Polynomial and Regular Images of $\mathbb{R}^n$

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Introduction

A map $f := (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is *polynomial* if its components f_k are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials g_k, h_k such that h_k never vanishes on $\mathbb{R}^n$. By Tarski-Seidenberg's principle the image of an either polynomial or regular map is a *semialgebraic set*, that is, it has a description by a finite boolean combination of polynomial equalities and inequalities. In 1990 *Oberwolfach reelle algebraische Geometrie* week Gamboa proposed:

Main Problem. Characterize the semialgebraic sets in $\mathbb{R}^m$ which are either polynomial or regular images of some $\mathbb{R}^n$.

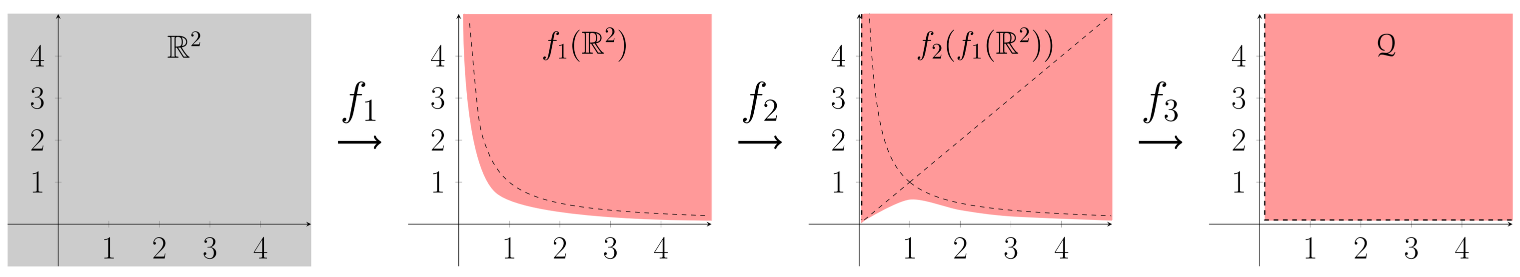
Two approaches to this problem: **(1) Explicit construction** of polynomial and regular representations for large families of semialgebraic sets, so far with **piecewise linear boundary**; and **(2) Search for obstructions** to be polynomial/regular images of $\mathbb{R}^n$. **Potential applications.** Optimization, Positivstellensätze or parametrizations of semialgebraic sets.

The Open Quadrant Problem

Is the set $\mathcal{Q} := \{x > 0, y > 0\} \subset \mathbb{R}^2$ a polynomial image of $\mathbb{R}^2$? Answer: **YES**

First solution. The initial answer was presented in 2002 *Oberwolfach reelle algebraische Geometrie* week. Required computer assistance for Sturm's algorithm.

Second solution. The shortest proof (sketched below).



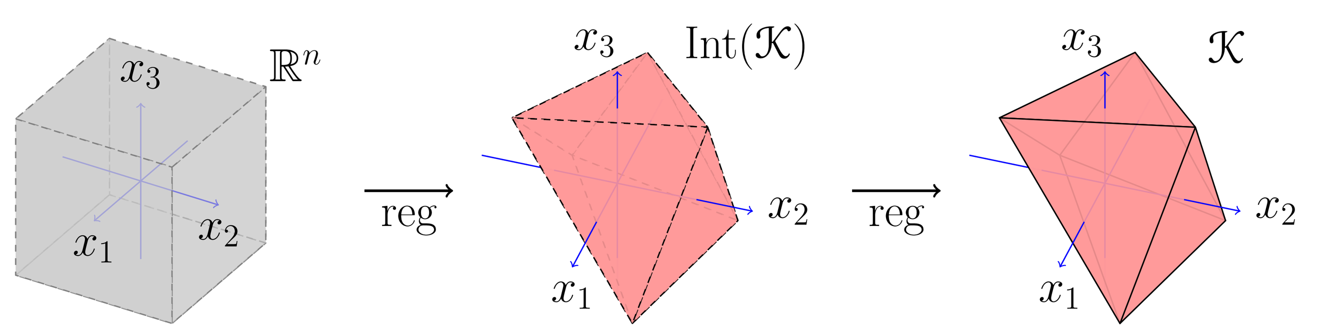
$f_1(x,y) := ((xy-1)^2 + x^2, (xy-1)^2 + y^2)$, $\quad f_2(x,y) := (x, y(xy-2)^2 + x(xy-1)^2)$, $\quad f_3(x,y) := (x(xy-2)^2 + \frac{1}{2}xy^2, y)$.

Third solution. The sparsest (known) polynomial map. A topological argument shows that the image of the map below is $\mathcal{Q}$.

$$f(x,y) := ((x^2y^4 + x^4y^2 - y^2 - 1)^2 + x^6y^4, (x^6y^2 + x^2y^2 - x^2 - 1)^2 + x^6y^4).$$

On Convex Polyhedra

Theorem 1. *An n-dimensional convex polyhedron and its interior are regular images of $\mathbb{R}^n$ $(n \geq 2)$.*



Definition. Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron. Its *recession cone* is

$$\vec{\mathcal{C}}(\mathcal{K}) := \{\vec{v} \in \mathbb{R}^n : p + \lambda\vec{v} \in \mathcal{K} \quad \forall p \in \mathcal{K}, \quad \lambda \geq 0\}.$$

Theorem 2. *Let $\mathcal{K} \subsetneq \mathbb{R}^n$ be an unbounded, n-dimensional convex polyhedron whose recession cone $\vec{\mathcal{C}}(\mathcal{K})$ is n-dimensional. Then $\mathcal{K}$ is a polynomial image of $\mathbb{R}^n$. In addition, if $\mathcal{K}$ has not bounded facets, then $\mathrm{Int}(\mathcal{K})$ is also a polynomial image of $\mathbb{R}^n$.*

Theorem 3. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron that is not affinely equivalent to a layer $[-a, a] \times \mathbb{R}^{n-1}$. Then the semialgebraic sets $\mathbb{R}^n \setminus \mathcal{K}$ and $\mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K})$ are polynomial images of $\mathbb{R}^n$.*

Full picture for convex polyhedra

Definition of p and r invariants:

$\mathrm{p}(\mathcal{S}) := \min\{n \in \mathbb{N} : \mathcal{S} = f(\mathbb{R}^n), f \ \text{ polynomial}\}$

$\mathrm{r}(\mathcal{S}) := \min\{n \in \mathbb{N} : \mathcal{S} = f(\mathbb{R}^n), f \ \text{ regular}\}$

<table>
<tr><th>$\mathcal{K}$ conv. pol.</th><th colspan="2">$\mathcal{K}$ bounded</th><th colspan="2">$\mathcal{K}$ unbounded</th></tr>
<tr><th>$\mathcal{S} = \mathbb{R}^n \setminus \mathcal{K}$</th><th>$n = 1$</th><th>$n \geq 2$</th><th>$n = 1$</th><th>$n \geq 2$</th></tr>
<tr><td>$\mathrm{r}(\mathcal{K})$</td><td>1</td><td rowspan="2">n</td><td>1</td><td rowspan="2">n</td></tr>
<tr><td>$\mathrm{r}(\mathrm{Int}(\mathcal{K}))$</td><td>2</td><td>2</td></tr>
<tr><td>$\mathrm{p}(\mathcal{K})$</td><td colspan="2" rowspan="2">$+\infty$</td><td>1</td><td>$n, +\infty$</td></tr>
<tr><td>$\mathrm{p}(\mathrm{Int}(\mathcal{K}))$</td><td>2</td><td>$n, n+1, +\infty$</td></tr>
<tr><td>$\mathrm{r}(\mathcal{S})$</td><td rowspan="4">$+\infty$</td><td rowspan="4">n</td><td>2</td><td rowspan="4">n</td></tr>
<tr><td>$\mathrm{r}(\overline{\mathcal{S}})$</td><td>1</td></tr>
<tr><td>$\mathrm{p}(\mathcal{S})$</td><td>2</td></tr>
<tr><td>$\mathrm{p}(\overline{\mathcal{S}})$</td><td>1</td></tr>
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General Properties

Basic properties. A regular image of $\mathbb{R}^n$ is **connected**, **irreducible** and **pure dimensional**. Polynomial images are in addition either **unbounded** or singletons and have either **unbounded** or singleton **projections**.

Advanced Properties. The *set of points at infinity* of $\mathcal{S} \subset \mathbb{R}^n \subset \mathbb{RP}^n$ is

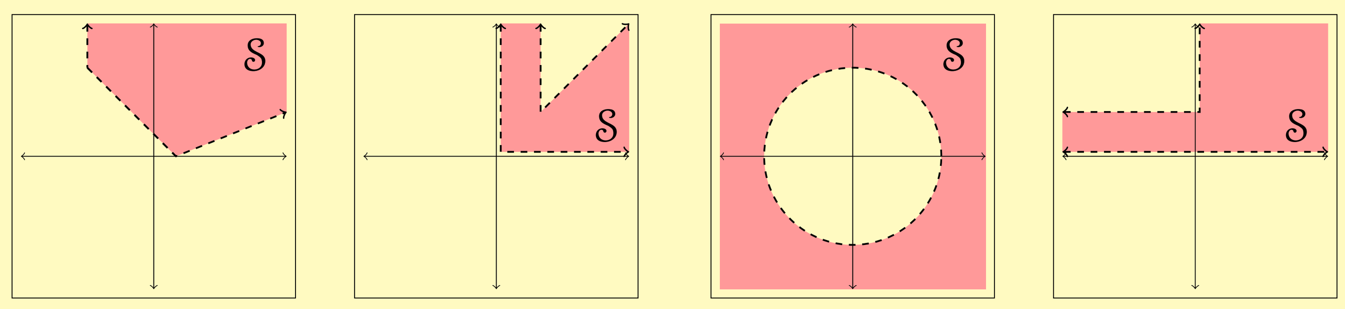
$$\mathcal{S}_\infty := \mathrm{Cl}_{\mathbb{RP}^n}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{R}) \qquad (\mathsf{H}_\infty(\mathbb{R}) \text{ hyperplane at infinity}).$$

Theorem 4. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a polynomial image of $\mathbb{R}^n$. Then $\mathcal{S}_\infty \neq \emptyset$ is connected.*

Remark. This condition does not hold in general for regular images.

Theorem 5. *Let $\mathcal{S} \subset \mathbb{R}^m$ be an n-dimensional polynomial image of $\mathbb{R}^n$. Let $\mathcal{T}$ be the set of points of dimension $n-1$ of $\mathrm{Cl}(\mathcal{S}) \setminus \mathcal{S}$. We have:*
(i) *For any $x \in \mathcal{T}$ there is a non-constant polynomial image Γ of $\mathbb{R}$ such that $x \in \Gamma \subset \overline{\mathcal{T}}^{\mathrm{zar}} \cap \mathrm{Cl}(\mathcal{S})$.*
(ii) *If $n = 2$, $\mathcal{T} \subset \bigcup_{i=1}^r \Gamma_i \subset \overline{\mathcal{T}}^{\mathrm{zar}} \cap \mathrm{Cl}(\mathcal{S})$ where each Γ_i is a polynomial image of $\mathbb{R}$.*

Which of the following open sets are polynomial images of $\mathbb{R}^2$?



ANS: The last one

Characterization for the 1-Dimensional Case

Let $\mathcal{S} \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set.

Theorem 6. *The following assertions are equivalent:*
(i) *$\mathcal{S}$ is a polynomial image of $\mathbb{R}^n$ for some $n \geq 1$.*
(ii) *$\mathcal{S}$ is irreducible, unbounded and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S})$ is an invariant rational curve such that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{C}) = \{p\}$ and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S})_p$ is irreducible.*

If that is the case, $\mathrm{p}(\mathcal{S}) \leq 2$. In addition, $\mathrm{p}(\mathcal{S}) = 1 \iff \mathcal{S}$ is closed in $\mathbb{R}^m$.

Theorem 7. *The following assertions are equivalent:*
(i) *$\mathcal{S}$ is a regular image of $\mathbb{R}^n$ for some $n \geq 1$.*
(ii) *$\mathcal{S}$ is irreducible and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(\mathcal{S})$ is a rational curve.*

If that is the case, then $\mathrm{r}(\mathcal{S}) \leq 2$. In addition, $\mathrm{r}(\mathcal{S}) = 1 \iff$ either $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) = \mathcal{S}$, or $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \setminus \mathcal{S} = \{p\}$ and the analytic closure of the germ $\mathcal{S}_p$ is irreducible.

$\mathcal{S}$	$\mathbb{R}$ or $[0, +\infty)$	$\nexists$	$[0, 1)$	$(0, +\infty)$	$(0, 1)$	Any non-rational algebraic curve
$\mathrm{r}(\mathcal{S})$	1	1	1	2	2	$+\infty$
$\mathrm{p}(\mathcal{S})$	1	2	$+\infty$	2	$+\infty$	$+\infty$

Related Problems

A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is *Nash* if each component of f is a *Nash function*, that is, a smooth function with semialgebraic graph. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d.

Shiota's conjecture. $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$ if and only if $\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0, 1] \to \mathcal{S}$ whose image meets all connected components of the set of regular points of $\mathcal{S}$.

Corollary 8. *Assume $\mathcal{S}$ is pure dimensional, irreducible and with arc-symmetric closure. Then $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*

Corollary 9. *Assume $\mathcal{S}$ is Nash path connected. Then $\mathcal{S}$ is the projection of an irreducible algebraic set $X \subset \mathbb{R}^n$ whose connected components are Nash diffeomorphic to $\mathbb{R}^d$. In addition, each connected component of X maps onto $\mathcal{S}$.*

Selected References

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