On the set of local extrema of a subanalytic function

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Introduction

Let $\mathfrak{F}$ be a category of subanalytic subsets of real analytic manifolds that is closed under elementary set-theoretical operations (locally finite unions, difference and product) and basic topological operations (taking connected components and closures). Fix a real analytic manifold M and denote $\mathfrak{F}(M)$ the family of the subsets of M that belong to $\mathfrak{F}$. Let $f : X \to \mathbb{R}$ be a subanalytic function on a subset $X \in \mathfrak{F}(M)$.

Main Problem. Under which conditions does the set $\operatorname{Max}(f)$ of local maxima of f belong to $\mathfrak{F}(M)$?

If $\mathfrak{F}$ is an o-minimal structure (so we forget about locally finite unions), the answer is clearly affirmative as $\operatorname{Max}(f)$ is a definable set. The solution to the problem requires further work for a category $\mathfrak{F}$ of subanalytic sets that does not constitute an o-minimal structure.

Basic definitions

Let $\mathfrak{M}$ be the class of real analytic manifolds. Let $M \in \mathfrak{M}$ and $S, X \subset M$.

- S is *semianalytic* if $S_x = \bigcup_{\text{finite}}\{f = 0, g_1 > 0, \dots, g_r > 0\}$, $f, g_i \in \mathcal{O}_{M,x}$, $x \in M$.
- S is *basic C-semianalytic* if $S := \{f = 0, g_1 > 0, \dots, g_r > 0\}$: $f, g_i \in \mathcal{O}(M)$.
- S is *C-semianalytic* if it is a locally finite union of basic C-semianalytic sets.
- X is *subanalytic* if it is locally a project. of a relatively compact semianalytic set.

A *weak category* $\mathfrak{F}$ is a collection of families $\mathfrak{F}(M)$ of subanalytic subsets of M satisfying:

- $M \in \mathfrak{F}(M)$.
- If $S_1, S_2 \in \mathfrak{F}(M)$, then $S_1 \setminus S_2 \in \mathfrak{F}(M)$.
- If $\{S_i\}_{i\in I} \subset \mathfrak{F}(M)$ is a locally finite family in M, then $\bigcup_{i\in I} S_i \in \mathfrak{F}(M)$.
- If $S \in \mathfrak{F}(M)$ and $T \in \mathfrak{F}(N)$, then $S \times T \in \mathfrak{F}(M \times N)$.
- If $S \in \mathfrak{F}(M)$, then its connected components and $\operatorname{Cl}(S)$ belong to $\mathfrak{F}(M)$.

$\mathfrak{F}$ *contains algebraic intersections* if $M \cap X \in \mathfrak{F}(M)$ $\forall X \subset \mathbb{R}^n$ alg. set, $\forall M \in \mathfrak{M}$.

Some weak categories that contain algebraic intersections: Subanalytic sets, Semianalytic sets and C-semianalytic sets.

Main results

Fix a weak category $\mathfrak{F}$ (e.g, subanalytic sets, semianalytic sets or C-semianalytic sets).

Definitions. Let $f : X \to \mathbb{R}$ be a function on $X \in \mathfrak{F}(M)$. We say that:

- f is an *$\mathfrak{F}$-function* if the graph $\Gamma(f) \in \mathfrak{F}(M \times \mathbb{R})$.
- f *behaves well on fibers* if inverse images under f of intervals belong to $\mathfrak{F}(M)$.
- f is *M-compact* *if $f(K \cap X)$ is bounded for each compact subset $K \subset M$.*

Remark. $\operatorname{Max}_\lambda(f) := \operatorname{Max}(f) \cap \{f = \lambda\} = \{f = \lambda\} \setminus \operatorname{Cl}(\{f > \lambda\}) \in \mathfrak{F}(M)$ if f behaves well on fibers. In addition, $\operatorname{Max}(f) = \bigsqcup_{\lambda\in\mathbb{R}} \operatorname{Max}_\lambda(f)$.

Theorem 1. *Let $f : X \to \mathbb{R}$ be a continuous subanalytic function on $X \in \mathfrak{F}(M)$ that behaves well on fibers. The following are equivalent:*

(i) $\operatorname{Max}(f) \in \mathfrak{F}(M)$.
(ii) *The family of the connected components of $\operatorname{Max}(f)$ is locally finite in M.*
(iii) *The family $\{\operatorname{Max}_\lambda(f)\}_{\lambda\in\mathbb{R}}$ is locally finite in M.*
(iv) *The family $\{\operatorname{Max}_\lambda(f)\}_{\lambda\in\mathbb{R}}$ is locally isolated in M.*

Theorem 2. *Let $f : X \to \mathbb{R}$ be an M-compact subanalytic function. Then f behaves well on fibers and $\{\operatorname{Max}_\lambda(f)\}_{\lambda\in\mathbb{R}}$ is locally finite in M. Consequently, each $\operatorname{Max}_\lambda(f)$ and $\operatorname{Max}(f)$ are subanalytic subsets of M.*

Lemma 2. *Let $f : X \to \mathbb{R}$ be a continuous subanalytic function on a closed subanalytic set. Then $\{\operatorname{Max}_\lambda(f)\}_{\lambda\in\mathbb{R}}$ is locally finite and each $\operatorname{Max}_\lambda(f)$ and $\operatorname{Max}(f)$ are subanalytic sets.*

Corollary 2. *Let $\mathfrak{F}$ be either the category of semianalytic or C-semianalytic sets and $f : X \to \mathbb{R}$ the restriction to $X \in \mathfrak{F}(M)$ of an analytic function on M. Then $\{\operatorname{Max}_\lambda(f)\}_{\lambda\in\mathbb{R}}$ is locally finite in M and $\operatorname{Max}_\lambda(f), \operatorname{Max}(f) \in \mathfrak{F}(M)$ for each $\lambda \in \mathbb{R}$.*

Proposition 1 (Sharpness of results)**.** *Let $\mathfrak{F}$ be a weak category that contains algebraic intersections and $X \in \mathfrak{F}(M)$ a non-closed subset of M. Then there exists a continuous $\mathfrak{F}$-function $f : X \to \mathbb{R}$ that behaves well on fibers but $\{\operatorname{Max}_\lambda(f)\}_{\lambda\in\mathbb{R}}$ is not locally finite in M. Thus, $\operatorname{Max}_\lambda(f) \in \mathfrak{F}(M)$ for each $\lambda \in \mathbb{R}$, whereas $\operatorname{Max}(f) \notin \mathfrak{F}(M)$.*

Some relevant examples

Lemma 1. *Let $\mathfrak{S}$ be an o-minimal structure. Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}$ be definable (w.r.t. $\mathfrak{S}$). Then $\operatorname{Max}(f)$ is definable and $f(\operatorname{Max}(f))$ is a finite set.*

Example 1 (o-minimal structure $\mathbb{R}_{\text{an}}$: **A main tool**)**.**

- $f : [-1, 1]^n \to \mathbb{R}$ is a *restricted analytic function* if it admits an analytic continuation to an open neighborhood of $[-1, 1]^n$ in $\mathbb{R}^n$.
- $X \subset \mathbb{R}^n$ is a *global subanalytic subset* of $\mathbb{R}^n$ if there exists a semialgebraic homeomorphism $g : \mathbb{R}^n \to (-1, 1)^n$ such that $g(X)$ is a subanalytic subset of $\mathbb{R}^n$.

The global subanalytic sets are the collection of definable sets in the o-minimal structure $\mathbb{R}_{\text{an}}$ generated by the restricted analytic functions.

Corollary 1. *If $X \subset \mathbb{R}^n$ is a global subanalytic set and $f : X \to \mathbb{R}$ is a definable function of $\mathbb{R}_{\text{an}}$, then $\operatorname{Max}(f) \in \mathbb{R}_{\text{an}}(\mathbb{R}^n)$ and $f(\operatorname{Max}(f))$ is a finite set.*

Example 2. Let $X_m := \{m\mathtt{x} \geq \mathtt{y} > (m-1)\mathtt{x} > 0\}$ and $X := \{\mathtt{x} > 0, \mathtt{y} > 0\} = \bigsqcup_{m\geq 1} X_m$. Consider the continuous function

$$g : X = \bigsqcup_{k\geq 1}(X_{2k} \cup X_{2k-1}) \to \mathbb{R},\ (x, y) \mapsto \begin{cases} \frac{y}{x} - k & \text{if } (x, y) \in X_{2k}, \\ k - 1 & \text{if } (x, y) \in X_{2k-1}, \end{cases}$$

whose graph is $\Gamma := \bigsqcup_{k\geq 1}(\Gamma_{2k} \cup \Gamma_{2k-1})$, where

$$\Gamma_{2k-1} := X_{2k-1} \times \{k-1\} \quad \text{and} \quad \Gamma_{2k} := \{2k\mathtt{x} \geq \mathtt{y} > (2k-1)\mathtt{x} > 0, \mathtt{xz} = \mathtt{y} - k\mathtt{x}\}.$$

Thus, g is C-semianalytic but $\operatorname{Max}(g)$ is not subanalytic.

Example 3. Let $g : \mathbb{R}^2 \to \mathbb{R}^3$, $(x_1, x_2) \mapsto (x_1, x_1x_2, x_1e^{x_2})$ and $X := g([-1, 1]^2) \setminus \{(0, 0, 0)\}$, which is not a semianalytic subset of $\mathbb{R}^2$. Consider the continuous function

$$f : X \to \mathbb{R},\ (x_1, x_2, x_3) \mapsto \frac{x_2}{x_1},$$

whose graph is a C-semianalytic subset of $\mathbb{R}^4$. Thus, f is a C-semianalytic function and it does not behave well on fibers (even as a subanalytic function).

Example 4. Define $Z_1 := \{\|\mathtt{x}\| \geq 1\}$, $Z_m := \{\frac{1}{m} \leq \|\mathtt{x}\| < \frac{1}{m-1}\}$ for $m \geq 2$ and $Z := \mathbb{R}^n \setminus \{0\} = \bigsqcup_{k\geq 1} Z_k$. Consider the non M-compact continuous function

$$g : Z \to \mathbb{R},\ x \mapsto \begin{cases} k - 1 & \text{if } x \in Z_{2k-1}, \\ \frac{1}{\|x\|} - k & \text{if } x \in Z_{2k}. \end{cases}$$

The graph of g is $\Gamma_g := \bigsqcup_{k\geq 1}(\Gamma_{2k} \cup \Gamma_{2k-1})$, where

$$\Gamma_{2k-1} := Z_{2k-1} \times \{k-1\} \quad \text{and} \quad \Gamma_{2k} := \{2k \geq \tfrac{1}{\|\mathtt{x}\|} > 2k-1, \mathtt{z} = \tfrac{1}{\|\mathtt{x}\|} - k\}.$$

The set Γ_g is a C-semianalytic subset of $\mathbb{R}^n$ because $\{\Gamma_{2k-1}, \Gamma_{2k}\}_{k\geq 1}$ is locally finite in $\mathbb{R}^n$. The set $\operatorname{Max}(g) = \bigsqcup_{k\geq 1} Z_{2k-1}$ is not subanalytic because the family of its connected components is not locally finite in $\mathbb{R}^n$.

Related Results

Let $f : X \to \mathbb{R}$ be a function. We say that f is *open* if it maps open subsets of X onto open subsets of $\mathbb{R}$. The function f is *open at $x \in X$* if $f(x)$ belongs to $\operatorname{Int}(f(U))$ for each open neighborhood $U \subset X$ of x. Thus, f is open if and only if it is open at every point $x \in X$.

Lemma 3 (Non-openness points, [2])**.** *If X is a locally connected topological space and a function $f : X \to \mathbb{R}$ is continuous, the set $\operatorname{Extr}(f)$ of points of local extrema of f coincides with the set $\operatorname{NOp}(f)$ of its non-openness points.*

If $\mathfrak{F}$ is a weak category and $f : X \to \mathbb{R}$ is a continuous subanalytic function on a closed set $X \in \mathfrak{F}(M)$ that behaves well on fibers, $\operatorname{NOp}(f) \in \mathfrak{F}(M)$ and $\operatorname{Op}(f) := X \setminus \operatorname{Extr}(f) \in \mathfrak{F}(M)$.

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