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Some results on global real analytic geometry

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Dedicated to Murray Marshall, in memoriam.

Abstract. In the first part of this survey we recall how the concept of (real) C-analytic space emerged, when trying to generalize the classical concept of complex analytic space with a clear red line: to keep the validity of Theorems A and B, which are crucial properties of a very important type of complex analytic spaces: the Stein spaces. Recall that closed analytic subspaces of Stein open subsets of $\mathbb{C}^n$ play the same role as affine algebraic varieties in the algebraic setting. The second part is devoted to the concept of *C-semianalytic subset* of a real analytic manifold. C-semianalytic sets can be understood as the natural generalization to the semianalytic setting of C-analytic sets. The family of C-semianalytic sets is closed under the same operations as the family of semianalytic sets: locally finite unions and intersections, complement, closure, interior, connected components, inverse images under analytic maps, sets of points of dimension k, etc. although they are defined involving only global analytic functions. In addition, *the image of a C-semianalytic set S under a proper holomorphic map between Stein spaces is again a C-semianalytic set*. The previous result allows to understand better the structure of the set $N(X)$ of points of non-coherence of a C-analytic subset X of a real analytic manifold M. It is also remarkable that subanalytic sets are the images under proper analytic maps of C-semianalytic sets. In the third part we introduce *amenable C-semianalytic sets*, that can be understood as C-semianalytic sets with a neat behavior with respect to Zariski closure. This fact allows us to develop a natural definition of *irreducibility* and the corresponding *theory of irreducible components* for this type of sets. These concepts generalize the parallel ones for: complex algebraic and analytic sets, C-analytic sets, Nash sets and semialgebraic sets. We end this survey with a general view towards Nullstellensätze in the complex and real global analytic settings. This requires not only algebraic operations but also topological. In the real case we take advantage of Łojasiewicz radical ideal, whose definition is inspired in the classical Łojasiewicz's inequality.

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1. Emergence of real analytic spaces

In the first part of this survey we provide a brief historical summary on how real analytic spaces arose. We begin recalling first some relevant facts concerning complex analytic spaces.

1.A. Complex analytic spaces. During the 50s' of last century the theory of (complex) analytic spaces was developed 'symbiotically' with complex algebraic geometry. The local approach became clear after the classical Weierstrass' Theorems (Preparation and Division) and Rückert's Nullstellensatz for the ring of holomorphic function germs. The two main research teams concerning this subject developed their activity in France (we highlight the names of Oka, Cartan, Serre, Grothendieck, inside the Séminaire Cartan) and in Germany (here we highlight the names of Rückert, Bencke, Stein, Remmert, Grauert). The definition of analytic space has local nature and one needs to define first *local models* (of analytic spaces). We write $\mathcal{O}_{\mathbb{C}^n}$ to denote the sheaf of germs of holomorphic function on $\mathbb{C}^n$.

Definition 1.1. A *local model* consists of

(i) an open set $\Omega \subset \mathbb{C}^n$,
(ii) the zero-set $Y \subset \Omega$ of finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^n}(\Omega)$,
(iii) the ringed space $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y$ is the quotient sheaf of $\mathcal{O}_{\mathbb{C}^n}|_\Omega$ by the sheaf of ideals $\mathcal{I}_Y$ of those germs of holomorphic functions vanishing identically on Y.

Now, we are ready to introduce the concept of (complex) analytic space. Let X be a Hausdorff paracompact topological space endowed with a sheaf of rings $\mathcal{O}_X$.

Definition 1.2. The pair $(X, \mathcal{O}_X)$ is an *analytic space* if for each point $x \in X$ there exist an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic as a ringed space to a local model $(Y, \mathcal{O}_Y)$.

Oka-Cartan's Theorem states that the sheaf of ideals $\mathcal{I}_Y$ of a local model $(Y, \mathcal{O}_Y)$ is *coherent* [**GR**, §.IV.B-D]. This is a key result that makes the theory of complex analytic spaces rather similar to the one of complex algebraic varieties. Among complex analytic spaces we stress *Stein spaces*, which are important because they have nice properties. Roughly speaking, a Stein space is a space with 'enough' holomorphic functions. A precise definition is the following.

Definition 1.3. An analytic space $(X, \mathcal{O}_X)$ is *a Stein space* if it satisfies the following properties:

(i) the ring $\mathcal{O}(X) := H^0(X, \mathcal{O}_X)$ of holomorphic functions on X separates points and provides local coordinates (that is, isomorphisms with local models),
(ii) it is holomorphically convex (that is, the holomorphic convex hull of a compact set in X is compact).

Among the properties of Stein spaces probably the most important one is that they satisfy Cartan's Theorems A and B. The first result states that the fiber $\mathcal{O}_{X,x}$ of the sheaf $\mathcal{O}_X$ at each point $x \in X$ is generated by global sections. The second asserts that given a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules the cohomology groups $H^q(X, \mathcal{F})$ vanish for each $q \geq 1$. In particular, if a short sequence of coherent

$\mathcal{O}_X$-modules $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact, the corresponding sequence of rings of global sections $0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to 0$ is also exact.

Recall that a subset Y of a Stein space $(X, \mathcal{O}_X)$ is a *closed analytic subset* if for each $x \in X$ there exist analytic function germs $f_{1,x}, \ldots, f_{r_x,x} \in \mathcal{O}_{X,x}$ such that the germ Y_x is the common zero-set germ of $f_{1,x}, \ldots, f_{r_x,x}$. As a matter of fact, a closed analytic subset Y of a Stein space provides the Stein space $(Y, \mathcal{O}_Y := \mathcal{O}_X/\mathcal{I}_Y)$ where $\mathcal{I}_Y$ is the ideal sheaf of germs of $\mathcal{O}_X$ vanishing identically on Y. As $(\mathbb{C}, \mathcal{O}_{\mathbb{C}^n})$ is a Stein space, each closed analytic set $X \subset \mathbb{C}^n$ provides itself a Stein space and, in addition, there exist finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(\mathbb{C}^n)$ such that $X = \{z \in \mathbb{C}^n : \; f_1(z) = 0, \ldots, f_k(z) = 0\}$. Recall also that a Stein space is compact if and only if it is a finite set. Consequently, compact analytic subsets of $\mathbb{C}^n$ are only finite sets.

Closed analytic subspaces of Stein open subsets of $\mathbb{C}^n$ play the same role as affine algebraic varieties in the algebraic setting. If the open set Ω of Definition 1.1 is a *polydisc*, then it is a Stein space. Consequently, Stein spaces provide local models for complex analytic spaces. This behavior reproduces what happens in complex Algebraic Geometry where algebraic varieties are unions of affine charts.

If we try to mimic in the real case the definition provided above for the complex case, we are led to a very different situation. For instance,

(1) The ideal sheaf $\mathcal{I}_Y$ of a local model needs not to be coherent.
(2) There are real prime ideals $\mathfrak{p} \subset \mathcal{O}_{\mathbb{R}^n,0}$ whose zero-set is not pure dimensional. In fact, this pathology already appears in the real algebraic setting.
(3) It is not possible to develop a reasonable theory of irreducible components as it is done in the complex analytic setting (Cartan [**C2**], Forster [**Fo**], Remmert-Stein [**RS**]).

Well-known examples of fact (2) are Whitney's and Cartan's umbrellas W_1 and W_2, given respectively by equations $f_1 := x^2 - zy^2 = 0$ and $f_2 := x^3 - z(x^2+y^2) = 0$. Observe that the polynomial f_i generates the global ideal of W_i, however f_i does not generate the ideal sheaf of W_i at the points of its *stick* or *tail* (the part of W_i where local dimension equals 1, see §2.C). One can find in [**BC1**, **BC2**, **WB**] many other smart examples where the notion of irreducible component cannot have the usual meaning and several other pathologies appear. These facts led to two opposite positions. Grothendieck [**Gr**, p.12, l.16-24] considered the real case not interesting:

> *"Lorsque k est algébriquement clos, il est probablement vrai que tout espace analytique réduit à un point est de la forme qu'on vient d'indiquer, ce qui serait une des variantes du "Nullstellensatz" analytique. Signalons par contre tout de suite que rien de tel n'est vrai si k n'est pas algébriquement clos, par exemple si k est le corps des réels* $\mathbb{R}$*. Ainsi, le sous-espace analytique de* $\mathbb{R}^2$ *défini par l'idéal engendré par* $x^2 + y^2$ *est réduit au point origine, mais son anneau local en ce point n'est pas artinien, mais de dimension de Krull égale à* 1*. L'intérêt des espaces analytiques, lorsque k n'est pas algébriquement clos, est d'ailleurs douteux."*

Cartan after a careful analysis of examples quoted above [**BC1**, **BC2**] stressed a smaller class of real analytic sets with a better behavior. They were called by Whitney-Bruhat C-analytic sets (as an abbreviation of Cartan real analytic sets).

1.B. C-analytic spaces. The purpose of Cartan was to keep valid in the real case Theorems A and B. He proved that both results are preserved by direct limits. The first result that appears in [**C3**] is that $\mathbb{R}^n$ has a fundamental system of open

Stein neighborhoods in $\mathbb{C}^n$. The same property holds true for closed real analytic subsets X of $\mathbb{R}^n$ defined as the zero-set of finitely many real analytic functions on $\mathbb{R}^n$. These sets are exactly the real analytic sets considered by Cartan. It holds that the same finitely many analytic equations that define the real analytic set X extend to holomorphic functions on a suitable open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and such extensions define a complex analytic set, which is a Stein space because it is a closed analytic subset of a Stein open set. Cartan provided in [**C3**] several equivalent conditions for a closed real analytic subset $X \subset \mathbb{R}^n$ to guarantee that it keeps Theorems A and B. Namely,

(i) To be the zero-set of finitely many analytic functions on $\mathbb{R}^n$.
(ii) To be the real part $Z \cap \mathbb{R}^n$ of a complex analytic subset Z of an open neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$.
(iii) To be the support of a coherent sheaf of $\mathcal{O}_{\mathbb{R}^n}$-modules.

Here $\mathcal{O}_{\mathbb{R}^n}$ denotes as usual the sheaf of germs of analytic function on $\mathbb{R}^n$. Condition (iii) is trivially true for *coherent* analytic sets, which are those real analytic sets X for which the sheaf of ideals $\mathcal{I}_X$ of $\mathcal{O}_{\mathbb{R}^n}$ is coherent. Recall that $\mathcal{I}_{X,x}$ is constituted, for each $x \in \mathbb{R}^n$, by those analytic function germs vanishing identically on X_x.

In order to prove that the class of C-analytic sets is smaller that those of real analytic sets (defined locally as zero-sets of finitely many real analytic functions), Cartan showed that there exist closed real analytic subsets $X \subset \mathbb{R}^n$ such that the only analytic function vanishing identically on X is the zero function.

Example 1.4 (Cartan). Define $X := \{(x, y, z) \in \mathbb{R}^3 :\ a(z)x^3 - z(x^2 + y^2) = 0\}$ where $a(z) := \exp(\frac{1}{z^2-1})$ for $-1 < z < 1$ and $a(z) := 0$ otherwise. The function $a(z)$ seen as a function in one complex variable has essential singularities at the points $z = 1$ and $z = -1$. Consequently, each real analytic function $f \in \mathcal{O}(\mathbb{R}^3)$ vanishing on X is identically zero on $\mathbb{R}^3$, see [**C3**, §11].

Cartan wrote in [**C4**, pag. 49] the following:

“. . . la seule notion de sous-ensemble analytique réel (d'une variété analytique-réelle V) qui ne conduise pas à des propriétés pathologiques doit se référer à l'espace complexe ambiant: il faut considérer les sous-ensembles fermés E de V tels qu'il existe une complexification W de V et un sous-ensemble analytique-complexe E' de W, de manière que $E = W \cap E'$. On démontre que ce sont aussi les sous-ensembles de V qui peuvent être définis globalement par un nombre fini d'équations analytiques. La notion de sous-ensemble analytique-réel a ainsi un caractère essentiellement global, contrairement à ce qui avait lieu pour les sous-ensembles analytiques-complexes.”

Following similar ideas to the ones exposed above Whitney-Bruhat generalized in [**WB**] Cartan's results for a real analytic manifold M. First of all they construct a *complexification* of M, that is, a complex analytic manifold N endowed with an antiholomorphic involution σ on N such that M is the fixed subset N^σ of N under σ. Then they proved that M has a fundamental system of open Stein neighborhoods inside N. Thus, closed real analytic subsets of M defined as the zero-set of finitely many analytic functions keep Theorems A and B. As commented above, they called these sets *C-analytic sets*. They also showed in [**WB**] that C-analytic subsets of M admit a unique irredundant decomposition into *irreducible components*, exactly as it happens with complex analytic sets. The irreducible components of a C-analytic subset $X \subset M$ are precisely the real parts of the irreducible components of the complex analytic set Y of a suitable Stein neighborhood $\Omega \subset N$ of M, which is defined by the same equations of X extended holomorphically to Ω. With these results, the analogy between complex analytic sets and C-analytic sets is complete. Following Whitney-Bruhat we define a *C-analytic subset* X of a real analytic manifold M as follows.

Definition 1.5. A set $X \subset M$ is *C-analytic* if there exist $f_1, \dots, f_k \in \mathcal{O}(M)$ such that $X = \{x \in M : f_1(x) = 0, \dots, f_k(x) = 0\}$.

Later Tognoli [**T**] extended the concept of complexification to real analytic spaces, which were not necessarily embedded inside a real analytic manifold. He studied three properties of real analytic sets:

(1) To have a complexification.
(2) To be locally the real part of a complex analytic set, that is, its local models are provided by C-analytic sets.
(3) To be the fixed set of an antiholomorphic involution on a complex analytic space.

We point out here that a real analytic space has a complexification if and only if it is coherent. Indeed, the complexification Y_x of a real analytic set germ X_x is unique as complex analytic set germ. Cartan proved in [**C3**, Prop.12] that X is coherent at the point x if and only if for each y close to x the complex germ Y_y (induced by a representative Y of Y_x) provides the complexification of the germ X_y. By definition Y is a complexification of X if for each $x \in X$ the set germ Y_x is the complexification of X_x. Consequently, the set X has to be coherent.

If a real local model is coherent, then it has a complexification, which is essentially unique. Consequently, to construct a complexification of a coherent real analytic space, it is enough to paste properly these complexifications of local models, see [**T**].

Properties (2) and (3) above are equivalent (see [**T**]) and characterize, what we call inspired by Whitney-Bruhat, *C-analytic spaces*, which however were called in [**T**] *supports of coherent sheaves*. If $(X, \mathcal{O}_X)$ is a real analytic space satisfying (2) and (3) there exists a complex analytic space $(Y, \mathcal{O}_Y)$ that contains X as a closed subspace and satisfies:

- X is the fixed part Y^σ of an antiholomorphic involution $\sigma : Y \to Y$,
- $\mathcal{O}_X$ is the restriction to X of the subsheaf of $\mathcal{O}_Y$ constituted by the invariant sections (with respect to σ),
- X has a fundamental system of open Stein neighborhoods inside Y.

In addition, the germ of $(Y, \mathcal{O}_Y)$ at X is unique up to an isomorphism [**T**]. For simplicity we will call Y the *complexification* of X, even if Y_x does not provide the complexification of the germ X_x for each $x \in X$.

2. Inequalities and the global approach

In Real Analytic Geometry it is natural to consider also inequalities. In this way arose the concept of *semianalytic set* due to Łojasiewicz [**Ł1**,**Ł2**], which generalized to the real analytic setting the concept of semialgebraic set mimicking the local definition of a real analytic set.

Definition 2.1 (Łojasiewicz). Let M be a real analytic manifold. A set $S \subset M$ is a *semianalytic subset of M* if for each $x \in M$ there is an open neighborhood

$U^x \subset M$ such that the intersection $S \cap U^x$ is a finite union of sets of the form $\{f = 0, g_1 > 0, \ldots, g_s > 0\}$ where $f, g_1, \ldots, g_s \in \mathcal{O}(U^x)$.

The class of semianalytic sets behaves well with respect to boolean and topological operations, but it is not stable under proper analytic maps. This fact led Łojasiewicz [**Ł1**] and Hironaka [**Hi1**, **Hi2**] between others to introduce and develop the theory of *subanalytic sets*.

There exists a great difference between the definitions of C-analytic set (of global nature) and semianalytic set (of local nature). There are certain remarkable subsets of a C-analytic subset X of a real analytic manifold M that are known to be semianalytic. For instance, the set of points of X where the local dimension of X is equal to a certain non-negative integer, or the set of points of X where X is non-coherent. The semianalytic nature of these sets makes no reference to the global nature of the C-analytic set X. Thus, it seems reasonable to ask whether there exists a notion of global semianalytic set that restricts the class of semianalytic sets and mimics the definition of C-analytic sets proposed by Cartan. More precisely, we wonder whether there exists a class of semianalytic sets defined using only (global) analytic functions defined on M, but having a similar behavior with respect to boolean and topological operations to that of Łojasiewicz semianalytic sets. In addition, we would like that such class satisfies also some reasonable properties with respect to images under proper analytic maps.

2.A. Global semianalytic sets. A first tentative in this direction was explored by Andradas-Broecker-Ruiz [**ABR1**, **Rz2**, **Rz3**, **Rz4**] under compactness assumptions and by Andradas-Castilla [**AC**] in the general approach but for low dimension. They defined a *global semianalytic subset* of a real analytic manifold M as a definable subset with respect to the ring $\mathcal{O}(M)$, that is, a finite boolean combination of equalities and inequalities involving global analytic functions on M. They showed that this notion behaves in the desired way when the dimension of M is 1 or 2 or when M is a compact manifold. There exists further information concerning closure (and interior) of a global semianalytic set if $\dim(M) = 3$ but nothing conclusive for higher dimension if the involved global semianalytic set has non-compact boundary. A main difficulty appears when determining for general dimension whether the closure and the connected components of an arbitrary global semianalytic set are still or not global semianalytic sets. Another problem concerns the lack of information when referring to images of global semianalytic sets under proper analytic maps. Nevertheless global semianalytic sets have the so called *finiteness property* like semialgebraic sets. Namely,

PROPOSITION 2.2 (Finiteness property). *Let $S \subset M$ be a global semianalytic set in a real analytic manifold M.*

(i) *Suppose S is open in M. Then it is a finite union of open basic global semianalytic sets, that is, M is a finite union of global semianalytic sets of the type $\{f_1 > 0, \ldots, f_r > 0\}$ where each $f_i \in \mathcal{O}(M)$.*
(ii) *Suppose S is closed in M. Then it is a finite union of closed basic global semianalytic sets, that is, M is a finite union of global semianalytic sets of the type $\{f_1 \geq 0, \ldots, f_r \geq 0\}$ where each $f_i \in \mathcal{O}(M)$.*

The proof of the previous result, that appears in [**ABS**], is based on a weak Łojasiewicz inequality. The classical Łojasiewicz's inequality for continuous semialgebraic functions [**BCR**, 2.6.7] states the following.

THEOREM 2.3 (Classical Łojasiewicz's inequality). *Let K be a compact semialgebraic set and let f, g be continuous semialgebraic functions on K such that $\{f = 0\} \subset \{g = 0\}$. Then there exist an integer $m \geq 1$ and a constant $c \in \mathbb{R}$ such that $|g|^m \leq c|f|$ on K.*

The result proved in [**ABS**] is slightly different. Let Z be a C-analytic subset of $\mathbb{R}^n$ and let $f, g \in \mathcal{O}(Z) := \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(Z)$ be such that $\{f = 0\} \subset \{g = 0\}$. Recall that $\mathcal{I}(Z)$ is the ideal of those analytic functions on $\mathbb{R}^n$ that vanish identically on Z. Fixed a compact set $K \subset Z$ there exist a proper C-analytic subset $X_1 \subset \{f = 0\} \setminus K$ and an open neighborhood U of $\{f = 0\} \setminus X_1$ on which $|g|^m \leq |f|$ for some integer $m \geq 1$. The precise statement of the weak Łojasiewicz's inequality is the following. We use $\overline{\cdot}$ to denote the Euclidean closure of a subset of $\mathbb{R}^n$.

THEOREM 2.4 (Weak Łojasiewicz's inequality). *Let Z be a C-analytic subset of $\mathbb{R}^n$ and let $A \subset Z$ be a global semianalytic subset. Let $f, g \in \mathcal{O}(Z)$ be such that $\{f = 0\} \cap \overline{A} \subset \{g = 0\} \cap \overline{A}$. Fix a compact set $K \subset Z$ and denote $X := \{f = 0\}$. Then there exist a proper C-analytic subset $X_1 \subset X$ such that $X_1 \cap K = \emptyset$, an open neighborhood $U \subset \mathbb{R}^n$ of $X \setminus X_1$ and a positive integer $m \geq 1$ such that $|g|^m < |f|$ on $U \cap \overline{A} \setminus X$.*

Given analytic functions $f, g \in \mathcal{O}(Z)$ such that $\{f = 0\} \subset \{g = 0\}$, the previous result provides a recursive procedure to construct an analytic function h on Z whose zero-set does not meet a compact set K and satisfies $|gh|^m \leq |f|$ for some integer $m \geq 1$, see [**ABF1**, Prop.4.3]. More precisely,

PROPOSITION 2.5. *Let Z be a C-analytic set in $\mathbb{R}^n$ and $f, g \in \mathcal{O}(Z)$ such that $\{f = 0\} \subset \{g = 0\}$. Let $K \subset Z$ be a compact set. Then there exist an integer $m \geq 1$ and an analytic function $h \in \mathcal{O}(Z)$ such that $|h| < 1$, $\{h = 0\} \cap K = \varnothing$ and $|f| \geq |hg|^m$.*

Proposition 2.5 is a key result to prove Nullstellensatz for ideals in the ring $\mathcal{O}(Z)$ of analytic functions on Z, see [**ABF1**].

2.B. An alternative class of globally defined semianalytic sets. In [**ABF2**] we propose the following class of globally defined semianalytic subsets of a real analytic manifold M.

DEFINITION 2.6. A subset $S \subset M$ is *C-semianalytic* if S is a locally finite union of global basic semianalytic sets, that is, sets of the form $\{f = 0, g_1 > 0, \ldots, g_s > 0\}$ where $f, g_j \in \mathcal{O}(M)$.

The previous definition is equivalent to the following one, which is more similar to the one provided by Łojasiewicz for classical semianalytic sets.

DEFINITION 2.7. A subset $S \subset M$ is C-semianalytic if and only if for each $x \in M$ there is an open neighborhood $U^x \subset M$ such that $S \cap U^x$ is a global semianalytic set in M (in the sense of §2.A).

The class of C-semianalytic sets in M is closed under the following boolean and topological operations [**ABF2**]

- locally finite unions, intersections and complement,
- inverse image under analytic maps between real analytic manifolds,
- taking closure, interior and considering connected components.

The C-semianalytic sets have a more relevant and deep property that extends the well-known *Direct Image* Remmert's Theorem [**N1**, VII.§2.Thm.2]. The previous result states that the family of complex analytic sets is stable under proper holomorphic maps between complex analytic spaces. The C-semianalytic sets satisfy an analogous result. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be reduced Stein spaces. Let $\sigma : X \to X$ and $\tau : Y \to Y$ be antiholomorphic involutions. Assume X^σ and Y^τ are non-empty sets. We denote the set of σ-invariant holomorphic functions of X restricted to X^σ with $\mathcal{A}(X^\sigma)$. We say that a C-semianalytic set $S \subset X^\sigma$ is $\mathcal{A}(X^\sigma)$-*definable* if for each $x \in X^\sigma$ there exists an open neighborhood U^x such that $S \cap U^x$ is a finite union of sets of the type $\{F = 0, G_1 > 0, \ldots, G_r > 0\}$ where $F, G_i \in \mathcal{A}(X^\sigma)$. In [**ABF2**] we prove:

THEOREM 2.8 (Direct image under proper holomorphic maps). *Let*

$$F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be an invariant proper holomorphic map, that is, $\tau \circ F = F \circ \sigma$. Let $S \subset X^\sigma$ be an $\mathcal{A}(X^\sigma)$-definable C-semianalytic set. We have

(i) *$F(S)$ is a C-semianalytic subset of Y^τ of the same dimension as S.*
(ii) *If $E := \overline{F^{-1}(Y^\tau) \setminus X^\sigma}$, then $F(E \cap S)$ is a C-semianalytic subset of Y^τ.*
(iii) *If S is a C-analytic set and $F^{-1}(Y^\tau) = X^\sigma$, then $F(S)$ is also a C-analytic subset of Y^τ.*

Theorem 2.8 generalizes the result of Galbiati collected in [**Ga2**], where she proved that if $f : X \to Y$ is a proper analytic map between real analytic spaces that admits a proper complexification $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ and Z is a C-analytic subset of X, then $f(X \setminus Z)$ is a semianalytic set. In [**Hi3**] Hironaka quoted this result and remarked that $f(X \setminus Z)$ is *'globally semianalytic in Y with respect to the given complexification $\widetilde{Y}$ of Y'* in the same line as Theorem 2.8.

The following result, which is the key to prove Theorem 2.8, analyzes the local structure of proper surjective holomorphic morphisms between Stein spaces and its proof is developed in [**ABF2**]. For each $x \in X$ we denote the maximal ideal of $\mathcal{O}(X)$ associated to x with $\mathfrak{m}_x$ and for each $y \in Y$ we denote the maximal ideal of $\mathcal{O}(Y)$ associated to y with $\mathfrak{n}_y$. Compact analytic subsets of a Stein space are finite sets, so the fibers of a proper holomorphic map between Stein spaces are finite sets. Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a surjective proper holomorphic map between reduced Stein spaces and write $F^*(\mathcal{O}(Y)) := \{G \circ F : \ G \in \mathcal{O}(Y)\} \subset \mathcal{O}(X)$ and

$$F^*(\mathcal{O}(Y)_{\mathfrak{n}_y}) = \Big\{\frac{G \circ F}{H \circ F} : \ G, H \in \mathcal{O}(Y) \text{ and } H \notin \mathfrak{n}_y\Big\}.$$

THEOREM 2.9 (Local structure of finite holomorphic morphisms). *Let $y_0 \in Y$ with fiber $F^{-1}(y_0) = \{x_1, \ldots, x_\ell\}$ and denote $\Sigma := \mathcal{O}(X) \setminus (\mathfrak{m}_{x_1} \cup \cdots \cup \mathfrak{m}_{x_\ell})$. Then $\Sigma^{-1}(\mathcal{O}(X))$ is a finitely generated $\mathcal{O}(Y)_{\mathfrak{n}_{y_0}}$-module and there exist invariant $H_1, \ldots, H_m \in \mathcal{O}(X)$ such that $\Sigma^{-1}(\mathcal{O}(X)) = F^*(\mathcal{O}(Y)_{\mathfrak{n}_{y_0}})[H_1, \ldots, H_m]$.*

Let P be a property concerning either C-semianalytic or C-analytic sets. We say that P is a C-*property* if the set of points of an either C-semianalytic or C-analytic set S satisfying P is a C-semianalytic set. Some examples are the following:

(i) The set of points where the dimension of the C-semianalytic set S is k is a C-semianalytic set, that is, 'to be a point of local dimension k' is a C-property.

(ii) The set of points of non-coherence of a C-analytic set is C-semianalytic, that is, 'to be a point of non-coherence' (or 'to be a point of coherence') are C-properties. We will provide below more details concerning the points of non-coherence of a C-analytic set.

We end this part explaining why we do not introduce a concept of *C-subanalytic sets*. The family of semianalytic sets is not closed under the image of proper analytic maps. The concept of subanalytic set was introduced to get rid of this problem. Let us recall the concept of *subanalytic set* following the definition proposed in[**BM**].

DEFINITION 2.10. A subset $S \subset M$ is *subanalytic* if each point $x \in M$ admits a neighborhood U^x such that $S \cap U^x$ is a projection of a relatively compact semianalytic set, that is, there exists a real analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $S \cap U^x = \pi(A)$ where $\pi : M \times N \to M$ is the projection.

It could sound reasonable to consider the family of C-subanalytic sets. However, this is useless because, as we proved in [**ABF2**], each subanalytic set is the image of a C-semianalytic set under a proper analytic map. Thus, one can replace semianalytic sets by C-semianalytic sets when one defines subanalytic sets.

THEOREM 2.11. *Let S be a subset of a real analytic manifold N. The following assertions are equivalent:*

(i) *S is subanalytic.*
(ii) *There exists a basic C-semianalytic subset T of a real analytic manifold M and an analytic map $f : M \to N$ such that $f|_{\overline{T}} : \overline{T} \to N$ is proper and $S = f(T)$.*
(iii) *There exists a C-semianalytic subset T of a real analytic manifold M and an analytic map $f : M \to N$ such that $f|_{\overline{T}} : \overline{T} \to N$ is proper and $S = f(T)$.*

2.C. The set of points where a C-analytic set is non-coherent. The set of points $N(X)$ where an analytic set $X \subset M$ is non-coherent was studied first by Fensch in [**F**, I.§2] where he proved that *it is contained in a semianalytic set of dimension* $\leq \dim(X) - 2$. This result was revisited by Galbiati in [**Ga1**] and she proved that *it is in fact a semianalytic set*. Thus, analytic curves are coherent and real analytic surfaces have only isolated points where they fail to be coherent. As coherence is an open condition, $N(X)$ is a closed set. Later Tancredi-Tognoli provided in [**TT**] a simpler proof of Galbiati's result. Their procedure has helped us to understand the global structure of the set of points of non-coherence of a C-analytic set and to prove in [**ABF2**] the following.

THEOREM 2.12. *The set $N(X)$ of points of non-coherence of a C-analytic set $X \subset M$ is a C-semianalytic set of dimension $\leq \dim(X) - 2$.*

Let us give some general ideals about how the set $N(X)$ arises. By Cartan's criterium [**C3**, Prop.12] a real analytic set X is non-coherent at the point $a \in X$ when there exists points b arbitrarily close to a such that the complexification of the set germ X_b is not induced by the complexification of the germ X_a. A branch of points $x \in X$ where the complexification of the germ X_x does not coincide with the germ at x of a complexification Y of X will be called a *tail*. Roughly speaking, a branch of real points become a branch of complex points when crossing a non-coherence point (as real roots can disappear when passing through a double root of a polynomial). Many times this translates on a drop of dimension and the points of non-coherence are those points of X where the drops of dimension arise. Classical examples of this situation are Whitney's umbrella $W_1 := \{x^2 - zy^2 = 0\} \subset \mathbb{R}^3$ and Cartan's umbrella $W_2 := \{x^3 - z(x^2 + y^2) = 0\} \subset \mathbb{R}^3$. Both examples are two dimensional C-analytic sets that have 1-dimensional tails and in both cases the point of non-coherence is the origin. However, it is also possible that the 'tail' is hidden inside the 2-dimensional part of X. An example of this situation is $W_3 := \{z(x+y)(x^2+y^2) - x^4 = 0\}$. The points of the 'tail' are those in the line $\ell := \{x = 0, y = 0\}$. In this case the point of non-coherence of W_3 is the origin, but if $b \in \ell$ is close to the origin, then $\dim(W_{3,b}) = \dim(W_{3,0})$. So we have to be careful with these hidden tails!

Tails (of type 1), which are obtained locally as intersections of complex conjugated analytic germs, cannot occur in a normal C-analytic set because such C-analytic sets are locally irreducible and their complexifications are also locally irreducible. There is another way to produce tails (of type 2). Let $X \subset Y$ be a C-analytic set inside its complexification Y. It could happen that there exist points $x \in X$ such that the germ X_x is a subset of the singular locus of Y_x and $\dim_{\mathbb{R}}(X_x) \leq \dim_{\mathbb{C}}(Y_x) - 2$. This situation is reproduced in the following example.

Example 2.13. Consider the pencil of conics given by $x^2 + y^2 = t$ where t is a real parameter. Then $X := \{(x, y, z, t) \in \mathbb{R}^4 : \; x^2 + y^2 - tz^2 = 0\}$ can be understood as the pencil of (double) cones of vertex the origin and basis the conics above. Consider the complex analytic set $Y \subset \mathbb{C}^4$ given by the same equation as X. It holds that X is a C-analytic set in $\mathbb{R}^4$ and it is the fixed part of Y. Write $p := (0, 0, 0, d)$. If $d \geq 0$ the germ X_p has dimension 3, while for $d < 0$ the germ X_p is the germ at the point $(0, 0, 0, d)$ of the line $\ell := \{x = 0, y = 0, z = 0\}$. Observe that the germ X_p is contained in the singular locus of Y_p and $\dim_{\mathbb{R}}(X_p) = 1 = \dim_{\mathbb{C}}(Y_p) - 2$. Thus, we have found a one dimensional tail which does not come from the intersection of two complex conjugate branches.

The set of singular points of Y is the complex analytic set $\operatorname{Sing}(Y) = \{x = 0, y = 0, z = 0\} \cup \{x = 0, y = 0, t = 0\} \subset \mathbb{C}^4$, which has codimension 2 in Y. As Y is a complex irreducible analytic hypersurface, we deduce by [**O**] that Y is a normal complex analytic set. Thus, X is a normal C-analytic set. As X is not pure dimensional, it is non-coherent.

Let us see in an intuitive way how we can characterize the set $N(X)$. For an accurate approach see [**ABF2**, §5]. Assume that X is an irreducible C-analytic subset of $\mathbb{R}^n$. Let $\widetilde{X}$ be a complexification of X that is an invariant complex analytic subset of an open Stein neighborhood $\Omega \subset \mathbb{C}^n$ of $\mathbb{R}^n$. Denote the restriction to $\widetilde{X}$ of the complex conjugation on $\mathbb{C}^n$ with $\sigma : \widetilde{X} \to \widetilde{X}$. It holds $d := \dim_{\mathbb{R}}(X) = \dim_{\mathbb{C}}(\widetilde{X})$ and $X = \{x \in \widetilde{X} : \; \sigma(x) = x\}$. Let $\pi : Y \to \widetilde{X}$ be the normalization of $\widetilde{X}$. As $\widetilde{X}$ is Stein, also Y is Stein [**N2**]. The complex conjugation of $\widetilde{X}$ extends to an antiholomorphic involution $\widehat{\sigma}$ on Y that makes the following diagram commutative

$$\begin{array}{ccccccc} & & Y^{\widehat{\sigma}} & \hookrightarrow & Y & \xrightarrow{\widehat{\sigma}} & Y \\ & & \Big\downarrow{\scriptstyle \pi|_{Y^{\widehat{\sigma}}}} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ X & = & \widetilde{X}^{\sigma} & \hookrightarrow & \widetilde{X} & \xrightarrow{\sigma} & \widetilde{X} \end{array}$$

Roughly speaking, the inverse images of 'tails' of X correspond to:

- The set $\pi^{-1}(X)\setminus Y^{\widehat{\sigma}}$ (this set can be understood intuitively as the inverse image of those tails of type 1, which disappear when irreducible local components of X are separated after we apply normalization).
- The own 'tails' of $Y^{\widehat{\sigma}}$ (which provide tails of type 2 in X, see Example 2.13 for further details).

The set $N_d(X)$ of points of X such that the germ X_x has a non-coherent irreducible component of dimension d is obtained as follows. Define

$$C_1 := \pi^{-1}(X)\setminus Y^{\widehat{\sigma}} \qquad \text{(preimages of the tails of type 1)}$$

$$C_2 := Y^{\widehat{\sigma}}\setminus \overline{Y^{\widehat{\sigma}}\setminus \operatorname{Sing}(Y^{\widehat{\sigma}})} \qquad \text{(preimages of the tails of type 2)}$$

and denote $A_i = \overline{C_i}\cap \overline{Y^{\widehat{\sigma}}\setminus \operatorname{Sing}(Y^{\widehat{\sigma}})}$ (points where the preimages of tails of type i attach to the d-dimensional part of $Y^{\widehat{\sigma}}$). Consequently, $N_d(X)=\pi(A_1)\cup\pi(A_2)$ and we deduce that $N_d(X)$ is a C-semianalytic set as a consequence of the Direct Image Theorem 2.8.

The construction of the full set $N(X)$ is much more involved, but it follows from the same kind of ideas. The case when X is not irreducible is even more complicated and requires a more careful discussion, which is done with full detail in [**ABF2**, §5].

3. Amenable C-semianalytic sets and irreducible components

Irreducibility and irreducible components are usual concepts in Geometry and Algebra. Both concepts are strongly related with prime ideals and primary decomposition of ideals. There is an important background concerning this matter in Algebraic and Analytic Geometry. These concepts has been satisfactorily developed for complex algebraic sets (Lasker-Noether [**La**]), complex analytic sets and Stein spaces (Cartan [**C2**], Forster [**Fo**], Remmert-Stein [**RS**]), C-analytic sets (Whitney-Bruhat [**WB**]), Nash sets (Efroymson [**E**], Mostowski [**Mo**], Risler [**R2**]) and semialgebraic sets (Fernando-Gamboa [**FG**]). The global behavior of real analytic sets could be wild as commented above and this blocks the possibility of having a reasonable concept of irreducibility. As we have already mentioned, C-analytic sets have a good global behavior that enables a consistent concept of irreducibility. An additional requirement to avoid pathologies in the semianalytic setting should be that 'Zariski closure preserve dimensions'. The *Zariski closure* of a subset $E\subset M$ is the smallest C-analytic subset X of M that contains E. We define the dimension of a C-semianalytic set $S\subset M$ as $\dim(S):=\sup_{x\in M}\{\dim(S_x)\}$ and refer the reader to [**ABR2**, VIII.2.11] for the dimension of semianalytic germs. The Zariski closure of a C-semianalytic set is in general a C-analytic set of higher dimension.

EXAMPLE 3.1. For $n\geq 1$ consider the basic C-semianalytic set

$$S_n := \{y=nx, n\leq x\leq n+1\}\subset\mathbb{R}^2.$$

The family $\{S_n\}_{n\geq 1}$ is locally finite, so $S:=\bigcup_{n\geq 1}S_n$ is a C-semianalytic set. If $x\in S$ and U^x is a small enough C-semianalytic neighborhood of x, the Zariski closure $\overline{S\cap U^x}^{\text{zar}}$ is a line. The collection $\{\overline{S_n}^{\text{zar}}\}$ of all these lines is not locally finite at the origin and $\overline{S}^{\text{zar}}=\mathbb{R}^2$.

3.A. Amenable C-semianalytic sets. To guarantee a satisfactory behavior of Zariski closure we need a more restrictive concept that we introduced in [**Fe**].

DEFINITION 3.2. A subset $S \subset M$ is an *amenable C-semianalytic set* if it is a finite union of C-semianalytic sets of the type $X \cap U$ where $X \subset M$ is a C-analytic set and $U \subset M$ is an open C-semianalytic set. In particular, the Zariski closure of S has the same dimension as S.

The family of amenable C-semianalytic sets is closed under the following operations: finite unions and intersections, interior, connected components, sets of points of pure dimension k and inverse images of analytic maps. However, it is not closed under: complement, closure, locally finite unions and sets of points of dimension k (see [**Fe**] for a clarifying collection of examples).

A C-semianalytic set $S \subset M$ is amenable if and only if it is a locally finite countable union of basic C-semianalytic sets S_i such that the family $\{\overline{S_i}^{\text{zar}}\}_{i\geq 1}$ of their Zariski closures is locally finite (after eliminating repetitions). As a consequence the union of a locally finite collection of amenable C-semianalytic sets whose Zariski closures constitute a locally finite family (after eliminating repetitions) is an amenable C-semianalytic set. Amenable C-semianalytic sets have the same reasonable behavior under proper holomorphic maps as C-semianalytic sets. If we are under the same hypotheses of Theorem 2.8 we have

THEOREM 3.3. *Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an invariant proper holomorphic map between reduced Stein spaces. Let $S \subset X^\sigma$ be a $\mathcal{A}(X^\sigma)$-definable and amenable C-semianalytic set and let $S' \subset Y^\tau$ be an amenable C-semianalytic set. We have:*

(i) *$F(S)$ is an amenable C-semianalytic subset of Y^τ of the same dimension as S.*

(ii) *If T is a union of connected components of $F^{-1}(S') \cap X^\sigma$, then $F(T)$ is an amenable C-semianalytic set.*

3.B. Irreducibility. In the algebraic, complex analytic, C-analytic and Nash settings a geometric object is irreducible if it is not the union of two proper geometric objects of the same nature. In the amenable C-semianalytic setting this definition does not work because every C-semianalytic set with at least two points would be reducible. Indeed, if $p, q \in S$ and W is open C-semianalytic neighborhood of p in M such that $q \notin W$, it holds $S = (S \cap W) \cup (S \setminus \{p\})$ where $S \cap W$ and $S \setminus \{p\}$ are amenable C-semianalytic sets.

In the previous settings the irreducibility of a geometric object X is equivalent to the fact that the corresponding ring of polynomial, analytic or Nash functions on X is an integral domain. This equivalence suggests us to attach to each amenable C-semianalytic set $S \subset M$ the ring $\mathcal{O}(S)$ of real valued functions on S that admit an analytic extension to an open neighborhood of S in M. We say that S is *irreducible* if and only if $\mathcal{O}(S)$ is an integral domain.

Our definition extends the notion of irreducibility for C-analytic, Nash and semialgebraic sets. We refer the reader to [**Fe**] for the precise notion of irreducible semialgebraic set. In addition, if $X \subset \mathbb{C}^n$ is a complex analytic set and $X^{\mathbb{R}} \subset \mathbb{R}^{2n}$ is its underlying real analytic structure, *X is irreducible as a complex analytic set if and only if $X^{\mathbb{R}}$ is irreducible as a C-semianalytic set.*

The irreducibility of an amenable C-semianalytic set S has a close relation with the connectedness of certain subset of the normalization of the Zariski closure of S. More precisely, let $S \subset M$ be an amenable C-semianalytic set and let X be its Zariski closure. Let $(\widetilde{X},\sigma)$ be a Stein complexification of X together with the antiholomorphic involution $\sigma:\widetilde{X}\to\widetilde{X}$ whose set of fixed points is X. Let (Y,π) be the normalization of $\widetilde{X}$ and let $\widehat{\sigma}:Y\to Y$ be the antiholomorphic involution induced by σ in Y, which satisifies $\pi\circ\widehat{\sigma}=\sigma\circ\pi$.

THEOREM 3.4. *The amenable C-semianalytic set S is irreducible if and only if there exists a connected component T of $\pi^{-1}(S)$ such that $\pi(T)=S$.*

3.C. Irreducible components. In [**Fe**] we present a satisfactory theory of irreducible components for amenable C-semianalytic sets. It holds that if S is either C-analytic, semialgebraic or Nash, then its irreducible components as a set of the corresponding type coincide with the irreducible components of S as an amenable C-semianalytic set. In addition, if $X\subset\mathbb{C}^n$ is a complex analytic set and $X^{\mathbb{R}}\subset\mathbb{R}^{2n}$ is its underlying real analytic structure, *the underlying real analytic structures of the irreducible components of X as a complex analytic set coincide with the irreducible components of $X^{\mathbb{R}}$ as a C-semianalytic set.*

DEFINITION 3.5 (Irreducible components). *Let $S\subset M$ be an amenable C-semianalytic set. A countable locally finite family $\{S_i\}_{i\geq1}$ of amenable C-semianalytic sets that are contained in S is* a family of irreducible components of S *if the following conditions are fulfilled:*

(1) *Each S_i is irreducible.*
(2) *If $S_i\subset T\subset S$ is an irreducible amenable C-semianalytic set, then $S_i=T$.*
(3) *$S_i\neq S_j$ if $i\neq j$.*
(4) *$S=\bigcup_{i\geq1}S_i$.*

The following result states the existence and uniqueness of irreducible components of an amenable C-semianalytic set $S\subset M$.

THEOREM 3.6 (Existence and uniqueness). *There exists a bijection between the irreducible components of an amenable C-semianalytic set $S\subset M$ and the minimal prime ideals of the ring $\mathcal{O}(S)$.*

The family $\{\overline{S_i}^{\text{zar}}\}_{i\geq1}$ of the Zariski closures of the irreducible components $\{S_i\}_{i\geq1}$ of an amenable C-semianalytic set is locally finite in M (after eliminating repetitions). Consequently, any union of irreducible components of an amenable C-semianalytic $S\subset M$ is an amenable C-analytic set.

4. Nullstellensätze

A main tool in complex and real algebraic and analytic geometry is the use of Nullstellensätze. The Nullstellensatz for the ring of analytic functions germs is well-known, both in the complex and in the real cases. The first one is due to Rückert [**Ru**] while the second is due to Risler [**R3**]. Their statements are analogous to those for rings of complex or real polynomials. Recall that $Z(\mathfrak{a})$ denotes the zero set of the ideal $\mathfrak{a}$ of a ring of functions or germs whereas $I(S)$ is the ideal of those elements of the corresponding ring of functions or germs that are identically zero on S. Given a ring A, the *real radical* of an ideal $\mathfrak{a}$ of A is the set $\sqrt[r]{\mathfrak{a}}:=\{f\in A: f^{2m}+a_1^2+\ldots+a_k^2\in\mathfrak{a}\quad\text{for some } a_i\in A\}$. We summarize next the classical results mentioned above.

(i) Let $\mathfrak{a}\subset\mathbb{C}[x]$ be an ideal. Then $I(Z(\mathfrak{a}))=\sqrt{\mathfrak{a}}$ (Hilbert, 1893, [**H**, p. 320]).
(ii) Let $\mathfrak{a}\subset\mathbb{C}\{x\}$ is an ideal. Then $I(Z(\mathfrak{a}))=\sqrt{\mathfrak{a}}$ (Rückert, 1933, [**Ru**]).

(iii) Let $\mathfrak{a} \subset \mathbb{R}[x]$ is an ideal. Then $I(Z(\mathfrak{a})) = \sqrt[r]{\mathfrak{a}}$ (Risler, 1970, [**R1**]).
(iv) Let $\mathfrak{a} \subset \mathbb{R}\{x\}$ is an ideal. Then $I(Z(\mathfrak{a})) = \sqrt[r]{\mathfrak{a}}$ (Risler, 1976, [**R3**]).

Note that the complex case was approached many year earlier than the real one.

Next we look at rings of global analytic functions. Several difficulties arise. First of all the rings $\mathcal{O}(\mathbb{C}^n)$ and $\mathcal{O}(\mathbb{R}^n)$ are neither noetherian nor unique factorization domains. There are at least two important obstructions: (1) the 'multiplicity' of an analytic function at the points of its zero-set can be unbounded; and (2) there exist prime ideals in $\mathcal{O}(\mathbb{C}^n)$ and real prime ideals in $\mathcal{O}(\mathbb{R}^n)$ with empty zero-set. Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$.

Example 4.1. Consider the following analytic functions in one variable:

$$f(x) := \prod_{n\geq 1}\Big(1 - \frac{x}{n^2}\Big) \quad \text{and} \quad g(x) := \prod_{n\geq 1}\Big(1 - \frac{x}{n^2}\Big)^n.$$

Both functions have the same zero-set $\{n^2 : n \geq 1\}$ but clearly no power of f can belong to the ideal generated by g in $\mathcal{O}(\mathbb{K})$.

Example 4.2. Let $\mathfrak{U}$ be an ultrafilter of subsets of $\mathbb{N}$ containing all cofinite subsets. For an analytic function $F \in \mathcal{O}(\mathbb{K})$ we denote the *multiplicity* of F at the point $z \in \mathbb{K}$ with $\operatorname{mult}_z(F)$. Put $M(F,m) := \{\ell \in \mathbb{N} : \ \operatorname{mult}_\ell(F) \geq m\}$. Consider the non-empty set $\mathfrak{p} := \{F \in \mathcal{O}(\mathbb{K}) : \ M(F,m) \in \mathfrak{U}\ \forall m \geq 0\}$. Let us check that $\mathfrak{p}$ is a prime ideal.

Indeed, let $F, G \in \mathfrak{p}$. Then $M(F,m) \cap M(G,m) \subset M(F+G,m)$ because $\operatorname{mult}_\ell(F+G) \geq \min\{\operatorname{mult}_\ell(F), \operatorname{mult}_\ell(G)\}$, so $M(F+G,m) \in \mathfrak{U}$ for all $m \geq 0$. On the other hand, if $F \in \mathfrak{p}$ and $G \in \mathcal{O}(K)$, then $\operatorname{mult}_\ell(FG) = \operatorname{mult}_\ell(F) + \operatorname{mult}_\ell(G)$, so $M(FG,m) \supset M(F,m) \in \mathfrak{U}$ for all $m \geq 0$.

Suppose $F_1F_2 \in \mathfrak{p}$ but $F_1, F_2 \notin \mathfrak{p}$. Then there exist $m_1, m_2 \geq 0$ such that

$$M(F_1,m_1), M(F_2,m_2) \notin \mathfrak{U}.$$

Take $m_0 := \max\{m_1,m_2\}$ and note $M(F_1,m_0), M(F_2,m_0) \notin \mathfrak{U}$; hence, $M(F_1,m_0) \cup M(F_2,m_0) \notin \mathfrak{U}$. On the other hand,

$$M(F_1,m_0) \cup M(F_2,m_0) \supset M(F_1F_2, 2m_0) \in \mathfrak{U},$$

so also $M(F_1,m_0) \cup M(F_2,m_0) \in \mathfrak{U}$, which is a contradiction.

Thus, $\mathfrak{p}$ is a prime ideal. In fact, one can check that when $\mathbb{K} = \mathbb{R}$, the it is in addition a real prime ideal. Finally, observe $Z(\mathfrak{p}) = \varnothing$. For each $k \geq 1$ let $G_k \in \mathcal{O}(\mathbb{K})$ be an analytic function such that $Z(G_k) = \{\ell \in \mathbb{N} : \ \ell \geq k\}$ and $\operatorname{mult}_\ell(G_k) = \ell$ for all $\ell \geq k$. Since $\mathfrak{U}$ contains all cofinite subsets, we deduce that each $G_k \in \mathfrak{p}$, so $Z(\mathfrak{p}) \subset \bigcap_{k\geq 1} Z(G_k) = \varnothing$.

4.A. Forster's results for Stein algebras. The first approach to the global problem was done by Forster [**Fo**] in 1964. To control the difficulties mentioned above first of all he considers only 'closed ideals' of a Stein algebra. He considers a Stein space $(X, \mathcal{O}_X)$, its algebra of global holomorphic functions $\mathcal{O}(X) := H^0(X, \mathcal{O}_X)$ and those ideals in $\mathcal{O}(X)$ that are closed with respect to the usual Fréchet's topology of $\mathcal{O}(X)$, see [**GR**, VIII.A]. Cartan proved in [**C1**, VIII.Thm.4, pag.60] that if $(X, \mathcal{O}_X)$ is a Stein space, the closure of an ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ coincides with its *saturation* $H^0(X, \mathfrak{a}\mathcal{O}_X) := \{F \in \mathcal{O}(X) : \ F_x \in \mathfrak{a}\mathcal{O}_{X,z}\ \forall x \in X\}$. Consequently, $\mathfrak{a}$ is closed if and only if $\mathfrak{a} = H^0(X, \mathfrak{a}\mathcal{O}_X)$. We present next a key result for Forster's Nullstellensatz [**Fo**] that relates the fact that a holomorphic function f belongs to a primary ideal $\mathfrak{q} \subset \mathcal{O}(X)$ with the fact the germ f_x belongs to the fiber $\mathfrak{q}\mathcal{O}_{X,x}$ of the ideal sheaf $\mathfrak{q}\mathcal{O}_X$ for some $x \in X$. Its proof is based on Cartan's Theorem B.

LEMMA 4.3. *Let $\mathfrak{q}$ be a closed primary ideal of $\mathcal{O}(X)$ and let $f \in \mathcal{O}(X)$. Then, $f \in \mathfrak{q}$ if and only if there exists a point $x \in Z(\mathfrak{q})$ such that $f_x \in \mathfrak{q}\mathcal{O}_{X,x}$.*

Two straightforward but relevant consequences of the previous result are the following.

COROLLARY 4.4 (Closed primary case). *Let $\mathfrak{q}$ be a closed primary ideal of $\mathcal{O}(X)$. We have:*

(i) *If $\mathfrak{q}$ is a closed proper primary ideal, then its zero-set is not empty.*
(ii) *$I(Z(\mathfrak{q})) = \sqrt{\mathfrak{q}}$ and there is an integer $m \geq 1$ such that $(\sqrt{\mathfrak{q}})^m \subset \mathfrak{q}$.*

Once the primary case was solved, to approach the general case Forster proved that a closed ideal $\mathfrak{a}$ admits a *normal primary decomposition*. Given a collection of ideals $\{\mathfrak{a}_i\}_{i\in I}$ of $\mathcal{O}(X)$, we say that it is *locally finite* if the family of their zero-sets $\{\mathcal{Z}(\mathfrak{a}_i)\}_{i\in I}$ is locally finite in X. A decomposition $\mathfrak{a} = \bigcap_{i\in I} \mathfrak{a}_i$ of an ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ is called *irredundant* if $\mathfrak{a} \neq \bigcap_{i\in K} \mathfrak{a}_i$ for each proper subset $K \subsetneq I$. Moreover, a primary decomposition $\mathfrak{a} = \bigcap_{i\in I} \mathfrak{q}_i$ of an ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ is called *normal* if it is locally finite, irredundant and the associated prime ideals $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ are pairwise distinct. As usual, a primary ideal $\mathfrak{q}_j$ is called an *isolated primary component* if $\mathfrak{p}_j$ is minimal among the primes $\{\mathfrak{p}_i\}_{i\in I}$. Otherwise, $\mathfrak{q}_j$ is an *immersed primary component*. Of course, a normal primary decomposition is not finite in general. Forster primary decomposition result for $\mathcal{O}(X)$ is the following.

PROPOSITION 4.5. ([**Fo**, §5]) *Let $\mathfrak{a} \subset \mathcal{O}(X)$ be a closed ideal of $\mathcal{O}(X)$. Then $\mathfrak{a}$ admits a normal primary decomposition $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ such that all primary ideals $\mathfrak{q}_i$ are closed. Moreover, the prime ideals $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ and the primary isolated components are uniquely determined by $\mathfrak{a}$ and do not depend on the normal primary decomposition of $\mathfrak{a}$.*

Using the previous fact and a nice application of Baire's Theorem to the Fréchet space $\mathcal{O}(X)$ Forster proved the following result.

THEOREM 4.6 (Closed general case). *Let $\mathfrak{a} \subset \mathcal{O}(X)$ be a closed ideal and let $\mathfrak{a} = \bigcap_{i\in I} \mathfrak{q}_i$ be a normal primary decomposition of $\mathfrak{a}$. For each $i \in I$ define*

$$\begin{aligned}\mathtt{h}(\mathfrak{q}_i, \mathfrak{a}) &:= \inf\left\{k \in \mathbb{N} : \ F^k \in \mathfrak{q}_i, \ \forall F \in \overline{\sqrt{\mathfrak{a}}}\right\},\\ \mathtt{h}(\mathfrak{q}_i) &:= \inf\{k \in \mathbb{N} : \ F^k \in \mathfrak{q}_i, \ \forall F \in \sqrt{\mathfrak{q}_i}\},\\ \mathtt{h}(\mathfrak{a}) &:= \inf\left\{k \in \mathbb{N} : \ F^k \in \mathfrak{a}, \ \forall F \in \overline{\sqrt{\mathfrak{a}}}\right\}.\end{aligned}$$

Then we have

(i) *$\mathtt{h}(\mathfrak{a}) = \sup_{i\in I}\{\mathtt{h}(\mathfrak{q}_i, \mathfrak{a})\}$ and $\sqrt{\mathfrak{a}}$ is closed if and only if $\mathtt{h}(\mathfrak{a}) < +\infty$;*
(iii) *If $\mathfrak{a}$ does not have immersed primary components, $\mathtt{h}(\mathfrak{a}) = \sup_{i\in I}\{\mathtt{h}(\mathfrak{q}_i)\}$;*
(iv) *$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ if and only if $\mathtt{h}(\mathfrak{a}) < +\infty$ and in such case $\sqrt{\mathfrak{a}}^{\mathtt{h}(\mathfrak{a})} \subset \mathfrak{a}$.*

In this context we extended Forster's Nullstellensatz in [**ABF1**] to the non-closed case as we state in the next result.

THEOREM 4.7 (Nullstellensatz). *Let $(X, \mathcal{O}_X)$ be a Stein space and $\mathfrak{a} \subset \mathcal{O}(X)$ an ideal. Then $I(Z(\mathfrak{a})) = \overline{\sqrt{\mathfrak{a}}}$.*

4.B. A real Nullstellensatz. Let $(X, \mathcal{O}_X)$ be a C-analytic set endowed with its natural structure of real analytic space and let $\mathcal{O}(X)$ be its algebra of global analytic functions. It seems really difficult to obtain a real Nullstellensatz for $\mathcal{O}(X)$ in the sense of Risler, so we tried an alternative way that involves a concept of 'convexity' for ideals [**ABF1**]. The ring $\mathcal{O}(X) := H^0(X, \mathcal{O}_X) = \mathcal{O}(\mathbb{R}^n)/\mathcal{I}(X)$ can be understood as a subset of the Stein algebra $\mathcal{O}(\widetilde{X})$ of its *complexification* $\widetilde{X}$ (understood as a complex analytic set germ at X). We stress that X needs not to be coherent as an analytic set, but it is the support of a coherent sheaf of $\mathcal{O}_{\mathbb{R}^n}$-modules. We endow $\mathcal{O}(X)$ with the topology induced by Fréchet's topology of $\mathcal{O}(\widetilde{X})$ and the saturation $\tilde{\mathfrak{a}} := H^0(X, \mathfrak{a}\mathcal{O}_X) = \{f \in \mathcal{O}(X) : \ f_x \in \mathfrak{a}\mathcal{O}_{X,x} \ \forall x \in X\}$ of an ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ is by [**dB2**] again its closure (with respect to this induced topology). In addition, $\mathfrak{a} \subset \tilde{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$.

As de Bartolomeis proved in [**dB1, dB2**], each *saturated ideal* $\mathfrak{a}$ *of* $\mathcal{O}(X)$ (that is, such that $\mathfrak{a} = \tilde{\mathfrak{a}}$) admits a *normal primary decomposition* similar to the one devised by Forster in the complex case. Note also that the previous definition of saturation coincides with the one proposed by Whitney for ideals in the ring of smooth functions over a real smooth manifold [**M**, II.1.3].

An ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ is *convex* if each $g \in \mathcal{O}(X)$ satisfying $|g| \leq f$ for some $f \in \mathfrak{a}$ belongs to $\mathfrak{a}$. We define the *convex hull* $\hat{\mathfrak{a}}$ *of an ideal* $\mathfrak{a}$ *of* $\mathcal{O}(X)$ by

$$\hat{\mathfrak{a}} := \{g \in \mathcal{O}(X) : \ \exists f \in \mathfrak{a} \text{ such that } |g| \leq f\}.$$

Notice that $\hat{\mathfrak{a}}$ is the smallest convex ideal of $\mathcal{O}(X)$ that contains $\mathfrak{a}$ and $\hat{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$. We define the *Łojasiewicz radical ideal* of an ideal $\mathfrak{a} \subset \mathcal{O}(X)$ as: $\sqrt[\text{Ł}]{\mathfrak{a}} := \sqrt{\hat{\mathfrak{a}}}$. In particular, Łojasiewicz's radical is a radical convex ideal. The notion of Łojasiewicz radical has been used by many authors to approach different problems mainly related to rings of germs, see for instance [**D**, p. 104], [**K**, 1.21] or [**DM**, §6] but also in the global smooth case [**ABN**]. Our main result in the global analytic context is the following [**ABF1**].

THEOREM 4.8 (Real Nullstellensatz). *Let* $X \subset \mathbb{R}^n$ *be a* C*-analytic set and* $\mathfrak{a}$ *an ideal of the ring* $\mathcal{O}(X)$*. Then* $I(Z(\mathfrak{a})) = \widetilde{\sqrt[\text{Ł}]{\mathfrak{a}}}$.

SKETCH OF THE PROOF. The proof of the previous result is based mainly in two main facts. The first one is Theorem 2.5. Let $f, g \in \mathcal{O}(X)$ be such that $Z(f) \subset Z(g)$. Fix a compact set $K \subset X$. Then, by Theorem 2.5 there exist an integer $m \geq 1$ and an analytic function $h \in \mathcal{O}(X)$ such that $Z(h) \cap K = \varnothing$ and $|f| \geq |hg|^m$. Consequently, $gh \in \sqrt[\text{Ł}]{f\mathcal{O}(X)}$, so $g_x \in (\sqrt[\text{Ł}]{f\mathcal{O}(X)})\mathcal{O}_{X,x}$ for each $x \in K$. As this holds true for each compact subset K of X, we conclude $g \in \widetilde{\sqrt[\text{Ł}]{f\mathcal{O}(X)}}$, so we have got the real Nullstellensatz for principal ideals!

The second fact consists of a reduction of the general problem to the case of a principal ideal. To that end we need, given an ideal $\mathfrak{a} \subset \mathcal{O}(X)$, an analytic function f having the same zero-set as $\mathfrak{a}$. Observe that $I(Z(\mathfrak{a})) = I(Z(f))$, so if such a function exists, then each $g \in I(Z(\mathfrak{a}))$ satisfies $Z(f) \subset Z(g)$, so $g \in \widetilde{\sqrt[\text{Ł}]{f\mathcal{O}(X)}}$. In case $f \in \tilde{\mathfrak{a}}$, we will have

$$g \in \widetilde{\sqrt[\text{Ł}]{f\mathcal{O}(X)}} \subset \widetilde{\sqrt[\text{Ł}]{\tilde{\mathfrak{a}}}} = \widetilde{\sqrt[\text{Ł}]{\mathfrak{a}}},$$

and the proof will be done.

In case $\mathfrak{a}$ is finitely generated, the function f is easily found as the sum of the squares of a finite system of generators of $\mathfrak{a}$. In case $\mathfrak{a}$ is not finitely generated, it admits a system of countably many generators a_k for $k \geq 1$. In addition, we may assume that all this generators extend holomorphically to a common open Stein neighborhood of X in its complexification $\widetilde{X}$. We choose now suitable positive coefficients to make the series $\sum_k c_k a_k^2$ converge to a real analytic function f. But there is a price to pay: the analytic function f does not belong in general to $\mathfrak{a}$ but to $\tilde{\mathfrak{a}}$. To prove this last fact, recall that $\tilde{\mathfrak{a}}$ is the closure of $\mathfrak{a}$ in $\mathcal{O}(X)$ with respect to the topology induced by Fréchet's topology of $\mathcal{O}(\widetilde{X})$. □

In general, if $\mathfrak{a} \subset \mathcal{O}(X)$ is an ideal, $\widetilde{\sqrt[r]{\mathfrak{a}}} \subset \widetilde{\sqrt[L]{\mathfrak{a}}}$ and it is a natural question to determine under which conditions both ideals coincide. This question has a close relation with Hilbert 17^{th} Problem for the ring of global analytic functions. Indeed, if we compare the radical ideals $\sqrt[r]{\mathfrak{a}}$ and $\sqrt[L]{\mathfrak{a}}$, we obtain the following:

- $g \in \sqrt[L]{\mathfrak{a}}$ if and only if there exist $f \in \mathfrak{a}$ and $m \geq 1$ such that $f - g^{2m} \geq 0$.
- $g \in \sqrt[r]{\mathfrak{a}}$ if and only if there exists $m \geq 1$ and $a_1, \dots, a_k \in \mathcal{O}(X)$ such that $g^{2m} + a_1^2 + \cdots + a_k^2 = f \in \mathfrak{a}$ or equivalently if there exist $f \in \mathfrak{a}$ and $m \geq 1$ such that $f - g^{2m}$ is a sum of squares in $\mathcal{O}(X)$.

Thus, we would have $\sqrt[L]{\mathfrak{a}} = \sqrt[r]{\mathfrak{a}}$, if any non-negative analytic function were a sum of squares. Unfortunately, this is not true even for polynomials and the best result one can afford is the following: *a polynomial* $f \in \mathbb{R}[x_1 \dots, x_n]$ *such that* $f(x) \geq 0$ *for each* $x \in \mathbb{R}^n$ *is a sum of squares in the field* $\mathbb{R}(x_1, \dots, x_n)$ *of rational functions* (Artin, 1927, [**Ar**]). In other words, in general denominators are needed to obtain representations as sum of squares (see Motzkin, 1967, [**Mz**] for the first explicit example). We formulate Hilbert 17^{th} Problem for analytic functions as follows.

Question 4.9. Let $f \in \mathcal{O}(\mathbb{R}^n)$ be such that $f(x) \geq 0$ for each $x \in \mathbb{R}^n$. Do there exist analytic functions $g, a_1, \dots a_k$ such that $Z(g) \subset Z(f)$ and $g^2 f = a_1^2 + \ldots + a_k^2$?

The answer is not known in general, but there are partial results related to the topological properties of zero-set of the given non-negative function $f \in \mathcal{O}(\mathbb{R}^n)$. Hilbert 17^{th} Problem has a positive solution when: (1) $Z(f)$ is a discrete set [**BKS**]; (2) $Z(f)$ is compact [**Rz1**,**Jw**]; (3) $Z(f)$ is discrete outside a compact set [**Jw**]; and (4) $Z(f)$ is a countable union of pairwise disjoint compact sets [**ABFR3**]. In the latter case the sum of squares could be an infinite convergent sum of squares (in a strong sense, [**ABFR3**]). This lack of global information suggest the following definition.

Definition 4.10. A C-analytic set $Z \subset \mathbb{R}^n$ is an H^{a}*-set* if each positive semidefinite analytic function $f \in \mathcal{O}(\mathbb{R}^n)$ whose zero-set is Z can be represented as a (possible infinite) sum of squares of meromorphic functions on $\mathbb{R}^n$.

Concerning this setting in [**ABF1**] we prove the following result. In order to consider infinite sum of squares one introduces naturally the real-analytic radical ideal $\sqrt[ra]{\cdot}$ which considers infinite convergent sum of squares instead of only finite sums of squares.

Theorem 4.11. *Let* $X \subset \mathbb{R}^n$ *be a* C*-analytic set and* $\mathfrak{a}$ *an ideal of* $\mathcal{O}(X)$ *such that* $Z(\mathfrak{a})$ *is a* H^{a}*-set. Then* $I(Z(\mathfrak{a})) = \widetilde{\sqrt[ra]{\mathfrak{a}}}$.

If X is either an analytic curve [**ABFR1**], a coherent analytic surface [**ABFR2**] or a C-analytic set whose connected components are all compact, then $Z(\mathfrak{a})$ is a

H^{a}-set for each ideal $\mathfrak{a} \subset \mathcal{O}(X)$. Thus, the previous result applies to this situations and the real Nullstellensatz holds for such an X in terms of the real radical (or the real-analytic radical).

Sketch of proof of Theorem 4.11. The proof of this result provided in [**ABF1**] is reduced after some work to the case when $\mathfrak{a} = \mathfrak{p}$ is a saturated real-analytic prime ideal of $\mathcal{O}(\mathbb{R}^n)$ and $Z := Z(\mathfrak{p})$ is an H^{a}-set. Let us roughly comment some general details concerning the proof of this case. Real-analytic means that $\mathfrak{p} = \sqrt[\mathrm{ra}]{\mathfrak{p}}$, that is, if $\sum_{k\geq 1} a_k^2 \in \mathfrak{p}$ with each $a_k \in \mathcal{O}(\mathbb{R}^n)$, then every $a_k \in \mathfrak{p}$.

Let $f \in \mathfrak{p}$ be a non-negative analytic function with the same zero-set as $\mathfrak{p}$ and take $g \in I(Z(\mathfrak{p})) = \widetilde{\sqrt[\mathrm{L}]{\mathfrak{p}}}$. Pick a point $x_0 \in Z(\mathfrak{p})$. Then, by Theorem 2.5 there exists $b \in \mathcal{O}(\mathbb{R}^n)$ such that $b(x_0) \neq 0$ and $f - (bg)^{2m} \geq 0$. Observe that $b \notin \mathfrak{p}$ because $b(x_0) \neq 0$. It is not clear that $f - (bg)^{2m}$ vanishes only on Z. However, this fact can be fixed using a straightforward trick, so let us assume $Z(f - (bg)^{2m}) = Z$. As Z is an H^{a}-set, there exists $h, a_k \in \mathcal{O}(\mathbb{R}^n)$ such that h is not identically zero on $\mathbb{R}^n$ and $h^2(f - (bg)^{2m}) = \sum_{k\geq 1} a_k^2$. Thus, $h(bg)^m \in \mathfrak{p}$, but we still do not know whether the denominator h belongs to $\mathfrak{p}$ or not. In order to get rid of h we need to push it a little 'without changing' the analytic function $f - (bg)^{2m}$. This can be done using a suitable analytic diffeomorphism close to the identity that:

- keeps $f - (bg)^{2m}$ invariant up to multiplication by a positive unit, but
- pushes the complex zero-set of an holomorphic extension of h away from the 'complex zero-set of $\mathfrak{p}$'. Recall that the real prime ideal $\mathfrak{p}$ has a natural holomorphic extension to a prime ideal of holomorphic functions defined on an open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$.

Thus, we may assume that $h \notin \mathfrak{p}$ because its holomorphic extension does not vanish identically on the complex zero-set of $\mathfrak{p}$. As $\mathfrak{p}$ is prime and $b \notin \mathfrak{p}$, we conclude $g \in \mathfrak{p}$, as required. □

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