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COHERENCE VERSUS ANALYTIC AND NASH EXTENSION PROPERTIES

by

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Abstract. — This survey presents an overview of our joint manuscript entitled "A Converse to Cartan's Theorem B: On the Extension Property for Global Analytic Sets and Nash Sets" (arXiv:2506.18347), which was presented by the first author at the congress "DDG40: Structures algébriques et ordonnées," held in August 2025 at the Observatoire Océanologique in Banyuls-sur-Mer, France. According to Cartan's Theorem B (1957), if $\Omega \subset \mathbb{R}^n$ is an open set and X is a coherent real analytic subset of Ω, then every real analytic function defined on X can be extended to a real analytic function on Ω (a property known as the analytic extension property). Recently, the converse of this implication has been established, employing an extensive application of the cohomology of analytic sheaves, as detailed in the aforementioned article. The classical literature only discusses very particular examples of non-coherent real analytic sets $X \subset \Omega$ that lack the analytic extension property, some of which can be traced back to the works of Cartan and Whitney.

We find that the categories of sets with the analytic extension property and coherent real analytic sets are the same. In the prior article, we have also extended the previous characterization to address the Nash case, which requires a distinct approach due to its finitary characteristics and its rather unsatisfactory interactions with the cohomology of sheaves of Nash function germs.

A foundational finding in Real Algebraic Geometry states that if $M \subset \mathbb{R}^n$ is a Nash manifold, the C^∞ semialgebraic functions on M coincides with Nash functions on that manifold. In our recent work, referenced above, we provided a comprehensive characterization (again focusing on coherence) of the semialgebraic sets $S \subset \Omega$ for which the C^∞ semialgebraic functions on S coincide with Nash functions on S.

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***Résumé* (Cohérence et propriétés dextension analytique et de Nash). —** Cette étude présente un aperçu de notre article commun intitulé « Réciproque du théorème B de Cartan : Sur la propriété d'extension pour les ensembles analytiques globaux et les ensembles de Nash » (arXiv:2506.18347), présenté par le premier auteur au congrès « DDG40 : Structures algébriques et ordonnées », qui s'est tenu en août 2025 à l'Observatoire océanologique de Banyuls-sur-Mer, en France. D'après le théorème B de Cartan (1957), si $\Omega \subset \mathbb{R}^n$ est un ouvert et X un sous-ensemble analytique réel cohérent de Ω, alors toute fonction analytique réelle définie sur X peut être prolongée en une fonction analytique réelle sur Ω (propriété dite d'extension analytique). Récemment, la réciproque de cette implication a été établie grâce à une application extensive de la cohomologie des faisceaux analytiques, comme détaillé dans l'article susmentionné. La littérature classique n'aborde que des exemples très particuliers d'ensembles analytiques réels non cohérents $X \subset \Omega$ dépourvus de la propriété d'extension analytique, dont certains remontent aux travaux de Cartan et Whitney.

Nous constatons que la catégories des ensembles possédant la propriété d'extension analytique et celle des ensembles analytiques réels cohérents sont identiques. Dans l'article précédent, nous avons également étendu cette caractérisation au cas de Nash, qui requiert une approche distincte en raison de ses propriétés finitaires et de ses interactions peu satisfaisantes avec la cohomologie des faisceaux de germes de fonctions de Nash.

Un résultat fondamental en géométrie algébrique réelle stipule que si $M \subset \mathbb{R}^n$ est une variété de Nash, les fonctions semi-algébriques C^∞ sur M coïncident avec les fonctions de Nash sur cette variété. Dans nos travaux récents, cités précédemment, nous avons fourni une caractérisation complète (axée à nouveau sur la cohérence) des ensembles semi-algébriques $S \subset \Omega$ pour lesquels les fonctions semi-algébriques C^∞ sur S coïncident avec les fonctions de Nash sur S.

1. Introduction

1.1. Analytic case. — Let Ω be an open subset of $\mathbb{R}^n$ and let X be a subset of Ω. We denote the set of natural numbers including 0 with $\mathbb{N} := \{0, 1, 2, \ldots\}$. Recall that a function $f : X \to \mathbb{R}$ is called *(real) analytic* if for each point $a \in X$ there exists an open neighborhood U of a in Ω such that $f|_{X \cap U}$ extends to an analytic function defined on U or, equivalently, if there exist an open neighborhood U of a in Ω and a (real) power series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha (x-a)^\alpha$ that converges (absolutely) at the points of U and such that the value $f(x)$ equals the sum of the convergent series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha (x-a)^\alpha$ in $\mathbb{R}$ for each $x \in X \cap U$. The set $X \subset \Omega$ has the *analytic extension property* if each analytic function $f : X \to \mathbb{R}$ extends to an analytic function defined on Ω. Contrary to what happens in differentiable or continuous contexts, in the analytic setting there exist no partitions of unity and this makes difficult to deal with problems relative to extension of functions. In this survey we explain following **[22]** how to solve the next problem. We use coherent analytic sheaves, which relate local and global approaches by means of analytic continuation results.

***Problem 1.1.* —** *Which sets $X \subset \Omega$ have the analytic extension property?*

This problem dates back to Cartan's 1957 article [**11**], where he proved several key results in the theory of coherent sheaves, including the well-known Theorem B. A set $X \subset \Omega$ is called *analytic* if it is closed in Ω and for each point $a \in X$, there is an open neighborhood U^a of a in Ω and finitely many analytic functions $f_1, \dots, f_r$ defined on U^a whose common zero set is $X \cap U^a$. Note that both U^a and the analytic functions $f_1, \dots, f_r$ are dependent on the point a. The set $X \subset \Omega$ is *global analytic* if there are finitely many analytic functions $f_1, \dots, f_r$ defined on all of Ω whose common zero set is X. Whitney and Bruhat referred to global analytic sets as *C-analytic sets* in [**39**], likely as a shortening of 'Cartan real analytic sets'. It is crucial to recognize that some analytic sets are not global analytic. For example, consider a modification of Cartan's umbrella [**11**, §11] or a compact Whitney-style umbrella [**1**, Ex. 3. 3(ii)].

Examples 1.2. **—** (i) Consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ given by the following formula:

$$a(z) := \begin{cases} \exp(\frac{1}{z^2-1}) & \text{if } |z| < 1, \\ 0 & \text{if } |z| \geq 1. \end{cases}$$

The function a is analytic on $\mathbb{R} \setminus \{-1, 1\}$. Let $Z \subset \mathbb{R}^3$ be the closed set given by the equation $z(x^2 + y^2) - x^3 a(z) = 0$. The intersection $S := Z \cap U$ is a C-analytic subset of the open set $U := \mathbb{R}^3 \setminus \{x = 0, y = 0, |z| \neq 1\}$. On the other hand,

$$Z \cap \{x^2 + y^2 < \varepsilon, \tfrac{1}{2} < |z| < \tfrac{3}{2}\} = \{x = 0, y = 0, \tfrac{1}{2} < |z| < \tfrac{3}{2}\}$$

for $\varepsilon > 0$ small enough. Consequently, *Z is an analytic set*. However, *it is not a C-analytic subset of any of its open neighborhoods in* $\mathbb{R}^3$, because by [**11**, Prop. 16] every analytic function f on a connected open neighborhood V of Z in $\mathbb{R}^3$ that vanishes at each point of Z is identically zero on V.

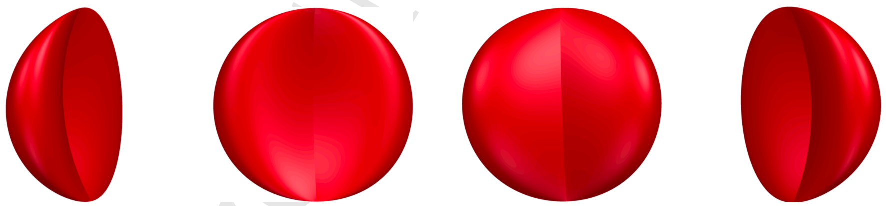

FIGURE 1. Several views of the two dimensional part $S_{(2)}$ of the seminanalytic set S in Example 1.2(ii)

(ii) Consider the closed subset $Z \subset \mathbb{R}^3$ defined by the equation

$$f := (1 - 4(x^2 + y^2 + z^2))x^2 - ((y-1)^2 + z^2 - 1)^2 a(z) = 0$$

where $a(z)$ is the same function as in (i). *It is an analytic set and it is compact.* Observe that S is the union of the circle $\{x = 0, (y-1)^2 + z^2 - 1 = 0\}$ with a two dimensional C-semianalytic set $S_{(2)}$ (Figure 1) contained in the ball $\{x^2 + y^2 + z^2 \leq \frac{1}{4}\}$. As before, one proves that *it is not a C-analytic subset of any of its open neighborhoods in* $\mathbb{R}^3$.

The global analyticity is an immediate necessary condition for the set $X \subset \Omega$ to have the analytic extension property. We borrow the following strategy from the proof of [**36**, Thm. 1].

Proof. — Suppose the set $X \subset \Omega$ has the analytic extension property, but it is not C-analytic. By [**39**, §6.Prop. 7 & Cor. 2] or [**23**, Cor.I. 8], the family of C-analytic subsets of Ω is stable under arbitrary intersections. Thus, there exists the smallest C-analytic subset Y of Ω that contains X. Suppose there exists a point $y \in Y \setminus X$. Consider the analytic function $f : X \to \mathbb{R}$, $x \mapsto \frac{1}{\|x-y\|^2}$ and an analytic function $F : \Omega \to \mathbb{R}$ that extends f. Define the analytic function $G : \Omega \to \mathbb{R}$, $x \mapsto 1 - F(x)\|x-y\|^2$. As G is identically zero on X, it vanishes identically on Y, but this is a contradiction, because $y \in Y$ and $G(y) = 1$. □

Global analyticity is still not enough to assure the analytic extension property. We next include the following variation of a well-known example due to Efroymson [**16**, p. 356].

Example 1.3. — Consider Cartan's umbrella $X := \{(x,y,z) \in \mathbb{R}^3 : z(x^2+y^2) - x^3 = 0\}$ and the analytic function

$$F_0 : \mathbb{R}^3 \setminus \{(0,0,1)\} \to \mathbb{R},\ (x,y,z) \mapsto \frac{x}{x^2+y^2+(z-1)^2}.$$

Consider the extension f by 0 of $F_0|_{X\setminus\{(0,0,1)\}}$ to X. Observe that $X \subset \mathbb{R}^3$ is a C-analytic set (even an algebraic set) and f *is an analytic function on* X.

The latter assertion can be proven as follows. Cartan's umbrella X decomposes as the union of the singular ruled surface $C := \{x^2+y^2 \neq 0, z = \frac{x^3}{x^2+y^2}\}$ (the cloth of the umbrella) and the z-axis $H := \{x = 0, y = 0\}$ (the handle of the umbrella). Observe that C meets H only at the origin O. As $(0,0,1) \in H \setminus \{O\}$ and f vanishes identically around $(0,0,1)$, the function f is analytic on X, because $\{x^2+y^2+(z-1)^2 = 0\} \cap X = \{(0,0,1)\}$. Define

$$X_{\mathbb{C}} := \{(x,y,z) \in \mathbb{C}^3 : z(x^2+y^2) - x^3 = 0\}$$

and

$$Y_{\mathbb{C}} := \{(x,y,z) \in \mathbb{C}^3 : x^2+y^2+(z-1)^2 = 0\},$$

which are irreducible complex algebraic sets. Thus, they are pure dimensional (because their sets of regular points are by [**34**, Cor.(4. 16)] connected) of dimension 2. Thus, the intersection $X_{\mathbb{C}} \cap Y_{\mathbb{C}}$ is by [**28**, Prop. I.7.1] pure dimensional of (complex) dimension 1. By the curve selection lemma [**10**, Prop. 8.1.13] there exists an analytic path $\alpha : (-1,1) \to X_{\mathbb{C}} \cap Y_{\mathbb{C}}$ such that $\alpha(0) = (0,0,1)$ and $\alpha((0,1)) \subset (X_{\mathbb{C}} \cap Y_{\mathbb{C}}) \setminus \{x = 0, y = 0, z \in \mathbb{R}\}$.

Suppose now that f has an analytic extension F to $\mathbb{R}^3$. Observe that $(x^2+y^2+(z-1)^2)F(x,y,z) = x$ on X. We can complexify the present geometric configuration using [**39**, §7] and obtain an open neighborhood V of $\mathbb{R}^3$ in $\mathbb{C}^3$ and a holomorphic function $F_{\mathbb{C}} : V \to \mathbb{C}$ such that $F_{\mathbb{C}}$ extends F and

$(x^2+y^2+(z-1)^2)F_{\mathbb{C}}(x,y,z)=x$ on $X_{\mathbb{C}}\cap V$. Thus, if $t>0$ is small enough, the point $\alpha(t)$ belongs to $X_{\mathbb{C}}\cap Y_{\mathbb{C}}\cap V$ and $0=0F_{\mathbb{C}}(\alpha(t))=\alpha_1(t)\neq 0$, which is a contradiction. Consequently, the analytic extension F of f does not exist.

***Remark 1.4.* —** The reader observes that the previous proof of the non-existence of F is based on the following fact: if we move on X from the cloth to the handle through the origin, the local (real) dimension of X decreases from 2 to 1, but the local (complex) dimension of the complexification $X_{\mathbb{C}}$ of X does not: it is constantly equal to 2. This phenomenon of dimensional decrease allows to 'hide in the handle of X' the poles of the meromorphic function $(x,y,z)\mapsto\frac{1}{x^2+y^2+(z-1)^2}$ after multiplying it by the analytic function $(x,y,z)\to x$, which vanishes identically on the handle but not on X. Thus, we obtain by restriction to the real part X of $X_{\mathbb{C}}$ an analytic function f on X with a unique '*hidden pole*' at p. On the contrary, the meromorphic function $(x,y,z)\mapsto\frac{x}{x^2+y^2+(z-1)^2}$ has poles on $X_{\mathbb{C}}$ exactly at the points $X_{\mathbb{C}}\cap Y_{\mathbb{C}}\cap V$, which is a complex analytic set of dimension 1.

Cartan's Theorem B offers a sufficient condition for a C-analytic set to have the analytic extension property. This condition is local in nature, and it is known as *coherence.*

1.1.1. *Coherent analytic sets.* — Let us explain intuitively what means coherence. Let $X\subset\Omega$ be an analytic set and let $a\in X$. Consider an open neighborhood U of a in Ω and analytic functions $f_1,\dots,f_r$ defined on U such that their common zero set is $X\cap U$. The functions $f_1,\dots,f_r$ are *good equations for X locally at a* if each local analytic equation of X at a is a local analytic combination of $f_1,\dots,f_r$. More precisely, if f is an analytic function defined on an open neighborhood U_1 of a in Ω that vanishes on $X\cap U_1$, there exist an open neighborhood U_2 of a in $U\cap U_1$ and analytic functions $g_1,\dots,g_r$ defined on U_2 such that $f|_{U_2}=g_1f_1|_{U_2}+\cdots+g_rf_r|_{U_2}$ on U_2. The analytic set $X\subset\Omega$ is said to be *coherent at $a\in X$* if there exist an open neighborhood U of a in Ω and analytic functions $f_1,\dots,f_r$ defined on U such that $f_1,\dots,f_r$ are good equations for X locally at each point $b\in X\cap U$. The analytic set $X\subset\Omega$ is *coherent* if it is coherent at each of its points. Non-singular analytic subsets of Ω and singular analytic curves of Ω are examples of coherent analytic set. Cartan proved that each coherent analytic set is C-analytic and that the above phenomenon of dimensional decrease cannot occur locally at any coherent point, see [**11**, §9&10] for further details. In particular, Cartan's umbrella and also Whitney's umbrella (given by the analytic equation $x^2-zy^2=0$ of $\mathbb{R}^3$) are not coherent at the origin. Actually, Cartan's Theorem B [**11**, §6, Thm. 3(B)] implies the following.

***Theorem 1.5* (Cartan, [11, §7(2)]). —** *Each coherent analytic set $X\subset\Omega$ has the analytic extension property.*

Coherence of analytic sets is a sufficient local condition that ensures the analytic extension property. In the real analytic context, Cartan's Theorem A [**11**, §6, Thm. 3(A)] indicates that coherence also has a global structural implication: *The local analytic structure of a coherent analytic set comes from its global analytic structure.*

Generally, the issue of finding local obstructions to having the analytic extension property has been discussed mostly for C-analytic sets, for which the previously mentioned phenomenon of dimensional decrease occurs. Many researchers believe that the non-coherence of C-analytic sets involves visible '*tails*', similar to Cartan's and Whitney's umbrella examples. However, this is not true in all cases. We reference several examples from **[22]** of pure dimensional C-analytic sets (specifically algebraic sets) that are not coherent. These examples are presented in Section 3 along with more details than in the original manuscript **[22]**. Recall that a non-empty C-analytic set $X \subset \mathbb{R}^n$ is *C-irreducible* if it is not the union of two proper C-analytic subsets. For a thorough discussion on C-irreducibility and the decomposition of a C-analytic subset X of an open set $\Omega \subset \mathbb{R}^n$ as a unique irredundant union of a locally finite family $\{X_i\}_{i\in I}$ of C-irreducible analytic subsets of Ω, we refer the reader to **[18, 39]**.

1.1.2. *Main results in the analytic setting.* — Next theorem is the first main result of **[22]**. It asserts that: coherence is also a necessary condition to have the analytic extension property.

Theorem 1.6 **([22, Thm. 1.4]). —** *A set $X \subset \Omega$ has the analytic extension property if and only if it is a coherent analytic set.*

To prove it we show in **[22]** the following result. The ring $M(X)$ of *meromorphic functions* on a C-analytic subset X of an open set $\Omega \subset \mathbb{R}^n$ is the total ring of fractions of the (reduced) ring constituted by the restrictions to X of the analytic functions on Ω. If $\{X_i\}_{i\in I}$ is the family of the C-analytic irreducible components of X, we have $M(X) := \left\{\frac{f}{g} : f, g \in O(\Omega),\ g|_{X_i} \not\equiv 0\ \forall i \in I\right\}$ (endowed with the usual sum and product).

Theorem 1.7 **([22, Thm. 1.5]). —** *Let $X \subset \Omega$ be a non-coherent C-analytic set. Then there exist 'many' meromorphic functions $\xi \in M(X)$ that define analytic functions on X, but have no analytic extensions to Ω.*

Remark 1.8. **—** As a different first approach to Theorem 1.6, we suggest the reader check out Kollar's lecture **[31]**. It offers a somewhat different proof for a single pole. See also Theorem 5.1 for the non-pure dimensional case.

1.1.3. *Analytic sheaves.* — The theory of sheaves is the appropriate setting where one can contextualize the concept of coherence and consequently the theorems stated above in the analytic case. Let Ω be an open subset of $\mathbb{R}^n$ and let $X \subset \Omega$ be an analytic set. The idea is to associate to each point $x \in \Omega$ the ring $O_{\Omega,x}$ of germs f_x at x of the analytic functions f on a neighborhood of x in Ω and the ideal $\mathcal{J}_{X,x}$ of $O_{\Omega,x}$ constituted by the germs h_x that vanish identically on the germ X_x of X at x. For each open set $\Delta \subset \Omega$ define $O_\Delta := \bigcup_{x\in\Delta} O_{\Omega,x}$ and the subset $\mathcal{J}_{X,\Delta} := \bigcup_{x\in\Delta} \mathcal{J}_{X,x}$ of O_Δ. Consider also the projection map $\pi : O_\Omega \to \Omega,\ f_x \mapsto x$ and the restriction

$\pi_\Delta := \pi|_{O_\Delta} : O_\Delta \to \Delta$. The set O_Ω admits a natural topology such that each analytic function $f : \Delta \to \mathbb{R}$ can be interpreted as a continuous section $\sigma : \Delta \to O_\Delta$ of the (restricted) projection map π_Δ (that is, $\pi_\Delta \circ \sigma = \mathrm{id}_\Delta$). Namely, if $f : \Delta \to \mathbb{R}$ is an analytic function, define the section $\sigma_f : \Delta \to O_\Delta$, $x \mapsto f_x$ of π_Δ induced by f, whereas if $\sigma : \Delta \to O_\Delta$ is a continuous section of π_Δ, define the analytic function $f_\sigma : \Delta \to \mathbb{R}$, $x \mapsto (\sigma(x))(x)$ associated to σ. It holds that $f_{\sigma_f} = f$ and $\sigma_{f_\sigma} = \sigma$. If we endow O_Ω with such a topology, we obtain the so-called *sheaf O_Ω of analytic functions on* Ω and $\mathcal{J}_X := \mathcal{J}_{X,\Omega}$ endowed with the induced topology is the *subsheaf of O_Ω of ideals of germs of analytic functions vanishing on X*. The subsheaf $\mathcal{J}_X$ of O_Ω is coherent at a point $a \in \Omega$ if there exist an open neighborhood U of a in Ω and analytic functions $f_1, \dots, f_r$ defined on U such that the germs $f_{1,b}, \dots, f_{r,b}$ generate the ideal $\mathcal{J}_{X,b}$ of the ring $O_{\Omega,b}$ for each $b \in U$. The analytic set $X \subset \Omega$ is *coherent* if $\mathcal{J}_X$ is coherent at each point of Ω. This 'sheaf version' of coherence of $X \subset \Omega$ coincides with the one given above.

1.2. Nash case. — Problem 1.1 has a counterpart in Nash geometry. To revisit the definition, a subset of $\mathbb{R}^n$ is termed *semialgebraic* if it can be expressed as a Boolean combination of sets defined by polynomial equalities and inequalities. Now, let us consider an open semialgebraic set $\Omega \subset \mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ is *Nash* if it is smooth and its graph is semialgebraic. Alternatively, as stated in [**10**, Prop. 8.1.8], it can also be characterized as an analytic function with a semialgebraic graph. For a subset X of Ω, we define a function $f : X \to \mathbb{R}$ to be *local Nash* if, for every point $a \in X$, there exists an open semialgebraic neighborhood U of a within Ω such that the restriction of f to $X \cap U$ can be extended to a Nash function defined on U. We identify a set $X \subset \Omega$ as possessing the *Nash extension property* if every local Nash function $f : X \to \mathbb{R}$ can be extended to a Nash function defined on the entirety of Ω. This leads us to formulate the Nash extension problem in a manner analogous to what we previously articulated for the analytic case. Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set.

***Problem 1.9.* —** *Which sets $X \subset \Omega$ have the Nash extension property?*

A set $X \subset \Omega$ is a *Nash set* if it is the common zero set of finitely many Nash functions defined on Ω. An adapted argument to the one exposed above [**36**, Thm. 1] for the analytic setting shows that *the Nash extension property is reserved for Nash sets.*

Proof. — Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set and let $X \subset \Omega$. Suppose that X has the Nash extension property, but it is not a Nash subset of Ω. As the ring $N(\Omega)$ of Nash functions on Ω is noetherian [**10**, Thm. 8.7.18], there exists the smallest Nash subset Y of Ω that contains X. Observe that Y is in particular a C-analytic set. Suppose there exists a point $y \in Y \setminus X$ and consider the local Nash function $f : X \to \mathbb{R}$, $x \mapsto \frac{1}{\|x-y\|^2}$, which is also an analytic function on X. As we have seen above f has no analytic extension to Ω, so it has no Nash extension to Ω, as well, which is a contradiction. Consequently, X is a Nash subset of Ω. □

The notion of coherent Nash set can be defined as in the analytic case: it is enough to replace ‘open neighborhood’ with ‘open semialgebraic neighborhood’ and ‘analytic’ with ‘Nash’, respectively. Recall that a Nash set $X \subset \Omega$ is coherent (as a Nash set) if and only if it is coherent as an analytic set [**6**, §2.B]. Problem 1.9 is solved by the second main result of [**22**].

Theorem 1.10 **([22, Thm. 1.7]).** — *A subset X of an open semialgebraic set $\Omega \subset \mathbb{R}^n$ has the Nash extension property if and only if $X \subset \Omega$ is a coherent Nash set.*

The ‘if’ implication was established in the notable works [**12, 14**] by Coste, Ruiz, and Shiota. Up until now, similar to the analytic scenario, the ‘only if’ implication has primarily been explored for Nash sets $X \subset \mathbb{R}^n$ with seeable ‘tails’. This changes with our article [**22**], where we demonstrate the following key result. The ring $M^\bullet(X)$ of *meromorphic Nash functions* on a Nash subset X of an open semialgebraic set $\Omega \subset \mathbb{R}^n$ is the total ring of fractions of the ring constituted by the restrictions to X of the Nash functions on Ω. If $\{X_i\}_{i\in I}$ is the family of the Nash irreducible components of X, then $M^\bullet(X) := \big\{\frac{f}{g} : f, g \in N(\Omega),\ g|_{X_i} \not\equiv 0\ \forall i \in I\big\}$ (endowed with the usual sum and product).

Theorem 1.11 **([22, Thm. 1.8]).** — *Let X be a non-coherent Nash subset of an open semialgebraic set $\Omega \subset \mathbb{R}^n$. Then there exist ‘many’ meromorphic Nash functions $\xi \in M^\bullet(X)$ that define local Nash functions on X, but have no Nash extensions to Ω.*

The proof of Theorem 1.11 closely follows that of Theorem 1.7, but it is important to account for the finiteness conditions that arise from the semialgebricity of Nash functions. In the context of Nash functions, passing from local results to global outcomes can be quite challenging due to their unexpected cohomological properties, even in seemingly straightforward scenarios. For instance, Hubbard demonstrated that $H^1(\mathbb{R}, N_{\mathbb{R}}) \neq 0$ in [**30**]. See also Theorem 5.1 for the non-pure dimensional case.

1.3. Smooth semialgebraic case. — Let $S \subset \mathbb{R}^n$ be a closed semialgebraic set, and let $f : S \to \mathbb{R}$ be a semialgebraic function. We assume that f can be extended to a function $F : \mathbb{R}^n \to \mathbb{R}$ that is of class $\mathcal{C}^p$ for some $p \in N$. A question posed by Bierstone and Milman asks whether it is feasible for F to also be semialgebraic. In the continuous case, an affirmative answer is provided in the work [**15**] of Delfs and Knebusch. Additionally, in [**4**] Aschenbrenner and Thamrongthanyalak addressed this question affirmatively for the case $p = 1$. Recently, Fefferman and Luli [**17**] resolved the scenario for $n = 2$ for any $p \geq 1$. Bierstone, Campesato, and Milman presented in [**7**] a weaker solution to this issue, implying only a loss of differentiability via a function $t : N \to N$ such that $t(p) \geq p$ for all $p \geq 1$. Thus, if the semialgebraic function $f : S \to \mathbb{R}$ extends to a function $F : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^{t(p)}$ for some $p \in N$, then f can be extended to a semialgebraic function $F' : \mathbb{R}^n \to \mathbb{R}$ that is $\mathcal{C}^p$ differentiable.

A well-known result in Nash geometry [**10**, Prop. 8.1.8] states that a function on a Nash manifold $M \subset \mathbb{R}^n$ is smooth and semialgebraic (as defined by Nash [**10**, Def. 2.9.3]) if and only if it can be represented as an analytic algebraic function on M. This leads to the natural question of which semialgebraic subsets $S \subset \mathbb{R}^n$ might share a similar characteristic. For an integer $p \geq 1$, we say a function on S is of class S^p if it is semialgebraic and, for every point $x \in S$, there exists an open neighborhood $U \subset \mathbb{R}^n$ of x and a C^p function $F : U \to \mathbb{R}$ such that when we restrict F to $S \cap U$, it agrees with f on that intersection. A function $f : S \to \mathbb{R}$ is $S^{(\infty)}$ if it is indeed an S^p function for every integer $p \geq 1$.

Problem 1.12. — *Which semialgebraic sets $S \subset \mathbb{R}^n$ satisfy that each $S^{(\infty)}$ function on S is the restriction to S of a Nash function defined on an open semialgebraic neighborhood of S in $\mathbb{R}^n$?*

In §2.3 we recall the *set $T(S)$ of 'tails'* of a semialgebraic set $S \subset \mathbb{R}^n$, whereas *coherent semialgebraic sets S* are those semialgebraic sets with $T(S) = \varnothing$. Problem 1.12 is solved by our third main result (Theorem 4.6), which involves again coherence and we state here as follows.

Theorem 1.13 ([**22**, Thm. 1.10]). — *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then every $S^{(\infty)}$ function on S is the restriction to S of a Nash function defined on an open semialgebraic neighborhood of S in $\mathbb{R}^n$ if and only if S is coherent or equivalently if $T(S) = \varnothing$.*

Structure of the article. — The article is organized as follows. In Section 2 we analyze the concepts of coherence and 'tails' in the contexts relevant for this article, namely: real analytic, Nash and semialgebraic sets. In Section 3 we recall and explain with care several known examples of pure dimensional real algebraic set that are neither coherent C-analytic sets nor coherent Nash sets. In Section 4.2 we survey the main statements of [**22**, §6-7] and in Section 5 we propose an original proof to Theorem 5.1 already treated before in [**36, 37**].

2. Coherence and tails of real analytic, Nash and semialgebraic sets

Given a function f or a set $\mathfrak{F}$ of functions on a set X, we denote their respective zero sets with $\mathcal{Z}(f) = \{x \in X : f(x) = 0\}$ and $\mathcal{Z}(\mathfrak{F}) = \bigcap_{f\in\mathfrak{F}} \mathcal{Z}(f)$. If X is a topological space and we have function germs $f_{1,x}, \dots, f_{r,x}$ at a point $x \in X$, represented by functions $f_1, \dots, f_r$ that are defined on a shared open neighborhood $V^x \subset X$ around x, we define $\mathcal{Z}(f_{1,x}, \dots, f_{r,x})$ as the set germ $\{y \in V^x : f_1(y) = 0, \dots, f_r(y) = 0\}_x$ at x. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and we will equip $\mathbb{K}^n$ with the Euclidean topology.

2.1. Coherence and 'tails' in the real analytic setting. — We compare the concept of coherence for complex analytic sets and real analytic sets and analyze its main consequences. We explore the notion of coherence in both complex analytic sets and real analytic sets, highlighting its key implications. Let $\Omega \subset \mathbb{K}^n$ denote an open set, and let $X \subset \Omega$ represent a closed set. Recall that X is an *analytic subset of* Ω if, for every point $x \in X$, there are analytic function germs $f_{1,x}, \dots, f_{r,x} \in O_{\mathbb{K}^n,x}$ such that $X_x = \mathcal{Z}(f_{1,x}, \dots, f_{r,x})$.

2.1.1. *Coherence of complex analytic sets.*— Let us consider the case where $\mathbb{K}=\mathbb{C}$ and X is a complex analytic subset within an open set $\Omega\subset\mathbb{C}^n$. According to Oka's coherence theorems [**35**, Ch.IV], every complex analytic set X within Ω is *coherent*. When Ω is a Stein space, the coherence of X leads to the following implications:

(i) X is the zero set of a finite number of analytic functions defined on Ω (refer to [**11**, Footnote (9) to the proof of (b) $\Longrightarrow$ (c) of Prop. 15]).

(ii) The ideal $\mathcal{I}(X)$, comprising analytic functions on Ω that vanish on X, generates the ideal $\mathcal{J}_{X,x}$ of analytic germs at each point $x\in X$ that are identically zero on X_x (Cartan's Theorem A).

(iii) Analytic functions defined on X (which are essentially the global sections of the sheaf $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}_X$) can be viewed as restrictions of analytic functions on Ω (Cartan's Theorem B).

It is important to highlight that the conditions for these properties to hold are global: Ω being a Stein space indicates that it possesses a structure conducive to the existence of a sufficient number of analytic functions.

2.1.2. *Coherence of real analytic sets.* — We refer readers to [**25, 26**] for an overview of the general theory surrounding coherent analytic sheaves. Using Whitney's analytic immersion theorem alongside analytic tubular neighborhoods for analytic manifolds, we can simplify our setting by assuming that $\Omega=\mathbb{R}^n$ and X is a C-analytic subset of $\mathbb{R}^n$. This means that X is the zero set of a finite number of analytic functions defined on $\mathbb{R}^n$. We define the ideal $\mathcal{I}(X):=\{f\in\mathcal{O}(\mathbb{R}^n): X\subset\mathcal{Z}(f)\}$, which comprises those analytic functions that vanish identically on X. We also consider two sheaves of $\mathcal{O}_{\mathbb{R}^n}$-ideals on $\mathbb{R}^n$: $\mathcal{J}_X$ and $\mathcal{I}_X$. We have:

$$\begin{cases}\mathcal{J}_{X,x}:=\{f_x\in\mathcal{O}_{\mathbb{R}^n,x}: X_x\subset\mathcal{Z}(f_x)\},\\ \mathcal{I}_{X,x}:=\mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x}.\end{cases}$$

In general, it holds that $\mathcal{I}_{X,x}\subset\mathcal{J}_{X,x}$ for every point $x\in X$. A real analytic set $X\subset\mathbb{R}^n$ is deemed *coherent* if the sheaf of $\mathcal{O}_{\mathbb{R}^n}$-ideals $\mathcal{J}_X$ is coherent. According to [**11**, Prop. 4], the sheaf of rings $\mathcal{O}_{\mathbb{R}^n}$ is coherent. This implies that a sheaf of $\mathcal{O}_{\mathbb{R}^n}$-ideals $\mathcal{F}$ is coherent when for each point $x\in\mathbb{R}^n$, there exists an open neighborhood $U^x\subset\mathbb{R}^n$ and finitely many analytic functions $f_1,\dots,f_r\in\mathcal{O}(U^x)$ such that $f_{1,y},\dots,f_{r,y}$ generate $\mathcal{F}_y$ for all $y\in U^x$, see [**27**, IV.B.Prop. 8]. Furthermore, the renowned Cartan's Theorems A and B ([**11**, Thm. 3]) articulate the local-global behavior of coherent sheaves $\mathcal{F}$ of $\mathcal{O}_{\mathbb{R}^n}$-modules:

(A) *The stalks of a coherent sheaf $\mathcal{F}$ are spanned by the global sections.*

(B) *Each p-cohomology group of a coherent sheaf $\mathcal{F}$ is trivial for each $p>0$.*

Remark 2.1. — The previous results regarding real analytic sets are derived from Cartan's Theorems A and B related to Stein spaces, along with an important fact: *each open set $U\subset\mathbb{R}^n$ possesses a basis of 'very nice' open neighborhoods in $\mathbb{C}^n$. These neighborhoods are: Stein open sets that remain unchanged under conjugation in $\mathbb{C}^n$,* *and have U as a deformation retract* [**11**, §1]. A key distinction between complex and real analytic sets lies in the following fact: while complex analytic sets are always coherent and they satisfy Cartan's Theorems A and B when they are in addition Stein spaces, real analytic sets always admit the mentioned basis of Stein open neighborhoods in $\mathbb{C}^n$ and they satisfy Cartan's Theorems A and B when they are in addition coherent.

Based on the results from [**11**, Prop. 14&15] and Cartan's Theorem A, we can outline an important characterization of coherent real analytic sets.

Corollary 2.2 **([22, Thm. 1.4]). —** *A real analytic set $X \subset \mathbb{R}^n$ is coherent if and only if it is a C-analytic set and $\mathcal{I}_{X,x} = \mathcal{J}_{X,x}$ for each $x \in X$.*

2.1.3. *Complexification of a C-analytic set.* — We recall the construction of the complexification of X. We start with the coherent sheaves of $\mathcal{O}_{\mathbb{C}^n}$-ideals, represented as $\mathcal{I}_X \otimes_{\mathbb{R}} \mathbb{C}$ on $\mathbb{R}^n$. According to [**11**, Prop. 2], we can find an open neighborhood $\Omega \subset \mathbb{C}^n$ of $\mathbb{R}^n$, along with a coherent sheaf F of $\mathcal{O}_{\mathbb{C}^n}$-ideals defined on Ω such that $F|_{\mathbb{R}^n} = \mathcal{I}_X \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{I}(X)\mathcal{O}_{\mathbb{C}^n}|_{\mathbb{R}^n}$. By adjusting Ω if needed, we can ensure that the support of F is an invariant complex analytic set, denoted as $\widetilde{X} \subset \Omega$. This set retains X as its fixed part with respect to conjugation in $\mathbb{C}^n$ and we refer to $\widetilde{X}$ as a complexification of X. By the results outlined in [**11**] (and also seen in [**2**, Thm. 1.2.12]), any two complexifications of X will coincide within an open neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$. Furthermore, by shrinking Ω, we can always suppose that it remains an invariant Stein open subset of $\mathbb{C}^n$ and that $\widetilde{X}$ is an invariant Stein complex analytic set.

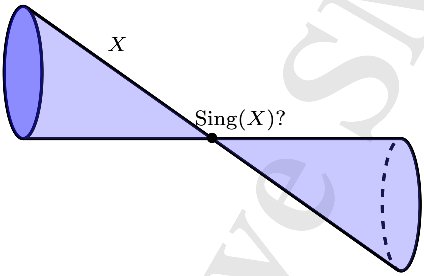

FIGURE 2. 'Real vision' of a C-analytic set X

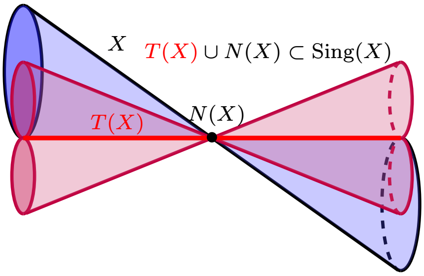

FIGURE 3. 'Imaginary vision' of X inside $\widetilde{X}$

2.1.4. *Set of 'tails' of a C-analytic set.* — Let $X \subset \mathbb{R}^n$ be a C-analytic set and let us denote $\mathrm{Reg}(X)$ the set of points $x \in X$ such that $\mathcal{O}_{\mathbb{R}^n,x}/\mathcal{I}_{X,x}$ is a regular local ring and denote $\mathrm{Sing}(X) := X \setminus \mathrm{Reg}(X)$. The *set of 'tails' of a C-analytic set* $X \subset \mathbb{R}^n$ is $T(X) := \{x \in X : \mathcal{J}_{X,x} \neq \mathcal{I}_{X,x}\}$, that is, the set of points of X where $\mathcal{J}_X$ and $\mathcal{I}_X$ differ. We define $N(X)$ as the set of points $x \in X$ such that $\mathcal{J}_X$ is not coherent at x.

A *C-semianalytic subset* S of $\mathbb{R}^n$ is a locally finite union in $\mathbb{R}^n$ of *basic C-semianalytic sets*, that is, subsets of $\mathbb{R}^n$ of the type $\{f=0, g_1>0,\dots,g_r>0\}$ where $r\geq 1$ and $f,g_i\in\mathcal{O}(\mathbb{R}^n)$ (see [**1**, §3]). In the following Cl denotes the closure with respect to the Euclidean topology. We next collect some properties of the previous sets:

(0) Both $N(X)$ and $T(X)$ are C-semianalytic sets contained in $\operatorname{Sing}(X)$, see [**1**, §5. 1] and [**22**, Lem. 3.6].

(1) If $\widetilde{X}$ is a complexification of X and $x\in X$, then $x\in T(X)$ if and only if there exists an irreducible component T_x of $\widetilde{X}_x$ such that $\dim_{\mathbb{R}}(T_x\cap\mathbb{R}^n_x)<\dim_{\mathbb{C}}(T_x)$ (Figures 2 and 3), see [**22**, Lem. 3.5, Cor.4.8].

(2) $\operatorname{Cl}(T(X))=\operatorname{Cl}(T(X)\setminus N(X))=T(X)\cup N(X)\subset\operatorname{Sing}(X)$. Thus,

$$\dim(N(X)_x)<\dim(T(X)_x)$$

for each $x\in N(X)$, see [**22**, Thm. 4.1].

(3) X is coherent if and only if $N(X)=\varnothing$ or equivalently if $T(X)=\varnothing$, see [**22**, Cor. 4.3].

(4) If S is a connected component of $\operatorname{Cl}(T(X))$, then $S\cap N(X)\neq\varnothing$, see [**22**, Cor. 4.4].

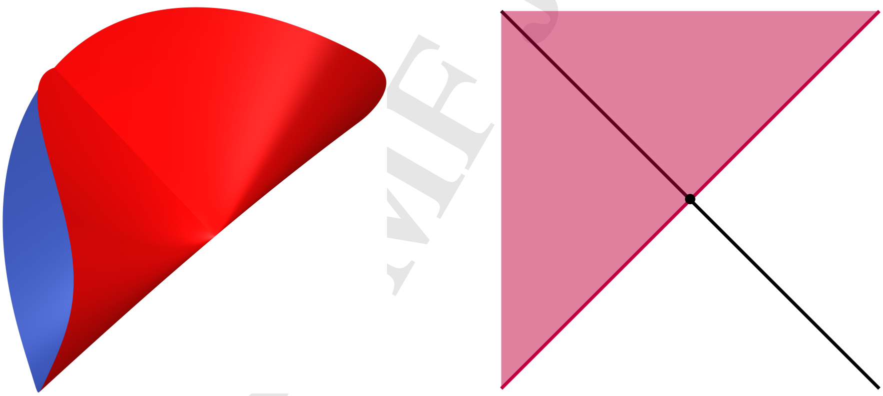

FIGURE 4. $X: y^4z-x^4-x^2y^2=0$

Example 2.3 **([22, Ex. 4.2]).** — If $X\subset\mathbb{R}^n$ is a C-analytic set, $N(X)$ and $T(X)$ may have non-empty intersection, even if X is C-irreducible.

Define $X:=\{y^4z-x^4-x^2y^2=0\}$, which is a slight modification of Whitney's umbrella (see (2.1)). Let us show: *X is a C-irreducible C-analytic set, $N(X)=\{(0,0,0)\}$ and $T(X)=\{x=0,y=0\}$.*

Let $\widetilde{X}:=\{y^4z-x^4-x^2y^2=0\}\subset\mathbb{C}^3$ be the (algebraic) complexification and consider $s:=\frac{x}{y}$, which satisfies the integral equation $s^4+s^2-z=0$. This element provides the normalization

$$\varphi:\mathbb{C}^2\to\widetilde{X},\ (s,y)\mapsto(sy,y,s^2+s^4). \tag{2.1}$$

We have $\varphi^{-1}(\{x=0,y=0\}) = \{(s_1+s_2\sqrt{-1},0)\in\mathbb{C}^2:\ s_1s_2(2s_1^2-2s2^2+1)=0\}$, which is a connected set, and $\operatorname{Cl}(\varphi^{-1}(\{x=0,y=0\})\setminus\mathbb{R}^2)\cap\mathbb{R}^2=\{(0,0)\}$. Consequently, $\varphi^{-1}(\{x=0,y=0\})$ is contained in the normalization of each open neighborhood W of X in the complexification $\widetilde{X}$ and $N(X)=\{\varphi(0,0)\}=\{(0,0,0)\}$. As $\varphi^{-1}(X)=\mathbb{R}^2\cup\varphi^{-1}(\{x=0,y=0\})$ is connected, X is by [**18**, Thm. 1.2] a C-irreducible C-analytic set. For each $a\in\mathbb{R}$ the fiber

$$\varphi^{-1}(\{(0,0,a)\})=\Big\{\Big(\pm\frac{1}{2}\Big(\sqrt{-2\pm2\sqrt{1+4a}}\Big),0\Big)\Big\}$$

contains exactly: two double imaginary points if $a=-\frac{1}{4}$, a double real point and two simple conjugated imaginary points if $a=0$, two simple real points and two conjugated imaginary points if $a>0$ and two (different) couples of conjugated imaginary points if $a<0$ and $a\neq-\frac{1}{4}$. Thus, the (complex) analytic germ $\widetilde{X}_{(0,0,a)}$ has in all cases 2-dimensional (complex) irreducible components that meet $\mathbb{R}^3_{(0,0,a)}$ in dimension ≤ 1. By §2.1.4(1) we deduce $(0,0,a)\in T(X)$ for each $a\in\mathbb{R}$, as required.

2.2. Coherence and 'tails' in the Nash setting. — In the Introduction, we have already introduced the concepts of semialgebraic sets, Nash functions, and Nash sets. A semialgebraic set $M\subset\mathbb{R}^n$ is termed an *affine Nash manifold* when it is additionally a smooth submanifold of an open semialgebraic subset of $\mathbb{R}^n$. According to [**10**, Prop. 8.1.8 & Cor.9.3.10], Nash manifolds are precisely the analytic submanifolds of an open semialgebraic subset of $\mathbb{R}^n$ that are themselves semialgebraic sets. Let us consider an open semialgebraic set $U\subset\mathbb{R}^n$ and denote the ring of Nash functions on U as $N(U)$. For a semialgebraic set $S\subset\mathbb{R}^n$, we define a function $f:S\to\mathbb{R}$ as a *Nash function* if there exists an open semialgebraic neighborhood $U\subset\mathbb{R}^n$ of S along with a Nash extension $F:U\to\mathbb{R}$ of f. We will denote the ring of Nash functions on S as $N(S)$. It has been proved in [**21**, Prop. 2.8(ii)] that Nash subsets of an open semialgebraic set $U\subset\mathbb{R}^n$ coincide with those C-analytic subsets of U that are also semialgebraic subsets of U. Therefore, the Nash category effectively merges the global analytic and semialgebraic categories.

2.2.1. *Coherence of Nash sets.* — Let X be a Nash subset of an open semialgebraic set $\Omega\subset\mathbb{R}^n$. Utilizing Mostowski's Lemma [**33**, Lem. 6] we can assume $\Omega=\mathbb{R}^n$. We obtain Nash functions on the Nash subset X of $\mathbb{R}^n$ by looking at global sections of a certain sheaf [**6**, §2.B]. Define the sheaf of $N_{\mathbb{R}^n}$-ideals as $\mathcal{J}_X^\bullet$ where $\mathcal{J}^\bullet_{X,x}:=\{f_x\in N_{\mathbb{R}^n,x}:X_x\subset\mathcal{Z}(f_x)\}$, which is supported on X. This consists of the germs of Nash functions in $\mathbb{R}^n_x$ that vanish identically on X_x for each $x\in\mathbb{R}^n$. Similar to the analytic case, the sheaf $N_{\mathbb{R}^n}/\mathcal{J}^\bullet_X$ can exhibit unfavorable local-global behavior, and the ring of its global sections may not align with the ring of restrictions to X of Nash functions defined on $\mathbb{R}^n$.

Let us denote the (finitely generated) ideal consisting of Nash functions vanishing identically on X as $\mathcal{I}^\bullet(X)=\{f\in N(\mathbb{R}^n):X\subset\mathcal{Z}(f)\}$. The sheaf $\mathcal{I}^\bullet_X:=\mathcal{I}^\bullet(X)N_{\mathbb{R}^n}$ plays a key role, characterized by strong properties that exceed the conditions of Nash coherence. According to [**21**, Prop. 2.8(i)], we find that $\mathcal{I}_X=\mathcal{I}^\bullet_X\mathcal{O}_{\mathbb{R}^n}$, and based on

[22, Lem. 2.6], X is coherent as a Nash set if and only if $\mathcal{J}_X^\bullet = \mathcal{I}_X^\bullet$. Additionally, [6, §2.B] reveals a natural isomorphism between the ring $N(X)$ of Nash functions on X and the global sections of the sheaf $N_{\mathbb{R}^n}/\mathcal{I}_X^\bullet$. By referring to [14, Thm. 1, Cor.2] and [13, Prop. 0.6, Prop.0.7], we arrive at a notable result in the style of Cartan.

Theorem 2.4. — *Let $X \subset \mathbb{R}^n$ be a coherent Nash set. Then:*

(A) *$\mathcal{I}^\bullet(X)$ generates the ideal $\mathcal{J}_{X,x}^\bullet$ for each $x \in X$.*

(B) *The global sections of the sheaf $N_{\mathbb{R}^n}/\mathcal{J}_X^\bullet$ are restrictions to X of Nash functions on $\mathbb{R}^n$.*

Remark 2.5 **(Set of 'tails' of a Nash set).** — Let $X \subset \mathbb{R}^n$ be a Nash set. As the inclusion homomorphism $N_{\mathbb{R}^n,x} \hookrightarrow O_{\mathbb{R}^n,x}$ is faithfully flat, $\mathcal{J}_{X,x} = \mathcal{J}_{X,x}^\bullet O_{\mathbb{R}^n,x}$ and $\mathcal{I}_{X,x} = \mathcal{I}_{X,x}^\bullet O_{\mathbb{R}^n,x}$ for each $x \in X$, we deduce that:

(1) $N_{\mathbb{R}^n,x}/\mathcal{I}_{X,x}^\bullet$ is a regular local ring if and only if $O_{\mathbb{R}^n,x}/\mathcal{I}_{X,x}$ is a regular local ring. This means that the set $\operatorname{Reg}(X)$ of regular points as a C-analytic set coincide with the set $\operatorname{Reg}^\bullet(X)$ of regular points as a Nash set. Consequently, the same happens with the sets $\operatorname{Sing}(X)$ and $\operatorname{Sing}^\bullet(X)$ of singular points of X.

(2) $\mathcal{J}_{X,x}^\bullet \neq \mathcal{I}_{X,x}^\bullet$ if and only if $\mathcal{J}_{X,x} \neq \mathcal{I}_{X,x}$. This means that the set of 'tails' $T(X) = \{x \in X : \mathcal{J}_{X,x} \neq \mathcal{I}_{X,x}\}$ of X understood as a C-analytic set coincides with the set of 'tails' $T^\bullet(X) := \{x \in X : \mathcal{J}_{X,x}^\bullet \neq \mathcal{I}_{X,x}^\bullet\}$ of X understood as a Nash set.

2.3. Coherence and 'tails' in the semialgebraic setting. — Let $S \subset \mathbb{R}^n$ be a semialgebraic set. According to [20, §(4. 10)], there exists an open semialgebraic set $U \subset \mathbb{R}^n$ such that the smallest Nash subset X of U containing S has the following property: *for every open semialgebraic neighborhood $V \subset U$ of S, we can find an open semialgebraic neighborhood $W \subset V$ of S where the smallest Nash subset X_V of V that includes S satisfies $X_V \cap W = X \cap W$.* Let $\mathcal{J}_S^\bullet$ denote the sheaf of $N_{\mathbb{R}^n}$-ideals defined by $\mathcal{J}_{S,x}^\bullet := \{f_x \in N_{\mathbb{R}^n,x} : S_x \subset \mathcal{Z}(f_x)\}$. For a given $x \in S$ this ideal consists of germs of Nash functions on $\mathbb{R}^n_x$ that vanish identically on S_x. As X is the smallest Nash subset of U that contains S, we conclude that $\mathcal{I}_{X,x}^\bullet \subset \mathcal{J}_{S,x}^\bullet$ for each $x \in S$.

Definitions 2.6 **([22, Def. 7.7]).** — The set $T(S) := \{x \in S : \mathcal{I}_{X,x}^\bullet \neq \mathcal{J}_{S,x}^\bullet\}$ is *the set of 'tails' of S*. We say that the semialgebraic set S is *coherent* if $T(S) = \varnothing$.

Examples 2.7. — Two insightful examples are the semialgebraic set

$$S_1 := \{x^2 - zy^2 = 0, z \geq 0\}$$

(Whitney's umbrella with the handle erased [22, §7.E. 2]), which has $T(S_1) = \varnothing$, and $S_2 := \{z(x^2+y^2) - x^3 = 0, x^2+y^2 \neq 0\} \cup \{(0,0)\}$ (Cartan's umbrella with the handle erased), which has $T(S_2) = \varnothing$.

3. Pure dimensional non-coherent examples

In Real Geometry, a general idea was that non-coherence occurs when the irreducible components of objects lack pure dimensionality. Moreover, non-coherence can stem from hidden 'tails' within these irreducible components. We will explore several pure dimensional examples that illustrate various scenarios regarding $\operatorname{Sing}(X)$, $T(X)$, and $N(X)$. In the realm of real analytic sets, it is crucial to differentiate between regular points and smooth points, which is not a concern in complex analysis, because both concepts are equivalent in such setting. Let $X \subset \mathbb{R}^n$ be a C-analytic set (or a Nash set) and take any point $x \in X$. We define x to be a *smooth point of* X if the analytic germ X_x equals the germ at x of a real analytic manifold. We denote the set of smooth points as $\mathtt{Smooth(X)}$ and the set of non-smooth points as $\mathtt{Non\text{-}Smooth(X)} := \mathtt{X} \setminus \mathtt{Smooth(X)}$. Observe that $\mathtt{Smooth(X)}$ *is always a subset of* $X \setminus N(X)$, since the germs at any point of a real analytic manifold maintain coherence. According to the Jacobian criterion, we know that $\operatorname{Reg}(X) \subset \mathtt{Smooth(X)}$, although this inclusion may be strict, as proved by Examples 3.1, 3.4, and 3.5 below. However, *if* X *is coherent, the equality* $\operatorname{Reg}(X) = \mathtt{Smooth(X)}$ *holds* [**22**, §3.E. 1].

Example 3.1. — A first family of examples is $X_{k,\ell} := \{z^{2k+1} - x^\ell y^{2k+1} = 0\}$ for $k, \ell \geq 1$ such that $2k+1$ and ℓ are coprime (Figures 5 and 6). The case $k = 1$ and $\ell = 2$ appears in [**29**] and [**9**, Ex. 2.1]. It holds $\operatorname{Sing}(X) = \{x^{\ell-1}y = 0, z = 0\}$ and $z^{2k+1} - x^\ell y^{2k+1} = \prod_{m=0}^{2k}(z - w^m x^{\ell/(2k+1)} y)$, where $w = \exp(\frac{2\pi\sqrt{-1}}{(2k+1)})$, so $X = \{z - x^{\ell/(2k+1)} y = 0\}$. Thus, X is semialgebraically homeomorphic to $\mathbb{R}^2$ via the semialgebraic homeomorphism $\varphi : \mathbb{R}^2 \to X,\ (x,y) \mapsto (x, y, x^{\ell/(2k+1)} y)$. Let $\widetilde{X} \subset \mathbb{C}^3$ be a complexification of X. At the points p of the real line $\{x = 0, z = 0\}$ the complex analytic germ $\widetilde{X}_p$ is irreducible and the analytic germ $X_p = \widetilde{X}_p \cap \mathbb{R}^3_p$ is 2-dimensional. By §2.1.4(1) we have $\{x^{\ell-1} = 0\} \subset \operatorname{Sing}(X) \setminus T(X)$. At the points p of the punctured real line $\{y = 0\} \setminus \{(0,0,0)\}$ the complex analytic germ $\widetilde{X}_p$ has $2k+1$ non-singular irreducible components and $2k$ of them meet $\mathbb{R}^3_p$ only at the line germ $\{y = 0, z = 0\}_p$. Thus, $\{y = 0, z = 0\} \setminus \{(0,0,0)\} \subset \mathtt{Smooth(X)} \setminus \operatorname{Reg}(\mathtt{X}) \subset \mathtt{T(X)} \setminus \mathtt{N(X)}$. As $T(X) \neq \varnothing$, we have $N(X) \subset \{(0,0,0)\}$ is non-empty, so $N(X) = \{(0,0,0)\}$ and $T(X) = \{y = 0, z = 0\} \setminus N(X)$.

Example 3.2. — Another relevant example from [**24**, Ex. 1.9] and [**2**, §5. 3.1, p.197] is $X := \{z(x+y)(x^2+y^2) - x^4 = 0\} \subset \mathbb{R}^3$ (Figure 7), which has

$$\operatorname{Sing}(X) = \{x = 0, y = 0\}.$$

Observe that X is the cone over the curve

$$Z := \{(x+y)(x^2+y^2) - x^4 = 0\} = X \cap \{z = 1\}.$$

Consider the polynomial map

$$\varphi : \mathbb{R}^2 \to X,\ (s,t) \mapsto (s(s^3 + s^2 t + st^2 + t^3), t(s^3 + s^2 t + st^2 + t^3), s^4),$$

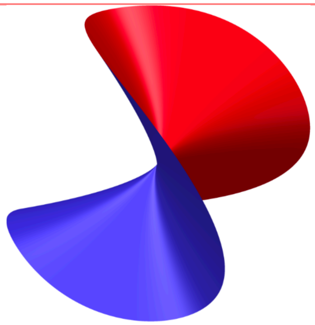

FIGURE 5. $X : z^3 - xy^3 = 0$

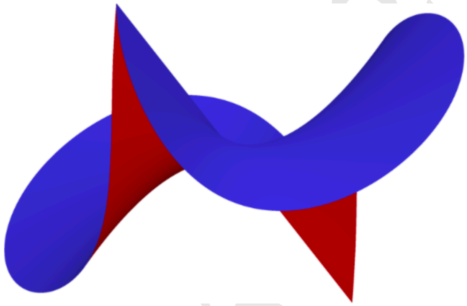

FIGURE 6. $X : z^3 - x^2y^3 = 0$

whose restriction to $\mathbb{R}^2 \setminus \{(0,0)\}$ is a Nash diffeomorphism onto its image $X \setminus \{(0,0,0)\}$ (use the corresponding Jacobian matrices). Thus, $X \setminus \{(0,0,0)\} \subset \mathtt{Smooth(X)}$, so $N(X) \subset \{(0,0,0)\}$. Let $\widetilde{X} \subset \mathbb{C}^3$ be a complexification of X. As the complex algebraic curve $Y := \{(x+y)(x^2+y^2) - x^4 = 0\} \subset \mathbb{C}^3$ has three non-singular branches at the origin, the complex analytic germ $\widetilde{X}_p$ has three non-singular irreducible components at the points p of the punctured real line $\{x = 0, y = 0\} \setminus \{(0,0,0)\}$ and exactly two of them meet $\mathbb{R}^3_p$ only at the line germ $\{x = 0, y = 0\}_p$. Thus, $\{x = 0, y = 0\} \setminus \{(0,0,0)\} \subset \mathtt{Smooth(X)} \setminus \mathrm{Reg}(\mathtt{X}) \subset \mathtt{T(X)} \setminus \mathtt{N(X)}$. As $T(X) \neq \varnothing$, we have $N(X) \subset \{(0,0,0)\}$ is non-empty, so $N(X) = \{(0,0,0)\}$ and $T(X) = \{x = 0, y = 0\} \setminus N(X)$.

***Example 3.3* ([22, Ex. 3.9]). —** If X is a C-analytic set (resp. a Nash set), the equality $\mathrm{Reg}(X) = \mathtt{Smooth(X)}$ does not guarantee that X is coherent. Let us modify Example 3.2. Consider $X := \{z^2(x+y)^2(x^2+y^2) - x^6 = 0\} \subset \mathbb{R}^3$ (Figure 8), which has $\mathrm{Sing}(X) = \{x = 0, yz = 0\}$. Observe that X is the cone over the curve $Z := \{(x+y)^2(x^2+y^2) - x^6 = 0\} = X \cap \{z = 1\}$.

Let $\widetilde{X} \subset \mathbb{C}^3$ be a complexification of X. At the points p of the punctured real line $\{x = 0, z = 0\} \setminus \{(0,0,0)\}$ the analytic germ $\widetilde{X}_p$ has two non-singular 2-dimensional (complex) irreducible components given by $\{z(x+y)\sqrt{(x^2+y^2)} - x^3 = 0\}_p$ and $\{z(x+y)\sqrt{(x^2+y^2)} + x^3\}_p$, which meet $\mathbb{R}^3_p$ into two non-singular 2-dimensional irreducible components. Thus, $\{x = 0, z = 0\} \setminus \{(0,0,0)\} \subset \mathrm{Sing}(X) \setminus T(X)$. As the complex algebraic curve $Y := \{(x+y)^2(x^2+y^2) - x^6 = 0\} \subset \mathbb{C}^3$ has four non-singular branches at the origin, the complex analytic germ $\widetilde{X}_p$ has four non-singular irreducible components at the points p of the punctured real line $\{x = 0, y = 0\} \setminus \{(0,0,0)\}$ and exactly two of them meet $\mathbb{R}^3_p$ only at the line germ $\{x = 0, y = 0\}_p$. Thus, $\{x = 0, y = 0\} \setminus \{(0,0,0)\} \subset T(X) \setminus N(X)$. As $T(X) \neq \varnothing$, we have $N(X) \subset \{(0,0,0)\}$ is non-empty, so $N(X) = \{(0,0,0)\}$ and $T(X) = \{x = 0, y = 0\} \setminus N(X)$.

We borrow the following example from [**8**, Rem. 7.3], [**9**, Ex. 2.2] and [**19**, Ex. 2.1].

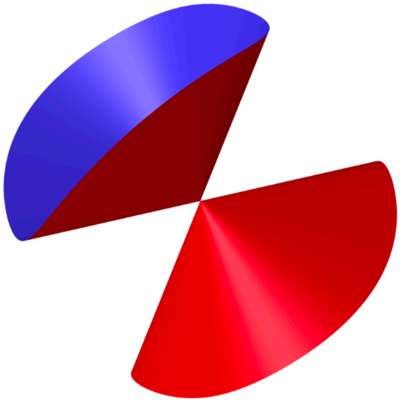

FIGURE 7. $X : z(x+y)(x^2+y^2) - x^4 = 0$

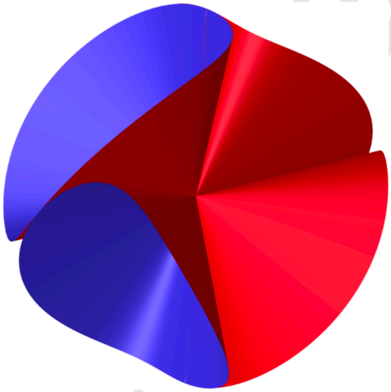

FIGURE 8. $X : z^2(x+y)^2(x^2+y^2) - x^6 = 0$

***Example 3.4* (Birdie non-coherent singularity). —** Consider the algebraic set $X := \{(x^2+zy^2)x - y^4 = 0\} \subset \mathbb{R}^3$ (Figure 9). The set of regular points of X is $X \setminus \{x = 0, y = 0\}$, whereas: *The set of smooth points of X is $X \setminus \{x = 0, y = 0, z \leq 0\}$.*

To prove that the points of the open half-line $\{x = 0, y = 0, z < 0\}$ are non-smooth we proceed by contradiction. Pick a point $p := (0, 0, -a^2) \in \{x = 0, y = 0, z < 0\}$ and assume that it is smooth. As the line $\{x = 0, y = 0\} \subset X$, the vector $(0, 0, 1)$ would be tangent to X at p, so the plane $z = -a^2$ would be transversal to X at p. Thus, the intersection $X \cap \{z = -a^2\}$ should be a curve that is smooth at p, but this is a contradiction, because such curve $\{(x^2 - (ay)^2)x - y^4 = 0, z = -a^2\}$ has three tangent lines at p, which are those lines of equations $\{x - ay = 0\}$, $\{x + ay = 0\}$ and $\{x = 0\}$ inside the plane $\{z = -a^2\}$. The origin cannot be a smooth point of X, because the set of smooth points of X is an open subset of X. Consequently, the set of non-smooth points of X contains the closed half-line $\{x = 0, y = 0, z \leq 0\}$.

Let us prove: *The points of the open half-line $\{x = 0, y = 0, z > 0\}$ are smooth* (and non-regular, so they are contained in $T(X)$ and $N(X) \neq \varnothing$). To that end, observe that the map

$$\varphi : \{(s, u) \in \mathbb{R}^2 : \ u > 0\} \to \mathbb{R}^3, \ (s, u) \mapsto ((s^2 + u)s^2, (s^2 + u)s, u)$$

is an analytic embedding whose image is $X \cap \{z > 0\}$. Observe (computing the Jacobian matrix of φ) that φ is a local analytic diffeomorphism onto its image. The analytic map φ extends to a holomorphic map

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^3, \ (v, w) \to ((v^2 + w)v^2, (v^2 + w)v, w),$$

which provides the normalization of the complexification $\widetilde{X} := \{(x^2 + zy^2)x - y^4 = 0\}$ of X. To that end, observe $v = \frac{x}{y}$ satisfies the integral equation ${}^3 + \mathsf{z} - \mathsf{y} = 0$ (and in addition $z = w$, $y = v^3 + zv$ and $x = vy$).

We next prove that $N(X)=\{(0,0,0)\}$, showing that $\varnothing\neq N(X)\subset\{(0,0,0)\}$. As $\operatorname{Reg}(X)\subsetneq \mathtt{Smooth(X)}=\mathtt{X}\setminus\{\mathtt{x}=\mathtt{0},\mathtt{y}=\mathtt{0},\mathtt{z}\leq\mathtt{0}\}\subset\mathtt{X}\setminus\mathtt{N(X)}$, let us check: *The points of the open half-line $\{x=0,y=0,z<0\}$ provide coherent analytic germs.* Pick a point $(0,0,t)$ with $t<0$. Then $\Phi^{-1}(\{(0,0,t)\})=\{(0,t),(\sqrt{-t},t),(-\sqrt{-t},t)\}$ and

$$\begin{aligned}
X_{(0,0,t)}&=\varphi(\mathbb{R}^2_{(0,t)})\cup\varphi(\mathbb{R}^2_{(\sqrt{-t},t)})\cup\varphi(\mathbb{R}^2_{(-\sqrt{-t},t)}),\\
\widetilde{X}_{(0,0,t)}&=\Phi(\mathbb{C}^2_{(0,t)})\cup\Phi(\mathbb{C}^2_{(\sqrt{-t},t)})\cup\Phi(\mathbb{C}^2_{(-\sqrt{-t},t)}).
\end{aligned}$$

Let $g\in\mathcal{J}_{X,(0,0,t)}$ and let $G\in O_{\mathbb{C}^n,(0,0,t)}$ be its natural holomorphic extension. As g vanishes identically on $\varphi(\mathbb{R}^2_{(0,t)})$, $\varphi(\mathbb{R}^2_{(\sqrt{-t},t)})$ and $\varphi(\mathbb{R}^2_{(-\sqrt{-t},t)})$, we deduce that G vanishes identically on $\Phi(\mathbb{C}^2_{(0,t)})$, $\Phi(\mathbb{C}^2_{(\sqrt{-t},t)})$ and $\Phi(\mathbb{C}^2_{(-\sqrt{-t},t)})$, because the latters are the complexifications of the analytic germs $\varphi(\mathbb{R}^2_{(0,t)})$, $\varphi(\mathbb{R}^2_{(\sqrt{-t},t)})$ and $\varphi(\mathbb{R}^2_{(-\sqrt{-t},t)})$. Thus, $G\in\mathcal{J}_{\widetilde{X},(0,0,t)}=((x^2+zy^2)x-y^4)O_{\mathbb{C}^n,(0,0,t)}$, so $(x^2+zy^2)x-y^4$ divides g and $\mathcal{J}_{X,(0,0,t)}=((x^2+zy^2)x-y^4)O_{\mathbb{C}^n,(0,0,t)}$. Consequently, $X_{(0,0,t)}$ is a coherent analytic germ for each $t<0$.

Thus, $T(X)=\operatorname{Sing}(X)\cap\{z<0\}$ (because the analytic germ $X_{(0,0,0)}$ is irreducible) and $N(X)=\{(0,0,0)\}$. The real parameter $s=\frac{x}{y}$ is a solution of the integral equation $^3+\mathtt{z}-\mathtt{y}=\mathtt{0}$. Using the solution of the general equation of degree 3 we find that

$$s=f(z,y):=\sqrt[3]{\frac{y}{2}+\sqrt{\frac{z^3}{27}+\frac{y^2}{4}}}+\sqrt[3]{\frac{y}{2}-\sqrt{\frac{z^3}{27}+\frac{y^2}{4}}}.$$

As $x=ys$, we deduce that $F(x,y,z):=x-yf(y,z)$ is an analytic equation of $X\setminus\{z\leq 0\}$ in $\mathbb{R}^3\setminus\{z\leq 0\}$, which is the complement of a C-semianalytic set with non-empty interior. The previous equation F generates the ideal $\mathcal{J}_{X,p}$ for each $p\in X\setminus\{z\leq 0\}$. By [**22**, Lem. 5.3] there is an analytic function $h\in O(\mathbb{R}^n\setminus N(X))$ such that h_x generates the ideal $\mathcal{J}_{X,x}$ for each $x\in\mathbb{R}^n\setminus N(X)$.

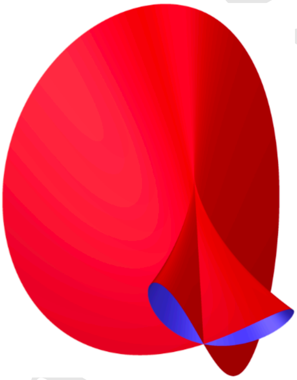

FIGURE 9. $X:(x^2+zy^2)x-y^4=0$

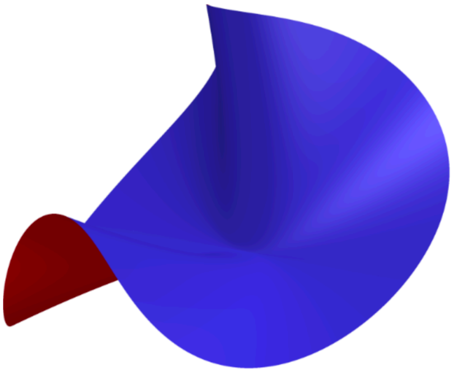

FIGURE 10. $X:(x^2+z^2y^2)x-y^4=0$

We modify the previous example to obtain the following.

***Example 3.5* (Fake blanket, [22, Ex. 3.11]). —** Consider the algebraic set $X := \{(x^2+z^2y^2)x - y^4 = 0\} \subset \mathbb{R}^3$ (Figure 10). The set of regular points of X is $X \setminus \{x=0, y=0\}$, whereas: *The set of smooth points of X is $X \setminus \{(0,0,0)\}$.*

To prove that the origin is a non-smooth point we proceed by contradiction. Assume it is a smooth point. As the line $\{x=0, y=0\} \subset X$, the vector $(0,0,1)$ would be tangent to X at p, so the plane $z=0$ would be transversal to X at p. Thus, the intersection $X \cap \{z=0\}$ should be a curve that is smooth at p, but this is a contradiction, because such curve is $\{x^3 - y^4 = 0, z = 0\}$.

We prove next that the points of $X \setminus \{(0,0,0)\}$ are smooth. To that end, observe that the map

$$\varphi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^3,\ (s,u) \mapsto ((s^2+u^2)s^2, (s^2+u^2)s, u)$$

is an analytic embedding whose image is $X \setminus \{(0,0,0)\}$. In fact, the previous map extends to a homeomorphism between $\mathbb{R}^2$ and X, whose inverse is the map

$$\psi : X \to \mathbb{R}^2,\ (x,y,z) \mapsto \begin{cases} (\frac{x}{y}, z) & \text{if } y \neq 0, \\ (0,z) & \text{if } y = 0. \end{cases}$$

As X is non-coherent, because $\operatorname{Reg}(X) \subsetneq \mathtt{Smooth}(\mathtt{X})$, and

$$X \setminus \{(0,0,0)\} \subset \mathtt{Smooth}(\mathtt{X}) \subset \mathtt{X} \setminus \mathtt{N}(\mathtt{X}) \subsetneq \mathtt{X},$$

we deduce $N(X) = \{(0,0,0)\}$. In addition, $T(X) = \operatorname{Sing}(X) \setminus N(X)$, because the analytic germ $\widetilde{X}_{(0,0,0)}$ is irreducible.

The analytic map φ extends to a holomorphic map

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^3,\ (v,w) \to ((v^2+w^2)v^2, (v^2+w^2)v, w),$$

which provides the normalization of the complexification $\widetilde{X} := \{(x^2+zy^2)x - y^4 = 0\}$ of X. To that end, observe $v = \frac{x}{y}$ satisfies the integral equation $^3 + \mathtt{z}^2 - \mathtt{y} = \mathtt{0}$ (an in addition $z = w$, $y = v^3 + z^2 v$ and $x = vy$).

The real parameter $s = \frac{x}{y}$ is a solution of the integral equation $^3 + \mathtt{z}^2 - \mathtt{y} = \mathtt{0}$. Using the solution of the general equation of degree 3 we find that

$$s = f(z,y) := \sqrt[3]{\frac{y}{2} + \sqrt{\frac{z^6}{27} + \frac{y^2}{4}}} + \sqrt[3]{\frac{y}{2} - \sqrt{\frac{z^6}{27} + \frac{y^2}{4}}}.$$

As $x = ys$, we deduce that $F(x,y,z) := x - yf(y,z)$ is an analytic equation of $X \setminus \{z=0\}$ in $\mathbb{R}^3 \setminus \{z=0\}$, which is the complement in $\mathbb{R}^3$ of a C-analytic set of dimension 2. The previous equation F generates the ideal $\mathcal{J}_{X,p}$ for each $p \in X \setminus \{z=0\}$. By [22, Lem. 5.3] there exists an analytic function $h \in O(\mathbb{R}^n \setminus N(X))$ such that h_x generates the ideal $\mathcal{J}_{X,x}$ for each $x \in \mathbb{R}^n \setminus N(X)$.

4. Main results presented in the seminar

We present in this section the main statements from [22, §6-7]. As the corresponding results are of a similar nature, we consider first the cases of real analytic and Nash sets and we provide a combined presentation.

4.1. Real analytic and Nash sets. — Let X be a C-analytic subset of $\mathbb{R}^n$, let $\zeta : X \to \mathbb{R}$ be a function and let $x \in X$. Let us analyze the conditions for ζ (in case it is either locally meromorphic at x or globally meromorphic on X) to have an analytic extension on an open neighborhood of x.

Remark 4.1 **(Local approach, [22, §6.A. 1]). —** Suppose there exist analytic germs $f_x, g_x \in \mathcal{O}_{\mathbb{R}^n,x}$ (depending on x) such that $\zeta_x = \frac{f_x}{g_x}$ and g_x does not belong to any minimal prime ideal of $\mathcal{J}_{X,x}$ in $\mathcal{O}_{\mathbb{R}^n,x}$. Observe that ζ is an analytic function at $x \in X$ if there exists $a_x \in \mathcal{O}_{\mathbb{R}^n,x}$ such that $\frac{f_x}{g_x} = a_x$ on X_x or, equivalently, if $f_x - a_x g_x \in \mathcal{J}_{X,x}$. This is equivalent to $f_x \in g_x \mathcal{O}_{\mathbb{R}^n,x} + \mathcal{J}_{X,x}$.

4.1.1. *Obstructing set of a meromorphic function.* — The *obstructing set* of $\frac{f}{g} \in \mathcal{M}(X)$ is defined as

$$\mathsf{O}\Big(\frac{f}{g}\Big) := \Big\{x \in X : f_x \notin g_x \mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x}\Big\},$$

see [22, Def. 6.2]. In case $X \subset \mathbb{R}^n$ is a Nash set and $\frac{f}{g} \in \mathcal{M}^\bullet(X)$, we define

$$\mathsf{O}^\bullet\Big(\frac{f}{g}\Big) := \Big\{x \in X : f_x \notin g_x \mathcal{O}^\bullet_{\mathbb{R}^n,x} + \mathcal{I}^\bullet_{X,x}\Big\}.$$

As the inclusion homomorphism $\mathcal{N}_{\mathbb{R}^n,x} \hookrightarrow \mathcal{O}_{\mathbb{R}^n,x}$ is faithfully flat, $\mathsf{O}^\bullet(\zeta) = \mathsf{O}(\zeta)$. The obstruction set of a meromorphic function has the following properties:

(1) The obstructing set $\mathsf{O}(\zeta)$ does not depend on the meromorphic expression of $\zeta \in \mathcal{M}(X)$, see [22, §6.A. 3].

(2) $\mathsf{O}(\zeta)$ is a closed subset of X, see [22, §6.A. 4].

(3) ζ has an analytic (resp. Nash) extension to $\mathbb{R}^n$ if and only if $\mathsf{O}(\zeta) = \varnothing$, see [22, §6.A. 4].

(4) The obstruction set $\mathsf{O}(\zeta)$ determines whether ζ admits or not an analytic (resp. Nash) extension to $\mathbb{R}^n$ and measures how far is ζ from having such analytic (resp. Nash) extension, see [22, §6.A. 4].

In [22] we have proved the following stronger versions of Theorems 1.7 and 1.11, whose statements together with Remarks 4.4 show the meaning of the word 'many' that appears in the statements of the previous results.

Theorem 4.2 ([22, Thm. 6.6]). **—** *Let $X \subset \mathbb{R}^n$ be a C-analytic set with $N(X) \neq \varnothing$. Let $Y \subset X$ be either a discrete subset of $\mathbb{R}^n$ or a C-analytic set of $\mathbb{R}^n$ that does not meet $N(X)$, contains no C-irreducible component of X and meets $T(X)$. Then there exists $\zeta \in \mathcal{M}(X)$ that defines an analytic function on X with $\mathsf{O}(\zeta) = Y \cap T(X)$.*

Theorem 4.3 **([22, Thm. 6.7]).** — *Let $X \subset \mathbb{R}^n$ be a Nash set with $N(X) \neq \varnothing$. Let $Y \subset X$ be either a finite set or a Nash set that does not meet $N(X)$, contains no Nash irreducible component of X and meets $T(X)$. Then there exists $\zeta \in \mathcal{M}^\bullet(X)$ that defines a local Nash function on X with $\mathtt{0}(\zeta) = Y \cap T(X)$.*

Remark 4.4 **([22, Rem. 6.8]).** — Let $X \subset \mathbb{R}^n$ be a d-dimensional C-analytic set with $T(X) \neq \varnothing$ and denote the $(n-1)$-dimensional sphere of $\mathbb{R}^n$ of center $p \in \mathbb{R}^n$ and radius $r > 0$ with $\mathbb{S}^{n-1}_{p,r}$. Consider a discrete closed subset $D := \{y_j\}_{j\in J}$ of $\mathrm{Cl}(T(X))$ contained in the C-semianalytic set $T(X) \setminus N(X)$ such that each y_j has an open neighborhood $U^{y_j} \subset \mathbb{R}^n$ satisfying that the intersection $M_j := U^{y_j} \cap T(X)$ is an analytic submanifold of U^{y_j}. Define a tuple $\varepsilon := \{\varepsilon_j\}_{j\in J}$ of positive real numbers such that: $T_j := T(X) \cap \mathbb{S}^{n-1}_{y_j,\varepsilon_j}$ is compact analytic subset of dimension $\dim(M_j) - 1$ of the analytic manifold M_j, $\mathbb{S}^{n-1}_{y_j,\varepsilon_j} \cap N(X) = \varnothing$ for each $j \in J$ and the family $\{T_j\}_{j\in J}$ is locally finite in $\mathrm{Cl}(T(X))$. Then $Y_{D,\varepsilon} := \mathrm{Sing}(X) \cap \bigcup_{j\in J} \mathbb{S}^{n-1}_{y_j,\varepsilon_j}$ is a C-analytic subset of X that does not meet $N(X)$ and such that $Y_{D,\varepsilon} \cap T(X) = \bigcup_{j\in J} T_j$. When varying pairs (D, ε) we obtain 'many' different intersections $Y_{D,\varepsilon} \cap T(X)$. In the Nash case we can picture similar constructions (changing discrete subsets by finite subsets).

Due to the existence of 'many' intersections $Y \cap T(X)$ for C-analytic subsets Y of X (contained in $X \setminus N(X)$), Theorems 4.2 and 4.3 imply that there are 'many' distinct meromorphic functions (or meromorphic Nash functions) on a non-coherent C-analytic set (or Nash set) X that are analytic (or local Nash) on X yet lack analytic (or Nash) extensions to $\mathbb{R}^n$ as stated in Theorems 1.7 and 1.11.

4.2. Smooth semialgebraic functions versus Nash functions. — The foundational definition of differentiable semialgebraic functions traces back to Whitney's extension theorem. We revisit an intrinsic perspective on these functions as discussed in [**4**] and treated in [**22**, §7].

4.2.1. *Differentiable semialgebraic functions.* — Recall that we denote $\mathbb{N} := \{0, 1, 2, \dots\}$ the set of natural numbers including 0. Let $S \subset \mathbb{R}^n$ be a semialgebraic set and denote $\mathtt{x} := (\mathtt{x}_1, \dots, \mathtt{x}_n)$.

Definitions 4.5 **(Def.1).** — (i) A *semialgebraic jet on S of order $p \geq 0$* is a collection of semialgebraic functions $F := (f_\alpha)_{|\alpha|\leq p}$ on S. Here we denote $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. For each $a \in S$ write $T^p_a F := \sum_{|\alpha|\leq p} \frac{f_\alpha(a)}{\alpha!}(\mathtt{x} - \mathtt{a})^\alpha$.

(ii) A continuous semialgebraic function $f : S \to \mathbb{R}$ is a *C^p semialgebraic function* if there exists a semialgebraic jet $F := (f_\alpha)_{|\alpha|\leq p}$ on S of order p such that $f_0 = f$ and for each point $a \in S$ there exists an increasing, continuous and concave function $\alpha_a : [0, +\infty) \to [0, +\infty)$ such that $\alpha_a(0) = 0$ and for each $z \in \mathbb{R}^n$ it holds

$$|T^p_x F(z) - T^p_y F(z)| \leq \alpha_a(\|x - y\|) \cdot (\|z - x\|^p + \|z - y\|^p)$$

for $x, y \in S$ when $x, y \to a$.

We denote with $S^p(S)$ the collection of all the C^p semialgebraic functions (S^p functions) on S. There are other alternatives to define S^p functions on S. We recall some of them:

(Def.2) There exist an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of S and a S^p function $F : U \to \mathbb{R}$ such that $F|_S = f$.

(Def.3) For each point $x \in S$, there exist an open semialgebraic neighborhood $U^x \subset \mathbb{R}^n$ of x and an S^p function $F_x : U^x \to \mathbb{R}$ such that $F_x|_{U^x \cap S} = f|_{U^x \cap S}$.

(Def.4) For each point $x \in S$, there exist an open neighborhood $U^x \subset \mathbb{R}^n$ of x and a C^p function $F_x : U^x \to \mathbb{R}$ such that $F_x|_{U^x \cap S} = f|_{U^x \cap S}$.

(Def.5) There exist an open neighborhood $U \subset \mathbb{R}^n$ of S and a C^p function $F : U \to \mathbb{R}$ such that $F|_S = f$.

In this context, we observe that (Def.2) leads to (Def.3), which in turn leads to (Def.4) if and only if (Def.5), while (Def.2) also directs to Definition 4.5. The implication from (Def.4) to (Def.5) can be established through the use of a C^p partition of unity. Similarly, for a compact semialgebraic set $S \subset \mathbb{R}^n$, one can demonstrate that (Def.3) implies (Def.2) by employing an S^p partition of unity. When $S \subset \mathbb{R}^n$ is a closed semialgebraic set, Definition 4.5 ensures (Def.2) as referenced in [**32, 38**]. Furthermore, in the specific case where $n = 2$, the authors in [**17**] establish that (Def.5) implies (Def.2), while in [**4**], it is shown that (Def.5) implies (Def.2) when $p = 1$ and $n \geq 1$. Additionally, a weak version of the implication '(Def.5) $\Longrightarrow$ (Def.2)' is discussed in [**7**, Cor. 1.7], utilizing an intermediary function $t : N \to N$ that captures a particular loss of differentiability. Specifically, if there exists an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of a closed semialgebraic set $S \subset \mathbb{R}^n$ alongside a $C^{t(p)}$ function $G : U \to \mathbb{R}$ satisfying $G|_S = f$, then we can find an S^p function $F : U \to \mathbb{R}$ such that $F|_S = f$. In the broader scenario, where $S \subset \mathbb{R}^n$ represents a general semialgebraic set, Definition 4.5 guarantees the existence of a S^{p-1} function F defined on an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of S that extends f, as stated in [**5**, Lem. 2.6].

4.2.2. *Rings of smooth semialgebraic and Nash functions.* — Let $S \subset \mathbb{R}^n$ be a semialgebraic set and define the *ring of smooth semialgebraic functions on* S as

$$S^{(\infty)}(S) := \bigcap_{p \geq 0} S^p(S), \tag{4.1}$$

where $S^p(S)$ is the ring of S^p functions on S following Definition 4.5. The elements of $S^{(\infty)}(S)$ satisfy the following properties:

(1) $f \in S^{(\infty)}(S)$ if and only if f is a semialgebraic function and for each $x \in S$ there exists a Nash function germ $F_x \in N_{\mathbb{R}^n,x}$ such that $F_x|_{S_x} = f_x$, see [**22**, Thm. 7.5, Appendix A].

(2) All the definitions above of S^p function on a semialgebraic set S provide the same definition of $S^{(\infty)}$ function germ at each point $x \in S$, see [**22**, Rem. 7.6].

Recall that the ring $N(S)$ of Nash functions on S is the set of functions $f : S \to \mathbb{R}$ for which there exist open semialgebraic neighborhoods $\Omega \subset \mathbb{R}^n$ and Nash functions $F : \Omega \to \mathbb{R}$ such that $F|_S = f$ (endowed with the usual product and sum of functions). The third main result of [**22**], that we have presented as Theorem 1.13 in the Introduction, can be restated as follows.

Theorem 4.6 **([22, Thm. 7.8]).** — *If $S \subset \mathbb{R}^n$ is a semialgebraic set, then $N(S) = S^{(\infty)}(S)$ if and only if $T(S) = \varnothing$.*

5. C-analytic sets with seeable 'tails': an alternative proof

The non pure dimensional case in the C-analytic and the Nash settings has been treated by several authors [**16, 36, 37**]. As quoted above, Cartan's umbrella $C := \{(x^2 + y^2)z - y^3 = 0\}$ and Whitney's umbrella $W := \{y^2 - zx^2 = 0\}$ are famous algebraic subsets of $\mathbb{R}^3$ with a seeable 'tail' that are in addition C-irreducible C-analytic sets (and irreducible Nash sets). The latter example has been managed by Efroymson in [**16**], when dealing with the extension theorem for Nash functions. These ideas where generalized in the presence of seeable 'tails' in [**36**] for C-analytic sets and in [**37**] for Nash sets. We provide in this section a short original alternative proof to Theorem 1.7 (resp. Theorem 1.11) for C-analytic sets (resp. Nash sets) with seeable 'tails'. The proof is different to those in [**36, 37**].

Theorem 5.1. — *Let $X \subset \mathbb{R}^n$ be a C-analytic set (resp. Nash set) with a seeable 'tail'. Then there exist meromorphic functions $\xi \in M(X)$ that define analytic functions on X, but have no analytic extensions to $\mathbb{R}^n$.*

For each $\lambda \in \mathbb{C}^n$ denote $P_\lambda := \lambda_1 \mathtt{x}_1^2 + \cdots + \lambda_n \mathtt{x}_n^2$. The proof of the following result is conducted similarly to the one of the density of transversality on manifolds. We include full details for the sake of completeness.

Lemma 5.2. — *Let $Z_x \subset \mathbb{C}^n_x$ be a complex analytic set germ at the origin $x = 0$. There exists a complex analytic subset $S \subsetneq \mathbb{C}^n$ such that for each $\lambda \in \mathbb{C}^n \setminus S$ the intersection $Z_x \cap \mathcal{Z}(P_\lambda)_x$ has dimension* $\dim(Z_x) - 1$.

Proof. — Let Z be a complex analytic subset of an open subset $\Omega \subset \mathbb{C}^n$ that is a representative of Z_x. Denote $y := (y_1, \ldots, y_n)$, $\lambda := (\lambda_1, \ldots, \lambda_n)$ and consider the complex analytic manifold $Z \setminus (\mathrm{Sing}(Z) \cup \{x\})$ and the analytic map

$$\Phi : (Z \setminus (\mathrm{Sing}(Z) \cup \{x\})) \times \mathbb{C}^n \to \mathbb{C},\ (y, \lambda) \mapsto \sum_{i=1}^n \lambda_i y_i^2,$$

Denote $u := (u_1, \ldots, u_n)$ and $\mu := (\mu_1, \ldots, \mu_n)$. If $(y, \lambda) \in (Z \setminus (\mathrm{Sing}(Z) \cup \{x\})) \times \mathbb{C}^n$, we have

$$D_{(y,\lambda)}\Phi : T_y(Z \setminus (\mathrm{Sing}(Z) \cup \{x\})) \times \mathbb{C}^n \to \mathbb{C},\ (v, \mu) \mapsto 2\sum_{i=1}^n \lambda_i y_i v_i + \sum_{i=1}^n \mu_i y_i^2,$$

which is surjective for each $(y,\lambda)\in (Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\times\mathbb{C}^n$. Thus, $\Phi^{-1}(0)$ is a complex analytic manifold. Consider the analytic map $\pi:\Phi^{-1}(0)\to\mathbb{C}^n$, $(y,\lambda)\mapsto\lambda$. Let $X:=\{(y,\lambda)\in\Phi^{-1}(0):\ \operatorname{rk}_{(y,\lambda)}(d_{(y,\lambda)}\pi)<n\}$, which is a complex analytic subset of $\Phi^{-1}(0)$. By Remmert's Theorem [**35**, VII.Thm. 2] the image $\pi(X)$ is a complex analytic subset of $\mathbb{C}^n$. By Sard's Theorem the dimension of $\pi(X)$ is strictly smaller than n. Pick a value $\lambda_0\in\mathbb{C}^n\setminus\pi(X)$ and observe that $D_{(y,\lambda_0)}\pi:T_{(y,\lambda_0)}\Phi^{-1}(0)\to\mathbb{C}^n$, $(v,\mu)\mapsto\mu$ is surjective for each $(y,\lambda_0)\in\Phi^{-1}(0)$. Let us check:

$$M_{\lambda_0}:=(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\cap\{P_{\lambda_0}=0\}$$

is a non-singular hypersurface of $Z\setminus(\operatorname{Sing}(Z)\cup\{x\})$.

It is enough to check:

$$D_y\Phi(\cdot,\lambda_0):T_y(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\to\mathbb{C},\ v\mapsto 2\sum_{i=1}^n\lambda_{0i}y_iv_i$$

is surjective for each $y\in M_{\lambda_0}$.

Pick $y_0\in M_{\lambda_0}$ and recall that

$$\begin{cases}D_{(y_0,\lambda_0)}\Phi:T_{y_0}(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\times\mathbb{C}^n\to\mathbb{C},\\ D_{(y_0,\lambda_0)}\pi:T_{(y_0,\lambda_0)}\Phi^{-1}(0)\to\mathbb{C}^n,\ (v,\mu)\mapsto\mu\end{cases}$$

are surjective. Let $w_0\in\mathbb{C}$ and let

$$(v_0,\mu_0)\in T_{(y_0,\lambda_0)}\Phi^{-1}(0)\subset T_{y_0}(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\times\mathbb{C}^n$$

be such that

$$D_{(y_0,\lambda_0)}\Phi(v_0,\mu_0)=2\sum_{i=1}^n\lambda_{0i}y_{0i}v_{0i}+\sum_{i=1}^n\mu_{0i}y_{i0}^2=w_0.$$

There exists $(u_0,\mu_0)\in T_{(y_0,\lambda_0)}\Phi^{-1}(0)\subset T_{y_0}(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\times\mathbb{C}^n$ such that

$$D_{(y_0,\lambda_0)}\pi(u_0,\mu_0)=\mu_0.$$

In addition, $D_{(y_0,\lambda_0)}\Phi(u_0,\mu_0)=0$, so

$$D_{(y_0,\lambda_0)}\Phi(u_0,\mu_0)=2\sum_{i=1}^n\lambda_{0i}y_{0i}u_{0i}+\sum_{i=1}^n\mu_{0i}y_{i0}^2=0\ \rightsquigarrow\ \sum_{i=1}^n\mu_{0i}y_{i0}^2=-2\sum_{i=1}^n\lambda_{0i}y_{0i}u_{0i}.$$

As $-u_0+v_0\in T_{y_0}(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))$, we conclude

$$D_{y_0}\Phi(\cdot,\lambda_0)(-u_0+v_0)=2\sum_{i=1}^n\lambda_{0i}y_{0i}(-u_{0i}+v_{0i})=2\sum_{i=1}^n\lambda_{0i}y_{0i}v_{0i}+\sum_{i=1}^n\mu_{0i}y_{i0}^2=w_0,$$

so $D_{y_0}\Phi(\cdot,\lambda_0):T_y(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\to\mathbb{C}$ is surjective.

The origin x is adherent to $(Z\setminus(\operatorname{Sing}(Z)\cup\{x\}))\cap\{P_{\lambda_0}=0\}$, so we can apply the previous argument to $Z\cap U^x$ for each open neighborhood $U^x\subset\mathbb{C}^n$ of x and X does not depend on U^x. Thus, the germ $Z_x\cap\mathcal{Z}(P_{\lambda_0})_x$ has dimension $\dim(Z_x)-1$ for each $\lambda_0\in\mathbb{C}^n\setminus\pi(X)$, as required. □

We include the following result (which is a particular case of [**22**, Lem. 6.3]). We denote the open orthant of $\mathbb{R}^n$ with $Q_n := \{\mathtt{y_1} > 0, \dots, \mathtt{y_n} > 0\} \subset \mathbb{R}^{\mathtt{n}}$.

Lemma 5.3. — *Let $y \in \mathbb{C}^n$ and let $Z_y \subset \mathbb{C}^n$ be a complex analytic germ. Then there exists a proper algebraic subset S of $\mathbb{R}^n$ such that ($Q_n \setminus S$ is dense in Q_n and) $\dim(Z_y \cap Z(P_{\lambda,y})) < \dim(Z_y)$ for each $\lambda \in Q_n \setminus S$.*

Proof. — Let $\Omega \subset \mathbb{C}^n$ be an open neighborhood of y and let $Z_1, \dots, Z_r$ be irreducible complex analytic subsets of Ω such that the irreducible components of Z_y are $Z_{1,y}, \dots, Z_{r,y}$. Pick a point $p_i \in Z_i$ for $i = 1, \dots, r$ and consider the polynomial in $\lambda := (\lambda_1, \dots, \lambda_n)$ given by $F := \prod_{i=1}^s P_\lambda(p_i)$. Let $S := \{\lambda \in \mathbb{R}^n : F(\lambda) = 0\}$. As the open orthant $Q_n \subset \mathbb{R}^n$ is Zariski dense in $\mathbb{R}^n$, we have that $Q_n \setminus S$ is dense in Q_n. For each $\lambda \in Q_n \setminus S$ we have $\dim(Z_i \cap Z(P_\lambda)) < \dim(Z_i)$ for $i = 1, \dots, s$, so $\dim(Z \cap Z(P_\lambda)) < \dim(Z)$ and $\dim(Z_y \cap Z(P_{\lambda,y})) < \dim(Z_y)$, as required. □

We are ready to prove Theorem 5.1

Proof of Theorem 5.1. — As X has a seeable 'tail', there exists an irreducible component Y of X and a point $x \in Y$ such that $\dim(X_x) < \dim(Y)$. In particular $1 \le \dim(Y_x) \le \dim(X_x) < \dim(Y)$. Thus, $x \in \operatorname{Sing}(Y)$ and $\dim(Y) \ge 2$. Define recursively $\operatorname{Sing}_\ell(Y) := \operatorname{Sing}(\operatorname{Sing}_{\ell-1}(Y))$ for each $\ell \ge 2$. Let $\ell \ge 1$ be the largest integer such that $\operatorname{Sing}_\ell(Y)_x = Y_x$. Let $g \in O(\mathbb{R}^n)$ be a non-negative analytic equation (resp. $g \in N(\mathbb{R}^n)$ be a non-negative Nash equation) of $\operatorname{Sing}_\ell(Y)$ and assume for simplicity $x = 0$. Recall at this point that if X is a Nash set, its Nash irreducible components and its C-analytic irreducible components coincide ([**21**, Prop. 2.8] and [**18**]). Let X' be the union of all the C-irreducible components of X different form Y and let $h \in O(\mathbb{R}^n)$ be a non-negative analytic equation of X' (resp. $h \in N(\mathbb{R}^n)$ be a non-negative Nash equation of X'). For each $\lambda := (\lambda_1, \dots, \lambda_n) \in Q_n$ consider

$$\xi_\lambda := \frac{P_\lambda}{gh + P_\lambda},$$

which defines an analytic function (resp. a Nash function on $\mathbb{R}^n$) on $\mathbb{R}^n \setminus \{0\}$. Observe that $X_x = Y_x \cup X'_x = \operatorname{Sing}_\ell(Y)_x \cup X'_x = \mathcal{Z}(gh)_x$, so the previous meromorphic function ξ_λ is identically 1 on the germ $X_x \setminus \{x\}$. Thus, $\xi_\lambda|_X$ extends to an analytic (resp. semialgebraic local Nash) function on X (which values 1 at x). Let us check: *There exists $\lambda \in Q$ such that $\xi_\lambda|_X$ has no analytic (resp. Nash) extension to $\mathbb{R}^n$.*

Assume that $\xi_\lambda|_X$ admits an analytic (resp. Nash) extension f to $\mathbb{R}^n$. Then, it admits an invariant holomorphic extension F to an open neighborhood $\Omega_0 \subset \mathbb{C}^n$ of $\mathbb{R}^n$. As Y is an irreducible component of X, it admits a complexification $\widetilde{Y}$ in an open Stein neighborhood $\Omega \subset \Omega_0$ of Y. We can also assume that g, h have invariant holomorphic extensions G, H to Ω. Denote again P_λ the invariant holomorphic extension of P_λ to $\mathbb{C}^n$. By the identity principle

$$F|_{\widetilde{Y}} = \frac{P_\lambda}{GH + P_\lambda}.$$

By [**39**, Cor. 2] we may assume $\widetilde{Y}$ is irreducible, so it is pure dimensional of complex dimension $\dim(Y) > \dim(\operatorname{Sing}_\ell(Y))$. Consider the complex analytic germ $\widetilde{Y}_x \cap \mathcal{Z}(GH)_x$ and let us check: *It has dimension strictly smaller that* $\dim(\widetilde{Y}_x) = \dim(\widetilde{Y})$.

Otherwise, as $\widetilde{Y}$ is irreducible, one deduces by the identity principle that $\widetilde{Y} \subset \mathcal{Z}(GH)$, so

$$Y = \widetilde{Y} \cap \mathbb{R}^n \subset \mathcal{Z}(GH) \cap \mathbb{R}^n = \mathcal{Z}(gh) = \operatorname{Sing}_\ell(Y) \cup X',$$

which is a contradiction. Consequently, $\dim(\widetilde{Y}_x \cap \mathcal{Z}(GH)_x) < \dim(\widetilde{Y}_x)$.

By Lemma 5.2 there exists a complex analytic subset $S \subsetneq \mathbb{C}^n$ such that for each $\lambda \in \mathbb{C}^n \setminus S$ the intersection $\widetilde{Y}_x \cap \mathcal{Z}(P_\lambda)_x$ has dimension $\dim(\widetilde{Y}_x) - 1$. The intersection $Q_n \cap S$ is a proper C-analytic subset of Q_n, so $Q_n \setminus S$ is dense in Q_n. By Lemma 5.3 there exists $\lambda \in Q_n \setminus S$ such that

$$\dim(\mathcal{Z}(GH)_x \cap \widetilde{Y}_x \cap \mathcal{Z}(P_\lambda)_x) < \dim(\mathcal{Z}(GH)_x \cap \widetilde{Y}_x) \leq \dim(\widetilde{Y}_x) - 1 = \dim(\widetilde{Y}_x \cap \mathcal{Z}(P_\lambda)_x).$$

By the curve selection lemma [**3**, Rem.VII. 4.3(b)] (applied to the underlying real structures of all the involved objects) there exists a real analytic arc $\alpha : (-1,1) \to \mathbb{C}^n$ such that $\alpha((0,1)) \subset (\widetilde{Y} \cap \mathcal{Z}(P_\lambda)) \setminus \mathcal{Z}(GH)$ and $\alpha(0) = x$. We have

$$F \circ \alpha|_{(0,1)} = \frac{P_\lambda}{GH + P_\lambda} \circ \alpha|_{(0,1)} = 0,$$

so $\xi_\lambda|_X(x) = 0$, which is a contradiction, because $\xi_\lambda|_{X_x} = 1$. Consequently, there exists no analytic (resp. Nash) extension of $\xi_\lambda|_X$ to $\mathbb{R}^n$, as required. □

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