset\mathcal{Z}_{\beta_{\rm s}^* M}(fg)$. By the Nullstellensatz for the ring $\mathcal{S}^*(M)$ [**11**, 3.11], the prime ideal $\mathfrak{q}_\lambda$ contains $fg\in\mathcal{S}^*(M)$ but it does not contain g; hence, $f\in\mathfrak{q}_\lambda$. Therefore, $\mathfrak{a}\subset\bigcap_{\lambda\in\Lambda}\mathfrak{q}_\lambda$.

Conversely, let $f\in\bigcap_{\lambda\in\Lambda}\mathfrak{q}_\lambda$ and let us denote a maximal ideal of $\mathcal{S}(M_{\rm lc})$ with $\mathfrak{b}_0$ such that $\beta_{\rm s}^*\,\mathtt{j}(\mathfrak{b}_0^*)=\mathfrak{m}^*$. Let $\mathfrak{b}_1^*$ be the immediate successor of $\mathfrak{b}_0\cap\mathcal{S}^*(M_{\rm lc})$ in $\operatorname{Spec}_{\rm s}^*(M_{\rm lc})$ and let us check first that $\operatorname{Spec}_{\rm s}^*(\mathtt{j})(\mathfrak{b}_1^*)=\mathfrak{q}_\lambda$ for some $\lambda\in\Lambda$. Since $\operatorname{Spec}_{\rm s}^*(\mathtt{j})(\mathfrak{b}_1^*)\subset\beta_{\rm s}^*\,\mathtt{j}(\mathfrak{b}_0^*)=\mathfrak{m}^*$, it is sufficient to check $h\in\operatorname{Spec}_{\rm s}^*(\mathtt{j})(\mathfrak{b}_1^*)$; in fact, by the Nullstellensatz [**11**, 3.11] for the ring $\mathcal{S}^*(M)$, it is enough to check that $\mathcal{Z}_{\beta_{\rm s}^* M}(a)=\partial M\cup Y$ for some $a\in\operatorname{Spec}_{\rm s}^*(\mathtt{j})(\mathfrak{b}_1^*)$. Since $\mathfrak{b}_1^*$ is the immediate successor of $\mathfrak{b}_0\cap\mathcal{S}^*(M_{\rm lc})$ in $\operatorname{Spec}_{\rm s}^*(M_{\rm lc})$, there exists, by Lemma 2.9 a bounded semialgebraic function $a_0\in\mathfrak{b}_1^*$ such that $Z_{M_{\rm lc}}(a_0)=\emptyset$. As a_0 is bounded and h vanishes identically on Y, the bounded semialgebraic function $a_0(h|_{M_{\rm lc}})\in\mathcal{S}^*(M_{\rm lc})$ admits a bounded semialgebraic extension $a\in\mathcal{S}(M)$ such that $Z_M(a)=Z_M(h)=Y$; in fact, one can check that $\mathcal{Z}_{\beta_{\rm s}^* M}(a)=\partial M\cup Y$ and because $a_0\in\mathfrak{b}_1^*$, it follows readily that $a\in\operatorname{Spec}_{\rm s}^*(\mathtt{j})(\mathfrak{b}_1^*)$, as desired.

In this way, we have proved the existence of $\lambda \in \Lambda$ such that $\operatorname{Spec}_s^*(\mathtt{j})(\mathfrak{b}_1^*) = \mathfrak{q}_\lambda$, so $f \in \operatorname{Spec}_s^*(\mathtt{j})(\mathfrak{b}_1^*)$. Thus, $f|_{M_{\rm lc}} \in \mathfrak{b}_1^*$ and by Theorem 6.1 there exists an open neighbourhood $V^{\mathfrak{b}_0^*} \subset \partial M_{\rm lc}$ of $\mathfrak{b}_0^*$ in $\partial M_{\rm lc}$ such that $V^{\mathfrak{b}_0^*} \subset \mathcal{Z}_{\beta_s^* M_{\rm lc}}(f|_{M_{\rm lc}})$.

Consider the closed subset $C := \beta_s^* M_{\rm lc} \setminus \bigcup_{\mathfrak{b}^* \in (\beta_s^* \mathtt{j})^{-1}(\mathfrak{m}^*)} V^{\mathfrak{b}^*}$ of $\beta_s^* M_{\rm lc}$ whose image $\beta_s^* \mathtt{j}(C)$ under the proper map $\beta_s^* \mathtt{j} : \beta_s^* M_{\rm lc} \to \beta_s^* M$ is a closed subset of $\beta_s^* \mathtt{j}(\partial M_{\rm lc}) = \beta_s^* M \setminus M_{\rm lc}$ that does not contain $\mathfrak{m}^*$. Moreover, since $\bigcup_{\mathfrak{b}^* \in (\beta_s^* \mathtt{j})^{-1}(\mathfrak{m}^*)} V^{\mathfrak{b}^*} \subset \mathcal{Z}_{\beta_s^* M_{\rm lc}}(f|_{M_{\rm lc}})$, it follows $\mathfrak{m}^* \in \partial M \setminus \beta_s^* \mathtt{j}(C) \subset \mathcal{Z}_{\beta_s^* M}(f)$ and so $f \in \mathfrak{a}$, as wanted. □

REMARKS 6.9 (i) Observe that $\operatorname{Cl}_{\beta_s^* M}(Y) = Y$ if and only if Y is compact. Thus, if such is the case, Theorem 6.8(ii) never happens. So for each free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$, the ideal $\mathfrak{a}$ of all bounded semialgebraic functions $f \in \mathcal{S}^*(M)$ whose extension $\hat{f}$ to $\beta_s^* M$ vanishes on a neighbourhood of $\mathfrak{m}^*$ in ∂M is the immediate successor of $\mathfrak{m} \cap \mathcal{S}^*(M)$ for every non-refinable chain of prime ideals ending at $\mathfrak{m}^*$.

(ii) If Y is not compact, there exist maximal ideals $\mathfrak{m}^* \in \beta_s^* M$ such that the description for the immediate successor of $\mathfrak{m} \cap \mathcal{S}^*(M)$ proposed in Theorem 6.1 does not work in any non-refinable chain of prime ideals in $\operatorname{Spec}_s^*(M)$ ending at $\mathfrak{m}^*$.

7. Maximal ideals of prefixed height

We are now in a position to prove that for each non-compact pure dimensional semialgebraic set M and for each $0 \leq r < d := \dim(M)$ there exists a free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$ such that $\operatorname{ht}(\mathfrak{m}) = r$ but $\operatorname{ht}(\mathfrak{m}^*) = d$ where, as above, $\mathfrak{m}^*$ is the unique maximal ideal of $\mathcal{S}^*(M)$ containing $\mathfrak{m}$. In fact, we prove the following stronger result, which is somehow related to Bröcker's ultrafilter theorem ([4, §4]) mainly if M is closed in $\mathbb{R}^m$ (see [10, 4.9]) and, as a consequence of [2, 2.2.9], also in the locally compact case. Recall that we denote the *local dimension of the semialgebraic set M at its point p with* $\dim_p(M)$ (see [2, 2.8.11] for further details).

THEOREM 7.1 (Maximal ideals of prefixed height) *Let $M \subset \mathbb{R}^m$ be a bounded non-compact semialgebraic set. Then*

(i) *The semialgebraic set* $\operatorname{Cl}_{\mathbb{R}^m}(M) \setminus (\operatorname{Cl}_{\mathbb{R}^m}(\rho_1(M)) \cup M)$ *is non-empty.*
(ii) *Let* $p \in \operatorname{Cl}_{\mathbb{R}^m}(M) \setminus (\operatorname{Cl}_{\mathbb{R}^m}(\rho_1(M)) \cup M)$ *and denote* $d := \dim_p \operatorname{Cl}_{\mathbb{R}^m}(M)$. *Let* $g \in \mathcal{S}^*(M)$ *be such that* $g(p) \neq 0$ *and let* $0 \leq k \leq d-1$. *Then there exists a free maximal ideal* $\mathfrak{m}_k$ *of* $\mathcal{S}(M)$ *such that* $\operatorname{ht}(\mathfrak{m}_k) = k$, $g \notin \mathfrak{m}_k^*$ *and* $\operatorname{ht}(\mathfrak{m}_k^*) = d$.

In particular, there exist free maximal ideals of $\mathcal{S}(M)$ of height zero, that is, they are also minimal prime ideals of $\mathcal{S}(M)$.

Proof of Part (i) *in Theorem* 7.1 Suppose by contradiction

$$\operatorname{Cl}_{\mathbb{R}^m}(M) = \operatorname{Cl}_{\mathbb{R}^m}(\rho_1(M)) \cup M.$$

Subtracting M on both sides, we obtain

$$\rho_0(M) = \operatorname{Cl}_{\mathbb{R}^m}(M) \setminus M = \operatorname{Cl}_{\mathbb{R}^m}(\rho_1(M)) \setminus M \subset \operatorname{Cl}_{\mathbb{R}^m}(\rho_1(M)) = \operatorname{Cl}_{\mathbb{R}^m}(\rho_0(\rho_0(M))).$$

But this is impossible because $\dim(\operatorname{Cl}_{\mathbb{R}^m}(\rho_0(\rho_0(M)))) = \dim(\rho_0(\rho_0(M))) < \dim(\rho_0(M))$ (see for instance [2, 2.8.13]). □

Next, we show Theorem 7.1(ii) under the assumption that M is a locally compact semialgebraic set. Afterwards, we approach the general case.

7.1. *Proof of Theorem* 7.1(ii) *for a locally compact M*

The proof of this result is conducted in several steps:
Step 1. We are going to find a bounded semialgebraic set M_1 that is semialgebraically homeomorphic to M such that $X_1 = \mathrm{Cl}_{\mathbb{R}^m}(M_1)$ contains a semialgebraic subset C_1 that is semialgebraically homeomorphic to the hypercube $I := [0, 1]^d$ and whose intersection $C_0 := M_1 \cap C_1$ with M_1 is semialgebraically homeomorphic to $[0, 1]^{d-1} \times (0, 1]$. We do this in such a way that there exist $h, h_1 \in \mathcal{S}^*(X_1)$ such that $h|_{C_1} \geq c > 0$ for some positive real number c, $\dim(Z_M(h_1)) = d$ and $hh_1 = 0$. Of course, $\mathcal{S}(M) \cong \mathcal{S}(M_1)$ and $\mathcal{S}^*(M) \cong \mathcal{S}^*(M_1)$.

Indeed, after a change of coordinates we may assume that p is the origin of $\mathbb{R}^m$. Since $\dim_p \mathrm{Cl}_{\mathbb{R}^m}(M) = d$, there exists a compact semialgebraic neighbourhood V of p in $\mathbb{R}^m$ such that $\dim(\mathrm{Cl}_{\mathbb{R}^m}(M) \cap V) = d$. By Lemma 2.1, there exist $g_1, g_2 \in \mathcal{S}^*(\mathbb{R}^m)$ such that $Z_{\mathbb{R}^m}(g_1) = V$ and $Z_{\mathbb{R}^m}(g_2) = \mathbb{R}^m \setminus \mathrm{Int}_{\mathbb{R}^m}(V)$. Substitute g by the product $(gg_2)^2$. Then $g(p) > 0$ and $gg_1 = 0$.

When applying [2, 9.3.6] to the semialgebraic set $E = M \cup \{p\}$, there exists $\varepsilon > 0$ and a semialgebraic homeomorphism $\varphi : \bar{\mathbb{B}}_m(p, \varepsilon) \to \bar{\mathbb{B}}_m(p, \varepsilon)$ such that

(i) $\|\varphi(y) - p\| = \|y - p\|$ for every $y \in \bar{\mathbb{B}}_m(p, \varepsilon)$,
(ii) $\varphi|_{\mathbb{S}^{m-1}(p,\varepsilon)}$ is the identity map,
(iii) $\varphi^{-1}(M \cap \bar{\mathbb{B}}_m(p, \varepsilon))$ is the cone with vertex p and basis $N := M \cap \mathbb{S}^{m-1}(p, \varepsilon)$ after taking out the vertex p.

Since $g(p) > 0$, we may assume $g(x) \geq g(p)/2 =: c > 0$ for every $x \in M \cap \bar{\mathbb{B}}_m(p, \varepsilon)$. We extend φ to a semialgebraic homeomorphism $\psi : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$\psi(x) := \begin{cases} x & \text{if } x \in \mathbb{R}^m \setminus \bar{\mathbb{B}}_m(p, \varepsilon), \\ \varphi(x) & \text{if } x \in \bar{\mathbb{B}}_m(p, \varepsilon). \end{cases}$$

In the following, we identify M with $\psi^{-1}(M)$. As $\dim_p \mathrm{Cl}_{\mathbb{R}^m}(M) = d$, it follows that N is a semialgebraic set of dimension $d - 1$. By [2, 2.3.6], N is a finite union of semialgebraic sets B_i, each of them semialgebraically homeomorphic to an open hypercube $(0, 1)^k \subset \mathbb{R}^k$ for some $0 \leq k \leq d - 1$, and by [2, 2.8.9] it holds $\dim(B_i) = d - 1$ for some index i. We denote $B := B_i$ and consider a compact semialgebraic subset $K \subset B$ that is semialgebraically homeomorphic to the hypercube $[0, 1]^{d-1}$ and the cone C of vertex p and basis K that satisfies $C \setminus \{p\} \subset M \cap \bar{\mathbb{B}}_m(p, \varepsilon)$. Thus, $g(x) \geq c > 0$ for each point $x \in C$. Consider the semialgebraic map

$$\eta : \mathbb{R}^m \setminus \mathbb{B}_m(p, 1) \to \mathbb{R}^m, \quad y \mapsto \left(1 - \frac{1}{\|y\|}\right) y$$

whose restriction θ to $\mathbb{R}^m \setminus \bar{\mathbb{B}}_m(p, 1)$ is a semialgebraic homeomorphism onto its image $\mathbb{R}^m \setminus \{p\}$ with inverse

$$\theta^{-1} : \mathbb{R}^m \setminus \{p\} \to \mathbb{R}^m \setminus \bar{\mathbb{B}}_m(p, 1), \quad x \mapsto \left(1 + \frac{1}{\|x\|}\right) x.$$

Observe that θ^{-1} preserves the lines through p and transforms spheres centred at p into spheres centred at p. Moreover, it transforms bounded subsets of $\mathbb{R}^m \setminus \{p\}$ into bounded subsets of $\mathbb{R}^m \setminus \bar{\mathbb{B}}_m(p, 1)$.

Define $M_1 := \theta^{-1}(M)$, which is semialgebraically homeomorphic to M. This set M_1 contains a semialgebraic set $C_0 := \theta^{-1}(C)$, which is semialgebraically homeomorphic to $[0, 1]^{d-1} \times (0, 1]$ and whose closure C_1 in $\mathbb{R}^m$, which is clearly contained in the compact semialgebraic set $X_1 := \mathrm{Cl}_{\mathbb{R}^m}(M_1)$, is semialgebraically homeomorphic to $I := [0, 1]^d$. Clearly, $C_0 = C_1 \cap M_1$.

Moreover, note that M_1 is a locally compact and bounded subset of $\mathbb{R}^m \setminus \bar{\mathbb{B}}_m(p, 1)$ and $h := g \circ \eta|_{X_1} \in \mathcal{S}^*(X_1)$ satisfies $h(x) \geq c > 0$ for every $x \in C_1$. On the other hand, since $\dim(\mathrm{Cl}_{\mathbb{R}^m}(M) \cap V) = d$ and $gg_1 = 0$, the semialgebraic function $h_1 := g_1 \circ \eta|_{X_1} \in \mathcal{S}^*(X_1)$ satisfies $\dim(Z_{M_1}(h_1)) = d$ and $hh_1 = 0$.

Step 2. *Construction of a suitable chain of polynomial prime ideals*. Let $0 \leq k \leq d$ and consider the independent linear forms

$$\mathtt{y}_i := \begin{cases} \mathtt{x}_i & \text{if } i = 1, \ldots, k, \\ \mathtt{x}_{d+k+1-i} & \text{if } i = k+1, \ldots, d. \end{cases}$$

Consider the chain of prime ideals of length d of the polynomial ring $A := \mathbb{R}[\mathtt{x}_1 \ldots, \mathtt{x}_d]$

$$(0) = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_k \subsetneq \cdots \subsetneq \mathfrak{p}_d,$$

where $\mathfrak{p}_i := (\mathtt{y}_1, \ldots, \mathtt{y}_i)A$ for $i = 1, \ldots, d$. For each $i = 1, \ldots, d$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}[\mathtt{y}_{i+1}, \ldots, \mathtt{y}_d] & \hookrightarrow & \mathbb{R}[\mathtt{y}_i, \ldots, \mathtt{y}_d] \\ \cong \uparrow & & \uparrow \cong \\ A/\mathfrak{p}_i & \overset{\phi_i}{\hookrightarrow} & A/\mathfrak{p}_{i-1} \end{array} \qquad \text{where} \quad \phi_i(\mathtt{y}_j + \mathfrak{p}_i) := \begin{cases} \mathfrak{p}_{i-1} & \text{if } 1 \leq j \leq i, \\ \mathtt{y}_j + \mathfrak{p}_{i-1} & \text{if } i+1 \leq j \leq d. \end{cases}$$

Let α_i be the cone of positive elements of an ordering of the quotient field $\kappa(\mathfrak{p}_i)$ of the domain $A/\mathfrak{p}_i$, chosen in such a way that $(\kappa(\mathfrak{p}_{i-1}), \alpha_{i-1})$ is an ordered extension of $(\kappa(\mathfrak{p}_i), \alpha_i)$ for $i = 0, \ldots, d$. By [**10**, 4.10], there exists a chain of prime ideals $\mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_d$ in $\mathcal{S}(I)$ such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ and $\mathrm{d}_I(\mathfrak{P}_i) = d - \mathrm{ht}(\mathfrak{P}_i) = d - i$ for $i = 0, \ldots, d$.

Step 3. *Construction of the maximal ideal* $\mathfrak{m}_k$. Observe that $\mathcal{S}(C_1)$ and $\mathcal{S}(I)$ are isomorphic and fix an isomorphism Ψ between them. Consider the ideal in $\mathcal{S}(C_1)$ that corresponds via Ψ to $\mathfrak{P}_i$ and denote it again with $\mathfrak{P}_i$. Recall that C_1 is a closed semialgebraic subset of X_1 and so the homomorphism $\phi : \mathcal{S}(X_1) \to \mathcal{S}(C_1),\ f \mapsto f|_{C_1}$ is by Theorem 2.3 surjective. In this way, we get a chain of prime ideals in $\mathcal{S}(X_1)$ of length d, namely

$$\mathfrak{Q}_0 \subsetneq \cdots \subsetneq \mathfrak{Q}_d \quad \text{where } \mathfrak{Q}_i := \phi^{-1}(\mathfrak{P}_i) \quad \text{for } 1 \leq i \leq d.$$

Note that if $q \in C_1 \subset X_1$ is the point corresponding to the origin 0 of I via the semialgebraic homeomorphism between I and C_1, then $d = \dim_0 I = \dim_q C_1 = \dim_q X_1$.

(7.1.1) Hence, $\mathrm{d}_{X_1}(\mathfrak{Q}_i) = d - i$ for $0 \leq i \leq d$ and the chain of prime ideals $\mathfrak{Q}_0 \subsetneq \cdots \subsetneq \mathfrak{Q}_d$ is non-refinable by Theorem 2.6 and Paragraph 2.8.1.

Consider now the monomorphisms $\mathcal{S}(X_1) \hookrightarrow \mathcal{S}^*(M_1) \hookrightarrow \mathcal{S}(M_1)$. By the construction of the ideals $\mathfrak{Q}_i$, we have

(7.1.2) $$Z_{X_1}(f) \cap M_1 \neq \emptyset \quad \forall f \in \mathfrak{Q}_i \iff 0 \leq i \leq k.$$

Thus, by Lemma 2.15(iv), we obtain that $\mathfrak{q}_0 := \mathfrak{Q}_0\mathcal{S}(M_1) \subsetneq \cdots \subsetneq \mathfrak{Q}_k\mathcal{S}(M_1)$ is a chain of prime ideals in $\mathcal{S}(M_1)$ of length k. Moreover, by Theorem 2.13(ii),

$$\mathrm{d}_{M_1}(\mathfrak{q}_0) = \mathrm{d}_{X_1}(\mathfrak{Q}_0) = d = \dim_q X_1 = \dim_q M_1 \quad (\star)$$

and we deduce from Theorem 4.1 that $\mathfrak{q}_0$ is a minimal prime ideal of $\mathcal{S}(M_1)$. In fact, we claim: $\mathfrak{q}_0$ *is a free prime ideal*.

Indeed, suppose there exists a point $y \in M_1$ such that $f(y) = 0$ for all $f \in \mathfrak{q}_0$. Hence, $f(y) = 0$ for all $f \in \mathfrak{Q}_0$, which contradicts Paragraph 2.5.4 because all functions in $\mathfrak{Q}_0$ vanish at the point $q \neq y$.

(7.1.3) By Paragraph 2.4.2, the collection of all prime ideals of $\mathcal{S}(M_1)$ containing $\mathfrak{q}_0$ constitutes a chain. Let $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r =: \mathfrak{m}$ be such a collection; observe $k \leq r$. As M_1 and X_1 are locally compact, the map $\operatorname{Spec_s}(\mathtt{j}) : \operatorname{Spec_s}(M_1) \to \operatorname{Spec_s}(X_1)$ induced by the inclusion $\mathtt{j} : M_1 \hookrightarrow X_1$ is by Theorem 2.13(iii) a homeomorphism onto its image $\operatorname{Spec_s}(X_1) \setminus \operatorname{Cl}_{\operatorname{Spec_s}(X_1)}(Y_1)$, where $Y_1 := X_1 \setminus M_1$. Thus,

$$\mathfrak{q}_0 \cap \mathcal{S}(X_1) \subsetneq \cdots \subsetneq \mathfrak{q}_r \cap \mathcal{S}(X_1) \quad (\star\star)$$

is a chain of prime ideals in $\mathcal{S}(X_1)$ such that $\mathfrak{q}_r \cap \mathcal{S}(X_1) \notin \operatorname{Cl}_{\operatorname{Spec_s}(X_1)}(Y_1)$. Since the chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r = \mathfrak{m}$ does not admit a refinement, the chain in $(\star\star)$ does not admit a refinement, either.

(7.1.4) We claim: $\mathfrak{Q}_0\mathcal{S}(M_1) \cap \mathcal{S}(X_1) = \mathfrak{Q}_0$. This implies: *Each* $\mathfrak{q}_i \cap \mathcal{S}(X_1) \notin \operatorname{Cl}_{\operatorname{Spec_s}(X_1)}(Y_1)$ *is a prime ideal of* $\mathcal{S}(X_1)$ *containing* $\mathfrak{Q}_0$; *in particular*, $r \leq k$.

By Theorem 2.13(ii), it is enough to see $\mathfrak{Q}_0 \notin \mathcal{L}(Y_1) = \operatorname{Cl}_{\operatorname{Spec_s}(X_1)}(Y_1)$, where the equality holds true because X_1 is locally compact. Suppose by contradiction $\mathfrak{Q}_0 \in \operatorname{Cl}_{\operatorname{Spec_s}(X_1)}(Y_1)$. Then there exists by Lemma 2.10(ii) a function $f \in \mathfrak{Q}_0$ such that $Z_{X_1}(f) \subset Y_1$. Since $\mathfrak{Q}_0$ is a minimal prime ideal of $\mathcal{S}(X_1)$, there exists a function $g \in \mathcal{S}(X_1) \setminus \mathfrak{Q}_0$ such that $fg = 0$. This implies $g = 0$, which is a contradiction, because

$$X_1 = \operatorname{Cl}_{X_1}(M_1) = \operatorname{Cl}_{X_1}(X_1 \setminus Y_1) \subset Z_{X_1}(g).$$

(7.1.5) Now we deduce from Paragraph 7.1.1 to Paragraph 7.1.4 that $r = k$, $\mathfrak{Q}_i = \mathfrak{q}_i \cap \mathcal{S}(X_1)$ and $\mathfrak{q}_i = \mathfrak{Q}_i\mathcal{S}(M_1)$ for $i = 1, \ldots, k$. In particular, $\mathfrak{m}_k = \mathfrak{q}_k$ is a maximal ideal of $\mathcal{S}(M_1)$ of height $\geq k$. On the other hand, since $\mathfrak{Q}_k = \operatorname{Spec_s}(\mathtt{j})(\mathfrak{q}_k)$, it follows from Theorem 2.13(i) that $\mathrm{d}_{M_1}(\mathfrak{q}_k) = \mathrm{d}_{X_1}(\mathfrak{Q}_k) = d - k$.

Step 4. *The equality* $\operatorname{ht}(\mathfrak{m}_k) = k$ *and further properties concerning* $\mathfrak{m}_k^*$. Let $\mathfrak{m}_k^*$ be the unique prime ideal of $\mathcal{S}^*(M_1)$ such that $\mathfrak{m}_k \cap \mathcal{S}^*(M_1) \subset \mathfrak{m}_k^*$. Let us check first that $h|_{M_1} = g \circ \eta|_{M_1} \notin \mathfrak{m}_k^*$. We keep the notation $\phi : \mathcal{S}(X_1) \to \mathcal{S}(C_1)$, $f \to f|_{C_1}$ introduced in Step 2 and consider the monomorphism $\psi : \mathcal{S}(X_1) \hookrightarrow \mathcal{S}^*(M_1)$, $f \mapsto f|_{M_1}$ induced by the inclusion map $\mathtt{j} : M_1 \hookrightarrow X_1$. From Paragraph 2.5.3, it follows that $\psi^{-1}(\mathfrak{m}_k^*) = \mathfrak{m}_k^* \cap \mathcal{S}(X_1)$ is a maximal ideal of $\mathcal{S}(X_1)$ containing $\mathfrak{m}_k \cap \mathcal{S}(X_1) = \mathfrak{q}_k \cap \mathcal{S}(X_1) = \mathfrak{Q}_k$. Thus, $\psi^{-1}(\mathfrak{m}_k^*) = \mathfrak{Q}_d$ and $\mathfrak{n} = \phi(\psi^{-1}(\mathfrak{m}_k^*))$ is a maximal ideal of $\mathcal{S}(C_1)$. Since $h|_{C_1} \geq c > 0$ (see Step 1), it follows from Paragraph 2.4.1 that $\phi(h)$ is a unit in $\mathcal{S}(C_1)$ and so $\phi(h) \notin \mathfrak{n}$; therefore, $h|_{M_1} \notin \mathfrak{m}_k^*$.

Now we are ready to check $\operatorname{ht}(\mathfrak{m}_k) = k$. By Paragraph 2.8.2(iii), $\operatorname{ht}(\mathfrak{m}_k)$ is bounded above by the maximum of the set

$$\begin{aligned}\mathcal{F} &:= \{\mathrm{d}_{M_1}(\mathfrak{p}) - \mathrm{d}_{M_1}(\mathfrak{m}_k) :\ \mathfrak{p} \subset \mathfrak{m}_k \text{ is a minimal prime ideal of } \mathcal{S}(M_1)\} \\ &= \{\mathrm{d}_{M_1}(\mathfrak{p}) + k - d :\ \mathfrak{p} \subset \mathfrak{m}_k \text{ is a minimal prime ideal of } \mathcal{S}(M_1)\}.\end{aligned}$$

Fix a minimal prime ideal $\mathfrak{p} \subset \mathfrak{m}_k$. Since $(h|_{M_1}) \cdot (h_1|_{M_1}) = 0$ and $h|_{M_1} \notin \mathfrak{m}_k$, we deduce $h_1|_{M_1} \in \mathfrak{p}$. Thus, $\mathrm{d}_{M_1}(\mathfrak{p}) \leq \dim(Z_{M_1}(h_1)) = d$ and therefore $\mathrm{ht}(\mathfrak{m}_k) \leq \max \mathcal{F} \leq k$.

To finish the proof of Theorem 7.1 in the locally compact case, we need to check the equality $\mathrm{ht}(\mathfrak{m}_k^*) = d$.

Step 5. *Height of* $\mathfrak{m}_k^*$. Since $\mathfrak{Q}_0$ and $\mathfrak{Q}_0\mathcal{S}(M_1) \cap \mathcal{S}^*(M_1)$ are minimal prime ideals of $\mathcal{S}(X_1)$ and $\mathcal{S}(M_1)$, respectively, it follows from [12, 5.10] that $\mathrm{ht}(\mathfrak{m}_k^*) \geq \mathrm{ht}(\mathfrak{Q}_d) = d$. To prove the converse inequality, let $X \subset \mathbb{R}^m$ be a semialgebraic compactification of M_1. Observe that $\mathfrak{m}_k^* \cap \mathcal{S}(X)$ is a maximal ideal of $\mathcal{S}(X)$ by Paragraph 2.5.3 and let $x \in X$ be such that $\mathfrak{m}_x = \mathfrak{m}_k^* \cap \mathcal{S}(X)$. Consider a chain of prime ideals in $\mathcal{S}(X)$ passing through the prime ideal $\mathfrak{m}_k \cap \mathcal{S}(X)$ (see Corollary 5.9). By Theorem 2.13(ii), $\mathfrak{m}_k = (\mathfrak{m}_k \cap \mathcal{S}(X))\mathcal{S}(M_1)$ and $\mathrm{d}_X(\mathfrak{m}_k \cap \mathcal{S}(X)) = \mathrm{d}_{M_1}(\mathfrak{m}_k) = d - k$.

Using Paragraph 2.8.2, the length of those chains of prime ideals of $\mathcal{S}(X)$ having $\mathfrak{m}_k \cap \mathcal{S}(X)$ as its first member is $\leq d - k$. On the other hand, by Lemma 2.15(iv) each chain of prime ideals of $\mathcal{S}(X)$ whose members are contained in $\mathfrak{m}_k \cap \mathcal{S}(X)$ can be extended to a chain of prime ideals of $\mathcal{S}(M_1)$ of the same length, which cannot be greater than $k = \mathrm{ht}(\mathfrak{m}_k)$. Thus, $\mathrm{h}_X(\mathfrak{m}_k^*) \leq d$ and consequently $\mathrm{ht}(\mathfrak{m}_k^*) \leq d$ (see Corollary 5.9). Hence, $\mathrm{ht}(\mathfrak{m}_k^*) = d$, as wanted.

7.2. *Proof of Theorem* 7.1 *for an arbitrary M*

Let U be a closed semialgebraic neighbourhood of p in $\mathbb{R}^m$ such that the dimension of the closed semialgebraic subset $N := U \cap M$ of M equals $d := \dim_p M$, and $U \cap \mathrm{Cl}_{\mathbb{R}^m}(\rho_1(M)) = \emptyset$. Consider the locally compact semialgebraic set $M_{\mathrm{lc}} := M \setminus \rho_1(M)$, see Theorem 3.6. Then $N = U \cap M_{\mathrm{lc}}$ is locally compact because it is a closed subset of a locally compact space. Note $p \in \mathrm{Cl}_{\mathbb{R}^m}(N) \setminus N$. By Lemma 2.1, there exists $h \in \mathcal{S}^*(\mathbb{R}^m)$ such that $Z_{\mathbb{R}^m}(h) = \mathbb{R}^m \setminus \mathrm{Int}_{\mathbb{R}^m}(U)$. Clearly, $h(p) \neq 0$.

As the result is already proved for the locally compact case, for each $0 \leq k \leq d - 1$, there exists a maximal ideal $\mathfrak{n}_k$ of $\mathcal{S}(N)$ such that $\mathrm{ht}(\mathfrak{n}_k) = k$, $(gh)|_N \notin \mathfrak{n}_k^*$ and $\mathrm{ht}(\mathfrak{n}_k^*) = d$.

On the other hand, since N is a closed semialgebraic subset of M, the homomorphism $\eta : \mathcal{S}(M) \to \mathcal{S}(N),\ f \mapsto f|_N$ is by Theorem 2.3 surjective. Thus, $\mathfrak{m}_k = \eta^{-1}(\mathfrak{n}_k)$ is a maximal ideal of $\mathcal{S}(M)$ of height $\geq k$. Let us see that the later inequality is in fact an equality and $\mathrm{ht}(\mathfrak{m}_k^*) = d$. Observe $(gh)|_M \notin \mathfrak{m}_k^*$, so $h|_M \notin \mathfrak{m}_k^*$ and $g|_M \notin \mathfrak{m}_k^*$.

As N is closed in M, the closure $\mathrm{Cl}_{\mathrm{Spec}_s^\diamond(M)}(N)$ is homeomorphic to $\mathrm{Spec}_s^\diamond(N)$ by Lemma 2.10(iii). Thus, everything is left to show that $\mathrm{Cl}_{\mathrm{Spec}_s^\diamond(M)}(N)$ contains all prime ideals contained in $\mathfrak{m}^\diamond{}_k$. Indeed, let $\mathfrak{p} \subset \mathfrak{m}^\diamond{}_k$ and $f \in \mathcal{S}^\diamond(M) \setminus \mathfrak{p}$, that is, $\mathfrak{p} \in \mathcal{D}_{\mathrm{Spec}_s^\diamond(M)}(f)$, and suppose $N \cap \mathcal{D}_{\mathrm{Spec}_s^\diamond(M)}(f) = \emptyset$. Then $f|_N \equiv 0$, which implies $(h|_M)f = 0 \in \mathfrak{p}$, and so $h|_M \in \mathfrak{p}$. This is false because $h|_M \notin \mathfrak{m}^\diamond{}_k$ and we are done.

7.3. *Ulterior consequences*

To finish, we show that the height operator behaves quite different in the case of rings of bounded semialgebraic functions. Recall that by [10, 4.1] $\dim \mathcal{S}^*(M) = \dim(M)$.

Corollary 7.2 *Let $M \subset \mathbb{R}^m$ be a semialgebraic set and $\mathfrak{m}^*$ a maximal ideal of $\mathcal{S}^*(M)$. Then* $\mathrm{ht}(\mathfrak{m}^*) = 0$ *if and only if there exists an isolated point $p \in M$ such that $\mathfrak{m}^* = \mathfrak{m}_p^*$.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{S}(M)$ such that $\mathfrak{m} \cap \mathcal{S}^*(M) \subset \mathfrak{m}^*$. If $\mathfrak{m}^*$ is a free maximal ideal, then by Paragraph 2.5.4 $\mathfrak{m} \cap \mathcal{S}^*(M) \subsetneq \mathfrak{m}^*$ and so $\mathrm{ht}(\mathfrak{m}^*) \geq 1$. Hence, $\mathfrak{m}^*$ is a fixed ideal and let $p \in M$ such that $\mathfrak{m}^* = \mathfrak{m}_p^*$. If p is not isolated in M, there exists by the Curve Selection Lemma [2, 2.5.5] a semialgebraic path $\gamma : [0,1] \to \mathbb{R}^m$ such that $\gamma(0) = p$ and $\gamma([0,1]) \subset M \setminus \{p\}$. Then the set

$$\mathfrak{p} := \{f \in \mathcal{S}^*(M) : (f \circ \gamma)|_{(0,\varepsilon)} \equiv 0 \text{ for some } 0 < \varepsilon < 1\}$$

is a prime ideal of $\mathcal{S}^*(M)$, which is strictly contained in $\mathfrak{m}_p^*$. Thus, $\operatorname{ht}(\mathfrak{m}_p^*) \geq 1$.

Conversely, it is straightforward to check that the fixed maximal ideal corresponding to an isolated point of M has height 0. □

Corollary 7.3 *Let $M \subset \mathbb{R}^m$ be a semialgebraic curve without isolated points. Then* $\operatorname{ht}(\mathfrak{m}^*) = 1$ *for every maximal ideal* $\mathfrak{m}^*$ *of* $\mathcal{S}^*(M)$.

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