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Tesi di dottorato

Nash images of closed balls and applications

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To my beloved
'Cinese cattivo'

Contents

Chapter 1

Introduction

Although it is usually said that the first work in Real Geometry is due to Harnack [Hr], who obtained an upper bound for the number of connected components of a non-singular real algebraic curve in terms of its genus, modern Real Algebraic Geometry was born with Tarski's article [T], where it is proved that a projection of a semi-algebraic set is a semi-algebraic set.

We are interested in studying what might be called the 'inverse problem' to Tarski's result. A map $f := (f_1, \ldots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$ is *polynomial* if its components $f_k \in \mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_m]$ are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials $g_k, h_k \in \mathbb{R}[\mathtt{x}]$ such that h_k never vanishes on $\mathbb{R}^m$.

1.1. State of the art. In the 1990 *Oberwolfach reelle algebraische Geometrie week* [G] Gamboa proposed (see also Eisenbud's survey [E, §3.IV, p.69]):

Problem 1.1. To characterize the (semi-algebraic) subsets of $\mathbb{R}^n$ that are either polynomial or regular images of $\mathbb{R}^m$.

As specific examples of open questions he stated in Oberwolfach, we have:

1. Is the set $\{\mathtt{x}^2 + \mathtt{y}^2 > 1\}$ a polynomial image of $\mathbb{R}^2$?
2. Is the open quadrant $\{\mathtt{x} > 0,\ \mathtt{y} > 0\}$ a regular image of $\mathbb{R}^2$?

In 2002 Fernando and Gamboa answered both these questions in [FG1]. It constituted the starting point of the systematic study of the problem of representing semi-algebraic sets as polynomial or regular images of Euclidean spaces. They, jointly with Ueno, have attempted to understand better polynomial and regular images of $\mathbb{R}^m$ in the last two decades with the following main objectives:

- To find obstructions to be either polynomial or regular images.
- To prove (constructively) that large families of semi-algebraic sets with piecewise linear boundary (convex polyhedra, their interiors, complements and the interiors of their complements) are either polynomial or regular images of Euclidean spaces.

In [FG1, FG2, FU1, FGU2] are presented first steps to approach Problem 1.1. The most relevant one [FU1] shows that *the 'set of points at infinity' of $\mathcal{S}$ is a connected set.* In [Fe1] appears a complete solution to Problem 1.1 for the 1-dimensional case, whereas in [FGU1, FGU3, FGU5, FU2, FU3, FU4, FU5, FU6, U1, U2] it is provided a constructive full answer for the representation as either polynomial or regular images of the semi-algebraic sets with piecewise linear boundary commented above [FU4, Table 1]. A survey concerning these topics, which provides the reader a global idea of the state of the art, can be found in [FGU4].

The rigidity of polynomial and regular maps on $\mathbb{R}^m$ makes really difficult to approach Problem 1.1 in its full generality. The following example shows that even for 'simple' (whatever simple means) semi-algebraic sets, we do not know how to answer Problem 1.1.

Example 1.2. The semi-algebraic set $\mathcal{S} := \{\mathtt{y}^2 - \mathtt{x}^2 < 1\} \subset \mathbb{R}^2$ is not a polynomial image of $\mathbb{R}^2$ (see [FG1, Rmk.1.3(2)]). It is not known whether $\mathcal{S}$ is a polynomial image of $\mathbb{R}^n$ for some $n \geq 3$ or not. For the regular case we know that $\mathcal{S}$ is a regular image of $\mathbb{R}^3$ but it is not known if $\mathcal{S}$ is a regular image of $\mathbb{R}^2$ or not.

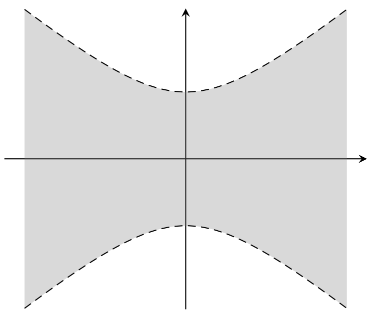

Figure 1.1: The semi-algebraic set $\mathcal{S} = \{\mathtt{y}^2 - \mathtt{x}^2 < 1\}$.

1.2. First alternative approach. At this point there are several possible ways to overcome the quoted difficulties. The first one is to change the polynomial and regular maps by more flexible maps like *Nash maps* (smooth semi-algebraic maps) [Fe4] or *regulous maps* (continuous rational maps) [FFQU]. Gamboa and Shiota proposed in 1990 to approach the following variant of Problem 1.1.

Problem 1.3. To characterize the (semi-algebraic) subsets of $\mathbb{R}^n$ that are Nash images of $\mathbb{R}^m$.

In 1990 Shiota proposed the following conjecture in order to provide a satisfactory answer to Problem 1.3.

Conjecture 1.4 (Shiota)**.** *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set of dimension d. Then, $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$ if and only if $\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all connected components of the set of regular points of $\mathcal{S}$.*

In [Fe4] Fernando provided a proof for Shiota's conjecture as a particular case of the following characterization of the semi-algebraic sets $\mathcal{S} \subset \mathbb{R}^n$ of dimension d that are images of affine spaces under Nash maps.

Theorem 1.5 (Nash images [Fe4, Thm.1.4]). *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set of dimension d. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*

(ii) *$\mathcal{S}$ is connected by Nash paths.*

(iii) *$\mathcal{S}$ is connected by analytic paths.*

(iv) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(v) *$\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(vi) *$\mathcal{S}$ is well-welded.*

The concept of well-welded semi-algebraic set will be recalled in Section 3.4.

1.3. Second alternative approach. Another possibility is to keep polynomial or regular functions and to change the domain of definition. If we consider a compact domain (and of course a compact image), we have more tools because for instance Weierstrass' polynomial approximation has an important role. The simplest compact semi-algebraic domains one can choose are either closed unit balls or unit spheres. In [KPS, §5.Prob.1] it is proposed the following concrete related problem:

Problem 1.6. Let $\mathcal{P}$ be an arbitrary (compact) convex polygon in $\mathbb{R}^2$. Construct explicit polynomials f and g in $\mathbb{R}[\mathtt{u},\mathtt{v},\mathtt{w}]$ such that $\mathcal{P} = (f,g)(\overline{\mathcal{B}}_3)$.

Sturmfels suggested Fernando and Ueno in 2018 to confront the previous problem, taking into account their knowledge in the subject of polynomial images of affine spaces. This suggestion was the starting point for the article [FU6], where it is made an extended study of the n-dimensional semi-algebraic subsets of $\mathbb{R}^n$ that are images under a polynomial map $f : \mathbb{R}^m \to \mathbb{R}^n$ of the m-dimensional closed unit ball $\overline{\mathcal{B}}_m$ for some $m \geq n$. A first main result in [FU6] is a strong generalization to arbitrary dimension of Problem 1.6.

Theorem 1.7 ([FU6, Thm.1.2]). *Let $\mathcal{S} \subset \mathbb{R}^n$ be the union of a finite family of n-dimensional convex (compact) polyhedra. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is connected by analytic paths.*

(ii) *There exists a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\overline{\mathcal{B}}_n) = \mathcal{S}$.*

The techniques involved to prove Theorem 1.7 are generalized in [FU6] to show the following result. A set $\mathcal{S} \subset \mathbb{R}^n$ is *strictly radially convex (with respect to a point $p \in \operatorname{Int}(\mathcal{S})$)* if for each ray ℓ with origin at p, the intersection $\ell \cap \mathcal{S}$ is a segment whose relative interior is contained in $\operatorname{Int}(\mathcal{S})$. Convex sets are particular examples of strictly radially convex sets (with respect to any of its interior points [Be, Lem.11.2.4]).

Theorem 1.8 ([FU6, Thm.1.3]). *Let $\mathcal{S} \subset \mathbb{R}^n$ be the union of a finite family of strictly radially convex semi-algebraic sets that are polynomial images of the closed unit ball $\overline{\mathcal{B}}_m$. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is connected by analytic paths.*

(ii) *There exists a polynomial map $f : \mathbb{R}^{m+1} \to \mathbb{R}^n$ such that $f(\overline{\mathcal{B}}_{m+1}) = \mathcal{S}$.*

1.4. First main result. Our starting point has been to combine both alternative approaches:

(1) To work with Nash maps instead of polynomial or regular maps.

(2) To work with closed unit balls instead of affine spaces.

Our first main result in this dissertation is the characterization of the compact semi-algebraic sets $\mathcal{S} \subset \mathbb{R}^n$ that are images of closed unit balls under Nash maps.

The statement of Theorem 1.5 does not take into account if $\mathcal{S}$ is compact or not and the involved Nash maps are rarely proper if $d \geq 2$. As closed unit balls $\overline{\mathcal{B}}_d$ are compact, the restrictions to $\overline{\mathcal{B}}_d$ of Nash maps are always proper maps.

Theorem 1.9 (Compact Nash images). *Let $\mathcal{S} \subset \mathbb{R}^n$ be a d-dimensional compact semi-algebraic set. The following assertions are equivalent:*

(i) *There exists a Nash map $f : \mathbb{R}^d \to \mathbb{R}^n$ such that $f(\overline{\mathcal{B}}_d) = \mathcal{S}$.*

(ii) *$\mathcal{S}$ is connected by Nash paths.*

(iii) *$\mathcal{S}$ is connected by analytic paths.*

(iv) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(v) *$\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(vi) *$\mathcal{S}$ is well-welded.*

Chapter 3 will be dedicated to prove this theorem. We start by showing in Section 3.1 that there exist polynomial maps $\mathbb{R}^d \to \mathbb{R}^d$ that transform:

(i) the standard sphere $\mathbb{S}^d$ onto the unit closed ball $\overline{\mathcal{B}}_d$ (Proposition 3.1.1),

(ii) the unit closed ball $\overline{\mathcal{B}}_d$ onto the cylinder $\overline{\mathcal{B}}_{d-1} \times [-1,1]$ (Proposition 3.1.2),

(iii) the cylinder $\overline{\mathcal{B}}_{d-1} \times [-1,1]$ onto the simplicial prism $\Delta_{d-1} \times [-1,1]$ (Corollary 3.1.4), where

$$\Delta_{d-1} := \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_{d-1} \geq 0,\ \mathtt{x}_1 + \cdots + \mathtt{x}_{d-1} \leq 1\} \subset \mathbb{R}^d,$$

(iv) the cylinder $\overline{\mathcal{B}}_{d-1} \times [-1,1]$ onto the hypercube $[-1,1]^d$ (Corollary 3.1.5).

In addition, there exist a polynomial map from the hypercube onto the unit closed ball (Proposition 3.1.6), a polynomial map from the simplicial prism onto the unit closed ball (Proposition 3.1.8) and a regular map from the hypercube onto the sphere (Proposition 3.1.9). Thus, we can take indistinctly as models to represent a compact semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ (connected by analytic paths) as image under Nash maps $\mathbb{R}^d \to \mathbb{R}^n$ any of the previous semi-algebraic sets in (i)-(iv). In fact, we will provide representations of $\mathcal{S}$ as image under Nash maps $\mathbb{R}^d \to \mathbb{R}^n$ of each of the previous models. The technicalities of the constructions we develop in this work make the simplicial prism $\Delta_{d-1} \times [-1, 1]$ the most suitable model to develop the main construction.

A main step to prove Theorem 1.9, which has interest by its own, is the following result:

Theorem 1.10. *Let $\mathcal{T} \subset \mathbb{R}^n$ be a compact checkerboard set of dimension d. Then, there exists a Nash map $f : \Delta_{d-1} \times [0, 1] \to \mathbb{R}^n$ such that*

$$f(\Delta_{d-1} \times [0, 1]) = \mathcal{T}.$$

The concept of checkerboard set will be recalled in Section 3.5. In particular, connected Nash manifolds $\mathcal{Q} \subset \mathbb{R}^n$ with (divisorial) corners can be embedded as checkerboard sets in some $\mathbb{R}^m$ [Fe4, Lem.8.3]. The precise definition of Nash manifold with (divisorial) corners appears in Section 2.5.3.

In Section 3.6.1 we treat separately the 1-dimensional case and we characterize 1-dimensional Nash images of closed balls in terms of their irreducibility. The ring $\mathcal{N}(\mathcal{S})$ of Nash functions on a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is a Noetherian ring [FG3, Thm.2.9] and we say that $\mathcal{S}$ is *irreducible* if and only if $\mathcal{N}(\mathcal{S})$ is an integral domain [FG3, §3].

Proposition 1.11 (The 1-dimensional case)**.** *Let $\mathcal{S} \subset \mathbb{R}^n$ be a 1-dimensional compact semi-algebraic set. Then $\mathcal{S}$ is a Nash image of some $\overline{\mathcal{B}}_m$ if and only if $\mathcal{S}$ is irreducible. In addition, if such is the case $\mathcal{S}$ is a Nash image of* $[-1, 1]$.

1.5. Consequences of the first main result. As consequences of Theorem 1.9 we will show in Chapter 3 the following.

1.5.1. General Nash images. Once we have completely characterised the Nash images of the closed ball, a natural question arises:

Problem 1.12. To determine all possible compact models that allow us to represent a compact semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ of dimension d connected by analytic paths as a Nash image.

This question is not trivial and different classes of semi-algebraic functions might have different answers. For instance, the family of polynomial images of the closed ball and the one of the sphere are different. In the Nash case we are able to provide a complete characterization of the compact models as a consequence of Theorem 1.9 and the following result (whose proof is contained in Section 3.7):

Theorem 1.13 (Bärchen-Schäfchen's Theorem)**.** *Let $\mathcal{T} \subset \mathbb{R}^m$ be any semi-algebraic set of dimension d. Then, there exists a regular map $f : \mathbb{R}^m \to \mathbb{R}^d$ such that $f(\mathcal{T}) = \overline{\mathcal{B}}_d$.*

It is natural now to wonder if the previous result extends to pairs of general semi-algebraic sets non necessarily compact.

Problem 1.14. To determine all possible models that allow us to represent a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ of dimension d connected by analytic paths as a Nash image.

If $\mathcal{S} \subset \mathbb{R}^n$ is non-compact and $\mathcal{T} \subset \mathbb{R}^m$ is compact, there exists no Nash map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $f(\mathcal{T}) = \mathcal{S}$. In Section 3.7 we prove the following:

Theorem 1.15. *Let $\mathcal{T} \subset \mathbb{R}^m$ be a semi-algebraic set and let $d \geq 2$. Assume that $\mathrm{Cl}(\mathcal{T}_d) \cap \mathcal{T}$ is not compact. Then, there exists a Nash map $f : \mathbb{R}^m \to \mathbb{R}^d$ such that $f(\mathcal{T}) = \mathbb{R}^d$.*

Combining the previous two results with Theorem 1.9 and [Fe4, Thm.1.4] we obtain the following satisfactory answer to Problem 1.14:

Theorem 1.16. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set of dimension $d \geq 2$ connected by analytic paths. For each semi-algebraic set $\mathcal{T} \subset \mathbb{R}^m$ with $d \leq \dim(\mathcal{T})$, such that $\mathrm{Cl}(\mathcal{T}_e) \cap \mathcal{T}$ is non-compact for some $d \leq e \leq \dim(\mathcal{T})$ in case $\mathcal{S}$ is non-compact, there exists a Nash map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $f(\mathcal{T}) = \mathcal{S}$.*

1.5.2. General surjective Nash maps between semi-algebraic sets Once established a satisfactory classification (both for the compact and non-compact case) of the possible models to represent semi-algebraic sets connected by analytic paths as Nash images, a natural question at this point is to determine until what extend we can represent *general* semi-algebraic sets as Nash images. Observe that the image of a semi-algebraic set connected by analytic paths under a Nash map is connected by analytic paths. In addition, the image of an irreducible semi-algebraic set under a Nash map is an irreducible semi-algebraic set [FG3, §3.1].

Thus, obstructions to construct a surjective Nash map $f : \mathcal{S} \to \mathcal{T}$ between arbitrary semi-algebraic sets $\mathcal{S} \subset \mathbb{R}^m$ and $\mathcal{T} \subset \mathbb{R}^n$concentrate on the configuration of the intersections of pairwise different analytic path-connected components $\{\mathcal{S}_i\}_{i=1}^r$ (resp. irreducible components $\{\mathcal{S}_j^*\}_{j=1}^\ell$) of $\mathcal{S}$ and the configuration of their images, which are semi-algebraic subsets $\mathcal{T}_i := f(\mathcal{S}_i)$ of $\mathcal{T}$ connected by analytic paths (resp. irreducible semi-algebraic subsets $\mathcal{T}_j^* := f(\mathcal{S}_j^*)$ of $\mathcal{T}$).

In order to soften these obstructions we will assume that each irreducible component $\mathcal{S}_i^*$ of $\mathcal{S}$ is mapped onto an analytic path-connected component $\mathcal{T}_i$ of $\mathcal{T}$ and that $\bigcap_{i=1}^r f(\mathcal{T}_i) \neq \varnothing$. Under this type of assumptions we propose the following characterization (whose proof is contained in Section 3.8.3).

Theorem 1.17 (Surjective Nash maps)**.** *Let $\mathcal{S} \subset \mathbb{R}^m$ and $\mathcal{T} \subset \mathbb{R}^n$ be semi-algebraic sets, let $\{\mathcal{S}_i^*\}_{i=1}^r$ be the irreducible components of $\mathcal{S}$ and let $\{\mathcal{T}_i\}_{i=1}^r$ be a family of (non-necessarily distinct) semi-algebraic subsets of $\mathcal{T}$ connected by analytic paths such that $\bigcap_{i=1}^r \mathcal{T}_i \neq \varnothing$. Denote $d_i := \dim(\mathcal{S}_i^*)$ and assume that the set $\mathcal{S}_i^{*,(d_i)}$ of points of $\mathcal{S}_i^*$ of dimension d_i is non-compact if $\mathcal{T}_i$ is non-compact for $i = 1, \ldots, r$. Then, there exists a Nash map $f : \mathcal{S} \to \mathcal{T}$ such that $f(\mathcal{S}_i^*) = \mathcal{T}_i$ for $i = 1, \ldots, r$ if and only if $e_i := \dim(\mathcal{T}_i) \leq \dim(\mathcal{S}_i^*) =: d_i$ for $i = 1, \ldots, r$.*

1.5.3. Representation of arc-symmetric semi-algebraic sets. Arc-symmetric semi-algebraic sets were introduced by Kurdyka in [K] and subsequently studied by many authors. Recall that a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *arc-symmetric* if for each analytic arc $\gamma : (-1,1) \to \mathbb{R}^n$ with $\gamma((-1,0)) \subset \mathcal{S}$ it holds that $\gamma((-1,1)) \subset \mathcal{S}$. In particular arc-symmetric semi-algebraic sets are closed subsets of $\mathbb{R}^n$. An arc-symmetric semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *irreducible* if it cannot be written as the union of two proper arc-symmetric semi-algebraic subsets [K, §2]. As a consequence of Theorem 3.2 and [K, Cor.2.8] we will show in Section 3.9.1 that a pure dimensional compact irreducible arc-symmetric semi-algebraic set is a Nash image of $\overline{\mathcal{B}}_d$ where $d := \dim(\mathcal{S})$.

Corollary 1.18. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional compact irreducible arc-symmetric semi-algebraic set of dimension d. Then $\mathcal{S}$ is a Nash image of $\overline{\mathcal{B}}_d$.*

1.5.4. Elimination of inequalities. A converse problem to Tarski's theorem is to find an algebraic set in $\mathbb{R}^{n+k}$ whose projection is a given semi-algebraic subset of $\mathbb{R}^n$. This is known as the *problem of eliminating inequalities*. Motzkin proved in [Mo] that this problem always has a solution for $k = 1$. However, his solution is rather complicated and is generally a reducible algebraic set. In another direction Andradas and Gamboa proved in [AG1, AG2] that if $\mathcal{S} \subset \mathbb{R}^n$ is a closed semi-algebraic set whose Zariski closure is irreducible, then $\mathcal{S}$ is the projection of an irreducible algebraic set in some $\mathbb{R}^{n+k}$. In [P] Pecker provides some improvements on both results: for the first by finding a construction of an algebraic set in $\mathbb{R}^{n+1}$ that projects onto the given semi-algebraic subset of $\mathbb{R}^n$, far simpler than the original construction of Motzkin; for the second by proving that if $\mathcal{S}$ is a locally closed semi-algebraic subset of $\mathbb{R}^n$ with an interior point, then $\mathcal{S}$ is the projection of an irreducible algebraic subset of $\mathbb{R}^{n+1}$. In [Fe4] it is proved that each semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is the image of a non-singular algebraic set $X \subset \mathbb{R}^{n+k}$ whose connected components are Nash diffeomorphic to affine spaces (maybe of different dimensions).

In this work we improve the previous result if $\mathcal{S}$ is compact and we prove that there exists an algebraic set $X \subset \mathbb{R}^{2d+1}$, where $d := \dim(\mathcal{S})$, that is Nash diffeomorphic to a finite pairwise disjoint union of spheres (maybe of different dimensions) that project onto $\mathcal{S}$. In Section 3.9.2 we show the following result that provides a non-singular compact algebraic set with the simplest possible topology that projects onto a compact semi-algebraic set.

Corollary 1.19. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact semi-algebraic set of dimension d. We have:*

(i) If $\mathcal{S}$ is connected by analytic paths, it is the projection of an irreducible compact non-singular algebraic set $X \subset \mathbb{R}^{n+k}$ (for some $k \geq 0$) that has at most two connected components Nash diffeomorphic to the sphere $\mathbb{S}^d$. In addition,

(1) Each connected component of X projects onto $\mathcal{S}$.

(2) There exists an automorphism of X that swaps both connected components of X.

(ii) In general $\mathcal{S}$ is the projection of an algebraic set $X \subset \mathbb{R}^{n+k}$ (for some $k \geq 0$) of dimension d that is Nash diffeomorphic to a finite pairwise disjoint union (of dimension d) of spheres (maybe of different dimensions).

Even for dimension 1, it is not possible to impose the connectedness of X (see Lemma 3.9.5 and Example 3.9.6). Contrast the previous result with [Fe4, Cor.1.8].

1.6. Second main result. Once we have completed a characterization of Nash images of closed balls, a natural question at this point is to determine until what extend we can represent semi-algebraic sets connected by analytic paths using polynomial maps. Polynomial images of models connected by polynomial paths (e.g. Euclidean spaces, closed balls etc.) are connected by polynomial paths. In general, semi-algebraic sets do not contain rational paths. By [C, V] a generic complex hypersurface Z of $\mathbb{CP}^m$ of degree $d \geq 2m-2$ for $m \geq 4$ and of degree $d \geq 2m-1$ for $m = 2, 3$ does not contain rational curves. If $\mathcal{S}$ is a semi-algebraic set whose Zariski closure in $\mathbb{RP}^m$ is a generic hypersurface of high enough degree, then its Zariski closure Z in $\mathbb{CP}^m$ does not contains rational curves, so $\mathcal{S}$ cannot contain rational paths. This means in particular that general semi-algebraic sets do not contain polynomial paths.

In Chapter 4 we show that if $\mathcal{S} \subset \mathbb{R}^m$ is a closed semi-algebraic set connected by analytic paths, then $\mathcal{S}$ is the image under a proper polynomial map of a Nash manifold with corners of the same dimension. In fact, there exists an algebraic set of smaller dimension such that the restriction of the polynomial map to the Nash manifold with corners minus this algebraic set is a Nash diffeomorphism onto its image.

Theorem 1.20. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a d-dimensional closed semi-algebraic set connected by analytic paths. Then there exist:*

(i) *A d-dimensional non-singular irreducible algebraic set $X \subset \mathbb{R}^n$ and a normal-crossings divisor $Y \subset X$.*

(ii) *A connected Nash manifold with corners $\mathcal{Q} \subset X$ (which is a closed subset of X) whose boundary $\partial\mathcal{Q}$ has Y as its Zariski closure.*

(iii) *A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that the restriction $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{S}$ is proper and $f(\mathcal{Q}) = \mathcal{S}$.*

(iv) *A closed semi-algebraic set $\mathcal{R} \subset \mathcal{S}$ of dimension strictly smaller than d such that $\mathcal{S} \setminus \mathcal{R}$ and $\mathcal{Q} \setminus f^{-1}(\mathcal{R})$ are Nash manifolds and the polynomial map $f|_{\mathcal{Q}\setminus f^{-1}(\mathcal{R})} : \mathcal{Q} \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism.*

If $\mathcal{S} \subset \mathbb{R}^m$ is a general semi-algebraic set connected by analytic paths, one can wonder if it is possible to provide a similar result that also works for $\mathcal{S}$. As the chosen Nash manifold with corners $\mathcal{Q}$ is closed in its Zariski closure and the chosen polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ restricts to a proper map $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^m$, its image $\mathcal{S}$ is a closed subset of $\mathbb{R}^m$. Thus, if $\mathcal{S}$ is not closed in $\mathbb{R}^m$, we should change the type of domain and/or the type of map. The second approach considering general Nash maps non-necessarily proper has been developed in [Fe4, Proof of Thm.1.4, §8.C.12] and it is shown that if the involved Nash map is not necessarily proper, then there exists a Nash manifold H with smooth boundary and a surjective Nash map $f : H \to \mathcal{S}$. If one wants to keep the properness condition, it is not possible to keep as domains Nash manifolds $\mathcal{Q}$ with corners because they are locally compact and images of locally compact subsets of $\mathbb{R}^n$ under proper maps are locally compact subsets of $\mathbb{R}^m$. Thus, we have to change the type of involved domains and we will consider semi-algebraic sets $\mathcal{T} \subset \mathbb{R}^n$ whose closure is a Nash manifold with corners $\mathcal{Q} \subset \mathbb{R}^n$ and $\mathcal{Q} \setminus \mathcal{T}$ is a union of some of the strata of a suitable stratification of $\partial\mathcal{Q}$.

Theorem 1.21. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a d-dimensional semi-algebraic set connected by analytic paths. Then there exist:*

(i) *A d-dimensional connected compact non-singular algebraic set $M \subset \mathbb{R}^n$ and a normal-crossings divisor $Y \subset M$.*

(ii) *A connected Nash quasi-manifold with corners $\mathcal{S}^\bullet \subset M$ that is a checkerboard set and whose closure in M is a compact connected Nash manifold with corners $\mathcal{Q}^\bullet \subset M$ whose boundary $\partial\mathcal{Q}^\bullet$ has Y as its Zarsiki closure.*

(iii) *A Nash map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that the restriction $f|_{\mathcal{S}^\bullet} : \mathcal{S}^\bullet \to \mathcal{S}$ is proper and $f(\mathcal{S}^\bullet) = \mathcal{S}$.*

(iv) *A closed semi-algebraic set $\mathcal{R} \subset \mathcal{S}$ of dimension strictly smaller than d such that $\mathcal{S} \setminus \mathcal{R}$ and $\mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R})$ are Nash manifolds and the Nash map $f|_{\mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R})} : \mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism.*

A Nash quasi-manifold with corners is a Nash manifold with corners with some faces erased (the precise definition is included in Section 4.3).

1.7. Third main result. The study of Nash images of closed balls took us to work closely with Nash manifolds with corners. In two recent papers Fernando and Ghiloni [FGh, FGh2] obtained approximations result in the semi-algebraic and smooth settings when the target space has singularities provided it admits 'nice' triangulations. Motivated by their work and our study of Nash manifolds with corners we dealt with the following problem:

Problem 1.22. Let $\mathcal{S} \subset \mathbb{R}^m$ be a locally compact semi-algebraic set, let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with (divisorial) corners and let $h : \mathcal{S} \to \mathcal{Q}$ be a proper continuous semi-algebraic map. Does there exist a Nash map $g : \mathcal{S} \to \mathcal{Q}$ arbitrarily close to h in the $\mathcal{C}^0$ semi-algebraic topology?

In the article [FGR] Fernando, Gamboa and Ruiz proved that given a Nash manifold $\mathcal{Q} \subset \mathbb{R}^n$ with corners it is contained as a closed subset in a Nash manifold $M \subset \mathbb{R}^n$ of the same dimension and the behaviour of the Nash closure of its boundary is the suitable one. We will show in Chapter 5 that the Nash manifold M can be 'folded' to reconstruct the manifold with corners $\mathcal{Q}$. That is, there exists a surjective Nash map $M \to \mathcal{Q}$ such that the restriction to $\mathcal{Q}$ is close to the identity and preserves the stratification of the boundary $\partial\mathcal{Q}$. The construction we present there, even if it requires some technicalities, is geometrical and neat.

Theorem 1.23 (Folding Nash manifolds)**.** *Let $\mathcal{Q} \subset \mathbb{R}^n$ be a d-dimensional Nash manifold with corners. Then, there exist*

(i) A d-dimensional Nash manifold $M \subset \mathbb{R}^n$ that contains $\mathcal{Q}$ as a closed subset.

(ii) A Nash normal-crossings divisor $Y \subset M$ that is the smallest Nash subset of M that contains $\partial\mathcal{Q}$ and satisfies $\mathcal{Q} \cap Y = \partial\mathcal{Q}$.

(iii) A Nash map $f : M \to \mathcal{Q}$ such that $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is a semi-algebraic homeomorphism close to the identity map and $f|_{\mathrm{Int}(\mathcal{Q})} : \mathrm{Int}(\mathcal{Q}) \to \mathrm{Int}(\mathcal{Q})$ is a Nash diffeomorphism.

In addition, for each $x \in \partial\mathcal{Q}$ there exist open semi-algebraic neighbourhoods $U, V \subset M$ of x equipped with Nash diffeomorphisms $\varphi : U \to \mathbb{R}^d$ and $\psi : V \to \mathbb{R}^d$ and $1 \le s \le d$ such that

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1, \dots, x_d) \mapsto (x_1^2, \dots, x_s^2, x_{s+1}, \dots, x_d).$$

This result, that has interest by its own, has remarkable consequences. In particular, it allows us to answer Problem 1.22.

Theorem 1.24 (Nash approximation)**.** *Let $\mathcal{S} \subset \mathbb{R}^m$ be a locally compact semi-algebraic set and let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners. Let $h : \mathcal{S} \to \mathcal{Q}$ be a proper continuous semi-algebraic map. Then there exist Nash maps $g : \mathcal{S} \to \mathcal{Q}$ arbitrarily close to h with respect to the $\mathcal{C}^0$ semi-algebraic topology.*

A second consequence of our construction is a variant of Theorem 1.20. A similar result changing $\mathcal{Q}$ by a Nash manifold with boundary seems difficult to be achieved if we want to keep that the map f is polynomial, so we will show in Section 5.3.1 that a closed semi-algebraic set $\mathcal{S}$ connected by analytic paths can be 'resolved' by a Nash manifold with boundary, up to consider Nash maps instead of polynomial ones.

Theorem 1.25. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a d-dimensional closed semi-algebraic set connected by analytic paths and let $\varepsilon > 0$. Then there exist:*

(i) *A d-dimensional non-singular algebraic set $X \subset \mathbb{R}^m$.*

(ii) *A Nash manifold with boundary $\mathcal{H}_\varepsilon \subset \mathbb{R}^m$ such that the Zariski closure Z_ε of $\partial\mathcal{H}_\varepsilon$ is a non-singular algebraic set contained in X of dimension $d-1$ and* $\mathrm{Int}(\mathcal{H}_\varepsilon)$ *is a connected component of $X \setminus Z_\varepsilon$.*

(iii) *A proper Nash map $f : \mathcal{H}_\varepsilon \to \mathcal{S}$ such that $f(\mathcal{H}_\varepsilon) = \mathcal{S}$.*

(iv) *The restriction $f|_{\mathcal{H}_\varepsilon \setminus f^{-1}(\mathcal{T}_\varepsilon)} : \mathcal{H}_\varepsilon \setminus f^{-1}(\mathcal{T}_\varepsilon) \to \mathcal{S} \setminus \mathcal{T}_\varepsilon$ is a Nash diffeomorphism, where $\mathcal{T}_\varepsilon := \{x \in \mathcal{S} :\ \mathrm{dist}(x, \mathcal{R}) \le \varepsilon\}$ for a certain closed semi-algebraic set $\mathcal{R} \subset \mathcal{S}$ of dimension strictly smaller than d.*

In Section 5.3.2 we will also provide an alternative characterization of the Nash images of the closed ball, taking advantage of this new technique of 'resolution' of semi-algebraic sets by Nash manifolds with boundary.

The results presented in this dissertation will be collected in the articles [CF1] (mainly those results in Chapter 3) and [CF2] (mainly those results in Chapters 4 and 5).

Chapter 2

Preliminaries

2.1 Real algebraic sets

The main goal of algebraic geometry is the study of those subsets of k^n defined as common zero sets of polynomials in $k[\mathtt{x}_1, \ldots, \mathtt{x}_n]$, where k is a field. The setting with a nicer behaviour arises when k is algebraically closed. In particular, the case $k = \mathbb{C}$ has deserved major attention, and complex algebraic geometry is a central part in mathematics. The non algebraically closed case often leads to surprising results - at least for classical algebraic geometers (see also [FGh3]). The reason is that the couple 'algebraic geometry-commutative algebra' does not have a happy marriage in the non algebraically closed case. In the following, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. A subset $V \subset \mathbb{K}^n$ is called an *algebraic set* if it can be represented as

$$V = \mathcal{Z}(S) := \{x \in \mathbb{K}^n : p(x) = 0 \ \ \forall p \in S\},$$

where $S \subset \mathbb{K}[\mathtt{x}] := \mathbb{K}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ is a non-empty subset.

If I is the ideal generated by a non-empty subset $S \subset \mathbb{K}[\mathtt{x}]$, it is straightforward to check that $\mathcal{Z}(I) = \mathcal{Z}(S)$. Hilbert's basis theorem [AM, Thm.7.5] asserts that $\mathbb{K}[\mathtt{x}]$ is a Noetherian ring. In particular each ideal $I \subset \mathbb{K}[\mathtt{x}]$ is finitely generated, so every algebraic set $V \subset \mathbb{K}^n$ is the zero set of a finite family of polynomials. If $\mathbb{K} = \mathbb{R}$, we can actually use a single equation by means of a sum of squares.

Proposition 2.1.2. *Let $V \subset \mathbb{R}^n$ be an algebraic set. Then, there exists a polynomial $p \in \mathbb{R}[\mathtt{x}]$ such that $V = \{p = 0\}$.*

Proof. Let $p_1, \ldots, p_k \in \mathbb{R}[\mathtt{x}]$ be polynomials such that $V = \{p_1 = 0, \ldots, p_k = 0\}$. Then $V = \{p_1^2 + \cdots + p_k^2 = 0\}$, as required. $\square$

Recall some standard facts about algebraic sets.

(i) The *vanishing ideal* of a subset $V \subset \mathbb{K}^n$ is

$$\mathcal{I}(V) := \{f \in \mathbb{K}[\mathtt{x}] : f(x) = 0 \ \ \forall x \in V\} \subset \mathbb{K}[\mathtt{x}].$$

(ii) Algebraic sets in $\mathbb{K}^n$ are the closed sets for a topology on $\mathbb{K}^n$, called the *Zariski topology.*

(iii) Given a subset $V \subset \mathbb{K}^n$ the *Zariski closure* $\overline{V}^{\text{zar}}$ of V is the smallest algebraic set $\overline{V}^{\text{zar}} \subset \mathbb{K}^n$ that contains V.

(iv) Every Zariski closed set in $\mathbb{K}^n$ is also closed in the standard Euclidean topology because polynomial functions are continuous with respect to the Euclidean topology.

(v) The Zariski topology of $\mathbb{K}^n$ is not Hausdorff, but points are closed.

We will say that an algebraic set $V \subset \mathbb{K}^n$ is *irreducible* if it cannot be decomposed as a finite union of strict algebraic subsets. That is, if $V_1, V_2 \subset \mathbb{K}^n$ are algebraic sets such that $V = V_1 \cup V_2$, then either $V = V_1$ or $V = V_2$. We say that the algebraic set V is *reducible* if it is not irreducible. An algebraic set $V \subset \mathbb{K}^n$ is irreducible if and only if $\mathcal{I}(V) \subset \mathbb{K}[\mathtt{x}]$ is a prime ideal. In fact, $V = V_1 \cup V_2$ with $V_1, V_2 \neq V$ if and only if there exist $f_1 \in \mathcal{I}(V_1)$ and $f_2 \in \mathcal{I}(V_2)$ such that $f_1, f_2 \notin \mathcal{I}(V)$ and $f_1 f_2 \in \mathcal{I}(V)$.

Algebraic sets admits a unique irredundant finite decomposition as the union of its irreducible algebraic components [BR, Prop.3.1.5].

Proposition 2.1.3 (Decomposition into irreducible components)**.** *Every algebraic set $V \subset \mathbb{K}^n$ can be decomposed as a finite union of irreducible algebraic sets, called irreducible components,*

$$V = V_1 \cup \cdots \cup V_k,$$

where $V_i \not\subset V_j$ if $i \neq j$. In addition, up to reordering the indices, this decomposition is unique.

The following examples show that real algebraic sets might present 'wilder' behaviours than complex algebraic sets.

Examples 2.1.4. (i) The ideal $I := (\mathtt{x}^2(\mathtt{x}^2 - 1) + \mathtt{y}^2)$ is a prime ideal of $\mathbb{R}[\mathtt{x}, \mathtt{y}]$, because the polynomial $\mathtt{x}^2(\mathtt{x}^2 - 1) + \mathtt{y}^2$ is irreducible, but

$$\mathcal{Z}(I) := \{(0,0), (1,0)\} \subset \mathbb{R}^2$$

is a reducible algebraic set.

(ii) $I := ((\mathtt{xy} - 1)^2 + \mathtt{x}^2) \subset \mathbb{R}[\mathtt{x}, \mathtt{y}]$ is a proper prime ideal, but the algebraic set $\mathcal{Z}(I) = \varnothing$.

(iii) The circle $\mathbb{S}^1 := \{\mathtt{x}^2 + \mathtt{y}^2 - 1 = 0\} \subset \mathbb{R}^2$ is a bounded irreducible algebraic set.

(iv) The set $\mathcal{Z}(\mathtt{y}^2 - \mathtt{x}^3 + \mathtt{x}) \subset \mathbb{R}^2$ is an irreducible algebraic set with two connected components. One of them is a bounded set (see Figure 2.1).

(v) The irreducible algebraic set $\mathcal{Z}(\mathtt{y}^2 - \mathtt{x}^3 + \mathtt{x}^2) \subset \mathbb{R}^2$ is not connected and one of its connected components is an isolated point (see Figure 2.1).

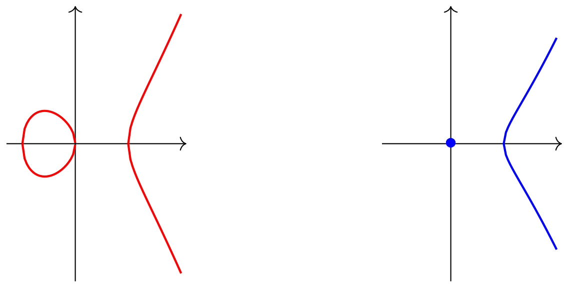

Figure 2.1: The cubic curves $\mathtt{y}^2 - \mathtt{x}^3 + \mathtt{x} = 0$ (left) and $\mathtt{y}^2 - \mathtt{x}^3 + \mathtt{x}^2 = 0$ (right).

2.2 Semi-algebraic sets

The field of real numbers $\mathbb{R}$ has a (unique) ordered structure. This ordered structure is intrinsically related to (some of) the topological properties of real algebraic sets. This is one of the reasons (not the only one) why it is 'natural' to consider a larger class of sets, larger than the one of algebraic sets, described involving both polynomial equalities and inequalities.

Examples 2.2.1. (i) Given $a, b \in \mathbb{R}$, the set $\mathcal{Z}(\mathtt{t}^2 + a\mathtt{t} + b) \subset \mathbb{R}$ is non-empty if and only if $a^2 - 4b \geq 0$.

(ii) A non-singular elliptic curve $\{\mathtt{y}^2 = \mathtt{x}^3 + a\mathtt{x} + b\} \subset \mathbb{R}^2$ is connected if and only if the quantity $4a^3 + 27b^2$ is positive.

A subset $\mathcal{S} \subset \mathbb{R}^n$ is called a *semi-algebraic set* if it can be described as a finite boolean combination of polynomial equalities and inequalities, that is, there exist polynomials $q_i, p_{ij} \in \mathbb{R}[\mathtt{x}]$ such that

$$\mathcal{S} = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : q_i(x) = 0,\, p_{ij}(x) > 0\}.$$

Example 2.2.2. The following sets are semi-algebraic subsets of $\mathbb{R}^n$.

(i) The *unit closed ball* $\overline{\mathcal{B}}_n := \{x \in \mathbb{R}^n \,:\, \|x\|^2 \leq 1\}$.

(ii) The *unit open ball* $\mathcal{B}_n := \{x \in \mathbb{R}^n \,:\, \|x\|^2 < 1\}$.

(iii) The *unit sphere* $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \,:\, \|x\|^2 = 1\}$.

The next theorem shows that the class of semi-algebraic sets is stable under taking projections [BCR, Thm.2.2.1].

Theorem 2.2.3 (Tarski)**.** *Let $\mathcal{S} \subset \mathbb{R}^{n+1}$ be a semi-algebraic set and consider the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$ onto the first n coordinates. Then, $\pi(\mathcal{S})$ is a semi-algebraic subset of $\mathbb{R}^n$.*

As a consequence of Tarski's theorem one obtains that the family of semi-algebraic sets is closed under usual topological operations, like taking closures (denoted by $\mathrm{Cl}(\cdot)$) and taking interiors (denoted by $\mathrm{Int}(\cdot)$).

The next example shows that, unlike the algebraic case, not all the semi-algebraic sets can be described without involving unions. Moreover, unions are needed to have a class of sets that includes algebraic sets and is closed under taking projections.

Example 2.2.4. The semi-algebraic set $\mathcal{S} := \{\mathtt{x} \leq 0\} \cup \{\mathtt{y} \geq 0\} \subset \mathbb{R}^2$ cannot be described as a basic semi-algebraic set $\mathcal{S} = \{g = 0, p_1 > 0, \ldots, p_k > 0\}$. The germ of signs at the origin

$$\begin{array}{c|c} + & + \\ \hline + & - \end{array}$$

provides an obstruction. Note that $\mathcal{S}$ is the projection of the algebraic set

$$\{(\mathtt{x} + \mathtt{z}^2)(\mathtt{y} - \mathtt{z}^2) = 0\} \subset \mathbb{R}^3.$$

Motzkin showed [Mo] that given a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ there exists an algebraic set $V \subset \mathbb{R}^{n+1}$ such that $\pi(V) = \mathcal{S}$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates. Combining the result of Motzkin and the theorem of Tarski we have: *The family of semi-algebraic sets is the smallest family of subsets of Euclidean spaces that contains algebraic sets and is closed under taking projections.*

2.2.1. Semi-algebraic maps. Let $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{T} \subset \mathbb{R}^m$ be two semi-algebraic sets. A (non-necessarily continuous) map $f : \mathcal{S} \to \mathcal{T}$ is *semi-algebraic* if its graph, that is, the set

$$\Gamma_f := \{(x, y) \in \mathcal{S} \times \mathcal{T} \,:\, y = f(x)\} \subset \mathbb{R}^{n+m},$$

is a semi-algebraic set.

If the semi-algebraic map $f : \mathcal{S} \to \mathcal{T}$ is invertible, its inverse $f^{-1} : \mathcal{T} \to \mathcal{S}$ is also semi-algebraic. As a straightforward consequence of Tarski's theorem we have the following:

Corollary 2.2.5. *Let $\mathcal{S} \subset \mathbb{R}^n$, $\mathcal{T} \subset \mathbb{R}^m$ and $\mathcal{U} \subset \mathbb{R}^p$ be semi-algebraic sets and let $f : \mathcal{S} \to \mathcal{T}$ and $g : \mathcal{T} \to \mathcal{U}$ be semi-algebraic maps.*

(i) The composition $g \circ f : \mathcal{S} \to \mathcal{U}$ is a semi-algebraic map.

(ii) If $A \subset \mathcal{S}$ is a semi-algebraic set, then $f(A)$ is a semi-algebraic set.

(iii) If $B \subset \mathcal{T}$ is a semi-algebraic set, then $f^{-1}(B)$ is a semi-algebraic set.

Let $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{T} \subset \mathbb{R}^m$ be semi-algebraic sets. A semi-algebraic map $f : \mathcal{S} \to \mathcal{T}$ is called a *semi-algebraic homeomorphism* if it is a homeomorphism between $\mathcal{S}$ and $\mathcal{T}$ (with respect to the Euclidean topologies inherited respectively from $\mathbb{R}^n$ and $\mathbb{R}^m$).

2.2.2. Regular functions on semi-algebraic sets. Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set. The function $f : \mathcal{S} \to \mathbb{R}$ is called a *regular function* if there exist polynomials $p, q \in \mathbb{R}[\mathtt{x}]$ such that $\{q = 0\} \cap \mathcal{S} = \varnothing$ and $f = p/q$. A map $g := (g_1, \ldots, g_m) : \mathcal{S} \to \mathbb{R}^m$ is called a *regular map* if its components g_i are regular functions. Note that regular maps are in particular semi-algebraic.

2.2.3. Dimension of a semi-algebraic set. The *dimension* of an algebraic set $V \subset \mathbb{R}^n$ is the Krull dimension of the ring $\mathbb{R}[\mathtt{x}]/\mathcal{I}(V)$ of polynomial functions on V. Recall that the *Krull dimension* of the ring $\mathbb{R}[\mathtt{x}]/\mathcal{I}(V)$ equals the maximal length of chains of prime ideals $\mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_k$ of $\mathbb{R}[\mathtt{x}]/\mathcal{I}(V)$. As $\mathbb{R}[\mathtt{x}]/\mathcal{I}(V)$ is a quotient of $\mathbb{R}[\mathtt{x}]$ its Krull dimension is smaller than or equal to the Krull dimension of $\mathbb{R}[\mathtt{x}]$, which is n (see [AM, Ex.7, pag.126]).

Definition 2.2.6. The *dimension* $\dim(\mathcal{S})$ of a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is the dimension $\dim(\overline{\mathcal{S}}^{\text{zar}})$ of its Zariski closure $\overline{\mathcal{S}}^{\text{zar}}$.

As the Euclidean topology is finer than the Zariski topology, it follows $\dim(\mathcal{S}) = \dim(\operatorname{Cl}(\mathcal{S}))$. Moreover, if $\mathcal{S} \subset \mathbb{R}^n$, then $\dim(\mathcal{S}) \leq n$.

We want to compare the dimension of a semi-algebraic set with the dimension of its image through a semi-algebraic map (see [BCR, Thm.2.8.8]).

Theorem 2.2.7. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set and let $f : \mathcal{S} \to \mathbb{R}^m$ be a semi-algebraic map. Then $\dim(f(\mathcal{S})) \leq \dim(\mathcal{S})$. In particular, if f is injective, then $\dim(f(\mathcal{S})) = \dim(\mathcal{S})$.*

We introduce now the notion of local dimension.

Definition 2.2.8. Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set and $x \in \mathcal{S}$ a point. The *local dimension of* $\mathcal{S}$ *at* x, denoted by $\dim(\mathcal{S}_x)$, is the minimum of the dimensions $\dim(U)$, where U is an open semi-algebraic neighbourhood of x in $\mathcal{S}$.

Let $U \subset \mathcal{S}$ be an open semi-algebraic neighbourhood of x in $\mathcal{S}$ such that $\dim(\mathcal{S}_x) = \dim(U)$. Then, for each open semi-algebraic neighbourhood $V \subset U$ of x in $\mathcal{S}$ it holds $\dim(\mathcal{S}_x) = \dim(V)$. Indeed, as $V \subset U$, it holds $\overline{V}^{\text{zar}} \subset \overline{U}^{\text{zar}}$. Thus,

$$\dim(\mathcal{S}_x) \leq \dim(V) = \dim(\overline{V}^{\text{zar}}) \leq \dim(\overline{U}^{\text{zar}}) = \dim(U) = \dim(\mathcal{S}_x).$$

We will say that $\mathcal{S}$ is *pure dimensional* if $\dim(\mathcal{S}_x) = \dim(\mathcal{S})$ for each $x \in \mathcal{S}$. For each $k \leq \dim(\mathcal{S})$ we will indicate with $\mathcal{S}^{(k)}$ the set of points of $\mathcal{S}$ of dimension k, that is, the set of points $x \in \mathcal{S}$ such that $\dim(\mathcal{S}_x) = k$.

Examples 2.2.9. (i) *Whitney's umbrella* $W := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0\} \subset \mathbb{R}^3$ is a connected and irreducible algebraic set but not pure dimensional (see Figure 2.2). Indeed, at each point p on the z-axis with $z < 0$ its local dimension is $\dim(W_p) = 1$, because

$$W \cap \{\mathtt{z} < 0\} = \{\mathtt{x} = 0,\ \mathtt{y} = 0,\ \mathtt{z} < 0\}.$$

(ii) *Cartan's umbrella* $\{\mathtt{z}(\mathtt{x}^2+\mathtt{y}^2)-\mathtt{x}^3 = 0\} \subset \mathbb{R}^3$ is connected and irreducible and it has a 'stick' (the line $\{\mathtt{x} = 0, \mathtt{y} = 0\}$) of dimension 1 (see Figure 2.2).

(iii) The surface $X := \{\mathtt{x}^2(1 - \mathtt{z}^2) - \mathtt{x}^4 - \mathtt{y}^2 = 0\} \subset \mathbb{R}^3$ is not bounded but the set $X^{(2)}$ of points where the local dimension is 2 is bounded.

The set $\mathcal{S}^{(d)}$ of points of dimension d of a semi-algebraic set $\mathcal{S}$ is a semi-algebraic subset of $\mathcal{S}$ (see [Fe2, §3.1]). As $S^{(0)}$ is a finite set, it is always compact. If $0 < d < \dim(\mathcal{S})$, in general $\mathcal{S}^{(d)}$ is not closed, but if $d = \dim(\mathcal{S})$ the set $\mathcal{S}^{(d)}$ of points of maximal dimension is a closed semi-algebraic subset of $\mathcal{S}$ (see [BCR, Prop.2.8.12]). More generally if $0 \leq e \leq d$, then $\bigcup_{k=e}^{d} \mathcal{S}^{(k)}$ is a closed subset of $\mathcal{S}$ because the local dimension of a semi-algebraic set is an upper semi-continuous function.

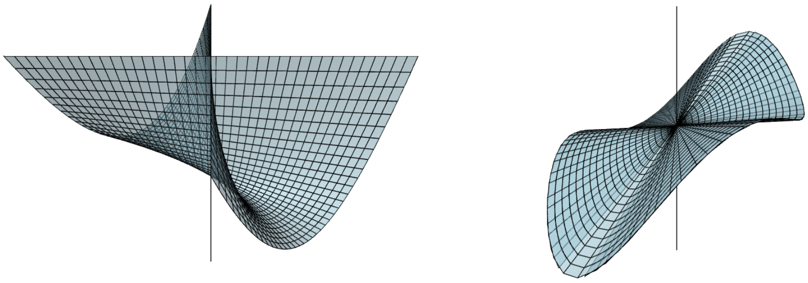

Figure 2.2: Whitney's (left) and Cartan's (right) umbrellas.

2.3 Affine Nash manifolds

In this section we will recall the definition of (affine) Nash manifold and of Nash maps between Nash manifolds. We also present (without proofs) some results of approximation for differentiable semi-algebraic functions by Nash functions that we need in the subsequent chapters. As main references we have used [Sh] and [BFR].

2.3.1. Differentiable semi-algebraic functions. Let $U \subset \mathbb{R}^n$ be an open semi-algebraic set. A function $f : U \to \mathbb{R}$ is a *Nash function* if it is semi-algebraic and smooth. In fact, by [BCR, Prop.8.1.8] f is a Nash function if and only if it is analytic and algebraic over $\mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$, that is, there exists a polynomial $P \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{t}] \setminus \{0\}$ such that $P(x, f(x)) = 0$ for each $x \in U$. We will deepen on this in Section 2.3.3. The ring of Nash functions on U is denoted by $\mathcal{N}(U)$.

Given a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ and $1 \leq \nu \leq \infty$ we say that the function $f : \mathcal{S} \to \mathbb{R}$ is *semi-algebraic of class* $\mathcal{C}^\nu$ if there exist an open semi-algebraic neighbourhood U of $\mathcal{S}$ and a semi-algebraic function $F : U \to \mathbb{R}$ of class $\mathcal{C}^\nu$ such that $F|_{\mathcal{S}} = f$. The ring of semi-algebraic functions of class $\mathcal{C}^\nu$ on $\mathcal{S}$ is denoted with $\mathcal{S}^\nu(\mathcal{S})$. If $\nu = \infty$, we call the $\mathcal{C}^\infty$ semi-algebraic functions on $\mathcal{S}$ *Nash functions* on $\mathcal{S}$ and we write $\mathcal{N}(\mathcal{S}) := \mathcal{S}^\infty(\mathcal{S})$. We denote the ring of continuous semi-algebraic functions on $\mathcal{S}$ with $\mathcal{S}^0(\mathcal{S})$.

Let $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{T} \subset \mathbb{R}^m$ be semi-algebraic sets. A map

$$f := (f_1, \dots, f_m) : \mathcal{S} \to \mathcal{T}$$

is an $\mathcal{S}^\nu$ *map* if each of its components $f_i \in \mathcal{S}^\nu(\mathcal{S})$. We indicate with the symbol $\mathcal{S}^\nu(\mathcal{S}, \mathcal{T})$ the set of $\mathcal{S}^\nu$ maps $f : \mathcal{S} \to \mathcal{T}$. An $\mathcal{S}^\nu$ map $f \in \mathcal{S}^\nu(\mathcal{S}, \mathcal{T})$ is an $\mathcal{S}^\nu$ *diffeomorphism* if it is invertible and the inverse $f^{-1} \in \mathcal{S}^\nu(\mathcal{T}, \mathcal{S})$.

Remark 2.3.1. If the semi-algebraic set $\mathcal{S}$ is compact, the definition of $\mathcal{S}^\nu$ function and an alternative definition in Whitney's style coincide. Indeed, if a function $f : \mathcal{S} \to \mathbb{R}$ is of class $\mathcal{S}^\nu$ according to our definition, it is clear that it admits local $\mathcal{S}^\nu$ extensions at any point of $\mathcal{S}$. Conversely, given $x \in \mathcal{S}$ let $U_x \subset \mathbb{R}^n$ be an open neighbourhood of x (non necessarily semi-algebraic) such that there exists an $\mathcal{S}^\nu$ local extension f_x of f, that is, an $\mathcal{S}^\nu$ function $f_x : U_x \to \mathbb{R}$ such that $f_x|_{U_x \cap \mathcal{S}} = f|_{U_x \cap \mathcal{S}}$. For every $x \in \mathcal{S}$ let V_x be an open ball centred at x such that $V_x \subset U_x$. The set $\{V_x \cap \mathcal{S}\}_{x \in \mathcal{S}}$ is an open covering of the compact set $\mathcal{S}$, so there exists a finite sub-covering $\{V_{x_1} \cap \mathcal{S}, \dots, V_{x_k} \cap \mathcal{S}\}$. Let $\{\rho_i\}_{i=1}^k \cup \{\rho\}$ be a semi-algebraic partition of unit of class $\mathcal{S}^\nu$ subordinated to the open semi-algebraic covering $\{V_{x_i}\}_{i=1}^k \cup \{\mathbb{R}^n \setminus \mathcal{S}\}$ of $\mathbb{R}^n$ (see [Sh, §II.2]). The set $V := V_{x_1} \cup \cdots \cup V_{x_k}$ is an open semi-algebraic neighbourhood of $\mathcal{S}$. Thus, the function

$$F := \rho_{x_1} f_{x_1} + \cdots + \rho_{x_k} f_{x_k} : V \to \mathbb{R},$$

is a semi-algebraic extension of f of class $\mathcal{C}^\nu$.

In the Nash setting the situation is different. Here local extendibility does not guarantee the existence of a global extension, even if we are dealing with compact semi-algebraic sets.

Example 2.3.2 ([BFG, Ex.5.10(i)]). Let $\mathcal{S} \subset \mathbb{R}^2$ be the compact semi-algebraic set

$$\mathcal{S} := \{(\mathtt{x}-2)^2 + \mathtt{y}^2 \leq 1\} \cup \{(\mathtt{x}+2)^2 + \mathtt{y}^2 \leq 1\} \cup \{\mathtt{y} = 0,\ -1 \leq \mathtt{x} \leq 1\}$$

and define $f : \mathcal{S} \to \mathbb{R}$ as $f(x,y) := y\sqrt{x^2+y^2}$. As f is the restriction to $\mathcal{S}$ of a Nash function on $\mathbb{R}^2 \setminus \{(0,0)\}$ and $f \equiv 0$ on $\{\mathtt{y} = 0,\ -1 \leq \mathtt{x} \leq 1\}$, it is clear that f admits local Nash extensions at any point of $\mathcal{S}$. By the identity principle f does not admit a Nash extension to an open semi-algebraic neighbourhood of $\mathcal{S}$ in $\mathbb{R}^2$.

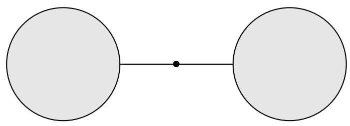

Figure 2.3: The semi-algebraic set $\mathcal{S}$.

2.3.2. Nash manifolds. Let $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{T} \subset \mathbb{R}^m$ be semi-algebraic sets. A map $f := (f_1, \dots, f_m) : \mathcal{S} \to \mathbb{R}^m$ is a *Nash map* if each of its components $f_i \in \mathcal{N}(S)$. We will indicate with the symbol $\mathcal{N}(\mathcal{S},\mathcal{T})$ the set of Nash maps $f : \mathcal{S} \to \mathcal{T}$. A Nash map $f \in \mathcal{N}(\mathcal{S},\mathcal{T})$ is a *Nash diffeomorphism* if it is invertible and the inverse $f^{-1} \in \mathcal{N}(\mathcal{T},\mathcal{S})$.

Definition 2.3.3. A semi-algebraic set $M \subset \mathbb{R}^n$ is called an *(affine) Nash manifold* of dimension d if it is a smooth submanifold of dimension d of (an open subset of) $\mathbb{R}^n$.

Let $\mathcal{B}_d(0,\varepsilon) \subset \mathbb{R}^d$ be the open ball of center the origin and radius $\varepsilon > 0$. The map $\psi : \mathcal{B}_d(0,\varepsilon) \to \mathbb{R}^d$ defined as

$$\psi(x) := \frac{1}{\sqrt{\varepsilon^2 - \|x\|^2}},$$

is a Nash diffeomorphism. Let $M \subset \mathbb{R}^n$ be a Nash manifold, let $p \in M$ and let $\pi : M \to T_pM$ be the projection into the tangent space T_pM of M at p. As π is a local Nash diffeomorphism (because Nash functions satisfy the implicit function theorem [BCR, Cor.2.9.8]) and composing with ψ if needed, it holds: *A semi-algebraic set $M \subset \mathbb{R}^n$ is a Nash manifold of dimension d if and only if every point $p \in M$ has an open semi-algebraic neighbourhood U equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ that maps p to the origin.* Even more, the

Nash manifold M can be covered by *finitely many* open semi-algebraic sets of this type [FGR, Lem.2.2].

The same argument works for Nash submanifolds of a Nash manifold. Let $N \subset M$ be a Nash submanifold of dimension e, then: *For each point $p \in N$ there exists an open semi-algebraic neighbourhood U of p in M equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ that maps p to the origin and such that $U \cap N = \{u_1 = 0, \dots, u_{d-e} = 0\}$.*

A *Nash vector bundle* over M is a vector bundle $(\mathcal{E}, \theta, M)$ such that $\mathcal{E}$ is an (affine) Nash manifold and the projection $\theta : \mathcal{E} \to M$ is a Nash map. Nash submanifolds admits *Nash tubular neighbourhoods* in the ambient Nash manifold where they are embedded (see [BCR, Cor.8.9.5] and [Sh, II.6.2]).

Theorem 2.3.4 (Nash tubular neighbourhood)**.** *There exist a Nash subbundle $(\mathcal{E}, \theta, N)$ of the trivial Nash bundle $(N \times \mathbb{R}^n, \eta, N)$, a strictly positive Nash function δ on N and a Nash diffeomorphism φ from a semi-algebraic neighbourhood V of N in M onto*

$$\mathcal{E}_\delta := \{(x, y) \in \mathcal{E} : \|y\| < \delta(x)\},$$

such that $\varphi|_N = (\mathrm{id}_N, 0)$. In addition, if M is an open subset of $\mathbb{R}^n$ we may assume $\operatorname{dist}(x, N) = \|x - \theta(x)\|$ for every $x \in \mathcal{E}_\delta$.

2.3.3. Algebraicity of Nash functions. Let $\mathcal{S} \subset \mathbb{R}^n$ be a non-empty semi-algebraic set and $f : \mathcal{S} \to \mathbb{R}$ a semi-algebraic function. Let $\Gamma_f \subset \mathbb{R}^{n+1}$ be the graph of f and let $X := \overline{\Gamma}_f^{\mathrm{zar}}$. The ideal $\mathcal{I}(X) \subset \mathbb{R}[\mathtt{x}, \mathtt{t}] := \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n, \mathtt{t}]$ is finitely generated and let $h_1, \dots, h_s \in \mathcal{I}(X)$ be generators. The polynomial $h := h_1^2 + \cdots + h_s^2$ is not identically zero on $\mathcal{S} \times \mathbb{R}$ and satisfies $h(x, f(x)) = 0$ for each $x \in \mathcal{S}$. In particular: *Nash functions are algebraic over the ring of polynomials* $\mathbb{R}[\mathtt{x}]$. It holds the following characterization for Nash functions (see [BCR, Prop.8.1.8]):

Proposition 2.3.5. *Let $U \subset \mathbb{R}^n$ be an open semi-algebraic set. A function $f : U \to \mathbb{R}$ is Nash if and only if it is analytic and algebraic on U.*

As a consequence of this result and the existence of Nash tubular neighbourhoods, we deduce: *An analytic function $f : U \to \mathbb{R}$ defined on an open connected semi-algebraic subset U of a connected Nash manifold $M \subset \mathbb{R}^n$ is Nash if and only if there exists a non-zero polynomial $h \in \mathbb{R}[\mathtt{x}, \mathtt{t}]$ such that $h(x, f(x)) = 0$ for all $x \in U$.*

2.3.4. Approximation of semi-algebraic maps. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d. We equip $\mathcal{S}^\nu$ with the *$\mathcal{S}^\nu$ semi-algebraic Whitney's topology ($\mathcal{S}^\nu$ topology* in short) [Sh, II.1]. If $\nu \geq 1$, let $\xi_1, \dots, \xi_s$ be semi-algebraic tangent fields on M that span the tangent bundle of M. For every strictly positive continuous semi-algebraic function $\varepsilon : M \to \mathbb{R}$ we denote by $\mathcal{U}_\varepsilon$ the set of all functions $g \in \mathcal{S}^\nu(M)$ such that

$$\begin{cases} |g| < \varepsilon & \text{if } \nu = 0, \\ |g| < \varepsilon \quad \text{and} \quad |\xi_{i_1} \cdots \xi_{i_\ell}(g)| < \varepsilon \text{ for } 1 \leq i_1, \dots, i_\ell \leq s, 1 \leq \ell \leq \nu & \text{if } \nu \geq 1. \end{cases}$$

These sets $\mathcal{U}_\epsilon$ form a basis of neighbourhoods of the zero function for a topology in $\mathcal{S}^\nu(M)$ (recall that $\mathcal{S}^\nu(M)$ is a topological ring), which does not depend on the choice of the tangent fields if $r \geq 1$.

Note that the obvious inclusions $\mathcal{S}^\nu(M) \subset \mathcal{S}^\mu(M)$, $\nu > \mu$ are continuous. Moreover, as semi-algebraic smooth functions are Nash on M (by the existence of tubular neighbourhoods), we have $\mathcal{N}(M) = \bigcap_\nu \mathcal{S}^\nu(M)$. The first important result is that the inclusion $\mathcal{N}(M) \subset \mathcal{S}^\nu(M)$ is dense.

Fact 2.3.6 ([Sh, II.4.1])**.** *Every $\mathcal{S}^\nu$ function on M can be approximated in the $\mathcal{S}^\nu$ topology by Nash functions.*

Let $N \subset \mathbb{R}^n$ be a Nash manifold. Recall that $\mathcal{S}^\nu(M, N)$ is the space of semi-algebraic maps $f : M \to N$ of class $\mathcal{C}^\nu$. We consider in $\mathcal{S}^\nu(M, N)$ the subspace topology given by the canonical inclusion in the following product space endowed with the product topology [Sh, Rmk.II.1.3]:

$$\mathcal{S}^\nu(M, N) \subset \mathcal{S}^\nu(M, \mathbb{R}^n) = \mathcal{S}^\nu(M, \mathbb{R}) \times \cdots \times \mathcal{S}^\nu(M, \mathbb{R}),\ f \mapsto (f_1, \ldots, f_n).$$

Roughly speaking, g is close to f when its components g_k are close to the components f_k of f.

Fix $0 \leq \nu \leq +\infty$. Let $M \subset \mathbb{R}^m$, $M' \subset \mathbb{R}^n$ and $M'' \subset \mathbb{R}^k$ be Nash manifolds and let $h : M' \to M''$ be an $\mathcal{S}^\nu$ map. Composing (on the left) induces the map

$$h_* : \mathcal{S}^\nu(M, M') \to \mathcal{S}^\nu(M, M''),\ f \mapsto h \circ f.$$

Fact 2.3.7 ([Sh, II.1.5])**.** *The map h_* is continuous with respect to the $\mathcal{S}^\nu$ topologies.*

We have an analogous situation composing on the right. Composing (on the right) induces the map

$$h^* : \mathcal{S}^\nu(M'', M) \to \mathcal{S}^\nu(M', M),\ f \mapsto f \circ h.$$

This map is non-necessarily continuous, but if $h : M' \to M''$ is proper (this happens for instance when M' is compact) it holds:

Fact 2.3.8 ([Sh, II.1.5])**.** *If $h : M' \to M''$ is proper, then the map h^* is continuous with respect to the $\mathcal{S}^\nu$ topologies.*

Another important result is that $\mathcal{S}^\nu$ diffeomorphisms between Nash manifolds constitute an open set in the $\mathcal{S}^\nu$ topology.

Fact 2.3.9 ([Sh, II.1.7])**.** *Let $h : M \to N$ be an $\mathcal{S}^\nu$ diffeomorphism of Nash manifolds. If an $\mathcal{S}^\nu$ map $g : M \to N$ is $\mathcal{S}^\nu$ close enough to h, then g is also an $\mathcal{S}^\nu$ diffeomorphism, and g^{-1} is $\mathcal{S}^\nu$ close to h^{-1}.*

From this and the existence of Nash tubular neighbourhoods we deduce that for all $\nu \geq 1$: *Every $\mathcal{S}^\nu$ diffeomorphism $f : M \to N$ can be approximated by Nash diffeomorphisms.* Thus, $\mathcal{S}^1$ and Nash classifications coincide for Nash manifolds.

The previous topologies can be extended to the set $\mathcal{S}^\nu(\mathcal{S}, \mathcal{T})$ of $\mathcal{S}^\nu$ semi-algebraic maps between two semi-algebraic sets $\mathcal{S}$ and $\mathcal{T}$ if $\mathcal{S} \subset M$ is closed (see [BFR, §2.D]). It holds that the restriction map is continuous.

Fact 2.3.10 ([BFR, §2.D]). *Let $M \subset \mathbb{R}^n$ be a Nash manifold and let $\mathcal{S}' \subset \mathcal{S} \subset M$ be closed semi-algebraic sets. For any $\mathcal{T} \subset \mathbb{R}^m$ the restriction map*

$$\mathcal{S}^\nu(\mathcal{S}, \mathcal{T}) \to \mathcal{S}^\nu(\mathcal{S}', \mathcal{T}),\ f \mapsto f|_{\mathcal{S}}$$

is continuous with respect to the $\mathcal{S}^\nu$ topologies.

We end this section recalling a result on absolute approximation for maps into Nash manifolds.

Fact 2.3.11 ([BFR, Prop.2.D.3]). *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be Nash manifolds and let $\mathcal{S} \subset M$ be a closed semi-algebraic set. Every $\mathcal{S}^\nu$ map $f : \mathcal{S} \to N$ can be $\mathcal{S}^\nu$ approximated by Nash maps.*

2.4 Back to semi-algebraic sets

In this section we collect some results and definitions (Hironaka's desingularization, regular points of a semi-algebraic set, irreducible components etc.) that we will use freely in the rest of this dissertation. We include these results here (without proofs) in order to keep this work as much self-contained as possible.

2.4.1. Regular points and smooth points. Let $Z \subset \mathbb{C}^n$ be a complex algebraic set and let $\mathcal{I}_{\mathbb{C}}(Z)$ be the ideal of all polynomials $F \in \mathbb{C}[\mathtt{x}]$ such that $F(z) = 0$ for each $z \in Z$. A point $z \in Z$ is *regular* if the localization of the polynomial ring $\mathbb{C}[\mathtt{x}]/\mathcal{I}_{\mathbb{C}}(Z)$ at the maximal ideal $\mathfrak{M}_z$ associated to z is a regular local ring. In this complex setting the Jacobian criterion and Hilbert's Nullstellensatz imply that $z \in Z$ is regular if and only if there exists an open neighbourhood $U \subset \mathbb{C}^n$ of z such that $U \cap Z$ is an analytic manifold. We denote $\operatorname{Reg}(Z)$ the set of regular points of Z and it is an open dense subset of Z. If Z is irreducible, it is pure dimensional and $\operatorname{Reg}(Z)$ is a connected analytic manifold. In case Z is not irreducible, then the connected components of $\operatorname{Reg}(Z)$ are finitely many analytic manifolds (possibly of different dimensions). We denote $\operatorname{Sing}(Z) := Z \setminus \operatorname{Reg}(Z)$ the *set of singular points of Z*.

Let $X \subset \mathbb{R}^n$ be a (real) algebraic set and let $\mathcal{I}_{\mathbb{R}}(X)$ be the ideal of all polynomials $f \in \mathbb{R}[\mathtt{x}]$ such that $f(x) = 0$ for each $x \in X$. A point $x \in X$ is *regular* if the localization of $\mathbb{R}[\mathtt{x}]/\mathcal{I}_{\mathbb{R}}(X)$ at the maximal ideal $\mathfrak{m}_x$ associated to x is a regular local ring [BCR, §3.3].

Let $\widetilde{X} \subset \mathbb{C}^n$ be the complex algebraic set that is the zero set of the extended ideal $\mathcal{I}_{\mathbb{R}}(X)\mathbb{C}[\mathtt{x}]$. We call $\widetilde{X}$ the *complexification of* X. The ideal $\mathcal{I}_{\mathbb{C}}(\widetilde{X})$ coincides with the tensorized ideal $\mathcal{I}_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C}$, so $\widetilde{X}$ is the smallest complex algebraic subset of $\mathbb{C}^n$ that contains X and

$$\mathbb{C}[\mathtt{x}]/\mathcal{I}_{\mathbb{C}}(\widetilde{X}) \cong (\mathbb{R}[\mathtt{x}]/\mathcal{I}_{\mathbb{R}}(X)) \otimes_{\mathbb{R}} \mathbb{C}.$$

The localization $(\mathbb{R}[\mathtt{x}]/\mathcal{I}_{\mathbb{R}}(X))_{\mathfrak{m}_x}$ is a regular local ring if and only if so is its complexification

$$(\mathbb{R}[\mathtt{x}]/\mathcal{I}_{\mathbb{R}}(X))_{\mathfrak{m}_x} \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C}[\mathtt{x}]/\mathcal{I}_{\mathbb{C}}(\widetilde{X}))_{\mathfrak{M}_x}.$$

Thus, the *set of regular points* of X is $\operatorname{Reg}(X) = \operatorname{Reg}(\widetilde{X}) \cap X$ and its *set of singular points* is $\operatorname{Sing}(X) := X \setminus \operatorname{Reg}(X) = \operatorname{Sing}(\widetilde{X}) \cap X$. The open semi-algebraic subset $\operatorname{Reg}(X)$ of X is a finite union of Nash manifolds (possibly of different dimensions).

Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set of dimension d. We define

$$\operatorname{Reg}(\mathcal{S}) := \operatorname{Int}_{\operatorname{Reg}(\overline{\mathcal{S}}^{\text{zar}})}(\mathcal{S} \setminus \operatorname{Sing}(\overline{\mathcal{S}}^{\text{zar}})) \quad \text{and} \quad \operatorname{Sing}(\mathcal{S}) := \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S}).$$

The open subset $\operatorname{Reg}(\mathcal{S})$ of $\overline{\mathcal{S}}^{\text{zar}}$ is a finite union of Nash manifolds (possibly of different dimensions) and $\operatorname{Sing}(\mathcal{S})$ is a semi-algebraic set of dimension $< d$, which is closed in $\mathcal{S}$. The set of points of dimension k of $\operatorname{Reg}(\mathcal{S})$ is either the empty-set or a Nash manifold of dimension k for each $k = 0, 1, \ldots, d$. If $\mathcal{S}$ is pure dimensional, $\operatorname{Reg}(\mathcal{S})$ is a dense subset of $\mathcal{S}$.

A point $x \in \mathcal{S}$ is *smooth* if there exists an open neighbourhood $U \subset \mathbb{R}^n$ of x such that $U \cap \mathcal{S}$ is a Nash manifold. It holds that each regular point is a smooth point, but the converse is not always true even if $\mathcal{S} = X$ is a real algebraic set, as shown in the following examples.

Examples 2.4.1. (i) [BCR, Ex.3.3.12(b)] Consider the irreducible curve $X \subset \mathbb{R}^2$ given by the equation $\mathtt{y}^3 + 2\mathtt{x}^2\mathtt{y} - \mathtt{x}^4 = 0$ (see Figure 2.4). The set of regular points of X is $X \setminus \{0\}$. As the germ $X_0 = \{\mathtt{x}^2 - \mathtt{y}(1 + \sqrt{1+\mathtt{y}}) = 0\}_0$, it follows by the implicit function theorem for Nash functions (see [BCR, Cor.2.9.8]) that the origin is a smooth point of X.

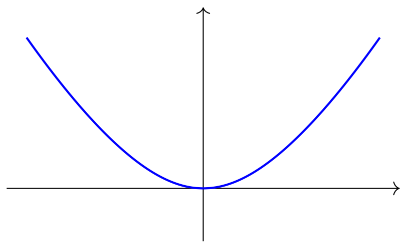

Figure 2.4: The curve $\mathtt{y}^3 + 2\mathtt{x}^2\mathtt{y} - \mathtt{x}^4 = 0$.

(ii) [Fe4, Ex.2.1] Consider the algebraic set $X := \{(\mathtt{x}^2 + \mathtt{z}\mathtt{y}^2)\mathtt{x} - \mathtt{y}^4 = 0\} \subset \mathbb{R}^3$. The set of regular points of X is $X \setminus \{\mathtt{x} = 0, \mathtt{y} = 0\}$, whereas the set of smooth points of X is $X \setminus \{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} \leq 0\}$ (see Figure 2.5).

To prove that the points of the open half-line $\{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} < 0\}$ are non-smooth we proceed by contradiction. Pick a point

$$p := (0, 0, -a^2) \in \{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} < 0\}$$

and assume that it is smooth. As the line $\{\mathtt{x} = 0, \mathtt{y} = 0\} \subset X$, the vector $(0, 0, 1)$ would be tangent to X at p, so the plane $\mathtt{z} = -a^2$ would be transversal to X at p. Thus, the intersection $X \cap \{\mathtt{z} = -a^2\}$ should be a curve that is smooth at p, but this is a contradiction because such curve $\{(\mathtt{x}^2 - (a\mathtt{y})^2)\mathtt{x} - \mathtt{y}^4 = 0, \mathtt{z} = -a^2\}$ has three tangent lines at p, which are those lines of equations $\{\mathtt{x} - a\mathtt{y} = 0\}$, $\{\mathtt{x} + a\mathtt{y} = 0\}$ and $\{\mathtt{x} = 0\}$ inside the plane $\{\mathtt{z} = -a^2\}$. The origin cannot be a smooth point of X because the set of smooth points of X is an open subset of X. Consequently, the set of non-smooth points of X contains the closed half-line $\{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} \leq 0\}$.

To finish we prove that the points of the open half-line $\{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} > 0\}$ are smooth. To that end, observe that the map

$$\varphi : \{(t, s) \in \mathbb{R}^2 : \ t > 0\} \to \mathbb{R}^3, (s, t) \mapsto ((s^2 + t^2)s^2, (s^2 + t^2)s, t^2)$$

is a Nash embedding whose image is $X \cap \{\mathtt{z} > 0\}$.

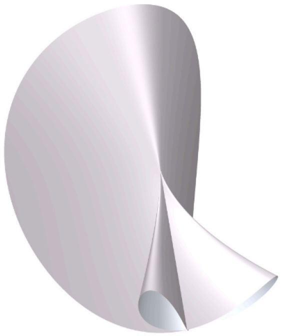

Figure 2.5: $X = \{(\mathtt{x}^2 + \mathtt{z}\mathtt{y}^2)\mathtt{x} - \mathtt{y}^4 = 0\}$ (figure borrowed from [Fe4, Fig.1.4]).

The set $\mathrm{Sth}(\mathcal{S})$ of smooth points of a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is by [St] a semi-algebraic subset of $\mathbb{R}^n$ (and consequently a union of Nash submanifolds of $\mathbb{R}^n$ possibly of different dimension), which contains $\mathrm{Reg}(\mathcal{S})$ (maybe as a proper subset as it happens in Example 2.4.1), and it is open in $\mathcal{S}$. The set of points of dimension k of $\mathrm{Sth}(\mathcal{S})$ is either the empty-set or a Nash manifold of dimension k for each $k = 0, 1, \ldots, d$. In particular, if $\mathcal{S}$ is pure dimensional $\mathrm{Sth}(\mathcal{S})$ is a Nash submanifold of $\mathbb{R}^n$. If X is an algebraic set, $\mathrm{Sing}(X)$ is always an algebraic subset of X whereas the set $X \setminus \mathrm{Sth}(X)$ of non-smooth points is in general only a semi-algebraic subset of X (see Example 2.4.1(ii)).

2.4.2. Desingularization of real algebraic sets. Let $X \subset Y \subset \mathbb{R}^n$ be algebraic sets such that Y is non-singular and has dimension d. Recall that X is a *normal-crossings divisor of* Y if for each point $x \in X$ there exists a regular system of parameters $\mathtt{x}_1, \ldots, \mathtt{x}_d$ for Y at x such that X is given on an open Zariski neighbourhood of x in Y by the equation $\mathtt{x}_1 \cdots \mathtt{x}_k = 0$ for some $k \leq d$. In particular, the irreducible components of X are non-singular and have codimension 1 in Y.

Hironaka's desingularization results [Hi1] are powerful tools that we will use fruitfully in the following sections. We recall here the results we need (see also Kollár's lecture notes [Ko]).

Theorem 2.4.2 (Desingularization). *Let $X \subset \mathbb{R}^n$ be an algebraic set. Then, there exist a non-singular algebraic set $X' \subset \mathbb{R}^m$ and a proper polynomial map $f : X' \to X$ such that*

$$f|_{X' \setminus f^{-1}(\mathrm{Sing}(X))} : X' \setminus f^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}\, X$$

is a diffeomorphism whose inverse map is a regular map.

Remark 2.4.3. If X is pure dimensional, $X \setminus \mathrm{Sing}\, X$ is dense in X. As f is proper, it is surjective.

Theorem 2.4.4 (Embedded desingularization). *Let $X \subset Y \subset \mathbb{R}^n$ be algebraic sets such that Y is non-singular. Then, there exists a non-singular algebraic set $Y' \subset \mathbb{R}^m$ and a proper surjective polynomial map $g : Y' \to Y$ such that $g^{-1}(X)$ is a normal-crossings divisor of Y' and the restriction*

$$g|_{Y' \setminus g^{-1}(X)} : Y' \setminus g^{-1}(X) \to Y \setminus X$$

is a diffeomorphism whose inverse map is a regular map.

2.4.3. Irreducible components of a semi-algebraic set. In classical literature a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is called irreducible if its Zariski closure $\overline{\mathcal{S}}^{\text{zar}}$ is an irreducible algebraic set (see for instance [GV]). With this definition the semi-algebraic sets

$$\mathcal{S}_1 := \{\mathtt{y}^2 - \mathtt{x}^2(\mathtt{x}+1) = 0\} \setminus \{(-1,0)\}, \ \mathcal{S}_2 := \{\mathtt{y}^2 - \mathtt{x}^2 = 1\}$$

are irreducible (see Figure 2.6). The feeling is that they 'should be reducible': $\mathcal{S}_1$ consists of two analytic branches and $\mathcal{S}_2$ is not connected.

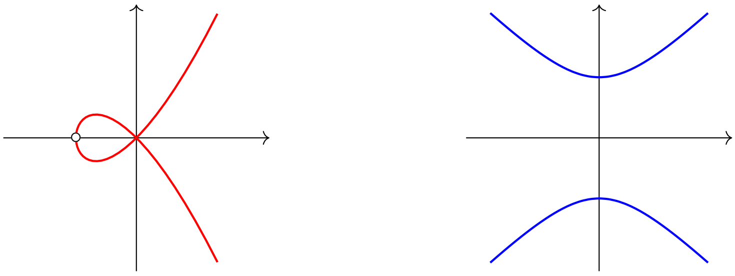

Figure 2.6: The semi-algebraic sets $\mathcal{S}_1$ (left) and $\mathcal{S}_2$ (right).

In [FG3] Fernando and Gamboa propose the following definition of irreducibility for semi-algebraic sets.

Definition 2.4.5. A semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *irreducible* if its ring of Nash functions $\mathcal{N}(\mathcal{S})$ is an integral domain.

One deduce straightforwardly the following facts concerning irreducibility.

(i) Irreducible semi-algebraic sets are connected, because the ring of Nash functions of a disconnected semi-algebraic set is the direct sum of the rings of Nash functions of its connected components.

(ii) The Zariski closure of an irreducible semi-algebraic set is irreducible as an algebraic set.

(iii) Let $\mathcal{S}, \mathcal{T} \subset \mathbb{R}^n$ be semi-algebraic sets. If $\mathcal{T} \subset \mathcal{S} \subset \operatorname{Cl}(\mathcal{T})$ and $\mathcal{T}$ is irreducible, then $\mathcal{S}$ is irreducible.

Example 2.4.6. The semi-algebraic set $\mathcal{S}_1 := \{\mathtt{y}^2 - \mathtt{x}^2(\mathtt{x}+1) = 0\} \setminus \{(-1,0)\}$ is reducible (see Figure 2.6). Indeed, the Nash functions $f_1, f_2 : \mathcal{S}_1 \to \mathbb{R}$ defined as

$$f_1(x,y) = y + x\sqrt{x+1}, \ \ f_2(x,y) = y - x\sqrt{x+1}$$

are not identically zero on $\mathcal{S}_1$, but $f_1 f_2 \equiv 0$ on $\mathcal{S}_1$.

We introduce now the irreducible components of a semi-algebraic set.

Definition 2.4.7. A semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ admits a decomposition into *irreducible components* if there exist semi-algebraic sets $\mathcal{S}_1, \dots, \mathcal{S}_r \subset \mathcal{S}$ such that:

(i) Each $\mathcal{S}_i$ is irreducible.

(ii) If $\mathcal{T} \subset \mathcal{S}$ is an irreducible semi-algebraic set that contains $\mathcal{S}_i$, then $\mathcal{S}_i = \mathcal{T}$.

(iii) $\mathcal{S}_i \not\subset \bigcup_{j\neq i} S_j$.

(iv) $\mathcal{S} = \bigcup_{i=1}^{r} \mathcal{S}_i$.

In [FG3, Thm.4.3, Rmk.4.4] Fernando and Gamboa present the following result concerning the irreducible components of a semi-algebraic set.

Theorem 2.4.8. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set. Then, $\mathcal{S}$ admits a decomposition into irreducible components and this decomposition is unique. In addition, the irreducible components of a semi-algebraic set are closed in $\mathcal{S}$.*

Remark 2.4.9. If $\mathcal{S} \subset \mathbb{R}^n$ is a semi-algebraic set and $\{\mathcal{S}_i\}_{i=1}^r$ is the family of its irreducible components, then $\dim(\mathcal{S}_i \cap \mathcal{S}_j) < \min\{\dim(\mathcal{S}_i), \dim(\mathcal{S}_j)\}$ if $i \neq j$.

For each $i = 1, \dots, r$ let $f_i : \mathcal{S} \to \mathbb{R}$ be a Nash function on $\mathcal{S}$ such that $\mathcal{S}_i = \{f_i = 0\}$ (see [FG3, Lem.2.4, Thm. 4.3]). As $\mathcal{S}_i$ is irreducible, we deduce by [FG3, Lem.3.6] that $\dim(\mathcal{S}_j \cap \mathcal{S}_i) = \dim(\{f_j = 0\} \cap \mathcal{S}_i) < \dim(\mathcal{S}_i)$ if $i \neq j$, because otherwise $\mathcal{S}_i \subset \{f_j = 0\} = \mathcal{S}_j$.

2.5 Affine Nash manifolds with corners

In this section we introduce the definition of Nash sets and of Nash manifold with (divisorial) corners. We collect here some results (without proofs) concerning Nash manifolds with corners and Nash normal-crossings that we will use later in this dissertation. The main reference is [FGR].

2.5.1. Nash subsets of a Nash manifold. Let $M \subset \mathbb{R}^n$ be a Nash manifold.

Definition 2.5.1. A *Nash subset* X of M is a set of the form

$$X = \mathcal{Z}_M(I) := \{x \in M : \forall f \in I \ f(x) = 0\} \subset M,$$

where $I \subset \mathcal{N}(M)$ is an ideal of the ring of global Nash functions on M.

As the ring $\mathcal{N}(M)$ is Noetherian (see [BCR, 8.7.18]), the ideal I is generated by finitely many global Nash functions $f_1, \dots, f_r : M \to \mathbb{R}$. Thus, the Nash set X is the zero set of the functions $f_1, \dots, f_r$ and in fact X is the zero set of the single global Nash function $f := f_1^2 + \cdots + f_r^2$.

Of course every Nash set is a semi-algebraic set. A first example of Nash subsets are (closed) Nash submanifolds of M.

Fact 2.5.2 ([Sh, II.5.4])**.** *Let $N \subset M$ be a closed Nash submanifold, then N is a Nash subset of M.*

A Nash subset $X \subset M$ is *irreducible* if whenever $X = X_1 \cup X_2$, where X_1 and X_2 are Nash subsets of M, then either $X = X_1$ or $X = X_2$. We have the following algebraic characterization of irreducibility: *X is irreducible if and only if the Nash vanishing ideal $\mathcal{I}(X) := \{f \in \mathcal{N}(M) : f|_X \equiv 0\}$ of X is a prime ideal of the ring $\mathcal{N}(M)$.* Indeed, $X = X_1 \cup X_2$ with $X_1, X_2 \neq X$ if and only if there exist $f_1 \in \mathcal{I}(X_1)$ and $f_2 \in \mathcal{I}(X_2)$ such that $f_1, f_2 \notin \mathcal{I}(X)$ and $f_1 f_2 \in \mathcal{I}(X)$.

As $\mathcal{N}(M)$ is a Noetherian ring, it follows: *Nash subsets of M have (unique) finite decompositions into Nash irreducible components.* In [FG3, §3.1] it is proved that a Nash set is irreducible if and only if it is irreducible as a semi-algebraic set and in [FG3, §4] it is shown that its decomposition into Nash irreducible components coincides with its decomposition into semi-algebraic irreducible components.

2.5.2. Nash normal-crossings. We introduce now the notion of Nash normal-crossings. It has two different aspects - the local one and the global one. We start with the local notion:

Definition 2.5.3. Let X be a Nash subset of a Nash manifold $M \subset \mathbb{R}^n$. We say that X *has only normal-crossings at a point* $x \in X$ if there exists an open semi-algebraic neighbourhood $U \subset M$ of x equipped with a Nash diffeomorphism $u := (u_1, \dots, u_d) : U \to \mathbb{R}^d$ such that the germ $X_x := \{\mathtt{u}_1 \cdots \mathtt{u}_r = 0\}_x$ for some $1 \leq r \leq d$. We say that X *has only normal-crossings in* $V \subset M$ if it has only normal-crossings at each $x \in V \cap X$.

The following result shows that if X has only normal-crossings in M there exists a *finite* number of local charts that cover X and provide a global picture of its local structure.

Theorem 2.5.4 ([FGR, Thm.1.6])**.** *Let X be a Nash subset of the Nash manifold $M \subset \mathbb{R}^n$. Suppose that X has only normal crossings in M. Then, X can be covered by finitely many open semi-algebraic subsets U_i of M equipped with Nash diffeomorphisms $u_i := (u_{i1}, \dots, u_{id}) : U_i \to \mathbb{R}^d$ such that*

$$U_i \cap X = \{\mathtt{u}_{i1} \cdots \mathtt{u}_{ir_i} = 0\}.$$

Next, we have the global version of normal-crossings:

Definition 2.5.5. Let $M \subset \mathbb{R}^n$ be a Nash manifold. A *Nash normal-crossings divisor* of M is a Nash subset $X \subset M$ whose irreducible components are non-singular Nash hypersurfaces $X_1, \dots, X_p$ of M in general position. This means that at every point $x \in X_{i_1} \cap \dots \cap X_{i_r}$ with $x \notin X_i$ for $i \neq i_k$ the tangent spaces $T_x X_{i_1}, \dots, T_x X_{i_r}$ are linearly independent in the tangent space $T_x M$, that is,

$$\dim(T_x X_{i_1} \cap \dots \cap T_x X_{i_r}) = \dim(T_x M) - r.$$

Let us confront the local notion with the global one. The following example shows the differences between these two notions.

Example 2.5.6. Consider the irreducible algebraic set (see Figure 2.6)

$$X := \{\mathtt{y}^2 - \mathtt{x}^2(\mathtt{x}+1) = 0\} \subset \mathbb{R}^2.$$

(i) X is an irreducible Nash subset of $\mathbb{R}^2$. As it is singular at the origin, it is not a Nash normal-crossings divisor of $\mathbb{R}^2$. However, X has only normal-crossings at all points of $\mathbb{R}^2$.

(ii) $Y := X \setminus \{(-1, 0)\}$ is a Nash subset of the Nash manifold $M := \{\mathtt{x} > -1\}$. The Nash irreducible components of Y in M are the non-singular hypersurfaces

$$Y_1 := \{(x, y) \in M : y - x\sqrt{x+1} = 0\},\ Y_2 := \{(x, y) \in M : y + x\sqrt{x+1} = 0\}$$

which meet transversally at the origin. Thus, Y is a Nash normal-crossings divisor of the Nash manifold M.

2.5.3. Nash manifolds with corners. We introduce now the definition of (Nash) manifold with corners.

Definition 2.5.7. A semi-algebraic set $\mathcal{Q} \subset \mathbb{R}^n$ is an *(affine) Nash manifold with corners* of dimension d if for each point $y \in \mathcal{Q}$ there exist an integer $0 \leq k \leq d$ and an open semi-algebraic neighbourhood U of y in $\mathcal{Q}$ equipped with a Nash diffeomorphism $\phi : U \to \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_k \geq 0\} \subset \mathbb{R}^d$ that maps y to the origin.

The set of *internal points* $\operatorname{Int}(\mathcal{Q})$ of $\mathcal{Q}$ is the set of points $x \in \mathcal{Q}$ at which $\mathcal{Q}_x$ is the germ of a Nash manifold, namely $\operatorname{Int}(\mathcal{Q}) := \operatorname{Sth}(\mathcal{Q})$. The *boundary* $\partial\mathcal{Q}$ of $\mathcal{Q}$ is $\partial\mathcal{Q} := \mathcal{Q} \setminus \operatorname{Int}(\mathcal{Q}) = \mathcal{Q} \setminus \operatorname{Sth}(\mathcal{Q})$. Observe that this definition coincide with the definition of internal points and boundary of $\mathcal{Q}$ seen as topological manifold with boundary.

Remark 2.5.8. In Section 3.5 we will introduce the definition of boundary $\partial\mathcal{S}$ for semi-algebraic sets $\mathcal{S} \subset \mathbb{R}^n$. The definition of boundary of Nash manifolds with corners given here does not coincide with the definition of boundary of semi-algebraic sets given in Section 3.5, if we regard $\mathcal{Q}$ as a semi-algebraic set (see for instance Example 2.4.1(i)). This (small) abuse of notation will not create any ambiguity in the subsequent sections, as the situation will be always clear from the context.

A Nash manifold with corners $\mathcal{Q} \subset \mathbb{R}^n$ is locally closed, hence $\mathcal{Q}$ is open in its closure $\operatorname{Cl}(\mathcal{Q})$. Thus, the set $C := \operatorname{Cl}(\mathcal{Q}) \setminus \mathcal{Q}$ is a closed semi-algebraic subset of $\mathbb{R}^n$. Consequently, $\mathcal{Q}$ *is a closed Nash submanifold with corners of the Nash manifold* $\mathbb{R}^n \setminus C$. In [FGR] Fernando, Gamboa and Ruiz showed that $\mathcal{Q}$ is a closed subset of an affine Nash manifold of the same dimension.

Theorem 2.5.9 ([FGR, Thm.1.11])**.** *Let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners of dimension d. There exists a d-dimensional Nash manifold $M \subset \mathbb{R}^n$ that contains $\mathcal{Q}$ as a closed subset and satisfies:*

(i) *The Nash closure Y of $\partial\mathcal{Q}$ in M has only Nash normal-crossings in M and $\mathcal{Q} \cap Y = \partial\mathcal{Q}$.*

(ii) *For every $x \in \partial\mathcal{Q}$ the analytic closure of the germ $\partial\mathcal{Q}_x$ is Y_x.*

(iii) *M can be covered by finitely many open semi-algebraic subsets U_i, for $i = 1, \dots, r$, equipped with Nash diffeomorphisms*

$$u_i := (u_{i1}, \dots, u_{id}) : U_i \to \mathbb{R}^d$$

such that:

$$\begin{cases} U_i \subset \operatorname{Int}(\mathcal{Q}) \text{ or } U_i \cap \mathcal{Q} = \varnothing, & \text{if } U_i \text{ does not meet } \partial\mathcal{Q}, \\ U_i \cap \mathcal{Q} = \{u_{i1} \geq 0, \dots, u_{ik_i} \geq 0\}, & \text{if } U_i \text{ meets } \partial\mathcal{Q} \text{ (for a suitable } k_i \geq 1). \end{cases}$$

The Nash manifold M is called a *Nash envelope* of $\mathcal{Q}$. In general it is not guaranteed that the Nash closure Y of $\partial\mathcal{Q}$ in M is a Nash normal-crossings divisor of M as shown in the following example.

Example 2.5.10. The *teardrop* is the Nash manifold with corners (see Figure 2.7) defined as

$$\mathcal{Q} := \{\mathtt{x} \geq 0,\ \mathtt{y}^2 \leq \mathtt{x}^2 - \mathtt{x}^4\} \subset \mathbb{R}^2.$$

Given any open semi-algebraic neighbourhood M of $\mathcal{Q}$ in $\mathbb{R}^2$ the Nash closure of $\partial\mathcal{Q}$ in M is not a Nash normal-crossings divisor.

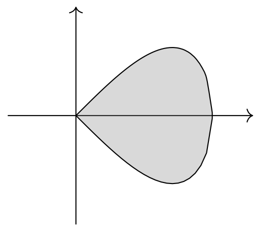

Figure 2.7: The teardrop.

We define now Nash manifolds with divisorial corners. For the rest of this dissertation, unless otherwise stated: *All the Nash manifolds with corners will be Nash manifolds with* ***divisorial corners***.

Definition 2.5.11. A Nash manifold with corners $\mathcal{Q} \subset \mathbb{R}^n$ is a manifold with *divisorial* corners if there exists a Nash envelope $M \subset \mathbb{R}^n$ such that the Nash closure of $\partial\mathcal{Q}$ in M is a Nash normal-crossings divisor.

A *facet* of a Nash manifold with corners $\mathcal{Q} \subset \mathbb{R}^n$ is the (topological) closure in $\mathcal{Q}$ of a connected component of $\mathrm{Sth}(\partial\mathcal{Q})$. Recall that $\mathrm{Sth}(\partial\mathcal{Q})$ is the set of points $x \in \partial\mathcal{Q}$ such that the germ $\partial\mathcal{Q}_x$ is the germ of a Nash manifold (see Section 2.4.1). As $\partial\mathcal{Q}$ is semi-algebraic, the facets are semi-algebraic and finitely many.

In [FGR] is shown the following characterization for Nash manifolds with divisorial corners:

Theorem 2.5.12 ([FGR, Thm.1.12, Cor.6.5])**.** *Let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners of dimension d. The following assertions are equivalent:*

(i) *There exists a Nash envelope $M \subset \mathbb{R}^n$ where the Nash closure of $\partial\mathcal{Q}$ is a Nash normal crossings divisor.*

(ii) *Every facet $\mathcal{F}$ of $\mathcal{Q}$ is contained in a Nash manifold $X \subset \mathbb{R}^n$ of dimension $d-1$.*

(iii) *The number of facets of $\mathcal{Q}$ that contain every given point $x \in \partial\mathcal{Q}$ coincides with the number of connected components of the germ $\mathrm{Sth}(\partial\mathcal{Q})_x$.*

(iv) *All the facets of $\mathcal{Q}$ are Nash manifold with divisorial corners.*

If that is the case, the Nash manifold M in (i) *can be chosen such that the Nash closure in M of every facet $\mathcal{F}$ of $\mathcal{Q}$ meets $\mathcal{Q}$ exactly along $\mathcal{F}$.*

Note that properties (iii) and (iv) are intrinsic properties of $\mathcal{Q}$ and do not depend on the Nash envelope M.

Chapter 3

Nash images of closed balls

In [Fe4] Fernando completely characterised the Nash images of Euclidean spaces. He proved the following theorem:

Theorem 3.1 (Nash images, [Fe4, Thm.1.4]). *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set of dimension d. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*

(ii) *$\mathcal{S}$ is connected by Nash paths.*

(iii) *$\mathcal{S}$ is connected by analytic paths.*

(iv) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(v) *$\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(vi) *$\mathcal{S}$ is well-welded.*

The concept of well-welded semi-algebraic set will be recalled in Section 3.4. Let $\overline{\mathcal{B}}_n := \{x \in \mathbb{R}^n : \|x\|^2 \leq 1\}$ be the unit closed ball. In this chapter we want to characterise the Nash images of the closed balls. That is to determine whether a given semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ is a Nash image of a closed ball $\overline{\mathcal{B}}_n$ or not. We will prove the following:

Theorem 3.2 (Compact Nash images). *Let $\mathcal{S} \subset \mathbb{R}^m$ be a d-dimensional compact semi-algebraic set. The following assertions are equivalent:*

(i) *There exists a Nash map $f : \mathbb{R}^d \to \mathbb{R}^m$ such that $f(\overline{\mathcal{B}}_d) = \mathcal{S}$.*

(ii) *$\mathcal{S}$ is connected by Nash paths.*

(iii) *$\mathcal{S}$ is connected by analytic paths.*

(iv) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.*

(v) $\mathcal{S}$ *is pure dimensional and there exists an analytic path* $\alpha : [0,1] \to \mathcal{S}$ *whose image meets all the connected components of the set of regular points of* $\mathcal{S}$.

(vi) $\mathcal{S}$ *is well-welded.*

Note that this theorem furnishes not only a characterization of the Nash images of the closed balls, but allows us to represent the desired semi-algebraic set as Nash image of the closed ball of smallest possible dimension. In fact, if $\mathcal{S} \subset \mathbb{R}^m$ is a semi-algebraic set of dimension d that is a Nash image of the closed ball $\overline{\mathcal{B}}_n$, by [BCR, Thm.2.8.8], one has $d \leq n$. Thus, by Theorem 3.2 $\mathcal{S}$ is a Nash image of the closed ball $\overline{\mathcal{B}}_d$ of dimension d, which is the smallest possible one.

3.1 Alternative compact models

The purpose of this section is to present simple alternative compact models to represent Nash images. We want to analyse some relationships between the models we will work with in the following sections. Most of the results of this section are borrowed from [FU6], where Fernando and Ueno found polynomial and regular relationships between the following compact models:

- the *standard sphere* $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\|^2 = 1\}$,
- the *unit closed ball* $\overline{\mathcal{B}}_d := \{x \in \mathbb{R}^d : \|x\|^2 \leq 1\}$,
- the *cylinder* $\mathcal{C}_d := \overline{\mathcal{B}}_{d-1} \times [-1,1]$,
- the *hypercube* $\mathcal{Q}_d := [-1,1]^d$,
- the *standard simplex*

$$\Delta_d := \{x \in \mathbb{R}^d : \mathtt{x}_1 \geq 0, \ldots, \mathtt{x}_d \geq 0,\, \mathtt{x}_1 + \ldots + \mathtt{x}_d \leq 1\},$$

- the *simplicial prism* $\Delta_{d-1} \times [-1,1]$.

We will include the proofs of their results in order to keep this work as much self-contained as possible.

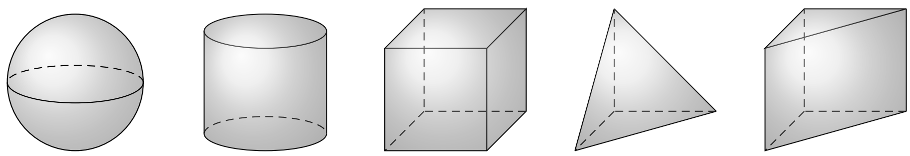

Figure 3.1: Alternative compact models.

Proposition 3.1.1. *The unit closed ball* $\overline{\mathcal{B}}_d$ *is a polynomial image of the standard sphere* $\mathbb{S}^d$.

Proof. The proof is straightforward. In fact, it is sufficient to consider the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$, $(x_1, \ldots, x_{d+1}) \mapsto (x_1, \ldots, x_d)$ onto the first d coordinates. $\square$

Proposition 3.1.2 (Cylinder, [FU6, Lem.2.1])**.** *The d-dimensional cylinder $\mathcal{C}_d := \overline{\mathcal{B}}_{d-1} \times [-1,1]$ is a polynomial image of the closed ball $\overline{\mathcal{B}}_d$.*

Proof. Define first the polynomials $g(\mathtt{t}) := \mathtt{t}(3-4\mathtt{t}^2)$ and $h(\mathtt{t}) := \sqrt{3}\left(1-\frac{4}{9}\mathtt{t}^2\right)$. Denote $x := (x_1,\dots,x_d)$, $x' := (x_1,\dots,x_{d-1})$ and define the polynomial maps

$$G : \mathbb{R}^d \to \mathbb{R}^d,\ x := (x', x_d) \mapsto (x', g(x_d)),$$
$$H : \mathbb{R}^d \to \mathbb{R}^d,\ x := (x', x_d) \mapsto (h(\|x'\|)x', x_d).$$

We claim: $G(\overline{\mathcal{B}}_d) \subset \mathcal{C}_d$. Define, for each $x' \in \overline{\mathcal{B}}_{d-1}$,

$$\mathcal{I}_{x'} := \{x'\} \times \left\{\mathtt{x}_d^2 \leq 1 - \|\mathtt{x}'\|^2\right\}$$

and observe that

$$G(\overline{\mathcal{B}}_d) = \bigcup_{x' \in \overline{\mathcal{B}}_{d-1}} G(\mathcal{I}_{x'}) = \bigcup_{x' \in \overline{\mathcal{B}}_{d-1}} \{x'\} \times g\left(\left\{\mathtt{x}_d^2 \leq 1 - \|\mathtt{x}'\|^2\right\}\right).$$

We distinguish two cases:

(i) If $\|x'\|^2 \leq \frac{3}{4}$, then $|x_d| \leq \frac{1}{2}$. The polynomial function g satisfies (see Figure 3.2)

$$g\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) = g([-1,1]) = [-1,1].$$

As $\{x'\} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \mathcal{I}_{x'} \subset \{x'\} \times [-1,1]$, we have $G(\mathcal{I}_{x'}) = \{x'\} \times [-1,1]$.

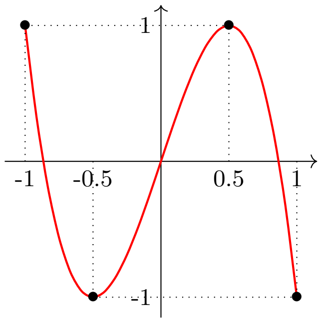

Figure 3.2: Graph of the polynomial function g.

(ii) If $\frac{3}{4} \leq \|x'\|^2 \leq 1$, then $\mathcal{I}_{x'} \subset \{x'\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]$. Define the polynomial function (see Figure 3.3)

$$g^*(\mathtt{t}) := g\left(\sqrt{1-\mathtt{t}^2}\right)^2 = (1-\mathtt{t}^2)(4\mathtt{t}^2-1)^2.$$

and notice that $g^*([0,1]) = [0,1]$. As g is odd and strictly increasing on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we have

$$G(\mathcal{I}_{x'}) = \{x'\} \times \{\mathtt{x}_d^2 \leq g^*(\|x'\|)\}$$

and the claim follows.

Note that in the proof of the claim we have also obtained the following equality

$$G(\overline{\mathcal{B}}_d) = \left\{\|\mathtt{x}'\|^2 \leq \frac{3}{4},\ |\mathtt{x}_d| \leq 1\right\} \cup \left\{\frac{3}{4} \leq \|\mathtt{x}'\|^2 \leq 1,\ \mathtt{x}_d^2 \leq g^*(\|\mathtt{x}'\|)\right\}. \quad (3.1.1)$$

Now we show that $H(\mathcal{C}_d) \subset \mathcal{C}_d$. Define the polynomial function $h^* := \mathtt{t}h$ (see Figure 3.3) and observe that $h^*([0,1]) = [0,1]$. Thus, fixed a point $(x', x_d) \in \mathcal{C}_d$, since $\|x'\| \leq 1$, we deduce $\|h(x')x'\| = |h^*(\|x'\|)| \leq 1$, so $H(x', x_d) \in \mathcal{C}_d$.

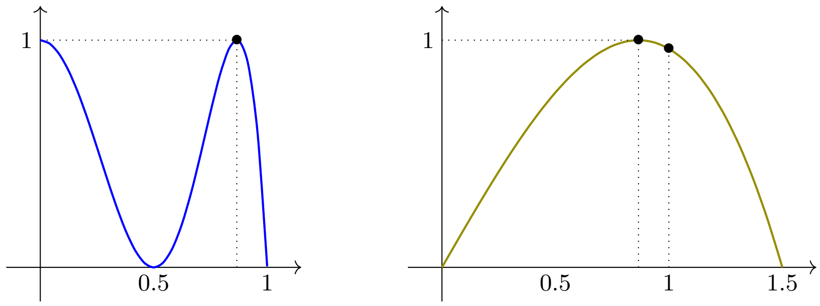

Figure 3.3: Graphs of the polynomial functions g^* (left) and h^* (right).

If we prove the equality $H\left(\left\{\|\mathtt{x}'\|^2 \leq \frac{3}{4},\ |\mathtt{x}_d| \leq 1\right\}\right) = \mathcal{C}_d$, the result follows. In fact using (3.1.1) and the fact that both $G(\overline{\mathcal{B}}_d), H(\mathcal{C}_d) \subset \mathcal{C}_d$, we obtain

$$H(G(\overline{\mathcal{B}}_d)) = \mathcal{C}_d.$$

To that end, note first that h^* has a global maximum at $t = \frac{\sqrt{3}}{2}$ on $[0,1]$ and verifies $h^*(0) = 0$ and $h^*(\frac{\sqrt{3}}{2}) = 1$. Fix a point $x' \in \partial\overline{\mathcal{B}}_{d-1}$ and observe that

$$\begin{aligned} &H\left(\left\{\lambda x' \,:\, \lambda \in \left[0, \frac{\sqrt{3}}{2}\right]\right\} \times [-1,1]\right) \\ &\quad = \left\{h^*(\lambda)x' \,:\, \lambda \in \left[0, \frac{\sqrt{3}}{2}\right]\right\} \times [-1,1] = \{\mu x' \,:\, \mu \in [0,1]\} \times [-1,1], \end{aligned}$$

so $H\left(\left\{\|\mathtt{x}'\|^2 \leq \frac{3}{4},\ |\mathtt{x}_d| \leq 1\right\}\right) = \mathcal{C}_d$, as required. $\square$

Lemma 3.1.3 (Standard simplex, [FU6, Lem.2.5])**.** *The d-dimensional simplex Δ_d is a polynomial image of the closed ball $\overline{\mathcal{B}}_d$.*

Proof. The polynomial map $f : \mathbb{R}^d \to \mathbb{R}^d$, $(x_1, \ldots, x_d) \mapsto (x_1^2, \ldots, x_d^2)$ satisfies $f(\overline{\mathcal{B}}_d) = \Delta_d$. $\square$

Corollary 3.1.4 (Simplicial prism, [FU6, Cor.2.8])**.** *The d-dimensional simplicial prism $\Delta_{d-1} \times [-1,1]$ is a polynomial image of the closed ball $\overline{\mathcal{B}}_d$.*

Proof. By Lemma 3.1.3 there exists a polynomial map $f_1 : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that $f_1(\overline{\mathcal{B}}_{d-1}) = \Delta_{d-1}$. By Proposition 3.1.2 there exists a polynomial map

$f_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that $f_0(\overline{\mathcal{B}}_d) = \overline{\mathcal{B}}_{d-1} \times [-1, 1]$. If we consider the polynomial map $f := (f_1, \mathtt{x}_d) \circ f_0 : \mathbb{R}^d \to \mathbb{R}^d$, it satisfies

$$f(\overline{\mathcal{B}}_d) = (f_1, \mathtt{x}_d)(\overline{\mathcal{B}}_{d-1} \times [-1, 1]) = \Delta_{d-1} \times [-1, 1],$$

as required. □

Corollary 3.1.5 (Hypercube, [FU6, Cor.2.9]). *The hypercube* $\mathcal{Q}_d := [-1, 1]^d$ *is a polynomial image of the closed ball* $\overline{\mathcal{B}}_d$*.*

Proof. We proceed by induction on the dimension $d \geq 1$. For $d = 1$ it holds $\overline{\mathcal{B}}_1 = [-1, 1] = \mathcal{Q}_1$. By induction hypothesis there exists a polynomial map $f_1 : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that $f_1(\overline{\mathcal{B}}_{d-1}) = \mathcal{Q}_{d-1}$. By Proposition 3.1.2 there exists a polynomial map $f_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that $f_0(\overline{\mathcal{B}}_d) = \overline{\mathcal{B}}_{d-1} \times [-1, 1]$. Thus the polynomial map $f := (f_1, \mathtt{x}_d) \circ f_0 : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$f(\overline{\mathcal{B}}_d) = (f_1, \mathtt{x}_d)(\overline{\mathcal{B}}_{d-1} \times [-1, 1]) = \mathcal{Q}_{d-1} \times [-1, 1] = \mathcal{Q}_d,$$

as required. □

Proposition 3.1.6 ([FU6, Lem.2.10]). *The d-dimensional closed ball* $\overline{\mathcal{B}}_d$ *is a polynomial image of the d-dimensional hypercube* $\mathcal{Q}_d := [-1, 1]^d$*.*

Proof. If $d = 1$ we have $\overline{\mathcal{B}}_1 = \mathcal{Q}_1 = [-1, 1]$, so we assume $d \geq 2$. Consider the non-negative univariate polynomial

$$h(\mathtt{t}) := \mathtt{t}^2 \frac{(\mathtt{t} - d)^{2(d-1)}}{(d-1)^{2(d-1)}} \in \mathbb{R}[\mathtt{t}],$$

and observe that $h(0) = h(d) = 0$, whereas $h(1) = 1$. The derivative

$$h'(\mathtt{t}) = \frac{2d\mathtt{t}(\mathtt{t} - d)^{2(d-1)-1}}{(d-1)^{2(d-1)}}(\mathtt{t} - 1)$$

is positive on $(0, 1)$ and negative on $(1, d)$. Thus, 1 is a global maximum of h on the interval $[0, d]$, so $0 \leq h(\mathtt{t}) \leq 1$ on $[0, d]$.

Recall that $\overline{\mathcal{B}}_d \subset \mathcal{Q}_d \subset \overline{\mathcal{B}}_d(0, \sqrt{d})$ and consider the polynomial map

$$f : \mathbb{R}^d \to \mathbb{R}^d,\ x \mapsto h(\|x\|^2)x.$$

Observe that $f(\overline{\mathcal{B}}_d) = f(\overline{\mathcal{B}}_d(0, \sqrt{d})) = \overline{\mathcal{B}}_d$, so $f(\mathcal{Q}_d) = \overline{\mathcal{B}}_d$, as required. □

Lemma 3.1.7. *The d-dimensional closed ball* $\overline{\mathcal{B}}_d$ *is a polynomial image of the d-dimensional simplex* Δ_d*.*

Proof. As $\Delta_1 = [0, 1]$ and $\overline{\mathcal{B}}_1 = [-1, 1]$, for $d = 1$ it is enough to consider the polynomial function $h(\mathtt{t}) = 2\mathtt{t} - 1$. We assume now $d \geq 2$. Proceeding in a similar way as in the proof of Proposition 3.1.6, we consider the univariate polynomial

$$h(\mathtt{t}) := \mathtt{t}^2 \frac{(\mathtt{t} - 2d^2)^{2(2d^2-1)}}{(2d^2-1)^{2(2d^2-1)}} \in \mathbb{R}[\mathtt{t}].$$

It satisfies $h(0) = h(2d^2) = 0$ and $h(1) = 1$. Moreover, h has a global maximum at $t = 1$ and it satisfies $0 \leq h(\mathtt{t}) \leq 1$ on the interval $[0, 2d^2]$.

Consider the simplex $\Delta'_d := \{\mathtt{x}_1 \geq -1, \ldots, \mathtt{x}_d \geq -1, \mathtt{x}_1 + \ldots + \mathtt{x}_d \leq \sqrt{d}\}$. As $2d^2 > d - 1 + (\sqrt{d} + d - 1)^2$, it holds $\overline{\mathcal{B}}_d \subset \Delta'_d \subset \overline{\mathcal{B}}_d(0, \sqrt{2}d)$. Consider the polynomial map

$$g : \mathbb{R}^d \to \mathbb{R}^d,\ x \mapsto h(\|x\|^2)x.$$

Observe that $g(\overline{\mathcal{B}}_d) = g(\overline{\mathcal{B}}_d(0, \sqrt{2}d)) = \overline{\mathcal{B}}_d$, so $g(\Delta'_d) = \overline{\mathcal{B}}_d$. Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be an affine map such that $h(\Delta_d) = \Delta'_d$, then the polynomial map $f := g \circ h$ satisfies $f(\Delta_d) = \overline{\mathcal{B}}_d$, as required. □

Proposition 3.1.8. *The d-dimensional closed ball $\overline{\mathcal{B}}_d$ is a polynomial image of the d-dimensional prism $\Delta_{d-1} \times [-1, 1]$.*

Proof. If $d = 1$, we have $\Delta_0 \times [-1, 1] = \overline{\mathcal{B}}_1 = [-1, 1]$. So we can consider the case $d \geq 2$. By Lemma 3.1.7 there exists a polynomial map $h_{d-1} : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that $h_{d-1}(\Delta_{d-1}) = \overline{\mathcal{B}}_{d-1}$. By Corollary 3.1.5 there exists a polynomial map $h'_{d-1} : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ that maps $\overline{\mathcal{B}}_{d-1}$ onto the hypercube $\mathcal{Q}_{d-1}$. By Proposition 3.1.6 there exists a polynomial map $g : \mathbb{R}^d \to \mathbb{R}^d$ such that $g(\mathcal{Q}_{d-1} \times [-1, 1]) = g(\mathcal{Q}_d) = \overline{\mathcal{B}}_d$. Thus, the polynomial map $f := g \circ (h'_{d-1} \circ h_{d-1}, \mathtt{x}_d) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $f(\Delta_{d-1} \times [-1, 1]) = \overline{\mathcal{B}}_d$. □

Proposition 3.1.9. *The d-dimensional sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is a regular image of the d-dimensional hypercube $\mathcal{Q}_d := [-1, 1]^d$.*

Proof. We proceed by induction on the dimension $d \geq 1$. For the 1-dimensional case, consider first the inverse of the stereographic projection

$$f_0 : \mathbb{R} \to \mathbb{S}^1 \setminus \{(0, 1)\},\ t \mapsto \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right).$$

We have $f_0([-1, 1]) = \mathbb{S}^1 \cap \{\mathtt{y} \leq 0\}$. Consider the polynomial map

$$f_1 : \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{C} \equiv \mathbb{R}^2,$$
$$(x, y) \equiv x + y\sqrt{-1} =: z \mapsto z^2 = x^2 - y^2 + 2xy\sqrt{-1} \equiv (x^2 - y^2, 2xy).$$

The image of $[-1, 1]$ under the regular map $\varphi := f_1 \circ f_0 : \mathbb{R} \to \mathbb{R}^2$ is $\mathbb{S}^1$.

By induction hypothesis there exists a regular map $g : \mathbb{R}^{d-1} \to \mathbb{R}^d$ such that $g([-1, 1]^{d-1}) = \mathbb{S}^{d-1}$ and let $\varphi := (\varphi_1, \varphi_2) : \mathbb{R} \to \mathbb{R}^2$ be the regular map described above, such that $\varphi([-1, 1]) = \mathbb{S}^1$. Denote $x' := (x_1, \ldots, x_{d-1})$ and $\mathtt{e}_{d+1} := (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$. Then the regular map

$$f : \mathbb{R}^d \to \mathbb{R}^{d+1},\ (x', x_d) \mapsto \varphi_1(x_d)(g(x'), 0) + \varphi_2(x_d)\mathtt{e}_{d+1},$$

satisfies $f([-1, 1]^d) = \mathbb{S}^d$. □

The following remark shows that the map in Proposition 3.1.9 cannot be taken polynomial. This implies that the family of the polynomial images of the d-dimensional closed unit ball is a proper sub-family of the family of polynomial images of the d-dimensional sphere (see also Proposition 3.1.1). However, the results proved in this section show that the family of the regular images of the d-dimensional closed unit ball and the family of the regular images of the d-sphere are the same.

Remark 3.1.10 ([FU6, §1.2]). *There exist no non-constant polynomial maps from the m-dimensional unit closed ball $\overline{\mathcal{B}}_m$ to the n-dimensional sphere $\mathbb{S}^n$.*

Consider a polynomial map $f := (f_1, \dots, f_{n+1}) : \mathbb{R}^m \to \mathbb{R}^{n+1}$ such that $f(\overline{\mathcal{B}}_m) \subset \mathbb{S}^n$. Thus, $f_1^2 + \cdots + f_{n+1}^2 = 1$ on the open ball $\mathcal{B}_m$. By the identity principle for polynomials, we deduce that $f_1^2 + \cdots + f_{n+1}^2 = 1$ on $\mathbb{R}^m$, so $\deg(f_1), \dots, \deg(f_{n+1}) \leq 0$. That is, the polynomial map f is a constant map.

Let us summarize the results of this section. We have proved that the closed ball $\overline{\mathcal{B}}_d$, the cylinder $\overline{\mathcal{B}}_{d-1} \times [-1, 1]$, the hypercube $[-1, 1]^d$, the simplicial prism $\Delta_{d-1} \times [-1, 1]$ and the simplex Δ_d are polynomial image one of each others. Moreover, each of them is a polynomial image of the sphere $\mathbb{S}^d$, but the sphere is a regular, but not a polynomial, image of the others compact models.

3.2 Necessary conditions

As the closed ball $\overline{\mathcal{B}}_d$ is convex it is connected by segments, so by Nash paths. *If $\mathcal{S} \subset \mathbb{R}^m$ is a semi-algebraic set and $f : \mathbb{R}^d \to \mathbb{R}^m$ is a Nash map such that $f(\overline{\mathcal{B}}_d) = \mathcal{S}$, then $\mathcal{S}$ is compact and connected by Nash paths.*

The fact that $\mathcal{S}$ is compact is straightforward. Fix now any two points $x, y \in \mathcal{S}$. As the Nash map f is surjective, there exist two points $\overline{x}, \overline{y} \in \overline{\mathcal{B}}_d$ such that $f(\overline{x}) = x$ and $f(\overline{y}) = y$. Let $\sigma : [0, 1] \to \overline{\mathcal{B}}_d$ be a segment between $\overline{x}$ and $\overline{y}$. Then $f \circ \sigma : [0, 1] \to \mathcal{S}$ is a Nash path between x and y.

Thus, we have the following necessary conditions for a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ to be Nash image of a closed ball:

Lemma 3.2.1 (Necessary conditions). *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set and $f : \mathbb{R}^d \to \mathbb{R}^m$ a Nash map such that $f(\overline{\mathcal{B}}_d) = \mathcal{S}$, then $\mathcal{S}$ is compact and connected by Nash paths.*

Let us see now some consequences of being connected by Nash paths. We will show that a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ connected by Nash paths is irreducible and pure dimensional.

Recall that a semi-algebraic set $\mathcal{S}$ is *irreducible* if its ring of Nash functions $\mathcal{N}(S)$ is an integral domain (see Definition 2.4.5). Recall also that $\mathcal{S}$ is *pure dimensional* if the dimension of the germ $\mathcal{S}_x$ is equal to the dimension of $\mathcal{S}$ for each $x \in \mathcal{S}$ (see Section 2.2.3).

Proposition 3.2.2. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set connected by Nash paths. Then $\mathcal{S}$ is irreducible.*

Proof. Suppose $\mathcal{S}$ is a reducible semi-algebraic set, that is, there exist Nash functions $f_1, f_2 : \mathcal{S} \to \mathbb{R}$ such that $f_1 f_2 \equiv 0$ on $\mathcal{S}$ but f_1 and f_2 are not identically zero. This implies that there exist two points $x, y \in \mathcal{S}$ such that $f_1(x) = 0$, $f_2(x) \neq 0$ and $f_1(y) \neq 0$, $f_2(y) = 0$. Consider a Nash path $\sigma : [0, 1] \to \mathbb{R}$ connecting x and y. As the semi-algebraic set $[0, 1]$ is irreducible and

$$(f_1 \circ \sigma) \cdot (f_2 \circ \sigma) \equiv 0,$$

we have either $f_1 \circ \sigma \equiv 0$ or $f_2 \circ \sigma \equiv 0$. We may assume, without loss of generality, that $\sigma([0, 1]) \subset \{f_1 = 0\}$. This is a contradiction, because $f_1(\sigma(1)) = f_1(y) \neq 0$. Thus, the semi-algebraic set $\mathcal{S}$ is irreducible. $\square$

Proposition 3.2.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set connected by Nash paths. Then $\mathcal{S}$ is pure dimensional.*

Proof. Assume $\mathcal{S}$ is not pure dimensional. Then there exists a point $y \in \mathcal{S}$ such that $\dim \mathcal{S}_y < \dim \mathcal{S}$. Let $B \subset \mathbb{R}^n$ be a small open ball centred at y such that $\dim(\mathcal{S} \cap B) < \dim \mathcal{S}$. Let Y be the Zariski closure of $\mathcal{S} \cap B$ and let $p \in \mathbb{R}[\mathtt{x}]$ be a polynomial equation of Y (see Proposition 2.1.2). Let $x \in \mathcal{S}$ be a point where the local dimension is maximal, that is, $\dim(\mathcal{S}_x) = \dim(\mathcal{S})$. The algebraic set Y has dimension strictly smaller than the dimension of $\mathcal{S}$, so we may assume $x \notin Y$. Consider a Nash path $\sigma : [0,1] \to \mathcal{S}$ such that $\sigma(0) = y$ and $\sigma(1) = x$. The Nash function $p \circ \sigma : [0,1] \to \mathbb{R}$ is identically zero on an open neighbourhood of 0. Thus, by the identity principle for Nash functions it is identically zero on $[0,1]$, that is, $\sigma([0,1]) \subset Y$. This is a contradiction because $\sigma(1) = x$ and $x \notin Y$. $\square$

In the following example we show that for a semi-algebraic set $\mathcal{S}$ being pure dimensional and irreducible is not enough to guarantee that $\mathcal{S}$ is connected by Nash paths. The example is borrowed from [Fe4, Ex.7.12], whereas the proof appears in [FU6, Ex.1.2].

Example 3.2.4 ([Fe4, Ex.7.12]). The irreducible and pure dimensional semi-algebraic set (see Figure 3.4)

$$\mathcal{S} := \{(4\mathtt{x}^2 - \mathtt{y}^2)(4\mathtt{y}^2 - \mathtt{x}^2) \geq 0, \ \mathtt{y} \geq 0\} \subset \mathbb{R}^2,$$

is not connected by Nash paths.

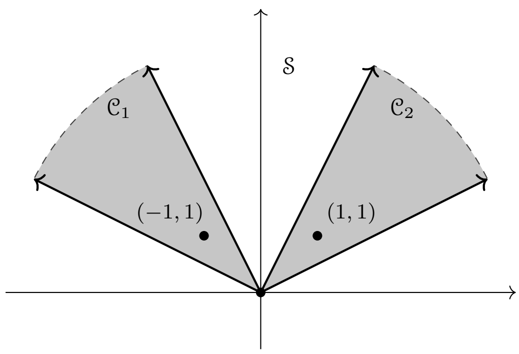

Figure 3.4: The semi-algebraic set $\mathcal{S}$ (figure borrowed from [FU6, Fig.1.1])

Proof. Pick the points $p_1 := (-1,1), p_2 := (1,1) \in \mathcal{S}$ and assume that there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = p_1$ and $\alpha(1) = p_2$. Consider the closed semi-algebraic sets $\mathcal{C}_1 := \mathcal{S} \cap \{\mathtt{x} \leq 0\}$ and $\mathcal{C}_2 := \mathcal{S} \cap \{\mathtt{x} \geq 0\}$, which satisfy $\mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2$. Both $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex, so they are connected by Nash paths and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{(0,0)\}$. Define $\mathcal{C}_i^* := \{\lambda w : \ w \in \mathcal{C}_i, \ \lambda \in \mathbb{R}\}$ for $i = 1,2$. Note that $\mathcal{S} \cap \{\mathtt{x} < 0\} = \mathcal{C}_1 \setminus \{(0,0)\}$ and $\mathcal{S} \cap \{\mathtt{x} > 0\} = \mathcal{C}_2 \setminus \{(0,0)\}$ are pairwise disjoint open subsets of $\mathcal{S}$. We have $0 \in \alpha^{-1}(\mathcal{C}_1 \setminus \{(0,0)\})$ and $1 \in \alpha^{-1}(\mathcal{C}_2 \setminus \{(0,0)\})$, so $t_0 := \inf(\alpha^{-1}(\mathcal{C}_2 \setminus \{(0,0)\})) > 0$. As α is a (non-constant) Nash path, $t_0 \in \operatorname{Cl}(\alpha^{-1}(\mathcal{C}_1 \setminus \{(0,0)\})) \cap \operatorname{Cl}(\alpha^{-1}(\mathcal{C}_2 \setminus \{(0,0)\}))$ and

$\alpha(t_0) = (0,0)$. As $\alpha^{-1}(\mathcal{C}_1 \setminus \{(0,0)\})$ and $\alpha^{-1}(\mathcal{C}_2 \setminus \{(0,0)\})$ are pairwise disjoint open subsets of $[0,1]$, there exists $\varepsilon > 0$ such that

$$\alpha((t_0 - \varepsilon, t_0)) \subset \mathcal{C}_1 \setminus \{(0,0)\} \quad \text{and} \quad \alpha((t_0, t_0 + \varepsilon)) \subset \mathcal{C}_2 \setminus \{(0,0)\}.$$

The tangent direction to $\alpha((t_0 - \varepsilon, t_0 + \varepsilon))$ at $\alpha(t_0) = (0,0)$ is the line generated by the vector

$$w = \lim_{t \to t_0} \frac{\alpha(t) - \alpha(t_0)}{(t - t_0)^k} = \begin{cases} \lim_{t \to t_0^+} \frac{\alpha(t)-(0,0)}{(t-t_0)^k} \in \mathcal{C}_1^* \setminus \{(0,0)\}, \\ \lim_{t \to t_0^-} \frac{\alpha(t)-(0,0)}{(t-t_0)^k} \in \mathcal{C}_2^* \setminus \{(0,0)\}, \end{cases}$$

where k is the multiplicity of t_0 as a root of $\|\alpha\|$. This is a contradiction (because $\mathcal{C}_1^* \cap \mathcal{C}_2^* = \{(0,0)\}$), so $\mathcal{S}$ is not connected by Nash paths. $\square$

3.3 Polynomial paths inside semi-algebraic sets

In this section we will present an improved polynomial curve selection lemma. It will allow us to approximate continuous semi-algebraic paths inside the closure of an open semi-algebraic set by polynomial paths, with strong control on the derivatives. This lemma will be one of the main ingredients in our proof of Theorem 3.2. In [Fe5, Thm.1.7] and [FU5, Thm.3.1] Fernando and Ueno have made an extended study of polynomial and Nash paths inside the closure of open semi-algebraic sets. For our purposes we need only a simplified version of the results obtained there.

We endow the space $\mathcal{C}^\nu([a,b],\mathbb{R})$ of differentiable functions of class $\mathcal{C}^\nu$ on the interval $[a,b]$ with the $\mathcal{C}^\nu$ compact-open topology. Recall that a basis of open neighbourhoods of $g \in \mathcal{C}^\nu([a,b],\mathbb{R})$ in this topology is constituted by the sets of the type:

$$\mathcal{U}^\nu_{g,\varepsilon} := \{f \in \mathcal{C}^\nu([a,b],\mathbb{R}) : \ \|f^{(\ell)} - g^{(\ell)}\|_{[a,b]} < \varepsilon : \ \ell = 0, \dots, \nu\}$$

where $\varepsilon > 0$ and $\|h\|_{[a,b]} := \max\{h(x) : x \in [a,b]\}$.

One has $\mathcal{C}^\nu([a,b],\mathbb{R}^n) = \mathcal{C}^\nu([a,b],\mathbb{R}) \times \cdots \times \mathcal{C}^\nu([a,b],\mathbb{R})$ and we endow this space with the product topology. If $X \subset [a,b]$, one defines analogously the $\mathcal{C}^\nu$ compact-open topology of the space $\mathcal{C}^\nu(X,\mathbb{R}^n)$. The following result is well-known and its proof follows straightforwardly from [H, §2.5. Ex.10, pp. 64-65] using standard arguments.

Lemma 3.3.1. *Let $U \subset \mathbb{R}^n$ be an open set and let $\varphi : U \to \mathbb{R}^m$ be a $\mathcal{C}^\ell$ map for some $0 \le \ell \le \nu$. Consider the map $\varphi_* : \mathcal{C}^\nu([a,b],U) \to \mathcal{C}^\ell([a,b],\mathbb{R}^m),\ f \mapsto \varphi \circ f$, where both spaces are endowed with their $\mathcal{C}^\ell$ compact-open topologies. Then φ_* is continuous.*

In addition, one has the following.

Lemma 3.3.2. *Let $X \subset [a,b]$ and consider the restriction map*

$$\rho : \mathcal{C}^\nu([a,b],\mathbb{R}^n) \to \mathcal{C}^\nu(X,\mathbb{R}^n),\ f \mapsto f|_X,$$

where the spaces are endowed with their respective $\mathcal{C}^\nu$ compact-open topologies. Then ρ is continuous and if in addition $X \subset [a,b]$ is closed, then ρ is surjective.

We borrow the following result from [B] that combines Weierstrass' polynomial approximation with Hermite's interpolation on a finite set.

Lemma 3.3.3. *Let $a < t_1 < \cdots < t_r < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a $\mathcal{C}^\nu$ function. Write $a_{ik} := f^{(k)}(t_i)$ for $i = 1, \ldots, r$ and $0 \le k \le \nu$. Fix $\varepsilon > 0$. Then there exists a polynomial $g \in \mathbb{R}[\mathtt{t}]$ such that:*

(i) $\|f^{(k)} - g^{(k)}\|_{[a,b]} < \varepsilon$ *for* $k = 0, \ldots, \nu$.

(ii) $g^{(k)}(t_i) = a_{ik}$ *for* $i = 1, \ldots, r$ *and* $0 \le k \le \nu$.

Proof. The proof is conducted in two steps:

STEP 1. *There exists a polynomial $h \in \mathbb{R}[\mathtt{t}]$ such that $\|h^{(k)} - f^{(k)}\|_{[a,b]} < \varepsilon$ for $k = 0, \ldots, \nu$.*

We proceed by induction on the integer $\nu \ge 0$. If $\nu = 0$, the result is classical Stone-Weierstrass' polynomial approximation theorem. Assume the result true for $\nu - 1 \ge 0$ and let us check that it is also true for ν.

Consider the $\mathcal{C}^{\nu-1}$ map f' on the interval $[a, b]$ and extend it as a $\mathcal{C}^{\nu-1}$ map to a bigger interval $[a', b']$ that contains $[a, b]$ in its interior. By induction hypothesis there exists a polynomial map $h_0 \in \mathbb{R}[\mathtt{t}]^n$ such that

$$|f^{(k+1)} - h_0^{(k)}| < \frac{\varepsilon}{1 + (b - a)}$$

on $[a', b']$ for $k = 0, \ldots, \nu - 1$. By Barrow's rule

$$f(t) = f(a) + \int_a^t f'(s)ds.$$

Define

$$h(t) := f(a) + \int_a^t h_0(s)ds,$$

which is a polynomial of $\mathbb{R}[\mathtt{t}]$. Observe that $h' = h_0$, so

$$\left\|f^{(k)} - h^{(k)}\right\|_{[a,b]} = \left\|f^{(k)} - h_0^{(k-1)}\right\|_{[a,b]} < \frac{\varepsilon}{1 + (b - a)} < \varepsilon$$

for $k = 1, \ldots, \nu$. In addition,

$$\begin{aligned}
\|h - f\|_{[a,b]} &= \left\|\int_a^t h_0(s)ds - \int_a^t f'(s)ds\right\|_{[a,b]} \\
&= \max_{[a,b]} \left\{\left|\int_a^t (h_0(s) - f'(s))ds\right|\right\} \\
&\le \max_{[a,b]} \left\{\int_a^t |h_0(s) - f'(s)|ds\right\} < (b - a)\frac{\varepsilon}{1 + (b - a)} < \varepsilon.
\end{aligned}$$

STEP 2. We show how to modify h in order to have also condition (ii).

Take polynomials P_{ik} such that

$$P_{ik}^{(\ell)}(t_j) = \begin{cases} 0 & \text{if } i \ne j \text{ or } k \ne \ell, \\ 1 & \text{if } i = j \text{ and } k = \ell, \end{cases}$$

for $i = 1, \dots, r$ and $0 \le k, \ell \le \nu$. For instance, *we may take*

$$P_{ik} := b_{ik}(\mathtt{t} - \mathtt{t}_i)^k \prod_{j \neq i}((\mathtt{t} - t_i)^{\nu+1} - (t_j - t_i)^{\nu+1})^{\nu+1}$$

for $b_{ik} := \frac{1}{k!} \frac{(-1)^{\nu+1}}{\prod_{j \neq i}(t_j - t_i)^{(\nu+1)^2}}$.

The Taylor expansion of P_{ik} at t_i has the form

$$P_{ik} = \frac{1}{k!}(\mathtt{t} - t_i)^k + e_{ik}(\mathtt{t} - t_i)^{\nu+1} + \cdots$$

for some $e_{ik} \in \mathbb{R}$, whereas the Taylor expansion of P_{ik} at t_j (for $j \neq i$) has the form

$$P_{ik} = \Big(b_{ik}(t_j - t_i)^k((\nu+1)(t_j - t_i)^{\nu})^{\nu+1} \prod_{m \neq i,j}((t_j - t_i)^{\nu+1} - (t_m - t_i)^{\nu+1})^{\nu+1}\Big)(\mathtt{t} - t_j)^{\nu+1} + \cdots.$$

Define

$$M := \max\{\|P_{ik}^{(\ell)}\|_{[a,b]} :\ 1 \le i \le r,\ 0 \le k,\ \ell \le \nu\} \quad \text{and} \quad \delta := \frac{\varepsilon}{1 + r(\nu+1)M}.$$

Let $h \in \mathbb{R}[\mathtt{t}]$ be a polynomial such that $\|h^{(k)} - f^{(k)}\|_{[a,b]} < \delta$ for $k = 0, \dots, \nu$. Define

$$g := h + \sum_{i=1}^{r}\sum_{k=0}^{\nu} c_{ik}P_{ik}$$

where $c_{ik} := a_{ik} - h^{(k)}(t_i) = f^{(k)}(t_i) - h^{(k)}(t_i)$ for $i = 1, \dots, r$ and $k = 0, \dots, \nu$. Thus,

$$g^{(\ell)}(t_j) = h^{(\ell)}(t_j) + \sum_{i=1}^{r}\sum_{k=0}^{\nu} c_{ik}P_{ik}^{(\ell)}(t_j) = h^{(\ell)}(t_j) + c_{j\ell} = a_{j\ell} = f^{(\ell)}(t_j)$$

for $j = 1, \dots, r$ and $\ell = 0, \dots, \nu$.

Observe that $|c_{ik}| = |f^{(k)}(t_i) - h^{(k)}(t_i)| < \delta$, so

$$\|g^{(\ell)} - f^{(\ell)}\|_{[a,b]} \le \|h^{(\ell)} - f^{(\ell)}\|_{[a,b]} + \sum_{i=1}^{r}\sum_{k=0}^{\nu}|c_{ik}|\|P_{ik}^{(\ell)}\|_{[a,b]} < \delta + r(\nu+1)M\delta = \varepsilon,$$

for each $\ell = 0, \dots, \nu$, as required. □

Lemma 3.3.4. *Let $\delta > 0$ and $f : [0, \delta] \to \mathbb{R}$ be a $\mathcal{C}^{k+1}$ function. Assume $f^{(\ell)}(0) = 0$ for each $\ell = 0, \dots, k-1$, $f^{(k)} > 0$ on $[0, \delta]$ and that $f|_{(0,\delta]}$ is strictly positive. Take $0 < \varepsilon < \min\{f^{(k)}|_{[0,\delta]}, f(\delta)\}$. If $g : [0, \delta] \to \mathbb{R}$ is a $\mathcal{C}^{k+1}$ function such that $g^{(\ell)}(0) = 0$ for $\ell = 0, \dots, k-1$, $g^{(k)}(0) = f^{(k)}(0)$, $|f - g|_{[0,\delta]} < \varepsilon$ and $|f^{(k)} - g^{(k)}|_{[0,\delta]} < \varepsilon$, then $g|_{(0,\delta]}$ is strictly positive.*

Proof. Using Taylor's expansion, we know that g around 0 has the form

$$g(\mathtt{t}) = \tfrac{g^{(k)}(0)}{k!}\mathtt{t}^k + \mathtt{t}^{k+1}\theta(\mathtt{t}) = \tfrac{f^{(k)}(0)}{k!}\mathtt{t}^k + \mathtt{t}^{k+1}\theta(\mathtt{t}),$$

where θ is a continuous map defined on an interval around 0. In particular, $g > 0$ for $t \in (0, \delta)$ close enough to 0. In addition, $g^{(k)} > 0$ on $[0, \delta]$ and $g(\delta) > 0$.

Suppose there exists $t^* \in (0, \delta)$ such that $g(t^*) \leq 0$. Then there exists $\xi_0 \in (0, \delta)$ such that $g(\xi_0) = 0$. Assume by induction on $\ell < k$ that there exists $0 < \xi_\ell < \cdots < \xi_0 < \delta$ such that $g^{(j)}(\xi_j) = 0$ for $j = 0, \ldots, \ell$. As $g^{(\ell)}(0) = 0$ and $g^{(\ell)}(\xi_\ell) = 0$, there exist $0 < \xi_{\ell+1} < \xi_\ell$ such that $g^{(\ell+1)}(\xi_{\ell+1}) = 0$. In particular, $g^{(k)}(\xi_k) = 0$ and $0 < \xi_k < \delta$, which contradicts the fact that $g^{(k)} > 0$ on $[0, \delta]$. Consequently, $g|_{(0,\delta]} > 0$, as required. □

If $\alpha : [a, b] \to \mathbb{R}^n$ is a continuous semi-algebraic path, by [BCR, Prop.2.9.10] there exists a finite set $\eta(\alpha) \subset [a, b]$ such that α is not Nash at the points of $\eta(\alpha)$, but $\alpha|_{[a,b]\setminus\eta(\alpha)}$ is a Nash map. We denote the Taylor expansion of degree $\ell \geq 1$ of α at $t_0 \in [a, b] \setminus \eta(\alpha)$ with $T^\ell_{t_0}\alpha := \sum_{k=0}^{\ell} \frac{1}{\ell!}\alpha^{(k)}(t_0)(\mathtt{t} - t_0)^k$.

Lemma 3.3.5 (Improved curve selection lemma)**.** *Let $\mathcal{S} \subset \mathbb{R}^n$ be an open semi-algebraic set and let $\{p_1, \ldots, p_r\} \subset \operatorname{Cl}(\mathcal{S})$ be any finite set of points, not necessarily distinct. Let $0 < t_1 < \cdots < t_r < 1$ and let $\alpha : [0, 1] \to \mathcal{S} \cup \{p_1, \ldots, p_r\}$ be a continuous semi-algebraic path such that $\alpha(t_i) = p_i$ for $i = 1, \ldots, r$ and satisfies $\eta(\alpha) \cap \{t_1, \ldots, t_r\} = \varnothing$ and $\alpha([0, 1] \setminus \{t_1, \ldots, t_r\}) \subset \mathcal{S}$. For each $\varepsilon > 0$ and each $m \geq 0$ there exists a polynomial path $\beta : [0, 1] \to \mathcal{S} \cup \{p_1, \ldots, p_r\}$ such that: $\|\alpha^{(k)} - \beta^{(k)}\| < \varepsilon$ for $k = 0, \ldots, m$, $T^m_{t_i}\beta = T^m_{t_i}\alpha$ for $i = 1, \ldots, r$ and $\beta([0, 1] \setminus \{t_1, \ldots, t_r\}) \subset \mathcal{S}$.*

Proof. Let $\delta > 0$ be such that $I := \bigcup_{i=1}^{r}[t_i - \delta, t_i + \delta] \subset [0, 1] \setminus \eta(\alpha)$. After making $\delta > 0$ smaller if necessary, we may choose polynomials $f_{ij}, g_{ij} \in \mathbb{R}[\mathtt{x}]$ such that:

$$\alpha([t_i - \delta, t_i)) \subset \{f_{i1} > 0, \ldots, f_{is} > 0\} \subset \mathcal{S},$$
$$\alpha((t_i, t_i + \delta]) \subset \{g_{i1} > 0, \ldots, g_{is} > 0\} \subset \mathcal{S},$$

Let $n_{ij}, p_{ij} \geq 1$ be such that

$$(f_{ij} \circ \alpha)(t_i - \mathtt{t}) = a_{ij}\mathtt{t}^{n_{ij}} + \cdots,$$
$$(g_{ij} \circ \alpha)(t_i + \mathtt{t}) = b_{ij}\mathtt{t}^{p_{ij}} + \cdots,$$

where $a_{ij} > 0$ and $b_{ij} > 0$. Let $\ell := \max\{n_{ij}, p_{ij}, m\} + 1$ and assume, taking a smaller $\delta > 0$ if necessary, that

$$(f_{ij} \circ \alpha)^{(n_{ij})}|_{[t_i-\delta,t_i+\delta]} > \frac{n_{ij}!a_{ij}}{2},$$
$$(g_{ij} \circ \alpha)^{(p_{ij})}|_{[t_i-\delta,t_i+\delta]} > \frac{p_{ij}!b_{ij}}{2}.$$

Define

$$\begin{aligned} 0 < \varepsilon_0 := \min\Big\{&\frac{n_{ij}!a_{ij}}{2}, \frac{p_{ij}!b_{ij}}{2}, (f_{ij} \circ \alpha)(t_i - \delta), \\ &(g_{ij} \circ \alpha)(t_i + \delta), \operatorname{dist}(\alpha([0, 1] \setminus I), \mathbb{R}^n \setminus \mathcal{S})\Big\}. \end{aligned} \tag{3.3.1}$$

By Lemmas 3.3.1 and 3.3.2 the maps

$$\varphi_{ij} : \mathcal{C}^\ell([0, 1], \mathbb{R}^n) \to \mathcal{C}^\ell([t_i - \delta, t_i + \delta], \mathbb{R}),\ \gamma \mapsto (f_{ij} \circ \gamma|_{[t_i-\delta,t_i+\delta]}),$$
$$\phi_{ij} : \mathcal{C}^\ell([0, 1], \mathbb{R}^n) \to \mathcal{C}^\ell([t_i - \delta, t_i + \delta], \mathbb{R}),\ \gamma \mapsto (g_{ij} \circ \gamma|_{[t_i-\delta,t_i+\delta]})$$

are continuous. Define

$$\begin{aligned}\mathcal{U}_0 := &\bigcap_{i=1}^{r}\bigcap_{j=1}^{s}\{\gamma\in\mathcal{C}^\ell([0,1],\mathbb{R}^n):\ \|\varphi_{ij}(\gamma)^{(n_{ij})}-\varphi_{ij}(\alpha)^{(n_{ij})}\|_{[t_i-\delta,t_i+\delta]}<\varepsilon_0\}\\ \cap&\bigcap_{i=1}^{r}\bigcap_{j=1}^{s}\{\gamma\in\mathcal{C}^\ell([0,1],\mathbb{R}^n):\ \|\phi_{ij}(\gamma)^{(p_{ij})}-\phi_{ij}(\alpha)^{(p_{ij})}\|_{[t_i-\delta,t_i+\delta]}<\varepsilon_0\},\end{aligned}$$

which is an open subset of $\mathcal{C}^\ell([0,1],\mathbb{R}^n)$. Then there exists $0<\varepsilon<\varepsilon_0$ such that

$$\mathcal{U}:=\{\gamma\in\mathcal{C}^\ell([0,1],\mathbb{R}^n):\ \|\gamma-\alpha\|_{[0,1]}<\varepsilon,\|\gamma^{(k)}-\alpha^{(k)}\|_I<\varepsilon,\ k=1,\dots,\ell\}\subset\mathcal{U}_0.$$

We claim: *Given $\gamma\in\mathcal{U}$ such that $T^\ell_{t_i}\alpha=T^\ell_{t_i}\gamma$ for $i=1,\dots,r$, we have $\gamma([0,1]\setminus\{t_1,\dots,t_r\})\subset\mathcal{S}$.*

As $\gamma\in\{\beta\in\mathcal{C}^\ell([0,1]):\ \|\beta-\alpha\|<\varepsilon\}$ and $0<\varepsilon<\operatorname{dist}(\alpha([0,1]\setminus I),\mathbb{R}^n\setminus\mathcal{S})$, we deduce $\operatorname{dist}(\gamma([0,1]\setminus I),\mathbb{R}^n\setminus\mathcal{S})>0$, so $\gamma([0,1]\setminus I)\subset\mathcal{S}$. Thus, to prove the claim it is enough to check:

$$\gamma([t_i-\delta,t_i))\subset\{f_{i1}>0,\dots,f_{is}>0\}\subset\mathcal{S},\tag{3.3.2}$$
$$\gamma((t_i,t_i+\delta])\subset\{g_{i1}>0,\dots,g_{is}>0\}\subset\mathcal{S}\tag{3.3.3}$$

We show only (3.3.2) because the proof of (3.3.3) is analogous.

Using Taylor's expansion, we know that γ around t_i has the form

$$\gamma(\mathtt{t})=T^{\ell-1}_{t_i}\gamma(\mathtt{t}-t_i)+(\mathtt{t}-t_i)^\ell\theta(\mathtt{t}-t_i)=T^{\ell-1}_{t_i}\alpha(\mathtt{t}-t_i)+(\mathtt{t}-t_i)^\ell\theta(\mathtt{t}-t_i)$$

where θ is a continuous map defined on an interval around 0. As α is analytic in a neighbourhood of t_i, there exists a tuple of analytic series $\tau\in\mathbb{R}\{\mathtt{t}\}^n$ such that

$$\alpha(\mathtt{t})=T^{\ell-1}_{t_i}\alpha(\mathtt{t}-t_i)+(\mathtt{t}-t_i)^\ell\tau(\mathtt{t}-t_i).$$

Thus, if $\zeta:=\theta-\tau$, which is a continuous function around 0, we deduce

$$\gamma(\mathtt{t})-\alpha(\mathtt{t})=(\mathtt{t}-t_i)^\ell\zeta(\mathtt{t}-t_i)\quad\rightsquigarrow\quad\gamma(t_i-\mathtt{t})-\alpha(t_i-\mathtt{t})=(-\mathtt{t})^\ell\zeta(-\mathtt{t}).$$

Write $\mathtt{x}:=(\mathtt{x}_1,\dots,\mathtt{x}_n)$, $\mathtt{y}:=(\mathtt{y}_1,\dots,\mathtt{y}_n)$ and let $\mathtt{z}$ be a single variable. As the polynomial $f_{ij}(\mathtt{x}+\mathtt{z}\mathtt{y})-f_{ij}(\mathtt{x})$ vanishes on the real algebraic set $\{\mathtt{z}=0\}$, there exists a polynomial $h_{ij}\in\mathbb{R}[\mathtt{x},\mathtt{y},\mathtt{z}]$ such that

$$f_{ij}(\mathtt{x}+\mathtt{z}\mathtt{y})=f_{ij}(\mathtt{x})+\mathtt{z}h_{ij}(\mathtt{x},\mathtt{y},\mathtt{z}).$$

As $\ell>n_{ij}$, we deduce

$$\begin{aligned}&f_{ij}(\gamma(t_i-\mathtt{t}))=f_{ij}(\alpha(t_i-\mathtt{t})+\gamma(t_i-\mathtt{t})-\alpha(t_i-\mathtt{t}))\\&\quad=f_{ij}(\alpha(t_i-\mathtt{t}))+(-1)^\ell\mathtt{t}^\ell h_{ij}(\alpha(t_i-\mathtt{t}),\zeta(-\mathtt{t}),(-1)^\ell\mathtt{t}^\ell)=a_{ij}\mathtt{t}^{n_{ij}}+\cdots.\end{aligned}$$

Consequently, $(f_{ij}\circ\gamma)^{(k)}(t_i)=0$ for $k=0,\dots,n_{ij}-1$ and

$$(f_{ij}\circ\gamma)^{(n_{ij})}(t_i)=n_{ij}!\,a_{ij}>0.$$

By Lemma 3.3.4 and (3.3.1) we obtain $(f_{ij}\circ\gamma)(t_i-t)>0$ for each $t\in(0,\delta]$ and $j=1,\dots,s$, that is, $\gamma(t)\in\{f_{i1}>0,\dots,f_{is}>0\}$ for each $t\in[t_i-\delta,t_i)$, as claimed.

To conclude, by Lemma 3.3.3 there exists a polynomial tuple $\beta\in\mathbb{R}[\mathtt{t}]^n$ such that $\|\alpha^{(k)}-\beta^{(k)}\|_{[0,1]}<\varepsilon$ for $k=0,\dots,\ell$ (that is, $\alpha\in\mathcal{U}$) and $\alpha^{(k)}(t_i)=\beta^{(k)}(t_i)$ for $i=1,\dots,r$ and $k=0,\dots,\ell$. We deduce $\beta([0,1]\setminus\{t_1,\dots,t_r\})\subset\mathcal{S}$, as required. $\square$

3.4 Well-welded semi-algebraic sets

Let $\sigma : [a,b] \to \mathbb{R}^m$ be a continuous semi-algebraic path and let $\eta(\sigma) \subset [a,b]$ be the finite set of points at which σ is not a Nash map already introduced before Lemma 3.3.5. By [BCR, Prop.8.1.12] and after reparametrizing (if necessary), we may assume that σ is Nash at a and b so $\eta(\sigma) \subset (a,b)$.

In his proof of the Shiota's conjecture [Fe4] Fernando introduced the concept of well-welded semi-algebraic sets. We recall here this definition.

Definition 3.4.1. A semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ is *well-welded* if $\mathcal{S}$ is pure dimensional and for each pair of points $x, y \in \mathcal{S}$ there exists a continuous semi-algebraic path $\sigma : [0,1] \to \mathcal{S}$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\eta(\sigma) \subset \operatorname{Reg}(\mathcal{S})$.

As a consequence of Theorem 3.1, we have the following:

Theorem 3.4.2. *Given a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ the following conditions are equivalent:*

(a) *$\mathcal{S}$ is connected by Nash paths.*

(b) *$\mathcal{S}$ is connected by analytic paths.*

(c) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\sigma : [0,1] \to \mathcal{S}$ that meets all the connected components of the set of regular points of $\mathcal{S}$.*

(d) *$\mathcal{S}$ is pure dimensional and there exists an analytic path $\sigma : [0,1] \to \mathcal{S}$ that meets all the connected components of the set of regular points of $\mathcal{S}$.*

(e) *$\mathcal{S}$ is well-welded.*

Theorem 3.1 is a very deep result, so we want to furnish an alternative proof of the equivalence of these implications that uses 'lighter' results.

The implications (a)⇒(b) and (c)⇒(d) are clear. We will prove the implication (d)⇒(e) in this section. Later in Section 3.5 we will prove the implications (e)⇒(a), (b)⇒(e) and (a)⇒(c).

3.4.1. Analytic arcs and well-welded sets. In order to prove the implication (d)⇒(e) we need the following lemma that allows us to modify an analytic arc by a Nash arc. This lemma has been proved by Fernando [Fe4, Lem.2.9] in a stronger version, but we only need a simplified version of his result.

Lemma 3.4.3. *Let $M \subset \mathbb{R}^m$ be a connected Nash manifold, let M_1, M_2 be open semi-algebraic subsets of M and let $\alpha : (-1,1) \to M_1 \cup M_2 \cup \{0\}$ be an analytic arc such that $\alpha(0) = 0$, $\alpha((0,1)) \subset M_1$ and $\alpha((-1,0)) \subset M_2$. Then there exist $\varepsilon > 0$ and a Nash arc $\beta : (-\varepsilon,\varepsilon) \to M_1 \cup M_2 \cup \{0\}$ such that $\beta((0,\varepsilon)) \subset M_1$ and $\beta((-\varepsilon,0)) \subset M_2$.*

Proof. For simplicity we can assume $M_1 \cap M_2 = \varnothing$ and $0 \notin M_1 \cup M_2$. Let $V \subset M$ be an open semi-algebraic neighbourhood of the origin equipped with a Nash diffeomorphism $\varphi : V \to \mathbb{R}^d$ such that $\varphi(0) = 0$. Shrinking the domain of α, we may assume $\alpha((-\delta,\delta)) \subset V$ for some $\delta > 0$. Denote $\widehat{\alpha} := \varphi \circ \alpha : (-\delta,\delta) \to \mathbb{R}^d$.

Shrinking M_i, V and the domain of $\widehat{\alpha}$, we may assume $0 \notin \varphi(M_i \cap V)$ and $\varphi(M_i \cap V) = \{g_{1i} > 0, \dots, g_{\ell i} > 0\}$ for some polynomials $g_{ji} \in \mathbb{R}[\mathtt{x}]$. Observe that the analytic series $(g_{j1} \circ \widehat{\alpha})(\mathtt{t})$ and $(g_{j2} \circ \widehat{\alpha})(-\mathtt{t})$ are positive on $(0, \delta)$. Thus, there exists $s \geq 1$ large enough such that if $\gamma \in \mathbb{R}\{\mathtt{t}\}^d$ and $\gamma - \widehat{\alpha} \in (\mathtt{t})^s \mathbb{R}\{\mathtt{t}\}^d$, we have $(g_{j1} \circ \gamma)(t) > 0$ and $(g_{j2} \circ \gamma)(-t) > 0$ for $t > 0$ small enough and $j = 1, \dots, \ell$. Let $\gamma_0 \in \mathbb{R}[\mathtt{t}]^d$ be a polynomial tuple such that $\gamma_0 - \widehat{\alpha} \in (\mathtt{t})^s \mathbb{R}\{\mathtt{t}\}^d$, and let $\varepsilon > 0$ be such that

$$g_{j1}(\gamma_0(t)) > 0, \quad g_{j2}(\gamma_0(-t)) > 0$$

for $0 < t < \varepsilon$. The Nash arc $\beta := \varphi^{-1} \circ \gamma_0 : (-\varepsilon, \varepsilon) \to M_1 \cup M_2 \cup \{0\}$ satisfies the required property. □

Proposition 3.4.4. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a pure dimensional semi-algebraic set. Assume there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all the connected components of* $\operatorname{Reg}(\mathcal{S})$*. Then $\mathcal{S}$ is well-welded.*

Proof. Let $M_1, \dots, M_r$ be the connected components of $\operatorname{Reg}(\mathcal{S})$. After repeating the connected components as many time as needed, we may assume that there exist $0 < s_1 < \ldots < s_r < 1$ and $\varepsilon_0 > 0$ such that

$$\alpha(s_i) \in \operatorname{Cl}(M_i) \cap \operatorname{Cl}(M_{i+1}) \text{ and } \alpha([s_i - \varepsilon_0, s_i)) \subset M_i,\ \alpha((s_i, s_i + \varepsilon_0]) \subset M_{i+1}.$$

The proof is conducted into two steps:

STEP 1. CONSTRUCTION OF NASH BRIDGES. Consider $X = \overline{\mathcal{S}}^{\text{zar}}$. By Theorem 2.4.2 there exist a non-singular algebraic set X_1 and a proper regular map $f : X_1 \to X$ such that $\mathcal{S} \subset f(X_1)$ (because $\mathcal{S}$ is pure dimensional) and

$$f|_{X_1 \setminus f^{-1}(\operatorname{Sing}(X))} : X_1 \setminus f^{-1}(\operatorname{Sing}(X)) \to X \setminus \operatorname{Sing}(X)$$

is a diffeomorphism whose inverse map is also regular.

Define $\Gamma_i := \alpha([s_i - \varepsilon_0, s_i + \varepsilon_0])$ and denote

$$\Lambda_i := \operatorname{Cl}(f^{-1}(\Gamma_i \setminus \operatorname{Sing}(X)) \cap f^{-1}(\Gamma_i)$$

and $N_i := f^{-1}(M_i)$, for $i = 1, \dots, r$. As $M_i \subset X \setminus \operatorname{Sing}(X)$, we have that $N_i \subset X_1 \setminus f^{-1}(\operatorname{Sing}(X))$ is a Nash manifold. After shrinking Γ_i if necessary, we may assume by [Fe4, Lem.B.2] that Λ_i is an analytic bridge between N_i and N_{i+1} such that $\Lambda_i \setminus \bigsqcup_{j=1}^r N_j = \{q_i\}$ and $f(q_i) = \alpha(s_i)$ for some $q_i \in X_1$.

We apply Lemma 3.4.3 to the Nash manifold X_1 and the open semi-algebraic subsets N_i and N_{i+1} so to find $0 < \varepsilon < \varepsilon_0$ and Nash curves $\sigma_i : [-\varepsilon, \varepsilon] \to X_1$ such that $\sigma_i([-\varepsilon, 0)] \subset N_i$, $\sigma_i((0, \varepsilon]) \subset N_{i+1}$ and $\sigma_i(0) = q_i$. Thus, the Nash arcs $\beta_i := f \circ \sigma_i : [-\varepsilon, \varepsilon] \to \mathcal{S}$ satisfy

$$\beta_i([-\varepsilon, 0)] \subset M_i,\ \beta_i((0, \varepsilon]) \subset M_{i+1} \text{ and } \beta_i(0) = \alpha(s_i),$$

STEP 2. CONSTRUCTION OF THE CONTINUOUS SEMI-ALGEBRAIC PATH. As $\mathcal{S}$ is pure dimensional, $\mathcal{S} = \bigcup_{i=1}^r \operatorname{Cl}(M_i) \cap \mathcal{S}$. Let $y_1, y_2 \in \mathcal{S}$ and assume $y_1 \in \operatorname{Cl}(M_i) \cap \mathcal{S}$ and $y_2 \in \operatorname{Cl}(M_j) \cap \mathcal{S}$ for some $i < j$. By the Nash curve selection lemma (see [BCR, Prop.8.1.13]) there exist Nash arcs $\alpha_k : (-1, 1) \to \mathbb{R}^m$ such that $\alpha_1((-1, 0)) \subset M_i$, $\alpha_2((-1, 0)) \subset M_j$ and $\alpha_k(0) = y_k$ for $k = 1, 2$. For each $\ell = i, \dots, j - 1$, denote $u_\ell := \beta_\ell(-\varepsilon)$ and $v_{\ell+1} := \beta_\ell(\varepsilon)$. As M_i and M_j

are connected Nash manifolds, there exist Nash paths $\gamma_1 : [0,1] \to M_i$ and $\gamma_2 : [0,1] \to M_j$ such that

$$\gamma_k(0) = z_k := \alpha_k\left(-\frac{1}{2}\right) \quad \text{and} \quad \gamma_k(1) = \begin{cases} u_i & \text{if } k=1, \\ v_j & \text{if } k=2. \end{cases}$$

Moreover, for each $i < \ell < j$, as the Nash manifold M_ℓ is connected, we can find a Nash path $\eta_\ell : [0,1] \to M_\ell$ such that $\eta_\ell(0) = v_\ell$, $\eta_\ell(1) = u_\ell$. The continuous semi-algebraic path

$$\lambda = \left(\alpha_1|_{[-\frac{1}{2},0]}\right)^{-1} * \gamma_1 * (\beta_i * \eta_i * \beta_{i+1} * \eta_{i+1} * \ldots * \eta_{j-1} * \beta_j) * \gamma_2^{-1} * \alpha_2|_{[-\frac{1}{2},0]}$$

connects the points y_1, y_2 and satisfies

$$\eta(\lambda) \subset \{z_1, z_2\} \cup \{u_i, \ldots, u_{j-1}\} \cup \{v_{i+1}, \ldots, v_j\} \subset \operatorname{Reg}(\mathcal{S}),$$

see Figure 3.5. Consequently, $\mathcal{S}$ is well-welded, as required. □

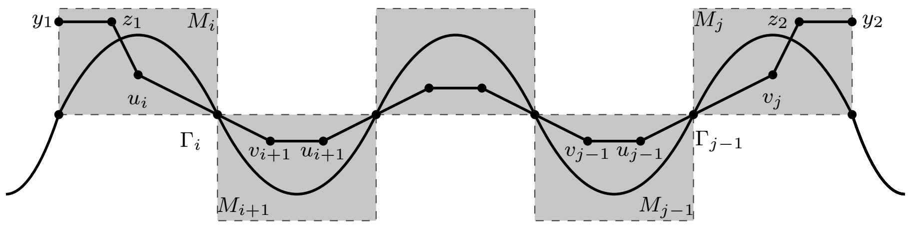

Figure 3.5: Sketch of proof of Proposition 3.4.4 (figure inspired by [Fe4, Fig.10]).

3.5 Checkerboard sets

If $\mathcal{S} \subset \mathbb{R}^m$ is a semi-algebraic set, we denote $\partial\mathcal{S} := \operatorname{Cl}(\mathcal{S}) \setminus \operatorname{Reg}(\mathcal{S})$. Observe that the set $\partial\mathcal{S}$ defined here is (in general) different from the set $\operatorname{Sing}(\mathcal{S}) := \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})$ defined in Section 2.4.1.

A pure dimensional semi-algebraic set $\mathcal{T} \subset \mathbb{R}^n$ is a *checkerboard set* if it satisfies the following properties:

- $\overline{\mathcal{T}}^{\text{zar}}$ is a non-singular algebraic set.
- $\overline{\partial\mathcal{T}}^{\text{zar}}$ is a normal-crossings divisor of $\overline{\mathcal{T}}^{\text{zar}}$.
- $\operatorname{Reg}(\mathcal{T})$ is connected.

We want to show that any checkerboard set is well-welded.

Proposition 3.5.1. *Let $\mathcal{T} \subset \mathbb{R}^n$ be a checkerboard set, then $\mathcal{T}$ is well-welded.*

Proof. As $\mathcal{T}$ is pure dimensional, $\mathcal{T} = \operatorname{Cl}(\operatorname{Reg}(\mathcal{T})) \cap \mathcal{T}$. Fix any two points $y_1, y_2 \in \mathcal{T}$. By the Nash curve selection lemma (see [BCR, Prop.8.1.13]) there exist Nash arcs $\alpha_k : (-1,1) \to \mathbb{R}^n$ such that $\alpha_1((0,1)), \alpha_2((-1,0)) \subset \operatorname{Reg}(\mathcal{T})$

and $\alpha_k(0) = y_k$ for $k = 1, 2$. As $\operatorname{Reg}(\mathcal{T})$ is a connected Nash manifold, there exists a Nash path $\gamma : [0, 1] \to \operatorname{Reg}(\mathcal{T})$ such that

$$\gamma(0) = z_1 := \alpha_1\left(\frac{1}{2}\right), \gamma(1) = z_2 := \alpha_2\left(-\frac{1}{2}\right).$$

The continuous semi-algebraic path $\beta := \alpha_1|_{[0,\frac{1}{2}]} * \gamma * \alpha_2|_{[-\frac{1}{2},0]}$ is a path between y_1 and y_2 and satisfies $\eta(\beta) \subset \{z_1, z_2\} \subset \operatorname{Reg}(\mathcal{T})$. Thus, $\mathcal{T}$ is well-welded, as required. □

In his proof of Shiota's conjecture Fernando proved the following result. We will use this result in an essential way in our proof of Theorem 3.2.

Theorem 3.5.2 ([Fe4, Thm.8.4])**.** *Let $\mathcal{S} \subset \mathbb{R}^m$ be a well-welded semi-algebraic set of dimension $d \geq 2$ and denote $X := \overline{\mathcal{S}}^{\text{zar}}$. Then there exists a checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ of dimension d and a proper regular map $f : Y := \overline{\mathcal{T}}^{\text{zar}} \to X$ such that $f(\mathcal{T}) = \mathcal{S}$.*

As the map f is proper, if the semi-algebraic set $\mathcal{S}$ is compact, we may assume that also the checkerboard set $\mathcal{T}$ is compact (see the proof of [Fe4, Thm.8.4]).

3.5.1. 1-dimensional semi-algebraic sets. To prove Theorem 3.4.2 without using Theorem 3.1 we will make an essential use of Theorem 3.5.2. This result holds for semi-algebraic sets of dimension $d \geq 2$, so we treat the 1-dimensional case separately.

Recall that a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ is *irreducible* if its ring of Nash functions $\mathcal{N}(\mathcal{S})$ is an integral domain (see Definition 2.4.5).

Proposition 3.5.3 (1-dimensional case)**.** *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set of dimension 1. The following conditions are equivalent*

(a) *$\mathcal{S}$ is connected by Nash paths.*

(b) *$\mathcal{S}$ is connected by analytic paths.*

(c) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\sigma : [0, 1] \to \mathcal{S}$ that meets all the connected components of the set of regular points of $\mathcal{S}$.*

(d) *$\mathcal{S}$ is pure dimensional and there exists an analytic path $\sigma : [0, 1] \to \mathcal{S}$ that meets all the connected components of the set of regular points of $\mathcal{S}$.*

(e) *$\mathcal{S}$ is well-welded.*

Proof. As $\dim(\mathcal{S}) = 1$, using the identity principle for analytic functions and standard arguments, it follows: if $\mathcal{S}$ satisfies one of the conditions in the statement, then $\mathcal{S}$ is irreducible. By [Fe4, Prop.1.6] if $\mathcal{S}$ is irreducible, then it is a Nash image of $\mathbb{R}$, so it verifies conditions (a), (b), (c) and (d). In particular $\mathcal{S}$ is connected by Nash paths, so it is also well-welded and (e) holds. □

3.5.2. Well-welded sets are connected by Nash paths. We want to show that a well-welded semi-algebraic set is connected by Nash paths without using Theorem 3.2. We will combine Lemma 3.3.5 and Theorem 3.5.2 in our argument.

Proposition 3.5.4. *If $\mathcal{S} \subset \mathbb{R}^m$ is a well-welded semi-algebraic set, then it is connected by Nash paths.*

Proof. By Proposition 3.5.3 we may assume $\dim \mathcal{S} \geq 2$. By Theorem 3.5.2 there exists a checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ and a Nash map $f : Y := \overline{\mathcal{T}}^{\text{zar}} \to X := \overline{\mathcal{S}}^{\text{zar}}$ such that $f(\mathcal{T}) = \mathcal{S}$.

Thus, in order to conclude, it is sufficient to show: *The checkerboard set $\mathcal{T}$ is connected by Nash paths.* Let (Ω, ν) be a Nash tubular neighbourhood for the Nash manifold $Y := \overline{\mathcal{T}}^{\text{zar}} \subset \mathbb{R}^m$. As $\text{Cl}(\mathcal{T}) \subset \overline{\mathcal{T}}^{\text{zar}}$, shrinking Ω if necessary, we may assume that ν admits a Nash extension to $\text{Cl}(\nu^{-1}(\mathcal{T}))$. Fix two points $p, q \in \mathcal{T}$ and consider the open semi-algebraic set $\mathcal{U} := \nu^{-1}(\text{Reg}(\mathcal{T})) \subset \mathbb{R}^n$. As $\mathcal{T}$ is pure dimensional, $p, q \in \text{Cl}(\text{Reg}(\mathcal{T})) \cap \mathcal{T}$, so $p, q \in \text{Cl}(\mathcal{U})$. By Lemma 3.3.5 there exist $\delta > 0$ and a polynomial path $\alpha : [-\delta, 1+\delta] \to \mathcal{U} \cup \{p, q\}$ such that $\alpha(0) = p$, $\alpha(1) = q$ and $\alpha([-\delta, 1+\delta] \setminus \{0, 1\}) \subset \mathcal{U}$. The image $\nu(\alpha([0, 1])) \subset \mathcal{T}$, because $\alpha(0), \alpha(1) \in \mathcal{T}$. Thus, $\widehat{\alpha} := \nu \circ \alpha|_{[0,1]} : [0, 1] \to \mathcal{T}$ is a Nash path between p and q, as required. □

3.5.3. Connection by analytic paths. We want to show that a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ connected by analytic paths is well-welded, which proves the implication (b)⇒(e) of Theorem 3.4.2.

Proposition 3.5.5. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set connected by analytic paths. Then $\mathcal{S}$ is well-welded.*

Proof. The proof of Proposition 3.2.3 works in the same way if $\mathcal{S}$ is connected by analytic paths. Thus, if such is the case, then $\mathcal{S}$ is pure dimensional.

By Proposition 3.5.3 we may assume $\dim \mathcal{S} \geq 2$. Let $M_1, \ldots, M_r$ be the connected component of $\text{Reg}(\mathcal{S})$. Let $x, y \in \mathcal{S}$ and let $\alpha : [0, 1] \to \mathcal{S}$ be an analytic path such that $\alpha(0) = x$ and $\alpha(1) = y$. As $\mathcal{S}$ is pure dimensional $\mathcal{S} = \bigcup_{i=1}^r \text{Cl}(M_i) \cap \mathcal{S}$. Let $i_1, \ldots, i_k \in \{1, \ldots, r\}$ be the indices such that $\alpha([0, 1]) \cap (\text{Cl}(M_{i_j}) \cap \mathcal{S}) \neq \varnothing$. Define the semi-algebraic set $\mathcal{T} := \bigcup_{j=1}^k \text{Cl}(M_{i_j}) \cap \mathcal{S}$. Proceeding as in the proof of Proposition 3.4.4 we can find a continuous semi-algebraic path $\beta : [0, 1] \to \mathcal{T}$ such that $\beta(0) = x$ and $\beta(1) = y$, that satisfies $\eta(\beta) \subset \text{Reg}(\mathcal{T}) \subset \text{Reg}(\mathcal{S})$. □

3.5.4. Nash paths through $\text{Reg}(\mathcal{S})$**.** We want to prove now that if a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$ is connected by Nash paths, there exists a Nash path that meets all the connected components of $\text{Reg}(\mathcal{S})$. This will complete the last implication of Theorem 3.4.2. As in the proof of Proposition 3.5.4 we will combine Lemma 3.3.5 and Theorem 3.5.2 in our argument.

Proposition 3.5.6. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set connected by Nash paths. Then $\mathcal{S}$ is pure dimensional and there exists a Nash path $\sigma : [0, 1] \to \mathcal{S}$ that meets all the connected components of* $\text{Reg}(\mathcal{S})$.

Proof. Let $d := \dim \mathcal{S}$. By Proposition 3.5.3 we may assume $d \geq 2$. As $\mathcal{S}$ is connected by analytic paths, by Proposition 3.5.5 $\mathcal{S}$ is well-welded. By Theorem 3.5.2 there exists a checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ of dimension d and a proper regular map $f : Y := \overline{\mathcal{T}}^{\text{zar}} \to X := \mathcal{S}^{\text{zar}}$ such that $f(\mathcal{T}) = \mathcal{S}$. Let now $M_1, \ldots, M_r$ be the connected components of $\operatorname{Reg}(\mathcal{S})$. Fix points $x_i \in M_i$ and let $y_i \in \mathcal{T}$ be such that $f(y_i) = x_i$. We claim: *There exists a continuous semi-algebraic path*

$$\alpha : [0,1] \to \operatorname{Reg}(\mathcal{T}) \cup \{y_1, \ldots, y_r\}$$

and $0 < t_1 < \ldots < t_r < 1$ *such that* $\alpha(t_i) = y_i$ *and* $\eta(\alpha) \cap \{t_1, \ldots, t_r\} = \varnothing$.

As $\operatorname{Reg}(\mathcal{T})$ is connected by Nash paths, in order to prove the claim it is enough to show: *For each point* $x \in \mathcal{T}$ *there exists a Nash arc* $\gamma : [-1,1] \to \mathcal{T}$ *such that* $\gamma(0) = x$ *and* $\gamma([-1,1]\setminus\{0\}) \subset \operatorname{Reg}(\mathcal{T})$. As $\operatorname{Reg}(\mathcal{T})$ is a Nash manifold, the claim is clear for the points $x \in \operatorname{Reg}(\mathcal{T})$. Suppose now $x \in \mathcal{T} \setminus \operatorname{Reg}(\mathcal{T})$. As $\mathcal{T}$ is pure dimensional, $x \in \operatorname{Cl}(\operatorname{Reg}(\mathcal{T})) \cap \mathcal{T}$. By the Nash curve selection lemma (see [BCR, Prop.8.1.13]), there exists a Nash path $\delta : [-1,1] \to \mathbb{R}^n$ such that $\delta(0) = x$ and $\sigma((0,1]) \subset \operatorname{Reg}(\mathcal{T})$. Thus, the Nash arc $\gamma : [-1,1] \to \operatorname{Reg}(\mathcal{T}) \cup \{x\}$ defined as $\gamma(t) := \delta(t^2)$, satisfies $\gamma(0) = x$.

Let (Ω, ν) be a Nash tubular neighbourhood for the Nash manifold $Y := \overline{\mathcal{T}}^{\text{zar}}$ and define $\Omega_{\mathcal{T}} := \nu^{-1}(\operatorname{Reg}(\mathcal{T}))$, which is an open semi-algebraic subset of $\mathbb{R}^n$. By Lemma 3.3.5 we approximate the continuous semi-algebraic path α by a polynomial path $\beta : [0,1] \to \Omega_{\mathcal{T}} \cup \{y_1, \ldots, y_r\}$ such that $\beta(t_i) = y_i$. Then the Nash path $\sigma := f \circ \nu \circ \beta : [0,1] \to \mathcal{S}$ meets all the connected components of $\operatorname{Reg}(\mathcal{S})$, because $\sigma(t_i) = x_i \in M_i$. □

3.5.5. Reduction to the case of checkerboard sets. By Theorem 3.4.2 in order to prove Theorem 3.2 we 'only' need to prove the following: *If* $\mathcal{S} \subset \mathbb{R}^m$ *is a compact well-welded semi-algebraic set of dimension d, then there exists a Nash map* $f : \mathbb{R}^d \to \mathbb{R}^m$ *such that* $f(\overline{\mathcal{B}}_d) = \mathcal{S}$.

We will prove Theorem 3.2 for dimension 1 in Section 3.6.1, so let us assume $\dim(\mathcal{S}) \geq 2$. In this case Theorem 3.5.2 provides a checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ and a proper regular map $f : \overline{\mathcal{T}}^{\text{zar}} \to \overline{\mathcal{S}}^{\text{zar}}$ such that $f(\mathcal{T}) = \mathcal{S}$. As the map f is proper, if the semi-algebraic set $\mathcal{S}$ is compact, we may assume that also the checkerboard set $\mathcal{T}$ is compact (see the proof of [Fe4, Thm.8.4]). Thus, we are reduced to prove the following:

Theorem 3.5.7. *Let* $\mathcal{T} \subset \mathbb{R}^n$ *be a compact checkerboard set of dimension* $d \geq 2$. *Then there exists a Nash map* $G : \mathbb{R}^d \to \mathbb{R}^n$ *such that* $G(\overline{\mathcal{B}}_d) = \mathcal{T}$.

By Corollary 3.1.4, there exists a polynomial map $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(\overline{\mathcal{B}}_d) = \Delta_{d-1} \times [0,1]$. Consider the inverse of the stereographic projection

$$\varphi : \mathbb{R}^d \to \mathbb{S}^d \setminus \{(0,\ldots,1)\},$$
$$x := (x_1, \ldots, x_d) \mapsto \left(\frac{2x_1}{1+\|x\|^2}, \ldots, \frac{2x_d}{1+\|x\|^2}, \frac{-1+\|x\|^2}{1+\|x\|^2}\right)$$

and let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection onto the first d coordinates. The regular map $g := \pi \circ \varphi : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $g(\mathbb{R}^d) = g(\overline{\mathcal{B}}_d) = \overline{\mathcal{B}}_d$. If there exists a Nash map $F : \Delta_{d-1} \times [0,1] \to \mathbb{R}^n$ such that $F(\Delta_{d-1} \times [0,1]) = \mathcal{T}$, then the composition $G := F \circ f \circ g : \mathbb{R}^d \to \mathbb{R}^d$ is a well-defined Nash map such that $G(\overline{\mathcal{B}}_d) = \mathcal{T}$.

Thus, in order to show Theorem 3.5.7 we can use the (more convenient) compact model $\Delta_{d-1} \times [0,1]$ and we are reduced to show the following:

Theorem 3.5.8. *Let $\mathcal{T} \subset \mathbb{R}^n$ be a compact checkerboard set of dimension $d \geq 2$. Then there exists a Nash map $F : \Delta_{d-1} \times [0,1] \to \mathbb{R}^n$ such that*

$$F(\Delta_{d-1} \times [0,1]) = \mathcal{T}.$$

3.6 Building Nash images with bare-hands

The purpose of this section is to prove Theorem 3.5.8, which provides a complete characterization of the Nash images of the closed ball. The proof is quite involved and intricate and we begin with some preliminary results to lighten the proof.

We will start with the 1-dimensional case, that requires a different proof. Then we will focus on the d-dimensional case for $d \geq 2$. For the general case we will take advantage of the fact that each checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ admits 'nice' triangulations. Roughly speaking, we 'build' $\mathcal{T}$ as Nash image of the prism $\Delta_{d-1} \times [0,1]$ 'simplex by simplex'.

To that end, we consider a suitable subset of the space of Nash maps $\mathcal{N}(\mathbb{R}^d, \mathbb{R}^n)$ and we will parametrize it (of course not in an injective way) using an open semi-algebraic set Θ_0 of a large Euclidean space. In this space, a continuous semi-algebraic path $\sigma : [0,1] \to \Theta_0$ will provide us a continuous semi-algebraic map $\Delta_{d-1} \times [0,1] \to \mathbb{R}^n$ that is Nash on the horizontal slices $\Delta_{d-1} \times \{t\}$. Using the results of Section 3.3 we approximate the path σ by a Nash path in order to obtain a Nash map $\Delta_{d-1} \times [0,1] \to \mathbb{R}^n$. All the construction is quite technical and requires care to guarantee that the obtained Nash map is surjective and that the target space is exactly $\mathcal{T}$.

Given a topological manifold X with boundary we denote its relative interior with $\operatorname{Int}(X)$ and its boundary $X \setminus \operatorname{Int}(X)$ with ∂X.

3.6.1. The 1-dimensional case. Nash images of closed balls contained in the real line are its compact intervals and all of them are affinely equivalent to the interval $\overline{\mathcal{B}}_1 := [-1,1]$. Nash images of closed balls contained in a circumference are its connected compact subsets and all of them are Nash images of $\overline{\mathcal{B}}_1$.

Examples 3.6.1. (i) The circumference $\mathbb{S}^1 := \{\mathtt{x}^2 + \mathtt{y}^2 = 1\}$ is a Nash image of $\overline{\mathcal{B}}_1$. Consider the inverse of the stereographic projection from the point $(0,1)$, which is the map

$$f : \mathbb{R} \to \mathbb{S}^1 \setminus \{(0,1)\},\ t \mapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right).$$

Next, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and the coordinates (x,y) with $x+\sqrt{-1}y$. Consider the map

$$g : \mathbb{C} \to \mathbb{C},\ z := x + \sqrt{-1}y \mapsto z^2 = (x^2 - y^2) + \sqrt{-1}(2xy).$$

The image of $\overline{\mathcal{B}}_1$ under $g \circ f$ is $\mathbb{S}^1$.

(ii) Any connected compact proper subset $\mathcal{S}$ of $\mathbb{S}^1$ that is not a point is a Nash image of $\overline{\mathcal{B}}_1$ because it is Nash diffeomorphic to $[-1,1]$.

We have the following:

Proposition 3.6.2 (1-dimensional case)**.** *Let $\mathcal{S}\subset\mathbb{R}^n$ be a 1-dimensional compact semi-algebraic set. Then $\mathcal{S}$ is a Nash image of some $\overline{\mathcal{B}}_m$ if and only if $\mathcal{S}$ is irreducible. In addition, if such is the case $\mathcal{S}$ is a Nash image of $[-1,1]$.*

Proof. Assume $\mathcal{S}$ is irreducible. Let X be the Zariski closure of $\mathcal{S}$ in $\mathbb{R}^n$ and let $\widetilde{X}$ be its complexification in $\mathbb{C}^n$. Let $(\widetilde{Y},\pi)$ be the normalization of $\widetilde{X}$ and let $\widehat{\sigma}$ be the involution of $\widetilde{Y}$ induced by the involution σ of $\widetilde{X}$ that arises from the restriction to $\widetilde{X}$ of the complex conjugation in $\mathbb{C}^n$. We may assume that $\widetilde{Y}\subset\mathbb{C}^m$ and that $\widehat{\sigma}$ is the restriction to $\widetilde{Y}$ of the complex conjugation of $\mathbb{C}^m$. By [FG3, Thm.3.15] and since $\mathcal{S}$ is irreducible, $\pi^{-1}(\mathcal{S})$ has a 1-dimensional connected component $\mathcal{T}$ such that $\pi(\mathcal{T})=\mathcal{S}$. As X has dimension 1, it is a coherent analytic set, so $\mathcal{T}\subset Y:=\widetilde{Y}\cap\mathbb{R}^m$. As $\widetilde{Y}$ is a normal-curve, Y is a non-singular real algebraic curve. We claim: *the connected components of Y are Nash diffeomorphic either to $\mathbb{S}^1$ or to the real line $\mathbb{R}$.*

By [Sh, VI.2.1] there exist a compact affine non-singular real algebraic curve Z, a finite set F which is empty if Y is compact and a union Y' of some connected components of $Z\setminus F$ such that Y is Nash diffeomorphic to Y' and $\operatorname{Cl}(Y')$ is a compact Nash curve with boundary F. As Z is a compact affine non-singular real algebraic curve, its connected components are diffeomorphic to $\mathbb{S}^1$, so by [Sh, VI.2.2] the connected components of Z are in fact Nash diffeomorphic to $\mathbb{S}^1$. Now, each connected component of Y is Nash diffeomorphic to an open connected subset of $\mathbb{S}^1$, that is, Nash diffeomorphic either to $\mathbb{S}^1$ or to the real line $\mathbb{R}$, as claimed.

As $\mathcal{T}$ is connected and 1-dimensional, it is Nash diffeomorphic to a connected compact 1-dimensional semi-algebraic subset of either $\mathbb{S}^1$ or $\mathbb{R}$. In the latter case $\mathcal{T}$ is Nash diffeomorphic to a compact interval of $\mathbb{R}$. By Examples 3.6.1 the semi-algebraic set $\mathcal{T}$ is a Nash image of $\overline{\mathcal{B}}_1$, so also $\mathcal{S}$ is a Nash image of $\overline{\mathcal{B}}_1$. The converse follows from Proposition 3.2.2. $\square$

3.6.2. Covering simplices with Nash maps. We start by proving some lemmas that will allow us to cover simplices with the images of suitable Nash maps. Denote

$$\Delta_{n-1}:=\left\{(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n:\ \lambda_1\geq 0,\dots,\lambda_n\geq 0,\ \sum_{k=1}^n\lambda_k=1\right\}. \tag{3.6.1}$$

The boundary $\partial\Delta_{n-1}=\bigcup_{i=1}^n(\Delta_{n-1}\cap\{\lambda_i=0\})$.

Lemma 3.6.3. *Consider an $(n-1)$-dimensional simplex $\sigma\subset\mathbb{R}^n$ of vertices $v_1,\dots,v_n$. Pick a point $p\in\mathbb{R}^n\setminus\sigma$ and consider the n-dimensional simplex $\widehat{\sigma}$ of vertices $\{p,v_1,\dots,v_n\}$. Let $F:\Delta_{n-1}\times[0,1]\to\mathbb{R}^n$ be a continuous semi-algebraic map such that $F|_{\Delta_{n-1}\times\{0\}}:\Delta_{n-1}\times\{0\}\to\sigma$ is a homeomorphism, $F(\partial\Delta_{n-1}\times(0,1))\subset\mathbb{R}^n\setminus\widehat{\sigma}$ and $F(\Delta_{n-1}\times\{1\})=\{p\}$. Then* $\operatorname{Int}(\widehat{\sigma})\subset F(\operatorname{Int}(\Delta_{n-1})\times(0,1))$ *and* $\widehat{\sigma}\subset F(\Delta_{n-1}\times[0,1])$.

Proof. As $\Delta_{n-1}\times[0,1]$ is compact and $\widehat{\sigma}=\operatorname{Cl}(\operatorname{Int}(\widehat{\sigma}))$, it is enough to check: $\operatorname{Int}(\widehat{\sigma})\subset F(\operatorname{Int}(\Delta_{n-1})\times(0,1))$.

Suppose there exists $z\in\operatorname{Int}(\widehat{\sigma})\setminus F(\Delta_{n-1}\times[0,1])$. Let us construct a (continuous) semi-algebraic retraction $\rho:\mathbb{R}^n\setminus\{z\}\to\partial\widehat{\sigma}$. For each $x\in\mathbb{R}^n\setminus\{z\}$ let ℓ_x be the ray $\{z+t\vec{zx}:\ t\in[0,+\infty)\}$. By [Be, 11.1.2.3, 11.1.2.7] $\ell_x\cap\partial\widehat{\sigma}=\{\rho(x)\}$ is a singleton and if $x\in\partial\widehat{\sigma}$, then $\rho(x)=x$. Define $\rho:\mathbb{R}^n\setminus\{z\}\to\partial\widehat{\sigma},\ x\mapsto\rho(x)$. Let $h_1,\dots,h_{n+1}\in\mathbb{R}[\mathtt{x}]$ be polynomials of degree 1 such that the hyperplanes $H_i:=\{h_i=0\}$ contain the facets of $\widehat{\sigma}$. Assume $\widehat{\sigma}\subset\{h_i\geq 0\}$. Note that $\rho(x)=z+\lambda\vec{zx}$, where λ is the smallest value $\mu>0$ such that $h_i(z+\mu\overline{zx})=0$ for some $i=1,\dots,n+1$. As $z\in\operatorname{Int}(\widehat{\sigma})$, we have $h_i(z)>0$ for $i=1,\dots,n+1$. Thus,

$$\frac{1}{\lambda}=\max\left\{\frac{h_i(z)-h_i(x)}{h_i(z)}\ :\ i=1,\dots,n+1\right\}>0.$$

Consequently,

$$\rho(x)=z+\frac{1}{\max\left\{\frac{h_i(z)-h_i(x)}{h_i(z)}\ :\ i=1,\dots,n+1\right\}}\vec{zx},$$

so $\rho:\mathbb{R}^n\setminus\{z\}\to\partial\widehat{\sigma}$ is a continuous map such that $\rho|_{\partial\widehat{\sigma}}=\operatorname{id}_{\partial\widehat{\sigma}}$, that is, ρ is a retraction.

Consider the continuous map $F^*:=\rho\circ F:\Delta_{n-1}\times[0,1]\to\partial\widehat{\sigma}$. *The restriction map* $F^*|_{\partial(\Delta_{n-1}\times[0,1])}:\partial(\Delta_{n-1}\times[0,1])\to\partial\widehat{\sigma}$ *has degree* 1 (as a continuous map between spheres of dimension $n-1$). Indeed, as

$$F^*|_{\Delta_{n-1}\times\{0\}}=F|_{\Delta_{n-1}\times\{0\}}:\Delta_{n-1}\times\{0\}\to\sigma$$

is a homeomorphism, then $(F^*)^{-1}(x)\cap\partial(\Delta_{n-1}\times[0,1])=(F|_{\Delta_{n-1}\times\{0\}})^{-1}(x)$ is a singleton and the restriction map $F^*|_{\partial(\Delta_{n-1}\times[0,1])}$ has degree 1.

By [H, Thm.5.1.6(b)], as $F^*|_{\partial(\Delta_{n-1}\times[0,1])}$ admits a continuous extension to $\Delta_{n-1}\times[0,1]$, we deduce that $F^*|_{\partial(\Delta_{n-1}\times[0,1])}$ has degree 0, which is a contradiction. Consequently, $\operatorname{Int}(\widehat{\sigma})\subset F(\operatorname{Int}(\Delta_{n-1})\times(0,1))$, as required. □

Given a polynomial $h\in\mathbb{R}[\mathtt{x}]$ of degree 1 and the hyperplane $H:=\{h=0\}$ of $\mathbb{R}^n$ denote the two subspaces determined by H as $H^+:=\{h\geq 0\}$ and $H^-:=\{h\leq 0\}$. Denote also $\vec{h}:=h-h(0)$. If $\mathcal{K}:=\{f_1\geq 0,\dots,f_m\geq 0\}\subset\mathbb{R}^n$ is an n-dimensional convex polyhedron, where each $f_i\in\mathbb{R}[\mathtt{x}]$ is a polynomial of degree 1, then $\operatorname{Int}(\mathcal{K})=\{f_1>0,\dots,f_m>0\}$. Thus, if $\mathcal{K}_1,\dots,\mathcal{K}_s\subset\mathbb{R}^n$ are n-dimensional convex polyhedra, then $\operatorname{Int}(\mathcal{K}_1\cap\ldots\cap\mathcal{K}_s)=\operatorname{Int}(\mathcal{K}_1)\cap\ldots\cap\operatorname{Int}(\mathcal{K}_s)$. The following construction will be useful for the following result.

Let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional convex polyhedron and let $\sigma\subset\partial\mathcal{K}$ be an $(n-1)$-dimensional simplex of vertices $v_1,\dots,v_n$. Let $p\in\operatorname{Int}(\mathcal{K})$ and let $\widehat{\sigma}$ be the n-simplex of vertices $\{p,v_1,\dots,v_n\}$. Let $H_1,\dots,H_n$ be the hyperplanes of $\mathbb{R}^n$ generated by the facets of $\widehat{\sigma}$ that contain p, which are those facets of $\widehat{\sigma}$ different from σ and suppose $v_i\notin H_i$. Assume that $\widehat{\sigma}\subset\bigcap_{j=1}^n H_j^+$ and consider the convex polyhedra $\mathcal{S}_j:=\mathcal{K}\cap\bigcap_{\ell\neq j}H_\ell^-$, (see Figure 3.6). Observe that $p\in\mathcal{S}_j$ and $\operatorname{Int}(\mathcal{S}_j)\neq\varnothing$ because $p\in\operatorname{Int}(\mathcal{K})$ and the hyperplanes $H_1,\dots,H_n$ are affinely independent.

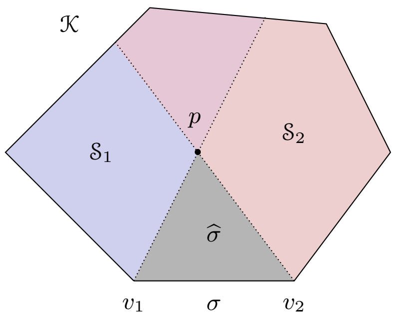

Figure 3.6: The polyhedra $\mathcal{S}_j$ (figure inspired by [FU5, Fig.4.2]).

Lemma 3.6.4. *Let $\mathcal{K} := \{g_1 \geq 0, \dots, g_s \geq 0\} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and let $\sigma \subset \partial\mathcal{K}$ be an $(n-1)$-dimensional simplex of vertices $v_1, \dots, v_n$. Fix $p \in \operatorname{Int}(\mathcal{K})$ and consider the simplex $\widehat{\sigma}$ of vertices $\{p, v_1, \dots, v_n\}$. Let $H_i := \{h_i = 0\}$ be the hyperplanes of $\mathbb{R}^n$ generated by the facets of $\widehat{\sigma}$ that contain p and assume $v_i \notin H_i$ and $\widehat{\sigma} \subset \bigcap_{i=1}^n H_i^+$. Let $h_0 \in \mathbb{R}[\mathtt{t}]$ be a polynomial of degree 1 such that $\sigma \subset \{h_0 = 0\}$ and $\widehat{\sigma} \subset \{h_0 \geq 0\}$. There exist continuous semi-algebraic paths $\alpha_i : [-\delta, 1+\delta] \to \mathcal{K}$ (for some $\delta > 0$) that are Nash on the compact neighbourhood $I := [-\delta, \delta] \cup [1-\delta, 1+\delta]$ of $\{0,1\}$ and satisfy $\alpha_i(0) = v_i$ and $\alpha_i(1) = p$ for $i = 1, \dots, n$ and $\varepsilon > 0$ such that the continuous semi-algebraic map*

$$F : \Delta_{n-1} \times [-\delta, 1+\delta] \to \mathcal{K},\ (\lambda_1, \dots, \lambda_n, t) \mapsto \sum_{i=1}^n \lambda_i \alpha_i(t),$$

(which is Nash on $\Delta_{n-1} \times I$) has the following property:

If $G : \Delta_{n-1} \times [-\delta, 1+\delta] \to \mathbb{R}^n$ is another continuous semi-algebraic map that is Nash on a neighbourhood $I' \subset I$ of $\Delta_{n-1} \times \{0,1\}$ and satisfies

$$\frac{\partial^\ell G}{\partial \mathtt{t}^\ell}(\lambda, 0) = \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}(\lambda, 0),\ \frac{\partial^\ell G}{\partial \mathtt{t}^\ell}(\lambda, 1) = \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}(\lambda, 1)$$

for each $\lambda \in \Delta_{n-1}$ and $\ell = 0, 1, 2, 3$, $\|G - F\| < \varepsilon$ and $\left\| \frac{\partial^\ell G}{\partial \mathtt{t}^\ell} - \frac{\partial^\ell F}{\partial \mathtt{t}^\ell} \right\|_{I'} < \varepsilon$ for $\ell = 1, 2, 3$, then $\widehat{\sigma} \subset G(\Delta_{n-1} \times [0,1]) \subset \mathcal{K}$ and $G(\Delta_{n-1} \times ([-\delta', 0] \cup [1, 1+\delta'])) \subset \widehat{\sigma}$ for some $0 < \delta' < \delta$ small enough.

Proof. The proof is conducted in several steps:

Initial preparation. Let us construct the continuous semi-algebraic paths $\alpha_i : [-\delta, 1+\delta] \to \mathcal{K}$. We claim: *There exist continuous semi-algebraic paths $\alpha_i : \mathbb{R} \to \mathcal{K}$ such that:*

(i) *α_i is Nash on I,*

(ii) *$\alpha_i(\mathtt{t}) = v_i + \mathtt{t}^2 u_i + \mathtt{t}^3 w + \cdots$ and $\alpha_i(1+\mathtt{t}) = p - \mathtt{t}^3 w + \cdots$,*

(iii) *$\alpha_i([-\delta, 0) \cup (1, 1+\delta]) \subset \operatorname{Int}(\widehat{\sigma})$,*

(iv) *$\alpha_i((0,1)) \subset \mathcal{S}_i := \operatorname{Int}(\mathcal{K} \cap \bigcap_{j \neq i} H_j^-)$,*

(v) $h_j \circ \alpha_i(\mathtt{t}) = -a_j\mathtt{t}^3 + \cdots$ *if* $i \neq j$ *and* $h_j \circ \alpha_i(1+\mathtt{t}) = a_j\mathtt{t}^3 + \cdots$, *where* $a_j > 0$ *and* $1 \leq i, j \leq n$,

(vi) $h_i \circ \alpha_i(\mathtt{t}) = h_i(v_i) + \vec{h}_i(u_i)\mathtt{t}^2 - a_i\mathtt{t}^3 + \cdots$ *where* $h_i(v_i) > 0$ *and* $\vec{h}_i(u_i) < 0$,

(vii) $h_0 \circ \alpha_i(\mathtt{t}) = b_{i0}\mathtt{t}^2 + \cdots$ *where* $b_{i0} > 0$ *and* $1 \leq i \leq n$,

(viii) $g_k \circ \alpha_i(\mathtt{t}) = c_{ik} + d_{ik}\mathtt{t}^2 + \cdots$ *where either* $c_{ik} > 0$ *or* $c_{ik} = 0$ *and* $d_{ik} > 0$,

(ix) $g_k \circ \alpha_i(1+\mathtt{t}) = e_{ik} + \cdots$ *where* $e_{ik} > 0$.

We construct each continuous semi-algebraic path α_i piecewisely. The open semi-algebraic set $\mathcal{S}_i$ defined in (iv) can be describes as

$$\mathcal{S}_i = \{g_1 > 0, \ldots, g_s > 0\} \cap \bigcap_{\substack{j \neq i \\ j \neq 0}} \{h_j < 0\}.$$

Define $u_i := \vec{v_i p}$ and observe that $\vec{h}_j(u_i) = 0$ if $1 \leq i, j \leq n$ and $i \neq j$. This is so because $h_j(p) = 0$ and $h_j(v_i) = 0$ if $1 \leq i, j \leq n$ and $i \neq j$. Recall that $h_i(v_i) > 0$ and $h_i(p) = 0$, so $\vec{h}_i(u_i) < 0$ for $1 \leq i \leq n$. In addition, $b_{i0} := \vec{h}_0(u_i) > 0$, because $h_0(p) > 0$ and $h_0(v_i) = 0$ for $1 \leq i \leq n$. As $g_k(v_i) \geq 0$ because $\widehat{\sigma} \subset \mathcal{K}$ and $g_k(v_i) + \vec{g}_k(u_i) = g_k(v_i + u_i) = g_k(p) > 0$ because $p \in \operatorname{Int}(\mathcal{K})$, we deduce that either $c_{ik} := g_k(v_i) > 0$ or $c_{ik} = 0$ and $d_{ik} := \vec{g}_k(u_i) > 0$.

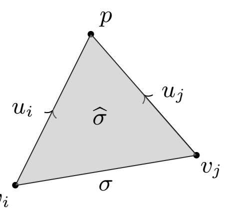

Figure 3.7: A picture of the situation.

As $\{\vec{h}_1, \ldots, \vec{h}_n\}$ are independent linear forms, the open semi-algebraic set $\bigcap_{j=1}^n \{\vec{h}_j < 0\} \neq \varnothing$. Pick a non-zero vector $w \in \bigcap_{j=1}^n \{\vec{h}_j < 0\}$ and write $a_j := -\vec{h}_j(w) > 0$ for $j = 1, \ldots, n$. Consider the polynomial path

$$\alpha_{i0} : \mathbb{R} \to \mathbb{R}^n, \ t \mapsto v_i + t^2 u_i + t^3 w.$$

As $h_0(v_i) = 0$ and $h_j(v_i) = 0$ for $1 \leq i, j \leq n$ if $i \neq j$, we deduce:

$$(h_0 \circ \alpha_{i0})(\mathtt{t}) = h_0(v_i) + \vec{h}_0(u_i)\mathtt{t}^2 + \vec{h}_0(w)\mathtt{t}^3 = b_{i0}\mathtt{t}^2 + \vec{h}_0(w)\mathtt{t}^3.$$

$$(h_j \circ \alpha_{i0})(\mathtt{t}) = h_j(v_i) + \vec{h}_j(u_i)\mathtt{t}^2 + \vec{h}_j(w)\mathtt{t}^3 = \vec{h}_j(w)\mathtt{t}^3 = -a_j\mathtt{t}^3 \quad \text{if } i \neq j,$$

$$(h_i \circ \alpha_{i0})(\mathtt{t}) = h_i(v_i) + \vec{h}_i(u_i)\mathtt{t}^2 + \vec{h}_i(w)\mathtt{t}^3 = h_i(v_i) + \vec{h}_i(u_i)\mathtt{t}^2 - a_i\mathtt{t}^3.$$

In addition,

$$(g_k \circ \alpha_{i0})(\mathtt{t}) = g_k(v_i) + \vec{g}_k(u_i)\mathtt{t}^2 + \vec{g}_k(w)\mathtt{t}^3 = c_{ik} + d_{ik}\mathtt{t}^2 + \vec{g}_k(w)\mathtt{t}^3,$$

where either $c_{ik} > 0$ or both $c_{ik} = 0$ and $d_{ik} > 0$.

Consider the polynomial path

$$\alpha_{i1} : \mathbb{R} \to \mathbb{R}^n, \ t \mapsto p - (t-1)^3 w.$$

Observe that

$$h_0 \circ \alpha_{i1}(1+\mathtt{t}) = h_0(p) - \vec{h}_0(w)\mathtt{t}^3,$$
$$h_j \circ \alpha_{i1}(1+\mathtt{t}) = h_j(p) - \vec{h}_j(w)\mathtt{t}^3 = a_j\mathtt{t}^3$$

for $1 \le i, j \le n$. As $a_j > 0$, we have

$$h_j \circ \alpha_{i0}(t) \begin{cases} < 0 & \text{if } t > 0, \\ > 0 & \text{if } t < 0, \end{cases} \text{ if } i \neq j \text{ and } h_j \circ \alpha_{i1}(1+t) \begin{cases} > 0 & \text{if } t > 0, \\ < 0 & \text{if } t < 0. \end{cases} \quad (3.6.2)$$

Denote $e_{ik} := g_k(p) > 0$ and observe that

$$g_k \circ \alpha_{i1}(1+\mathtt{t}) = g_k(p) - \vec{g}_k(w)\mathtt{t}^3 = e_{ik} - \vec{g}_k(w)\mathtt{t}^3.$$

Let $0 < \delta < \frac{1}{2}$ be such that $(h_0 \circ \alpha_{i0})(t) > 0$, $h_0 \circ \alpha_{i1}(1+t) > 0$, $(g_k \circ \alpha_{i0})(t) > 0$ and $g_k \circ \alpha_{i1}(1+t) > 0$ for $t \in [-\delta, \delta]$ (recall that $h_0(p) > 0$). Thus, by (3.6.2), $\alpha_{i0}([-\delta, 0)), \alpha_{i1}((1, 1+\delta]) \subset \operatorname{Int}(\widehat{\sigma})$ and $\alpha_{i0}((0, \delta]), \alpha_{i1}([1-\delta, 1)) \subset \mathcal{S}_i$.

As $\mathcal{S}_i$ is a convex set and $\alpha_{i0}(\delta), \alpha_{i1}(1-\delta) \in \mathcal{S}_i$, the segment that connects both points is contained in $\mathcal{S}_i$. Let

$$\alpha_{i2} : [\delta, 1-\delta] \to \mathcal{S}_i,\ t \mapsto \frac{(1-\delta)-t}{1-2\delta}\alpha_{i0}(\delta) + \frac{t-\delta}{1-2\delta}\alpha_{i1}(1-\delta)$$

be a parametrization of such segment. Define the continuous semi-algebraic path

$$\alpha_i := \alpha_{i0}|_{[-\delta,\delta]} * \alpha_{i2} * \alpha_{i1}|_{[1-\delta,1+\delta]} : [-\delta, 1+\delta] \to \mathcal{K},$$

which satisfies $\alpha_i([-\delta, 0) \cup (1, 1+\delta]) \subset \operatorname{Int}(\widehat{\sigma})$, $\alpha_i((0,1)) \subset \mathcal{S}_i$ and in fact all the required conditions (i)-(ix).

Step 1. *We have the following inclusions:* $\widehat{\sigma} \subset F(\Delta_{n-1} \times [0,1]) \subset \mathcal{K}$ *and* $F(\Delta_{n-1} \times ([-\delta, 0) \cup (1, 1+\delta])) \subset \operatorname{Int}(\widehat{\sigma}) \subset \mathcal{K}$.

Observe that $F(\Delta_{n-1} \times (0,1)) \subset \mathcal{K}$ because $\mathcal{K}$ is convex and $\alpha_i((0,1)) \in \mathcal{K}$. In addition, $\alpha_i(1) = p$ for each i, so $F(\Delta_{n-1} \times \{1\}) = p$, and

$$F(\lambda_1, \ldots, \lambda_n, 0) = \sum_{i=1}^{n} \lambda_i v_i,$$

so $F(\Delta_{n-1} \times \{0\}) = \sigma \subset \mathcal{K}$ and $F|_{\Delta_{n-1}\times\{0\}}$ is a homeomorphism. Thus, $F(\Delta_{n-1} \times [0,1]) \subset \mathcal{K}$.

Let us analyse the restriction map $F|_{\partial\Delta_{n-1}\times(0,1)} : \partial\Delta_{n-1} \times (0,1) \to \mathcal{K}$. Recall that $\partial\Delta_{n-1} = \bigcup_{i=1}^{n}(\Delta_{n-1} \cap \{\lambda_i = 0\})$.

Fix an index $i = 1, \ldots, n$ and write $\lambda^{(i)} := (\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i-1}, \ldots, \lambda_n)$ where $\sum_{j\neq i} \lambda_j = 1$ and each $\lambda_j \ge 0$ if $j \neq i$. We have $\operatorname{Int}(H_i^-) \subset \mathbb{R}^n \setminus \widehat{\sigma}$ and

$$F(\lambda^{(i)}, t) = \sum_{j\neq i} \lambda_j \alpha_j(t) \in \operatorname{Int}(\mathcal{K} \cap H_i^-) = \operatorname{Int}(\mathcal{K}) \cap \operatorname{Int}(H_i^-) \subset \mathcal{K} \setminus \widehat{\sigma} \quad (3.6.3)$$

for $t \in (0,1)$, because if $j \neq i$ each $\alpha_j(t) \in \operatorname{Int}(\mathcal{K}) \cap \operatorname{Int}(H_i^-)$ and the latter is convex. Thus, $F(\partial\Delta_{n-1} \times (0,1)) \subset \mathcal{K} \setminus \widehat{\sigma}$. By Lemma 3.6.3 $\widehat{\sigma} \subset F(\Delta_{n-1} \times [0,1])$.

As $\alpha_i([-\delta,0)\cup(1,1+\delta])\subset\operatorname{Int}(\widehat{\sigma})$ and $\operatorname{Int}(\widehat{\sigma})$ is convex, one concludes that $F(\Delta_{n-1}\times([-\delta,0)\cup(1,1+\delta]))\subset\operatorname{Int}(\widehat{\sigma})$.

Let us construct $\varepsilon>0$ such that: *if G is under the hypothesis of the statement, then $\widehat{\sigma}\subset G(\Delta_{n-1}\times[0,1])\subset\mathcal{K}$ and $G(\Delta_{n-1}\times([-\delta',0]\cup[1,1+\delta']))\subset\widehat{\sigma}$ for some $0<\delta'<\delta$ small enough.*

Step 2. *Choice of $\varepsilon>0$.* Recall that by (3.6.3) $F(\lambda,t)\in\mathbb{R}^n\setminus\widehat{\sigma}$ for each $(\lambda,t)\in\partial\Delta_{n-1}\times(0,1)$. For each $0<\rho<\frac{1}{2}$ define

$$\varepsilon_\rho:=\tfrac{1}{2}\min\{\operatorname{dist}(F(\lambda,t),\widehat{\sigma}):\ (\lambda,t)\in\partial\Delta_{n-1}\times[\rho,1-\rho]\}>0.$$

Observe that if $G:\Delta_{n-1}\times[\rho,1-\rho]\to\mathbb{R}^n$ satisfies $\|F-G\|<\varepsilon_\rho$, then $G((\partial\Delta_{n-1}\times[\rho,1-\rho])\subset\mathbb{R}^n\setminus\widehat{\sigma}$.

See assertions (i)-(ix) above for the definition of a_j,b_{i0},c_{ik},d_{ik} and e_{ik}. Consider

$$c^*_{ik}:=\begin{cases}c_{ik} & \text{if } c_{ik}>0,\\ d_{ik} & \text{if } c_{ik}=0.\end{cases}$$

and define

$$\varepsilon_0:=\tfrac{1}{2}\min\{a_j,b_{i0},c^*_{ik},e_{ik}:\ \forall\, i,j,k\}>0.$$

By hypothesis (v) and (vi):

$$\begin{aligned}h_j\circ\alpha_i(\mathtt{t})&=-a_j\mathtt{t}^3+\cdots\ \text{ if } i\neq j,\\ h_i\circ\alpha_i(\mathtt{t})&=h_i(v_i)+\vec{h}_i(u_i)\mathtt{t}^2-a_i\mathtt{t}^3+\cdots,\\ h_j\circ\alpha_i(1+\mathtt{t})&=a_j\mathtt{t}^3+\cdots.\end{aligned}$$

So there exists $0<\rho_0<\delta$ such that

$$-(h_j\circ\alpha_i)'''|_{[-\rho_0,\rho_0]}\geq\varepsilon_0\quad\text{and}\quad(h_j\circ\alpha_i)'''|_{[1-\rho_0,1+\rho_0]}\geq\varepsilon_0.$$

As h_i is a polynomial of degree 1, for $i=1,\dots,n$,

$$\frac{\partial^\ell}{\partial\mathtt{t}^\ell}((h_i\circ F)(\lambda,\mathtt{t}))=\sum_{j=1}^n\lambda_j(h_i\circ\alpha_j)^{(\ell)}(\mathtt{t})$$

for each $\ell\geq 0$. Consequently,

$$-\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)|_{\Delta_{n-1}\times[-\rho_0,\rho_0]}\geq\varepsilon_0,\qquad(3.6.4)$$

$$\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)|_{\Delta_{n-1}\times[1-\rho_0,1+\rho_0]}\geq\varepsilon_0\qquad(3.6.5)$$

for $i=1,\dots,n$.

For each $k=1,\dots,s$ define $\mathfrak{F}_k:=\{i=1,\dots,n:\ c_{ik}\neq 0\}$. We have

$$g_k(F(\lambda,\mathtt{t}))=\sum_{i\in\mathfrak{F}_k}\lambda_ic_{ik}+\sum_{i\in\mathfrak{F}_k}\lambda_id_{ik}\mathtt{t}^2+\sum_{i\notin\mathfrak{F}_k}\lambda_id_{ik}\mathtt{t}^2+\cdots$$

Define $\Gamma_k:=\{\lambda\in\Delta_{n-1}:\ \lambda_i=0,\ i\in\mathfrak{F}_k\}$. If $\mathfrak{F}_k\neq\{1,\dots,n\}$, then $\Gamma_k\neq\varnothing$ and $\mu_{0k}:=\min\{d_{ik}\ :\ i\notin\mathfrak{F}_k\}>0$. If $\mathfrak{F}_k=\{1,\dots,n\}$, define $\mu_{0k}:=1$.

If $\mathfrak{F}_k \neq \{1,\dots,n\}$ and $\lambda := (\lambda_1,\dots,\lambda_n) \in \Gamma_k$, then $\sum_{i\notin\mathfrak{F}_k} \lambda_i = 1$ and each $\lambda_i \geq 0$, so $\sum_{i\notin\mathfrak{F}_k} \lambda_i d_{ik} \geq \mu_{0k}$. Define

$$V_k := \Big\{\lambda \in \Delta_{n-1} : \sum_{i\in\mathfrak{F}_k} \lambda_i < \frac{1}{4}, \ \Big|\sum_{i\in\mathfrak{F}_k} \lambda_i d_{ik}\Big| < \frac{1}{4}\mu_{0k}\Big\}$$

if $\mathfrak{F}_k \neq \varnothing, \{1,\dots,n\}$, $V_k := \Delta_{n-1}$ if $\mathfrak{F}_k = \varnothing$ and $V_k := \varnothing$ if $\mathfrak{F}_k = \{1,\dots,n\}$. Observe that in the latter case $\Gamma_k = \varnothing$.

If $\mathfrak{F}_k \neq \varnothing, \{1,\dots,n\}$, then $V_k \neq \varnothing$ and if $\lambda \in V_k$, we have $\sum_{i\notin\mathfrak{F}_k} \lambda_i > \frac{3}{4}$, so $\sum_{i\notin\mathfrak{F}_k} \lambda_i d_{ik} > \frac{3}{4}\mu_{0k}$ and

$$\begin{aligned}\frac{1}{2}\frac{\partial^2(g_k\circ F)}{\partial \mathtt{t}^2}(\lambda,0) &= \sum_{i=1}^n \lambda_i d_{ik} = \sum_{i\in\mathfrak{F}_k} \lambda_i d_{ik} + \sum_{i\notin\mathfrak{F}_k} \lambda_i d_{ik}\\ &> -\frac{\mu_{0k}}{4} + \frac{3\mu_{0k}}{4} = \frac{1}{2}\mu_{0k},\end{aligned} \tag{3.6.6}$$

therefore,

$$\frac{\partial^2(g_k\circ F)}{\partial \mathtt{t}^2}(\lambda,0) > \mu_{0k}.$$

If $\mathfrak{F}_k = \varnothing$, then $\sum_{i=1}^n \lambda_i = 1$ and

$$\frac{1}{2}\frac{\partial^2(g_k\circ F)}{\partial \mathtt{t}^2}(\lambda,0) = \sum_{i=1}^n \lambda_i d_{ik} \geq \mu_{0k} > \frac{1}{2}\mu_{0k}. \tag{3.6.7}$$

If $\mathfrak{F}_k \neq \varnothing$, define

$$\mu_{1k} := \min\Big\{(g_k\circ F)(\lambda,0) = \sum_{i\in\mathfrak{F}_k} \lambda_i c_{ik} : \ \lambda := (\lambda_1,\dots,\lambda_n) \in \Delta_{n-1}\setminus V_k\Big\} > 0. \tag{3.6.8}$$

If $\mathfrak{F}_k = \varnothing$, define $\mu_{1k} := 1$.

Let $0 < \rho < \rho_0$ be such that

$$\begin{cases} |g_k(F(\lambda,t)) - g_k(F(\lambda,0))| < \frac{\mu_{1k}}{2} & \text{if } (\lambda,t) \in (\Delta_{n-1}\setminus V_k)\times[-\rho,\rho],\\ \frac{\partial^2(g_k\circ F)}{\partial \mathtt{t}^2}(\lambda,t) > \frac{\mu_{0k}}{2} & \text{if } (\lambda,t) \in V_k\times[-\rho,\rho]. \end{cases} \tag{3.6.9}$$

Observe that $F(\Delta_{n-1}\times[\rho,1+\rho]) \subset \operatorname{Int}(\mathcal{K})$ because $\operatorname{Int}(\mathcal{K})$ is convex and $\alpha_i([\rho,1+\rho]) \subset \operatorname{Int}(\mathcal{K})$ for $i = 1,\dots,n$, so

$$\varepsilon'_\rho := \tfrac{1}{2}\min\{\operatorname{dist}(F(\lambda,t),\mathbb{R}^n\setminus\operatorname{Int}(\mathcal{K})) : \ (\lambda,t) \in \Delta_{n-1}\times[\rho,1+\rho]\} > 0. \tag{3.6.10}$$

Thus, if $G : \Delta_{n-1}\times[\rho,1-\rho] \to \mathbb{R}^n$ and $\|F - G\| < \varepsilon'_\rho$, then

$$G((\Delta_{n-1}\times[\rho,1+\rho]) \subset \operatorname{Int}(\mathcal{K}).$$

Denote $\varepsilon'_0 := \min\{\varepsilon_0, \mu_{1k}, \frac{\mu_{0k}}{2} : k = 1,\dots,s\}$. The maps

$$\Psi_k : \mathcal{S}^3(\Delta_{n-1}\times[-\rho,1+\rho],\mathbb{R}^n) \to \mathcal{S}^3(\Delta_{n-1}\times[-\rho,1+\rho],\mathbb{R}),\ H \mapsto g_k\circ H,$$
$$\Theta_i : \mathcal{S}^3(\Delta_{n-1}\times[-\rho,1+\rho],\mathbb{R}^n) \to \mathcal{S}^3(\Delta_{n-1}\times[-\rho,1+\rho],\mathbb{R}),\ H \mapsto h_i\circ H$$

are continuous with respect to the $\mathcal{S}^3$ topology. Let $0<\varepsilon<\min\{\varepsilon'_\rho,\varepsilon_\rho\}$ be such that if $\|\frac{\partial^\ell}{\partial\mathtt{t}^\ell}F-\frac{\partial^\ell}{\partial\mathtt{t}^\ell}G\|<\varepsilon$ for $\ell=0,1,2,3$, then

$$\Big|\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(g_k\circ F)-\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(g_k\circ G)\Big|=\Big|\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(\Psi_k(F))-\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(\Psi_k(G))\Big|<\frac{\varepsilon'_0}{2},\tag{3.6.11}$$

$$\Big|\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(h_i\circ F)-\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(h_i\circ G)\Big|=\Big|\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(\Theta_i(F))-\frac{\partial^\ell}{\partial\mathtt{t}^\ell}(\Theta_i(G))\Big|<\frac{\varepsilon'_0}{2}\tag{3.6.12}$$

for $\ell=0,1,2,3$ and $G\in\mathcal{S}^3(\Delta_{n-1}\times[-\rho,1+\rho],\mathbb{R}^n)$. The chosen value $\varepsilon>0$ depends only on $\mathcal{K}$, F and $\rho>0$.

Let us check next: *$\varepsilon>0$ satisfies the conditions in the statement.* Let $G:\Delta_{n-1}\times[-\rho,1+\rho]\to\mathbb{R}^n$ be a continuous semi-algebraic map satisfying the conditions in the statement. We have to prove: *$\widehat{\sigma}\subset G(\Delta_{n-1}\times[0,1])\subset\mathcal{K}$ and $G(\Delta_{n-1}\times([-\delta',0]\cup[1,1+\delta']))\subset\widehat{\sigma}$ for some $0<\delta'<\delta$.*

Step 3. We prove first $\widehat{\sigma}\subset G(\Delta_{n-1}\times[0,1])$ as an application of Lemma 3.6.3. Observe that $G|_{\Delta_{n-1}\times\{0\}}=F|_{\Delta_{n-1}\times\{0\}}$ is a homeomorphism and $G|_{\Delta_{n-1}\times\{1\}}=F|_{\Delta_{n-1}\times\{1\}}=p$. Let us show: $G(\partial\Delta_{n-1}\times[0,1])\subset\mathbb{R}^n\setminus\widehat{\sigma}$.

As $\|F-G\|<\varepsilon\le\varepsilon_\rho$, we have $G(\partial\Delta_{n-1}\times[\rho,1-\rho])\subset\mathbb{R}^n\setminus\widehat{\sigma}$. We fix $\lambda^{(i)}\in\Delta_{n-1}\cap\{\lambda_i=0\}$ and claim: $G(\lambda^{(i)},t)\in\operatorname{Int}(H_i^-)=\{h_i<0\}$ *for each* $t\in(0,\rho]\cup[1-\rho,1)$.

Denote $\varphi_i:=h_i(G(\lambda^{(i)},\cdot))$. Suppose there exists $t_0\in(0,\rho]$ such that $\varphi_i(t_0)\ge0$. As

$$h_i(F(\lambda^{(i)},\mathtt{t}))=\sum_{j\neq i}\lambda_j(h_i\circ\alpha_j)(\mathtt{t})=\sum_{j\neq i}\lambda_j(-a_i\mathtt{t}^3+\cdots)$$

and $F(\lambda^{(i)},\mathtt{t})-G(\lambda^{(i)},\mathtt{t})\in(\mathtt{t})^4\mathbb{R}[[\mathtt{t}]]$, we have

$$\varphi_i(\mathtt{t})=h_i(G(\lambda^{(i)},\mathtt{t}))=\sum_{j\neq i}\lambda_j(-a_i\mathtt{t}^3+\cdots).$$

Thus, $\varphi_i(t)<0$ for $t>0$ close to 0, so we may assume $\varphi_i(t_0)=0$. Consequently, as $\varphi_i(0)=0$, there exists by Rolle's theorem $t_1\in(0,t_0)$ such that $\varphi_i'(t_1)=0$. As $\varphi_i'(0)=0$, there exists $t_2\in(0,t_1)$ satisfying $\varphi_i''(t_2)=0$. As $\varphi_i''(0)=0$, there exists $t_3\in(0,t_2)$ such that $\varphi_i'''(t_3)=0$. We have by (3.6.4) and (3.6.12)

$$\begin{aligned}\varepsilon_0\le\Big|\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda^{(i)},t_3)\Big|&=\Big|\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda^{(i)},t_3)-\varphi_i'''(t_3)\Big|\\&=\Big|\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda^{(i)},t_3)-\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ G)(\lambda^{(i)},t_3)\Big|\le\frac{\varepsilon_0}{2},\end{aligned}$$

which is a contradiction. Consequently, $\varphi_i(t)<0$ for each $t\in(0,\rho]$.

Analogously, one shows $\varphi_i(t)<0$ for each $t\in[1-\rho,1)$ and $i=1,\dots,n$. Thus,

$$G(\partial\Delta_{n-1}\times(0,1))=\bigcup_{i=1}^n G((\Delta_{n-1}\cap\{\lambda_i=0\})\times(0,1))\subset\bigcup_{i=1}^n\{h_i<0\}=\mathbb{R}^n\setminus\widehat{\sigma}.$$

By Lemma 3.6.3 we have $\widehat{\sigma} \subset G(\Delta_{n-1} \times [0,1])$.

STEP 4. We prove next: $G(\Delta_{n-1} \times ([-\rho, 1+\rho] \setminus \{0\})) \subset \operatorname{Int}(\mathcal{K})$.

Since $\|F - G\| < \varepsilon \leq \varepsilon'_\rho$ (see (3.6.10) for the definition of ε'_ρ), we have

$$G(\Delta_{n-1} \times [\rho, 1+\rho]) \subset \operatorname{Int}(\mathcal{K}).$$

Fix $k = 1, \dots, s$ and let $\lambda \in \Delta_{n-1}$. Let us check:

$$G(\lambda, t) \in \operatorname{Int}(\mathcal{K}) = \{g_1 > 0, \dots, g_s > 0\}$$

for each $t \in [-\rho, \rho] \setminus \{0\}$.

We distinguish two cases:

CASE 1. $\lambda \in \Delta_{n-1} \setminus V_k$. Observe that if $\mathfrak{F}_k = \varnothing$, then $\Delta_{n-1} \setminus V_k = \varnothing$. By (3.6.9) and (3.6.11)

$$|g_k \circ G - g_k \circ F| < \frac{\mu_{1k}}{2} \quad \text{and} \quad |g_k(F(\lambda, t)) - g_k(F(\lambda, 0))| < \frac{\mu_{1k}}{2},$$

if $t \in [-\rho, \rho]$. By (3.6.8) we deduce

$$\begin{aligned}(g_k \circ G)(\lambda, t) &= (g_k \circ F)(\lambda, 0) + (g_k \circ G)(\lambda, t) - (g_k \circ F)(\lambda, t) \\ &\qquad + (g_k \circ F)(\lambda, t) - (g_k \circ F)(\lambda, 0) > \mu_{1k} - \frac{\mu_{1k}}{2} - \frac{\mu_{1k}}{2} = 0\end{aligned}$$

for each $t \in [-\rho, \rho]$. Thus, $g_k(G(\lambda, t)) > 0$ for $t \in [-\rho, \rho]$ and $k = 1, \dots, s$, that is, $G(\lambda, t) \in \operatorname{Int}(\mathcal{K})$ for $t \in [-\rho, \rho]$.

CASE 2. $V_k \neq \varnothing$ and $\lambda \in V_k$. Then $0 \leq \sum_{i \in \mathfrak{F}_k} \lambda_i c_{ik}$ and $\sum_{i=1}^n \lambda_i d_{ik} \geq \frac{1}{2}\mu_{0k} > 0$ (see (3.6.6) and (3.6.7)). As

$$g_k(F(\lambda, \mathtt{t})) = \sum_{i=1}^n \lambda_i (g_k \circ \alpha_i)(\mathtt{t}) = \sum_{i \in \mathfrak{F}_k} \lambda_i c_{ik} + \sum_{i=1}^n \lambda_i (d_{ik}\mathtt{t}^2 + \cdots)$$

and $F(\lambda, \mathtt{t}) - G(\lambda, \mathtt{t}) \in (\mathtt{t})^4 \mathbb{R}[[\mathtt{t}]]$, we have

$$g_k(G(\lambda, \mathtt{t})) = \sum_{i \in \mathfrak{F}_k} \lambda_i c_{ik} + \sum_{i=1}^n \lambda_i (d_{ik}\mathtt{t}^2 + \cdots).$$

Define $\theta_k := g_k(G(\lambda, \cdot)) - \sum_{i \in \mathfrak{F}_k} \lambda_i c_{ik}$. Suppose there exists $t_0 \in [-\rho, \rho] \setminus \{0\}$ such that $\theta_k(t_0) \leq 0$. As $\theta_k(t) > 0$ for t close to 0, we may assume $\theta_k(t_0) = 0$. By Rolle's theorem there exists $t_1 \in (0, t_0)$ (or $t_1 \in (t_0, 0)$) such that $\theta'_k(t_1) = 0$. As $\theta'_k(0) = 0$, there exists $t_2 \in (0, t_1)$ (or $t_2 \in (t_1, 0)$) satisfying $\theta''_k(t_2) = 0$. We have by (3.6.9) and (3.6.11)

$$\begin{aligned}\frac{\mu_{0k}}{2} \leq \left|\frac{\partial^2}{\partial \mathtt{t}^2}(g_k \circ F)(\lambda, t_2)\right| &= \left|\frac{\partial^2}{\partial \mathtt{t}^2}(g_k \circ F)(\lambda, t_2) - \theta''_k(t_2)\right| \\ &= \left|\frac{\partial^2}{\partial \mathtt{t}^2}(g_k \circ F)(\lambda, t_2) - \frac{\partial^2}{\partial \mathtt{t}^2}(g_k \circ G)(\lambda, t_2)\right| < \frac{\varepsilon'_0}{2} \leq \frac{\mu_{0k}}{4},\end{aligned}$$

which is a contradiction. Consequently, $\theta_k(t) > 0$ for each $t \in [-\rho, \rho] \setminus \{0\}$ and $k = 1, \dots, s$. Thus $G(\lambda, t) \in \operatorname{Int}(\mathcal{K})$ for $t \in [-\rho, \rho] \setminus \{0\}$.

STEP 5. Observe that

$$G(\Delta_{n-1}\times\{0\})=F(\Delta_{n-1}\times\{0\})=\sigma\subset\widehat{\sigma},$$
$$G(\Delta_{n-1}\times\{1\})=F(\Delta_{n-1}\times\{1\})=\{p\}\subset\widehat{\sigma}.$$

Finally, we show: $G(\Delta_{n-1}\times([-\delta',0)\cup(1,1+\delta']))\subset\widehat{\sigma}$ *for some* $0<\delta'\leq\rho<\delta$.

For each $\lambda:=(\lambda_1,\ldots,\lambda_n)\in\Delta_{n-1}$ consider

$$h_i(F(\lambda,\mathtt{t}))=\sum_{j=1}^n\lambda_j(h_i\circ\alpha_j)(\mathtt{t})=(h_i\circ\alpha_i)(\mathtt{t})+\sum_{j\neq i}\lambda_j(-a_j\mathtt{t}^3+\cdots)$$

As $F(\lambda,\mathtt{t})-G(\lambda,\mathtt{t})\in(\mathtt{t})^4\mathbb{R}[[\mathtt{t}]]$, we have

$$\begin{aligned}h_i(G(\lambda,\mathtt{t}))&=\lambda_i(h_i(v_i)+\vec{h}_i(u_i)\mathtt{t}^2-a_i\mathtt{t}^3+\cdots)+\sum_{j\neq i}\lambda_j(-a_j\mathtt{t}^3+\cdots)\\&=\lambda_i(h_i(v_i)+\vec{h}_i(u_i))\mathtt{t}^2-\sum_{j=1}^n\lambda_ja_j\mathtt{t}^3+\cdots.\end{aligned}$$

Thus, $h_i(G(\lambda,t))>0$ for $t<0$ close enough to 0.

Pick $\lambda\in\Delta_{n-1}$ and define

$$\psi_i(\mathtt{t}):=h_i(G(\lambda,\mathtt{t}))-\lambda_i(h_i(v_i)+\vec{h}_i(u_i))\mathtt{t}^2=-\sum_{j=1}^n\lambda_ja_j\mathtt{t}^3+\cdots.$$

Suppose that there exists $t_0\in[-\rho,0)$ such that $\psi_i(t_0)\leq0$. Observe that $\psi_i(0)=0$ and $\psi_i(t)>0$ for $t<0$ close enough to 0. Thus, we may assume $\psi_i(t_0)=0$. As $\psi_i(0)=0$, by Rolle's theorem there exists $t_1\in(t_0,0)$ such that $\psi_i'(t_1)=0$. As $\psi_i'(0)=0$, there exists $t_2\in(t_1,0)$ satisfying $\psi_i''(t_2)=0$. As $\psi_i''(0)=0$, there exists $t_3\in(t_2,0)$ such that $\psi_i'''(t_3)=0$. Recall that by (3.6.4)

$$\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda,\mathtt{t})=-6\sum_{j=1}^n\lambda_ja_j+\cdots\leq-\varepsilon_0$$

Thus, we have by (3.6.12)

$$\begin{aligned}\varepsilon_0\leq\left|\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda,t_3)\right|&=\left|\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda,t_3)-\psi_i'''(t_3)\right|\\&=\left|\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ F)(\lambda,t_3)-\frac{\partial^3}{\partial\mathtt{t}^3}(h_i\circ G)(\lambda,t_3)\right|\leq\frac{\varepsilon_0}{2},\end{aligned}$$

which is a contradiction. Consequently, $\psi_i(t)>0$ for each $t\in[-\rho,0)$. As $h_i(v_i)>0$, there exists $0<\delta'\leq\rho<\delta$ such that $h_i(v_i)+\mathtt{t}^2\vec{h}_i(u_i)>0$ on $[-\delta',0)$ for $i=1,\ldots,n$. Thus, $h_i(G(\lambda,t))>0$ on $\Delta_{n-1}\times[-\delta',0)$ for $i=1,\ldots,n$.

Let us show $h_i(G(\lambda,t))>0$ for $(\lambda,t)\in(1,1+\rho]$ and $i=1,\ldots,n$. We have

$$h_i(F(\lambda,1+\mathtt{t}))=\sum_{j=1}^n\lambda_j(h_i\circ\alpha_j)=\sum_{j=1}^n\lambda_ja_j\mathtt{t}^3+\cdots$$

and we can repeat the previous argument taking $\phi_i(\mathtt{t}) := h_i(G(\lambda, 1+\mathtt{t})$. As $F(\lambda, 1+\mathtt{t}) - G(\lambda, 1+\mathtt{t}) \in (\mathtt{t})^4\mathbb{R}[[\mathtt{t}]]$,

$$\phi_i(\mathtt{t}) = h_i(G(\lambda, 1+\mathtt{t}) = \sum_{j=1}^n \lambda_j(h_i \circ \alpha_j) = \sum_{j=1}^n \lambda_j a_j \mathtt{t}^3 + \cdots .$$

Suppose that there exists $t_0 \in (0, \rho]$ such that $\phi_i(t_0) \leq 0$. Observe that $\phi_i(0) = 0$ and $\phi_i(t) > 0$ for $t > 0$ close enough to 0. Thus, we may assume $\phi_i(t_0) = 0$. As $\phi_i(0) = 0$, by Rolle's theorem, there exists $t_1 \in (0, t_0)$ such that $\phi_i'(t_1) = 0$. As $\phi_i'(0) = 0$, there exists $t_2 \in (0, t_1)$ satisfying $\phi_i''(t_2) = 0$. As $\psi_i''(0) = 0$, there exists $t_3 \in (0, t_2)$ such that $\phi_i'''(t_3) = 0$. We have by (3.6.5) and (3.6.12)

$$\begin{aligned}\varepsilon_0 \leq \left|\frac{\partial^3}{\partial \mathtt{t}^3}(h_i \circ F)(\lambda, 1+t_3)\right| &= \left|\frac{\partial^3}{\partial \mathtt{t}^3}(h_i \circ F)(\lambda, 1+t_3) - \psi_i'''(t_3)\right| \\ &= \left|\frac{\partial^3}{\partial \mathtt{t}^3}(h_i \circ F)(\lambda, 1+t_3) - \frac{\partial^3}{\partial \mathtt{t}^3}(h_i \circ G)(\lambda, 1+t_3)\right| \leq \frac{\varepsilon_0}{2},\end{aligned}$$

which is a contradiction. Thus $\phi_i(t) > 0$ for each $t \in (0, \rho]$, so $h_i(G(\lambda, t)) > 0$ on $\Delta_{n-1} \times (1, 1+\rho]$.

We conclude $G(\Delta_{n-1} \times ([-\delta', 0) \cup (1, 1+\delta'])) \subset \widehat{\sigma}$, as required. □

We will use the technical Lemma 3.6.4 to 'cover' simplices with Nash maps. To that end, we approximate first the continuous semi-algebraic paths by Nash paths. Let us check that for (close enough) approximations we obtain the desired result.

Remark 3.6.5. For each $i = 1, \ldots, n$ let $\alpha_i^* : [-\delta, 1+\delta] \to \mathcal{K}$ be a continuous semi-algebraic path such that $\alpha_i^*|_I$ is a Nash map, α_i^* is close to α_i, $(\alpha_i^*|_I)^{(\ell)}$ is close to $(\alpha_i|_I)^{(\ell)}$ for $\ell = 1, 2, 3$, $(\alpha_i^*)^{(\ell)}(0) = (\alpha_i)^{(\ell)}(0)$ and $(\alpha_i^*)^{(\ell)}(1) = (\alpha_i)^{(\ell)}(1)$ for $\ell = 0, 1, 2, 3$ (recall that $I := [-\delta, \delta] \cup [1-\delta, 1+\delta]$).

(i) Then there exists $\varepsilon^* > 0$ and

$$F^* : \Delta_{n-1} \times [-\delta, 1+\delta] \to \mathcal{K},\ (\lambda, t) \mapsto \sum_{i=1}^n \lambda_i \alpha_i^*(t)$$

that satisfy the same conditions as ε and F in the statement of Theorem 3.6.4.

Observe that

$$\begin{aligned}(F - F^*)(\lambda, t) &= \sum_{i=1}^n \lambda_i(\alpha_i(t) - \alpha_i^*(t)), \\ \left(\frac{\partial^\ell F}{\partial \mathtt{t}^\ell} - \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}\right)(\lambda, t) &= \sum_{i=1}^n \lambda_i(\alpha_i^{(\ell)}(t) - (\alpha_i^*)^{(\ell)}(t))\end{aligned}$$

for $\ell = 1, 2, 3$. In addition, for $\ell = 0, 1, 2, 3$,

$$\begin{aligned}\frac{\partial^\ell F}{\partial \mathtt{t}^\ell}(\lambda, 0) &= \sum_{i=1}^n \lambda_i \alpha_i^{(\ell)}(0) = \sum_{i=1}^n \lambda_i (\alpha_i^*)^{(\ell)}(0) = \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}(\lambda, 0), \\ \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}(\lambda, 1) &= \sum_{i=1}^n \lambda_i \alpha_i^{(\ell)}(1) = \sum_{i=1}^n \lambda_i (\alpha_i^*)^{(\ell)}(1) = \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}(\lambda, 1).\end{aligned}$$

Take $\varepsilon^* := \frac{\varepsilon}{2} > 0$ and assume that $\|\alpha_i - \alpha_i^*\| < \varepsilon^*$ and $\|\alpha_i^{(\ell)} - (\alpha_i^*)^{(\ell)}\|_I < \varepsilon^*$ for $\ell = 1,2,3$. Let $G : \Delta_{n-1} \times [-\delta, 1+\delta] \to \mathbb{R}^n$ be a continuous semi-algebraic map that is Nash on a neighbourhood $I' \times \Delta_{n-1} \subset I \times \Delta_{n-1}$ of $\Delta_{n-1} \times \{0,1\}$ and satisfies $\frac{\partial^\ell G}{\partial \mathtt{t}^\ell}(\lambda,0) = \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}(\lambda,0)$, $\frac{\partial^\ell G}{\partial \mathtt{t}^\ell}(\lambda,1) = \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}(\lambda,1)$ for each $\lambda \in \Delta_{n-1}$ and $\ell = 0,1,2,3$, $\|G - F^*\| < \varepsilon^*$ and $\|\frac{\partial^\ell G}{\partial \mathtt{t}^\ell} - \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}\|_{I'} < \varepsilon^*$ for $\ell = 1,2,3$. Then

$$\frac{\partial^\ell G}{\partial \mathtt{t}^\ell}(\lambda,0) = \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}(\lambda,0) = \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}(\lambda,0),$$
$$\frac{\partial^\ell G}{\partial \mathtt{t}^\ell}(\lambda,1) = \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}(\lambda,1) = \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}(\lambda,1)$$

for each $\lambda \in \Delta_{n-1}$ and $\ell = 0,1,2,3$, and

$$\|G - F\| \le \|G - F^*\| + \|F^* - F\| < \varepsilon^* + \varepsilon^* = \varepsilon,$$
$$\left\|\frac{\partial^\ell G}{\partial \mathtt{t}^\ell} - \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}\right\|_{I'} \le \left\|\frac{\partial^\ell G}{\partial \mathtt{t}^\ell} - \frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell}\right\|_{I'} + \left\|\frac{\partial^\ell F^*}{\partial \mathtt{t}^\ell} - \frac{\partial^\ell F}{\partial \mathtt{t}^\ell}\right\|_{I'} < \varepsilon^* + \varepsilon^* = \varepsilon$$

for $\ell = 1,2,3$. By Theorem 3.6.4 we have that $\widehat{\sigma} \subset G(\Delta_{n-1} \times [0,1]) \subset \mathcal{K}$ and $G(\Delta_{n-1} \times ([-\rho,0] \cup [1,1+\rho])) \subset \widehat{\sigma}$ for some $0 < \rho < \delta$ small enough, as required.

(ii) By (i) and Lemma 3.3.5 we may assume that each path $\alpha_i : [-\delta, 1+\delta] \to \mathcal{K}$ in the statement of Theorem 3.6.4 is Nash on $[-\delta, 1+\delta]$.

The following result provides sufficient conditions to guarantee that the high order derivatives of two continuous semi-algebraic functions on $\mathbb{R}^d \times [-1,1]$ that are Nash on a neighbourhood of a semi-algebraic set $\mathcal{S} \times \{0\}$ are equal at the points of $\mathcal{S} \times \{0\}$. This provides a sufficient condition to decide when the approximating maps satisfy the hypothesis of Lemma 3.6.4.

Lemma 3.6.6. *Let $\mathcal{S} \subset \mathbb{R}^d$ be a non-empty semi-algebraic set. Let $F, G : \mathbb{R}^d \times [-1,1] \to \mathbb{R}^m$ be two continuous semi-algebraic maps that are Nash on a neighbourhood of $\mathcal{S} \times \{0\}$ and suppose that there exists a Nash function $\lambda : [-1,1] \to \mathbb{R}$ such that $\|F - G\|_{\mathcal{S} \times [-1,1]} < |\lambda|_{[-1,1]}$ and that $\lambda(\mathtt{t}) = a_{k+1}\mathtt{t}^{k+1}u^2(\mathtt{t})$ where $a_{k+1} \neq 0$ and $u \in \mathbb{R}[[\mathtt{t}]]_{\rm alg}$ is a Nash series such that $u(0) = 1$. Then, for each $x \in \mathcal{S}$ we have*

$$\frac{\partial F}{\partial \mathtt{t}^\ell}(x,0) = \frac{\partial G}{\partial \mathtt{t}^\ell}(x,0)$$

for $\ell = 0, \dots, k$.

Proof. Pick $x \in \mathcal{S}$ and write

$$F(x,\mathtt{t}) := \sum_{\ell \ge 0} \frac{1}{\ell!}\frac{\partial F}{\partial \mathtt{t}^\ell}(x,0)\mathtt{t}^\ell,$$
$$G(x,\mathtt{t}) := \sum_{\ell \ge 0} \frac{1}{\ell!}\frac{\partial G}{\partial \mathtt{t}^\ell}(x,0)\mathtt{t}^\ell.$$

Thus, we have the following inequalities in the ring $\mathbb{R}[[\mathtt{t}]]_{\rm alg}$ of Nash series with respect to any of its two orders (the one characterised by $\mathtt{t} > 0$ and the other one by $\mathtt{t} < 0$):

$$\Big\|\sum_{\ell \ge 0} \frac{1}{\ell!}\left(\frac{\partial F}{\partial \mathtt{t}^\ell}(x,0) - \frac{\partial G}{\partial \mathtt{t}^\ell}(x,0)\right)\mathtt{t}^\ell\Big\| \le \|F(x,\mathtt{t}) - G(x,\mathtt{t})\|$$
$$\le |a_{k+1}||\mathtt{t}^{k+1}|u^2 \le |a_{k+1}\mathtt{t}^{k+1} + \cdots|.$$

Consequently, the series

$$\sum_{\ell\geq 0}\frac{1}{\ell!}\left(\frac{\partial F}{\partial \mathtt{t}^\ell}(x,0)-\frac{\partial G}{\partial \mathtt{t}^\ell}(x,0)\right)\mathtt{t}^\ell$$

is a series of order $\geq k+1$, so

$$\frac{\partial F}{\partial \mathtt{t}^\ell}(x,0)-\frac{\partial G}{\partial \mathtt{t}^\ell}(x,0)=0$$

for $\ell=0,\dots,k$, as required. □

3.6.3. Local charts and tubular neighbourhoods. Let $\mathcal{T}\subset\mathbb{R}^n$ be a compact checkerboard set of dimension $d\geq 2$. The algebraic set $M:=\overline{\mathcal{T}}^{\mathrm{zar}}\subset\mathbb{R}^n$ is a Nash manifold. By [FGR, Thm.1.6] the Nash normal-crossings divisor $\overline{\partial\mathcal{T}}^{\mathrm{zar}}\subset M$ can be covered by finitely many open semi-algebraic subset $U\subset M$ endowed with Nash diffeomorphisms $\psi_U:=(u_1,\dots,u_d):U\to\mathbb{R}^d$ such that $U\cap\overline{\partial\mathcal{T}}^{\mathrm{zar}}=\{u_1\cdots u_s=0\}$ for some s depending on U. As $\mathcal{T}$ is compact, there exist finitely many Nash diffeomorphisms $\phi_i:\mathbb{R}^d\to U_i\subset M$ for $i=1,\dots,r$ such that $\phi_i(\Lambda_{k_i})\subset\mathcal{T}$, where $\Lambda_{k_i}:=\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_{k_i}\geq 0\}\subset\mathbb{R}^d$ for some $0\leq k_i\leq d$, and $\{\phi_i(\Lambda_{k_i})\}_{i=1}^r$ is a finite covering for $\mathcal{T}$, that is, $\mathcal{T}=\bigcup_i\phi_i(\Lambda_{k_i})$. Moreover we may assume that, if $U_i\cap\overline{\partial\mathcal{T}}^{\mathrm{zar}}\neq\varnothing$ then $U_i\cap\overline{\partial\mathcal{T}}^{\mathrm{zar}}=\{u_1\cdots u_{k_i}=0\}$, where $\phi_i^{-1}:=(u_1,\dots,u_d)$. In particular, $\bigcup_i\phi_i(\mathrm{Int}(\Lambda_{k_i}))\subset\mathrm{Reg}(\mathcal{T})=\mathcal{T}\setminus\partial\mathcal{T}$.

Let (Ω,ν) be a Nash tubular neighbourhood for the Nash manifold $M:=\overline{\mathcal{T}}^{\mathrm{zar}}$ endowed with a retraction ν such that $\mathrm{dist}(z,M)=\|\nu(z)-z\|$ for each $z\in\Omega$ (see [BCR, Cor.8.9.5]). When $\mathcal{T}$ is compact, shrinking Ω if necessary, we may assume $\mathrm{Cl}(\nu^{-1}(\mathcal{T}))$ is compact and ν admits a Nash extension to $\mathrm{Cl}(\nu^{-1}(\mathcal{T}))$.

3.6.4. Some preliminary estimations. We want to provide some estimations in order to apply Lemma 3.6.6 later in our construction. Let $x\in M$ and $y\in\mathbb{R}^n$ be such that $x+y\in\Omega$, then

$$\begin{aligned}\|\nu(x+y)-x\|&\leq\|\nu(x+y)-(x+y)\|+\|y\|\\&=\mathrm{dist}(x+y,M)+\|y\|\leq\|x+y-x\|+\|y\|=2\|y\|.\end{aligned}$$

Let $\mathcal{F}:=\{\psi:\mathbb{R}^d\to\mathbb{R}^d\text{ linear}\}\equiv(\mathbb{R}^{d,*})^d$ and let $\psi_1,\dots,\psi_r\in\mathcal{F}$. If $w\in\mathbb{R}^d$ and $\lambda_1,\dots,\lambda_r\in\mathbb{R}$ are such that $(\phi_1\circ\psi_1)(w)+\sum_{i=1}^r\lambda_i(\phi_i\circ\psi_i)(w)\in\Omega$, then

$$\begin{aligned}&\Big\|\nu\Big((\phi_1\circ\psi_1)(w)+\sum_{i=1}^r\lambda_i(\phi_i\circ\psi_i)(w)\Big)-(\phi_1\circ\psi_1)(w)\Big\|\\&\qquad\leq 2\Big\|\sum_{i=1}^r\lambda_i(\phi_i\circ\psi_i)(w)\Big\|\leq 2\sum_{i=1}^r|\lambda_i|\|(\phi_i\circ\psi_i)(w)\|.\end{aligned}\tag{3.6.13}$$

Recall that $\overline{\mathcal{B}}_d(0,\varepsilon)$ (resp. $\mathcal{B}_d(0,\varepsilon)$) denotes the closed ball (resp. open ball) of $\mathbb{R}^d$ of centre the origin and radius $\varepsilon>0$.

Lemma 3.6.7. *Let $M\subset\mathbb{R}^n$ be a Nash manifold and consider a Nash chart $\theta:=(\theta_1,\dots,\theta_n):\mathbb{R}^d\to M$. Let $\pi:\mathbb{R}^n\to\mathbb{R}^d$ be the projection onto the first d coordinates and denote $W:=(\pi\circ\theta)(\mathbb{R}^d)$. Assume that W is open and that* *the map* $\theta' := \pi \circ \theta : \mathbb{R}^d \to W$ *is a Nash diffeomorphism. For each* $t > 0$ *there exists a constant* $L_t > 0$ *such that* $\|\theta^{-1}(x) - \theta^{-1}(y)\| \leq L_t\|x - y\|$ *for each* $x, y \in \theta(\overline{\mathcal{B}}_d(0, t))$.

Proof. Define $f := (\theta_{d+1}, \dots, \theta_n) \circ \theta'^{-1} : W \to \mathbb{R}^{n-d}$ and observe that we have $\theta \circ \theta'^{-1} : W \to \theta(\mathbb{R}^d),\ z \mapsto (z, f(z))$. Thus,

$$\|z - w\| \leq \|(z, f(z)) - (w, f(w))\| = \|(\theta \circ \theta'^{-1})(z) - (\theta \circ \theta'^{-1})(w)\|$$

for each $z, w \in W$. Consequently, writing $z = (\theta' \circ \theta^{-1})(x)$ and $w = (\theta' \circ \theta^{-1})(y)$, we deduce

$$\|(\theta' \circ \theta^{-1})(x) - \theta' \circ \theta^{-1}(y)\| \leq \|x - y\|$$

for each $x, y \in \theta(\mathbb{R}^d) = (\theta \circ \theta'^{-1})(W)$.

By the mean value theorem there exists a constant $L_t > 0$ such that

$$\|\theta'^{-1}(z) - \theta'^{-1}(w)\| \leq L_t\|z - w\|$$

for each $z, w \in \theta'(\overline{\mathcal{B}}_d(0, t))$. Thus,

$$\begin{aligned}\|\theta^{-1}(x) - \theta^{-1}(y)\| &= \|\theta'^{-1}((\theta' \circ \theta^{-1})(x)) - \theta'^{-1}((\theta' \circ \theta^{-1})(y))\| \\ &\leq L_t\|(\theta' \circ \theta^{-1})(x) - \theta' \circ \theta^{-1}(y)\| \leq L_t\|x - y\|\end{aligned}$$

for each $x, y \in \theta(\overline{\mathcal{B}}_d(0, t))$, as required. □

As $\mathcal{T}$ is compact, we may assume $\mathcal{T} \subset \bigcup_{i=1}^r \phi_i(\mathcal{B}_d(0, 1)) = M$. We may also assume, using the compactness of $\mathcal{T}$, that each ϕ_i is under the hypothesis of Lemma 3.6.7. Define $K := \max\{\|x\| : x \in \mathcal{T}\} > 0$. If $w \in \mathcal{B}_d(0, 1)$ and $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ are such that $\nu((\phi_1 \circ \psi_1)(w) + \sum_{i=1}^r \lambda_i(\phi_i \circ \psi_i)(w)) \in \phi_1(\mathcal{B}_d(0, 1))$, then by (3.6.13) and Lemma 3.6.7 there exists $L > 0$ such that

$$\begin{aligned}&\Big\|\phi_1^{-1}\Big(\nu\Big((\phi_1 \circ \psi_1)(w) + \sum_{i=1}^r \lambda_i(\phi_i \circ \psi_i)(w)\Big)\Big) - \psi_1(w)\Big\| \\ &\quad \leq L\Big\|\nu\Big((\phi_1 \circ \psi_1)(w) + \sum_{i=1}^r \lambda_i(\phi_i \circ \psi_i)(w)\Big) - (\phi_1 \circ \psi_1)(w)\Big\| \qquad (3.6.14)\\ &\quad \leq 2L\sum_{i=1}^r |\lambda_i|\|(\phi_i \circ \psi_i)(w)\| \leq 2LK\sum_{i=1}^r |\lambda_i|.\end{aligned}$$

3.6.5. Decomposition as a finite union of 'simplices'. Consider the vectors of the standard basis $\mathtt{e}_i := (0, \dots, 0, 1, 0, \dots, 0)$, for $i = 1, \dots, d$, of $\mathbb{R}^d$. Fix $k = 1, \dots, d$ and consider the convex polyhedron $\mathcal{K}_k$ that is the convex hull of the origin and the points

$$\mathtt{e}_1, \dots, \mathtt{e}_k, \mathtt{e}_{k+1}, -\mathtt{e}_{k+1}, \dots, \mathtt{e}_d, -\mathtt{e}_d.$$

We have that the polyhedron $\mathcal{K}_k$ is a compact neighbourhood of the origin in $\Lambda_k = \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_k \geq 0\}$, $\mathcal{K}_k \cap \partial\Lambda_k = \partial\mathcal{K}_k \cap \partial\Lambda_k$ and $\operatorname{Int}(\mathcal{K}_k) = \mathcal{K}_k \cap \operatorname{Int}(\Lambda_k)$. Observe that $\mathcal{K}_k$ is the union of the simplices $\Delta(\varepsilon_{k+1}, \dots, \varepsilon_d)$ of vertices the origin and the points

$$\mathtt{e}_1, \dots, \mathtt{e}_k, \varepsilon_{k+1}\mathtt{e}_{k+1}, \dots, \varepsilon_d\mathtt{e}_d,$$

where $\varepsilon_{k+1},\dots,\varepsilon_d=\pm1$. Let $\mathfrak{T}_k$ be the collection of the proper faces of the simplices $\Delta(\varepsilon_{k+1},\dots,\varepsilon_d)$ that are contained in $\partial\mathcal{K}_k$. Observe that $\mathfrak{T}_k$ provides a triangulation of $\partial\mathcal{K}_k$. Define

$$\begin{aligned}p_k&:=\sum_{j=1}^k\frac{1}{2d-k+1}\mathsf{e}_j\\&=\sum_{j=1}^k\frac{1}{2d-k+1}\mathsf{e}_j+\sum_{j=k+1}^d\frac{1}{2d-k+1}\mathsf{e}_j+\sum_{j=k+1}^d\frac{1}{2d-k+1}(-\mathsf{e}_j)+\frac{1}{2d-k+1}\mathbf{0},\end{aligned}$$

which belongs to $\operatorname{Int}(\mathcal{K}_k)$. Observe that if $k=0$ then p_0 is the origin. For each $\sigma\in\mathfrak{T}_k$ define $\widehat{\sigma}$ as the convex hull of $\sigma\cup\{p_k\}$, which is a simplex. Observe that $\widehat{\mathfrak{T}}_k:=\{\widehat{\sigma}:\ \sigma\in\mathfrak{T}_k\}$ is a triangulation of $\mathcal{K}_k$ such that

$$\widehat{\sigma}\cap\partial\Lambda_k=\widehat{\sigma}\cap\mathcal{K}_k\cap\partial\Lambda_k=\widehat{\sigma}\cap\partial\mathcal{K}_k\cap\partial\Lambda_k=\sigma\cap\partial\Lambda_k,$$

which is either the empty set or a face of σ (see Figure 3.8). Let $\mathcal{F}_k$ be the collection of the simplices $\widehat{\sigma}\in\widehat{\mathfrak{T}}_k$ of dimension d.

We retake here the Nash atlas $\{\phi_i\}_{i=1}^r$ of $M=\overline{\mathcal{T}}^{\mathrm{zar}}$ introduced in 3.6.3 and we keep all the hypothesis concerning $\{\phi_i\}_{i=1}^r$ already introduced there. We may assume that $\{\phi_i(\mathcal{K}_{k_i})\}_{i=1}^r$ is a covering of the compact checkerboard set $\mathcal{T}$ introduced in 3.6.3. For each i consider the finite family $\mathcal{F}_{k_i}$ of simplices of dimension d. Note that we are considering the families $\mathcal{F}_{k_{i_1}}$ and $\mathcal{F}_{k_{i_2}}$ as different families of simplices when $i_1\neq i_2$, even if $\mathcal{K}_{i_1}=\mathcal{K}_{i_2}$ as subsets of $\mathbb{R}^d$. We consider all the pairs (ϕ_i,τ) where $\tau\in\mathcal{F}_{k_i}$. Observe that τ is the convex hull of $\sigma\cup\{p_{k_i}\}$ for some $\sigma\in\mathfrak{T}_{k_i}$.

Repeating the diffeomorphisms ϕ_i as many times as needed and reordering the diffeomorphisms ϕ_i, we may assume that $\{\phi_i(\tau_i)\}_{i=1}^r$ is a covering of $\mathcal{T}$ such that τ_i is a d-dimensional simplex of $\mathbb{R}^d$ and $\tau_i\cap\phi_i^{-1}(\partial\mathcal{T})$ is either the empty set or a proper face of τ_i. Let σ_i be the $(d-1)$-dimensional face of τ_i that does not contain $p_i:=p_{k_i}$. Observe that $\tau_i\cap\phi_i^{-1}(\partial\mathcal{T})\subset\sigma_i$. Note that p_i belongs to $\phi_i^{-1}(\operatorname{Reg}(\mathcal{T}))$ because $p_i\in\operatorname{Int}(\Lambda_{k_i})\subset\phi_i^{-1}(\operatorname{Reg}(\mathcal{T}))$.

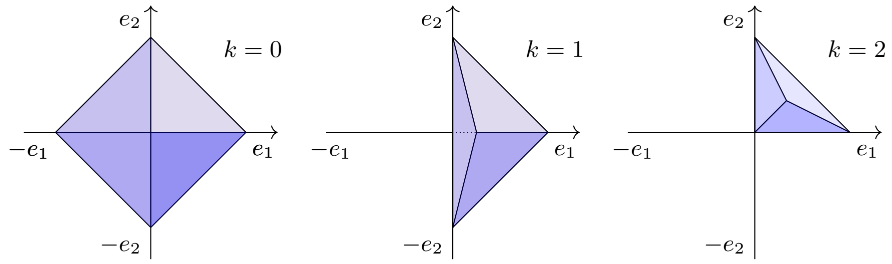

Figure 3.8: Triangulations $\widehat{\mathfrak{T}}_k$ of the polyhedra $\mathcal{K}_k$ for $d=2$.

3.6.6. Shrewd set of maps. Consider the vectors of the standard basis

$$\mathsf{e}_i:=(0,\dots,0,1,0,\dots,0),$$

for $i=1,\dots,r$, of $\mathbb{R}^r$. Let $\mathcal{F}:=\{\psi:\mathbb{R}^d\to\mathbb{R}^d\text{ linear}\}\equiv(\mathbb{R}^{d,*})^d$ and write $\mu:=(\mu_1,\dots,\mu_r)\in\mathbb{R}^r$ and $\psi:=(\psi_1,\dots,\psi_r)\in\mathcal{F}^r$. Let

$$\Gamma:\mathbb{R}^r\times\mathcal{F}^r\to\mathcal{N}(\mathbb{R}^d,\mathbb{R}^n),\ (\mu;\psi)\mapsto\sum_{i=1}^r\mu_i(\phi_i\circ\psi_i),$$

which is a continuous map if both spaces $\mathcal{F}$ and $\mathcal{N}(\mathbb{R}^d,\mathbb{R}^n)$ are endowed with the compact-open topology. The compact-open topology of $\mathcal{F}$ coincides with the topology of $\mathcal{F}$ induced by the Euclidean topology in the coefficients of $(\mathbb{R}^{d,*})^d$. Recall that

$$\Delta_{d-1} := \{\lambda_1 \geq 0, \dots, \lambda_d \geq 0,\ \lambda_1 + \dots + \lambda_d = 1\} \subset \mathbb{R}^d.$$

Note that an element of $\mathbb{R}^{d,*}$ is determined by the images of the vertices of Δ_{d-1}, which is a (compact) finite set.

Define $\Theta_0 := \{(\mu;\psi) \in \mathbb{R}^r \times \mathcal{F}^r :\ \Gamma(\mu,\psi)(\Delta_{d-1}) \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))\}$ and let us prove that it is an open semi-algebraic set. The objects $\mathcal{T}$ and ν were already introduced in 3.6.3.

Proposition 3.6.8. *The set $\Theta_0 \subset \mathbb{R}^r \times \mathcal{F}^r$ is open and semi-algebraic.*

Proof. The fact that Θ_0 is semi-algebraic follows by the Tarski-Seidenberg principle (see for instance [Co, Thm.2.6]), because it can be described as

$$\Theta_0 = \{x \in \mathbb{R}^r \times \mathcal{F}^r\ :\ \Psi(x)\},$$

where $\Psi(\mathtt{x})$ is a first order formula in the language of ordered fields.

Let us show now that Θ_0 is open. Recall that $\mathrm{Reg}(\mathcal{T})$ is an open semi-algebraic subset of $\mathcal{T}$. As $\mathcal{T}$ is pure dimensional, $\mathrm{Reg}(\mathcal{T})$ is open in the Nash manifold $M := \overline{\mathcal{T}}^{\mathrm{zar}}$. Thus $\nu^{-1}(\mathrm{Reg}(\mathcal{T}))$ is an open subset of $\mathbb{R}^n$. Consider now the set

$$\{F \in \mathcal{N}(\mathbb{R}^d,\mathbb{R}^n) :\ F(\Delta_{d-1}) \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))\},$$

which is an open subset of the open-compact topology of $\mathcal{N}(\mathbb{R}^d,\mathbb{R}^n)$. Thus, the set $\Theta_0 = \Gamma^{-1}(\{F \in \mathcal{N}(\mathbb{R}^d,\mathbb{R}^n) :\ F(\Delta_{n-1}) \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))\})$ is an open subset of $\mathbb{R}^r \times \mathcal{F}^r$, because the map Γ is continuous. □

3.6.7. Properties of Θ_0. For each $w \in \mathbb{R}^d$, define the linear map

$$\psi_w : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1,\dots,x_d) \mapsto (x_1 + \dots + x_d)w.$$

Observe that $\psi_w|_{\{\mathtt{x}_1+\dots+\mathtt{x}_d=1\}}$ is the constant map w and recall that the simplex $\Delta_{d-1} \subset \{\mathtt{x}_1 + \dots + \mathtt{x}_d = 1\}$. Let us analyse some properties of Θ_0:

(1) *If $\phi_i(w_i) \in \mathrm{Reg}(\mathcal{T})$, then $(\mathtt{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0) \in \Theta_0$.*

As $\Gamma(\mathtt{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0)(\Delta_{d-1}) = \{\phi_i(w_i)\} \subset \mathrm{Reg}(\mathcal{T}) \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))$, we have

$$(\mathtt{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0) \in \Theta_0.$$

(2) *Given $1 \leq i,j \leq r$, let $w_i \in \mathrm{Int}(\Lambda_{k_i})$ and $z_j \in \mathrm{Int}(\Lambda_{k_j})$ be such that $\phi_i(w_i) = \phi_j(z_j)$. Then*

$$(\mathtt{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0) \textit{ and } (\mathtt{e}_j; 0,\dots,0,\overset{(j)}{\psi_{w_j}},0,\dots,0)$$

belong to the same connected component of Θ_0.

Observe that

$$\phi_i(w_i) = (1-t)\phi_i(w_i) + t\phi_j(z_j) = \phi_j(z_j) \in \mathrm{Reg}(\mathcal{T}) \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))$$

for $t \in [0,1]$. Thus,

$$\begin{aligned}\Gamma(((1-t)\mathsf{e}_i + t\mathsf{e}_j; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0,\overset{(j)}{\psi_{z_j}},0,\dots,0))(\Delta_{d-1})&\\ = ((1-t)(\phi_i\circ\psi_{w_i}) + t(\phi_j\circ\psi_{z_j}))(\Delta_{d-1})&\\ = \{(1-t)\phi_i(w_i) + t\phi_j(z_j)\} \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T})),&\end{aligned}$$

so $((1-t)\mathsf{e}_i + t\mathsf{e}_j; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0,\overset{(j)}{\psi_{z_j}},0,\dots,0) \in \Theta_0$ for $t \in [0,1]$. Consequently, the connected set

$$\mathcal{C}_1 := \{((1-t)\mathsf{e}_i + t\mathsf{e}_j; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0,\overset{(j)}{\psi_{z_j}},0,\dots,0)\ :\ t\in[0,1]\},$$

is contained in one of the connected components of Θ_0. In addition, for $t \in [0,1]$

$$\Gamma(\mathsf{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0,\overset{(j)}{\psi_{tz_j}},0,\dots,0)(\Delta_{d-1}) = \{\phi_i(w_i)\} \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))$$
$$\Gamma(\mathsf{e}_j; 0,\dots,0,\overset{(i)}{\psi_{tw_i}},0,\dots,0,\overset{(j)}{\psi_{z_j}},0,\dots,0)(\Delta_{d-1}) = \{\phi_j(z_j)\} \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))$$

As $\psi_{tz_j} = t\psi_{z_j}$ and $\psi_{tw_i} = t\psi_{w_i}$ for $t\in[0,1]$, we deduce

$$(\mathsf{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0,\overset{(j)}{t\psi_{z_j}},0,\dots,0) \in \Theta_0 \quad \text{for } t\in[0,1],$$
$$(\mathsf{e}_j; 0,\dots,0,\overset{(i)}{t\psi_{w_i}},0,\dots,0,\overset{(j)}{\psi_{z_j}},0,\dots,0) \in \Theta_0 \quad \text{for } t\in[0,1].$$

Thus, the connected sets

$$\mathcal{C}_2 := \{(\mathsf{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0,\overset{(j)}{t\psi_{z_j}},0,\dots,0)\ :\ t\in[0,1]\},$$
$$\mathcal{C}_3 := \{(\mathsf{e}_j; 0,\dots,0,\overset{(i)}{t\psi_{w_i}},0,\dots,0,\overset{(j)}{\psi_{z_j}},0,\dots,0)\ :\ t\in[0,1]\}$$

are contained in a connected component of Θ_0. As

$$\mathcal{C}_1\cap\mathcal{C}_2 = \{(\mathsf{e}_i; 0,\dots,0,\overset{(i)}{\psi}_{w_i},0,\dots,0,\overset{(j)}{\psi}_{z_j},0,\dots,0)\},$$
$$\mathcal{C}_1\cap\mathcal{C}_3 = \{(\mathsf{e}_j; 0,\dots,0,\overset{(i)}{\psi}_{w_i},0,\dots,0,\overset{(j)}{\psi}_{z_j},0,\dots,0)\},$$

we deduce $\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$ is a connected subset of Θ_0 contained in one of its connected components.

We conclude that

$$(\mathsf{e}_i; 0,\dots,0,\overset{(i)}{\psi_{w_i}},0,\dots,0) \in \mathcal{C}_2 \text{ and } (\mathsf{e}_j; 0,\dots,0,\overset{(j)}{\psi_{w_j}},0,\dots,0) \in \mathcal{C}_3$$

belong to the same connected component of Θ_0.

(3) *If* $w_i \in \operatorname{Int}(\Lambda_{k_i})$ *and* $z_j \in \operatorname{Int}(\Lambda_{k_j})$*, then*

$$(\mathsf{e}_i; 0, \ldots, 0, \overset{(i)}{\psi_{w_i}}, 0, \ldots, 0) \quad \textit{and} \quad (\mathsf{e}_j; 0, \ldots, 0, \overset{(j)}{\psi_{z_j}}, 0, \ldots, 0)$$

belong to the same connected component of Θ_0.

As $\mathcal{T} = \bigcup_{i=1}^r \phi_i(\Lambda_{k_i})$ is connected and $\operatorname{Int}(\phi_i(\Lambda_{k_i}))$ is dense in $\phi_i(\Lambda_{k_i})$, given $1 \le i, j \le r$ there exists a chain $\{\phi_{i_\ell}(\Lambda_{k_{i_\ell}})\}_{\ell=1}^s$ such that $i = i_1$, $j = i_s$ and $\phi_{i_\ell}(\operatorname{Int}(\Lambda_{k_{i_\ell}})) \cap \phi_{i_{\ell+1}}(\operatorname{Int}(\Lambda_{k_{i_{\ell+1}}})) \neq \varnothing$ for each ℓ. Observe that

$$\bigcup_{i=1}^r \phi_i(\operatorname{Int}(\Lambda_{k_i})) \subset \operatorname{Reg}(\mathcal{T}),$$

see Section 3.6.3. So let us consider the case $\phi_i(\operatorname{Int}(\Lambda_{k_i})) \cap \phi_j(\operatorname{Int}(\Lambda_{k_j})) \neq \varnothing$. By (2) it is enough to consider the case $i = j$. Observe that

$$t\psi_{w_i} + (1-t)\psi_{z_i} = \psi_{tw_i + (1-t)z_i}$$

for each $t \in [0,1]$. As $w_i, z_i \in \operatorname{Int}(\Lambda_{k_i})$ and the latter is convex, we have that $tw_i + (1-t)z_i \in \operatorname{Int}(\Lambda_{k_i})$ for each $t \in [0,1]$, so

$$\begin{aligned}\Gamma((\mathsf{e}_i; 0, \ldots, 0, t\psi_{w_i} + \overset{(i)}{(1-t)}\psi_{z_i}, 0, \ldots, 0))& \\ = \phi_i(tw_i + (1-t)z_i) \in \operatorname{Reg}(\mathcal{T}) \subset \nu^{-1}(\operatorname{Reg}(\mathcal{T})).&\end{aligned}$$

Thus,

$$\{(\mathsf{e}_i; 0, \ldots, 0, t\psi_{w_i} + \overset{(i)}{(1-t)}\psi_{z_i}, 0, \ldots, 0) : \ t \in [0,1]\} \subset \Theta_0$$

is connected, so

$$(\mathsf{e}_i; 0, \ldots, 0, \overset{(i)}{\psi_{w_i}}, 0, \ldots, 0) \quad \text{and} \quad (\mathsf{e}_i; 0, \ldots, 0, \overset{(i)}{\psi_{z_i}}, 0, \ldots, 0)$$

belong to the same connected component of Θ_0.

(4) *There exists a connected component* Θ *of* Θ_0 *that contains the connected set* $\{e_i\} \times \mathcal{F}^{i-1} \times \{\psi_{w_i} : \ w_i \in \operatorname{Int}(\Lambda_{k_i})\} \times \mathcal{F}^{r-i}$ *for each* $i = 1, \ldots, r$.

Observe that

$$\begin{aligned}\Gamma((\mathsf{e}_i; \psi_1, \ldots, \psi_{i-1}, \psi_{w_i}, \psi_{i+1}, \ldots, \psi_r))(\Delta_{d-1})& \\ = \phi_i(\psi_{w_i})(\Delta_{d-1}) = \phi_i(w_i) \in \operatorname{Reg}(\mathcal{T}) \subset \nu^{-1}(\operatorname{Reg}(\mathcal{T})),&\end{aligned}$$

so $\{e_i\} \times \mathcal{F}^{i-1} \times \{\psi_{w_i} : \ w_i \in \operatorname{Int}(\Lambda_{k_i})\} \times \mathcal{F}^{r-i} \subset \Theta_0$ and it is a connected set, because it is a finite product of connected sets.

(5) Let Θ be the connected component of Θ_0 introduced in (4). *If* $\psi_i \in \mathcal{F}$ *satisfies* $\psi_i(\Delta_{d-1}) \subset \Lambda_{k_i}$*, then* $(\mathsf{e}_i; 0, \ldots, 0, \overset{(i)}{\psi_i}, 0, \ldots, 0) \in \operatorname{Cl}(\Theta)$*. If in addition* $\psi_i(\Delta_{d-1}) \subset \operatorname{Int}(\Lambda_{k_i})$*, then* $(\mathsf{e}_i; 0, \ldots, 0, \overset{(i)}{\psi_i}, 0, \ldots, 0) \in \Theta$.

Let $w_i \in \operatorname{Int}(\Lambda_{k_i})$. Recall that by (4)

$$(\mathsf{e}_i; 0, \ldots, 0, \overset{(i)}{\psi_{tw_i}}, 0, \ldots, 0) \in \Theta$$

for each $t \in (0,1]$, because Λ_{k_i} is an open cone, so $tw_i \in \operatorname{Int}(\Lambda_{k_i})$ for each $t \in (0,1]$. Consider the Nash path

$$\alpha : (0,1] \to \mathbb{R}^r \times \mathcal{F}^r,\ t \mapsto (e_i; 0, \dots, 0, \overset{(i)}{(1-t)\psi_i + t\psi_{w_i}}, 0, \dots, 0).$$

We claim: $\Gamma(\alpha(t))(\Delta_{d-1}) \subset \operatorname{Reg}(\mathcal{T})$ *for* $t \in (0,1]$.

This is so because $tw_i \in \operatorname{Int}(\Lambda_{k_i})$ and $(1-t)\psi_i(\Delta_{d-1}) \subset \Lambda_{k_i}$ for $t \in (0,1]$, so $((1-t)\psi_i + t\psi_{w_i})(\Delta_{d-1}) \subset \operatorname{Int}(\Lambda_{k_i})$ for $t \in (0,1]$. Thus, $\alpha(t) \in \Theta_0$ for each $t \in (0,1]$ (see (4)). As $\alpha(1) = \psi_{w_i} \in \Theta$, we conclude $\psi_i \in \operatorname{Cl}(\Theta)$.

If in addition $\psi_i(\Delta_{d-1}) \subset \operatorname{Int}(\Lambda_{k_i})$, then

$$(\mathtt{e}_i; 0, \dots, 0, \overset{(i)}{\psi_i}, 0, \dots, 0) \in \Theta_0,$$

so $(\mathtt{e}_i; 0, \dots, 0, \overset{(i)}{\psi_i}, 0, \dots, 0) \in \Theta_0 \cap \operatorname{Cl}(\Theta) = \Theta$.

(6)*If* $(\mu;\psi) \in \operatorname{Cl}(\Theta_0)$, *then* $\Gamma(\mu;\psi)(\Delta_{d-1}) \subset \operatorname{Cl}(\nu^{-1}(\mathcal{T}))$.

By the curve selection lemma (see [BCR, Thm.2.5.5]) there exists a continuous semi-algebraic path $\alpha : [0,1] \to \operatorname{Cl}(\Theta_0)$ such that $\alpha(0) = (\mu;\psi)$ and $\alpha((0,1]) \subset \Theta_0$. This means that for each $t \in (0,1]$ one has

$$\Gamma(\alpha(t))(\Delta_{d-1}) \subset \nu^{-1}(\operatorname{Reg}(\mathcal{T})).$$

If $x \in \Delta_{d-1}$, then $\Gamma(\alpha(t))(x) : [0,1] \to \mathbb{R}^n$ is a continuous semi-algebraic path such that $\Gamma(\alpha(t))(x) \subset \nu^{-1}(\operatorname{Reg}(\mathcal{T}))$ for each $t \in (0,1]$, so

$$\Gamma(\alpha(0))(x) \in \operatorname{Cl}(\nu^{-1}(\operatorname{Reg}(\mathcal{T}))) \subset \operatorname{Cl}(\nu^{-1}(\mathcal{T})).$$

Thus, $\Gamma(\mu;\psi)(\Delta_{d-1}) \subset \operatorname{Cl}(\nu^{-1}(\mathcal{T}))$.

(7) Recall that $\operatorname{Cl}(\nu^{-1}(\mathcal{T}))$ is a compact set and ν admits a Nash extension to $\operatorname{Cl}(\nu^{-1}(\mathcal{T}))$ (see Section 3.6.3). *As* $\nu|_{\operatorname{Cl}(\nu^{-1}(\mathcal{T}))}$ *is proper,*

$$\nu(\Gamma(\mu;\psi)(\Delta_{d-1})) \subset \nu(\operatorname{Cl}(\nu^{-1}(\mathcal{T}))) = \operatorname{Cl}(\mathcal{T}) = \mathcal{T}.$$

(8) Let $\mathtt{j} : \mathcal{N}(\mathbb{R}^d, \mathbb{R}^n) \to \mathcal{N}(\Delta_{d-1}, \mathbb{R}^n),\ f \mapsto f|_{\Delta_{n-1}}$, which is continuous if we endow both spaces with the compact-open topology (see Facts 2.3.7 and 2.3.10). *Then the composition*

$$\nu_* \circ \mathtt{j} \circ \Gamma : \operatorname{Cl}(\Theta) \to \mathcal{N}(\Delta_{d-1}, \mathcal{T}),\ (\mu;\psi) \mapsto \nu \circ (\Gamma((\mu;\psi))|_{\Delta_{d-1}})$$

is continuous.

(9) *If* $\beta : [0,1] \to \operatorname{Cl}(\Theta_0)$ *is a Nash path, then*

$$B : [0,1] \times \Delta_{d-1} \to \mathcal{T},\ (t,x) \mapsto \nu \circ (\Gamma(\beta(t))(x)$$

is a Nash map.

3.6.8. Nash images of the closed ball. We are finally ready to prove Theorem 3.2. As seen in Section 3.5.5 it is sufficient to prove Theorem 3.5.8. Let us prove: *Given a compact checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ of dimension $d \geq 2$, there exists a Nash map $F : \Delta_{d-1} \times [0,1] \to \mathbb{R}^n$ such that $F(\Delta_{d-1} \times [0,1]) = \mathcal{T}$.*

Proof of Theorem 3.5.8. We keep all the notations introduced in Sections 3.6.3, 3.6.5 and 3.6.6. We also keep all the assumptions done along these subsections. Recall that $\mathcal{T} = \bigcup_{i=1}^r \phi_i(\tau_i) \subset \bigcup_{i=1}^r \phi_i(\mathcal{B}_d(0,1))$, where τ_i is a d-dimensional simplex such that $\tau_i \cap \phi_i^{-1}(\partial\mathcal{T})$ is either empty or a proper face of τ_i contained in a $(d-1)$-dimensional face σ_i of τ_i. If $\tau_i \cap \phi_i^{-1}(\partial\mathcal{T}) = \varnothing$ we denote with σ_i the facet of τ_i that does not contain the origin of $\mathbb{R}^d$ (this situation corresponds to the case $k = 0$ in Section 3.6.5 and the origin is the point p_0 introduced there). In both cases the remaining vertex p_i of τ_i belongs to $\phi_i^{-1}(\operatorname{Reg}(\mathcal{T}))$ and τ_i is the convex hull of $\sigma_i \cup \{p_i\}$, see Section 3.6.5.

Denote v_{ij} for $j = 1, \ldots, d$ the vertices of σ_i. Let H_{ik} be the hyperplanes generated by the facets of τ_i that contains the vertex p_i and assume $v_{ij} \notin H_{ij}$ and $\tau_i \subset \bigcap_{j=1}^d H_{ij}^+$.

Let $\alpha_{ij} : [-\delta, 1+\delta] \to \mathbb{R}^d$ be Nash paths satisfying the conditions of Lemma 3.6.4 (see also Remark 3.6.5(ii)). Consider the Nash path

$$A_i : [-\delta, 1+\delta] \to \{\mathtt{e}_i\} \times \mathcal{F}^r,\ t \mapsto \Big(\mathtt{e}_i; 0, \ldots, 0, \sum_{j=1}^d \alpha_{ij}(t)\mathtt{x}_j, 0, \ldots, 0\Big)$$

and observe that

$$\Gamma(A_i(t))(\Delta_{d-1}) \subset \begin{cases} \phi_i(\tau_i) \subset \phi_i(\Lambda_{k_i}) \subset \mathcal{T} & \text{if } t = 0, \\ \phi_i(\operatorname{Int}(\tau_i)) \subset \phi_i(\operatorname{Int}(\Lambda_{k_i})) \subset \operatorname{Reg}(\mathcal{T}) & \text{if } t \in [-\delta, 1+\delta] \setminus \{0\}. \end{cases}$$

Thus, $A_i(t) \in \Theta$ if $t \in [-\delta, 1+\delta] \setminus \{0\}$ and $\zeta_i := A_i(0) \in \operatorname{Cl}(\Theta)$, see property 3.6.7(5). Define the linear maps $\eta_i := \sum_{j=1}^d \alpha_{ij}(-\delta)\mathtt{x}_j \in \mathcal{F}$ and $\xi_i := \sum_{j=1}^d \alpha_{ij}(1+\delta)\mathtt{x}_j \in \mathcal{F}$. As $\eta_i(\Delta_{d-1}), \xi_i(\Delta_{d-1}) \subset \operatorname{Int}(\tau_i) \subset \operatorname{Int}(\Lambda_{k_i})$, we deduce by property 3.6.7(5)

$$(\mathtt{e}_i; 0, \ldots, 0, \overset{(i)}{\eta_i}, 0, \ldots, 0),\ (\mathtt{e}_i; 0, \ldots, 0, \overset{(i)}{\xi_i}, 0, \ldots, 0) \in \Theta.$$

Up to repeating the charts ϕ_i as many times as needed, we may assume

$$\phi_i(\Lambda_{k_i}) \cap \phi_{i+1}(\Lambda_{k_{i+1}}) \neq \varnothing$$

and let $\phi_i(w_i) = \phi_{i+1}(z_{i+1}) \in \phi_i(\Lambda_{k_i}) \cap \phi_{i+1}(\Lambda_{k_{i+1}})$. Consider the Nash paths

$$\begin{aligned} B_i &: [0,1] \to \Theta_0,\ t \mapsto (\mathtt{e}_i; 0, \ldots, 0, (1-t)\xi_i + t\psi_{w_i}, t\psi_{z_{i+1}}, 0, \ldots, 0), \\ C_i &: [0,1] \to \Theta_0,\ t \mapsto ((1-t)\mathtt{e}_i + t\mathtt{e}_{i+1}; 0, \ldots, 0, \psi_{w_i}, \psi_{z_{i+1}}, 0, \ldots, 0), \\ D_i &: [0,1] \to \Theta_0,\ t \mapsto (\mathtt{e}_{i+1}; 0, \ldots, 0, (1-t)\psi_{w_i}, (1-t)\psi_{z_{i+1}} + t\eta_{i+1}, 0, \ldots, 0). \end{aligned}$$

We have $B_i([0,1]) \subset \Theta_0$ because $B_i(0) = (\mathtt{e}_i; 0, \ldots, 0, \overset{(i)}{\xi_i}, 0, \ldots, 0) \in \Theta_0$ and $B_i((0,1]) \subset \Theta_0$ by property 3.6.7(4). In addition, $C_i([0,1]) \subset \Theta_0$ by the proof of property 3.6.7(2). Moreover, $D_i([0,1]) \subset \Theta_0$ because

$$D_i(1) = (\mathtt{e}_{i+1}; 0, \ldots, 0, \overset{(i+1)}{\eta_{i+1}}, 0, \ldots, 0) \in \Theta_0$$

and $D_i([0,1)) \subset \Theta_0$ by property 3.6.7(4). Observe that

$$B_i(0) = (\mathtt{e}_i; 0, \dots, 0, \overset{(i)}{\xi_i}, 0, \dots, 0) \in \Theta$$

and $B_i([0,1])$ is connected so $B_i([0,1]) \subset \Theta$. Analogously, $C_i(0) = B_i(1) \in \Theta$ and $C_i([0,1])$ is connected so $C_i([0,1]) \subset \Theta$. In addition, $D_i(0) = C_i(1) \in \Theta$ and $D_i([0,1])$ is connected so $D_i([0,1]) \subset \Theta$.

Fix times $0 < t_1 < s_1 < \dots < t_r < s_r < 1$ and denote

$$\chi_i := A_i(1) = (\mathtt{e}_i, 0, \dots, 0, (\mathtt{x}_1 + \overset{(i)}{\cdots} + \mathtt{x}_d)p_i, 0, \dots, 0) \in \Theta.$$

Observe that $(\mathtt{x}_1 + \ldots + \mathtt{x}_d)p_i$ is the linear map $\mathbb{R}^d \to \mathbb{R}^d$ that takes the constant value $p_i \in \operatorname{Int}(\Lambda_{k_i})$ on the hyperplane $\{\mathtt{x}_1 + \cdots + \mathtt{x}_d = 1\}$. Consider the continuous semi-algebraic path obtained concatenating the previous paths:

$$E := \mathop{\ast}_{i=1}^{r} (A_i * B_i * C_i * D_i) : [0,1] \to \Theta \cup \{\zeta_1, \dots, \zeta_r\}$$

and assume, after reparametrizing the paths if necessary, $E(t_i) = \zeta_i$, $E_j(s_i) = \chi_i$ and $E|_{[t_i,s_i]}$ is an affine reparametrization of $A_i|_{[0,1]}$. Let $\rho > 0$ be such that E is Nash on

$$I := \bigcup_{i=1}^{r} \left([t_i - \rho, t_i + \rho] \cup [s_i - \rho, s_i + \rho]\right).$$

By Lemma 3.3.5 we can approximate the continuous semi-algebraic path E by a polynomial path $\gamma : [0,1] \to \Theta \cup \{\zeta_1, \dots, \zeta_r\}$, such that:

(i) $\gamma(t_i) = E(t_i) = \zeta_i$, $\gamma'(t_i) = E'(t_i)$, $\gamma''(t_i) = E''(t_i)$ and $\gamma'''(t_i) = E'''(t_i)$ for each $i = 1, \dots, r$.

(ii) $\gamma(s_i) = E(s_i) = \chi_i$, $\gamma'(s_i) = E'(s_i)$, $\gamma''(s_i) = E''(s_i)$ and $\gamma'''(s_i) = E'''(s_i)$ for each $i = 1, \dots, r$.

(iii) $\|\gamma - E\|$, $\|\gamma' - E'\|_I$, $\|\gamma'' - E''\|_I$ and $\|\gamma''' - E'''\|_I$ are small enough.

Write $\gamma := (\mu; \psi_1, \dots, \psi_r)$ and $\mu := (\mu_1, \dots, \mu_r)$. As

$$A_i(t) = \left(\mathtt{e}_i; 0, \dots, 0, \sum_{j=1}^{d} \alpha_{ij}(t)\mathtt{x}_j, 0, \dots, 0\right),$$

we deduce by (i) and (ii) above that

$$\mu_i(s_j) = \mu_i(t_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j, \end{cases}$$

$\mu_i'(t_i) = \mu_i'(s_i) = 0$, $\mu_i''(t_i) = \mu_i''(s_i) = 0$ and $\mu_i'''(t_i) = \mu_i'''(s_i) = 0$. By Lemma 3.6.4 $\tau_i \subset \psi_i([t_i, s_i] \times \Delta_{d-1}) \subset \Lambda_{k_i}$. Consider

$$\Gamma(\gamma) = \sum_{i=1}^{r} \mu_i(t)(\phi_i \circ \psi_i)(t, \mathtt{x}) : [0,1] \times \mathbb{R}^d \to \mathbb{R}^n.$$

We have $\Gamma(\gamma)(\{t\} \times \Delta_{d-1}) \subset \nu^{-1}(\mathrm{Reg}(\mathcal{T}))$ if $t \in [0,1] \setminus \{t_1, \dots, t_r\}$, whereas $\Gamma(\gamma)(\{t_i\} \times \Delta_{d-1}) = \phi(\sigma_i) \in \mathcal{T}$ for $i = 1, \dots, r$. This means that

$$\nu(\Gamma(\gamma)([0,1] \times \Delta_{d-1})) \subset \mathcal{T}. \tag{3.6.15}$$

Fix $i = 1, \dots, r$ and denote

$$\lambda_{ij} := \begin{cases} \mu_i - 1 & \text{if } j = i, \\ \mu_j & \text{if } i \neq j. \end{cases}$$

Observe that $\lambda_{ij}^{(\ell)}(t_i) = \lambda_{ij}^{(\ell)}(s_i) = 0$ for $1 \leq i, j \leq r$, $\ell = 0, 1, 2, 3$ and each λ_{ij} is close to zero. By (3.6.14) there exist $L, K > 0$ such that

$$\Big\| \phi_i^{-1} \Big(\nu \Big((\phi_i \circ \psi_i)(t,x) + \sum_{j=1}^{r} \lambda_{ij}(t) (\phi_j \circ \psi_j)(t,x) \Big) \Big) - \psi_i(t,x) \Big\| \\ \leq 2LK \sum_{j=1}^{r} |\lambda_{ij}(t)|.$$

By Lemmas 3.6.4 and 3.6.6, we deduce $\tau_i \subset \phi_i^{-1}(\nu(\Gamma(\gamma)([t_i, s_i] \times \Delta_{d-1}))) \subset \Lambda_{k_i}$ because $\tau_i \subset \psi_i([t_i, s_i] \times \Delta_{d-1}) \subset \Lambda_{k_i}$. Thus,

$$\phi_i(\tau_i) \subset \nu(\Gamma(\gamma)([t_i, s_i] \times \Delta_{d-1})) \subset \phi_i(\Lambda_{k_i}) \subset \mathcal{T}$$

for $i = 1, \dots, r$, so by (3.6.15)

$$\mathcal{T} = \bigcup_{i=1}^{r} \phi_i(\tau_i) \subset \nu(\Gamma(\gamma)(I \times \Delta_{d-1})) \subset \nu(\Gamma(\gamma)([0,1] \times \Delta_{d-1})) \subset \mathcal{T}.$$

Consequently, $\nu(\Gamma(\gamma)([0,1] \times \Delta_{d-1})) = \mathcal{T}$, as required. □

3.7 General Nash images.

Once we have completely characterised the Nash images of the closed ball, a natural question arises: To determine all possible compact models that allow us to represent a compact semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ of dimension d connected by analytic paths as a Nash image. This question is not trivial, and different classes of semi-algebraic functions might have different answers. For instance, we have seen in Remark 3.1.10 that the family of polynomial images of the closed ball and the one of sphere are different.

In the Nash case we are able to give a complete characterization of the compact models. Combining Theorem 3.2 and the next result we will show that: *If* $\mathcal{S} \subset \mathbb{R}^n$ *and* $\mathcal{T} \subset \mathbb{R}^m$ *are two semi-algebraic sets such that* $\mathcal{S} \subset \mathbb{R}^n$ *is compact, connected by analytic paths and* $\dim(\mathcal{S}) \leq \dim(\mathcal{T})$*, there exists a Nash map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *such that* $f(\mathcal{T}) = \mathcal{S}$. This means that we can use any semi-algebraic set $\mathcal{T}$ of dimension d to represent d-dimensional compact semi-algebraic sets connected by analytic paths as Nash images of $\mathcal{T}$. For instance, a 'semi-algebraic Teddy bear' can be mapped onto a 'semi-algebraic sheep' by means of a Nash map and vice versa (see Figure 3.9). Denote $\overline{\mathcal{B}}_m(p, \varepsilon)$ the closed ball of $\mathbb{R}^m$ of centre p and radius $\varepsilon > 0$.

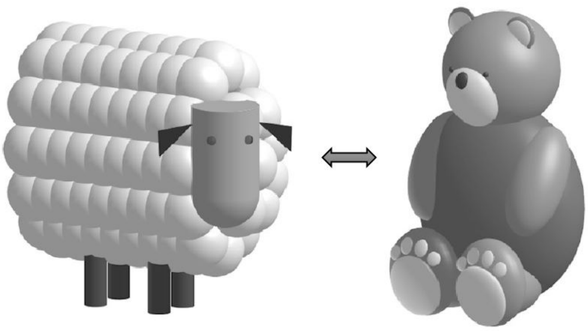

Figure 3.9: A sheep and a Teddy bear (figure borrowed from [FU6, Fig.1.3]).

Theorem 3.7.1 (Bärchen-Schäfchen's Theorem)**.** *Let $\mathcal{T} \subset \mathbb{R}^m$ be any semi-algebraic set of dimension d. Then, there exists a regular map $f : \mathbb{R}^m \to \mathbb{R}^d$ such that $f(\mathcal{T}) = \overline{\mathcal{B}}_d$.*

Proof. Let $p \in \mathcal{T}$ be a regular point of $\mathcal{T}$ such that $\dim(\mathcal{T}_p) = d$, let $p + T_p\mathcal{T}$ be the affine tangent space to $\operatorname{Reg}(\mathcal{T})$ at p and let $\pi : \mathbb{R}^m \to p + T_p\mathcal{T}$ be the orthogonal projection of $\mathbb{R}^m$ onto $p + T_p\mathcal{T}$. There exist $\varepsilon > 0$ and a compact neighbourhood $W^p \subset \operatorname{Reg}(\mathcal{T})$ of p such that $\pi|_{W^p} : W^p \to \overline{\mathcal{B}}_m(p, \varepsilon) \cap (p + T_p\mathcal{T})$ is a Nash diffeomorphism. For simplicity we assume that p is the origin and $\varepsilon = 1$, so that $\overline{\mathcal{B}}_m(p, \varepsilon) \cap (p + T_p\mathcal{T})$ is isometric to the unit closed ball $\overline{\mathcal{B}}_d$. Consider the inverse of the stereographic projection

$$\begin{aligned} \varphi : \mathbb{R}^d &\to \mathbb{S}^d \setminus \{(0, \dots, 1)\}, \\ x := (x_1, \dots, x_d) &\mapsto \left(\frac{2x_1}{1 + \|x\|^2}, \dots, \frac{2x_d}{1 + \|x\|^2}, \frac{-1 + \|x\|^2}{1 + \|x\|^2}\right). \end{aligned}$$

Let $\pi' : \mathbb{R}^{d+1} \to \mathbb{R}^d$, $(x_1, \dots, x_{d+1}) \mapsto (x_1, \dots, x_d)$ be the projection onto the first d coordinates and observe that $\pi' \circ \varphi : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$(\pi' \circ \varphi)(\mathbb{R}^d) = (\pi' \circ \varphi)(\overline{\mathcal{B}}_d) = \overline{\mathcal{B}}_d.$$

After taking suitable coordinates and considering the previous regular map, there exists a surjective regular map $g : p + T_p\mathcal{T} \to \overline{\mathcal{B}}_d$ such that

$$g(\overline{\mathcal{B}}_m(p, \varepsilon) \cap (p + T_p\mathcal{T})) = g(p + T_p\mathcal{T}) = \overline{\mathcal{B}}_d \subset \mathbb{R}^d.$$

In particular, $g(A) = \overline{\mathcal{B}}_d$ for each A such that $\overline{\mathcal{B}}_m(p, \varepsilon) \cap (p + T_p\mathcal{T}) \subset A \subset p + T_p\mathcal{T}$. Thus, the composition $g \circ \pi : \mathbb{R}^m \to \mathbb{R}^d$ is a regular map satisfying

$$(g \circ \pi)(\mathcal{T}) = g(\pi(\mathcal{T})) = g(\pi(W^p)) = g(\overline{\mathcal{B}}_m(p, \varepsilon) \cap (p + T_p\mathcal{T})) = \overline{\mathcal{B}}_d,$$

as required. □

It is natural now to wonder if the previous result extends to pairs of general semi-algebraic sets non necessarily compact. If $\mathcal{S} \subset \mathbb{R}^n$ is non-compact and $\mathcal{T} \subset \mathbb{R}^m$ is compact, there exists no Nash map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $f(\mathcal{T}) = \mathcal{S}$.

Let $\mathcal{T}_d$ be the set of points of $\mathcal{T}$ of dimension d, which is a semi-algebraic set (see [Fe2, §3.1]). *If $\mathcal{T}$ has dimension $d \geq 2$, the semi-algebraic set* $\operatorname{Cl}(\mathcal{T}_d) \cap \mathcal{T}$ *is* *not compact and* $\mathcal{S} \subset \mathbb{R}^n$ *is connected by analytic paths with* $\dim(\mathcal{S}) \leq d$*, then there exists a Nash map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *such that* $f(\mathcal{T}) = \mathcal{S}$. By Theorem 3.1 it is enough to consider the case $\mathcal{S} = \mathbb{R}^d$ for $d \geq 2$.

Theorem 3.7.2. *Let* $\mathcal{T} \subset \mathbb{R}^m$ *be a semi-algebraic set and let* $d \geq 2$*. Assume that* $\operatorname{Cl}(\mathcal{T}_d) \cap \mathcal{T}$ *is not compact. Then, there exists a Nash map* $f : \mathbb{R}^m \to \mathbb{R}^d$ *such that* $f(\mathcal{T}) = \mathbb{R}^d$*.*

Proof. If $m = d$ and $\mathcal{T} = \mathbb{R}^d$ there is nothing to prove. Thus, let us assume $\mathbb{R}^m \setminus \mathcal{T} \neq \varnothing$. We may assume that $\mathcal{T}_d$ is unbounded. Otherwise, $\mathcal{T}_d$ is bounded and not closed (because $\operatorname{Cl}(\mathcal{T}_d) \cap \mathcal{T}$ is not compact), so there exists $p \in \operatorname{Cl}(\mathcal{T}_d) \setminus \mathcal{T}$. Consider the Nash map

$$h : \mathbb{R}^m \setminus \{p\} \to \mathbb{R}^{m+1},\ x \mapsto \left(x, \frac{1}{\|x - p\|}\right),$$

which is a Nash diffeomorphism onto its image. Observe that $h(\mathcal{T}_d) \subset h(\mathcal{T}) \subset \mathbb{R}^{m+1}$ is unbounded. We identify $h(\mathcal{T})$ with $\mathcal{T}$ and $h(\mathcal{T}_d)$ with $\mathcal{T}_d$.

Consider the immersion $\psi_1 : \mathbb{R}^m \to \mathbb{RP}^m$, $x \mapsto [1 : x]$. As $\mathcal{T}_d$ is unbounded, we may assume

$$[0 : \ldots : 0 : \overset{(d+1)}{1} : 0 : \ldots : 0] \in \operatorname{Cl}_{\mathbb{RP}^m}(\mathcal{T}_d).$$

Consider the projection

$$\widehat{\pi} : \mathbb{RP}^m \to \mathbb{RP}^d,\ [x_0 : x_1 : \ldots : x_m] \mapsto [x_0 : x_1 : \ldots : x_d]$$

whose restriction to $\mathbb{R}^m$ is the projection $\pi : \mathbb{R}^m \to \mathbb{R}^d$, $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_d)$. As

$$\widehat{\pi}([0 : \ldots : 0 : 1 : 0 : \ldots : 0]) = [0 : \ldots : 0 : 1] \in \operatorname{Cl}_{\mathbb{RP}^d}(\pi(\mathcal{T}_d)),$$

we deduce $\pi(\mathcal{T})$ is not bounded. Thus, taking $\pi(\mathcal{T})$ instead of $\mathcal{T}$ we may assume $m = d$, $\dim(\mathcal{T}) = d$ and $\mathcal{T}_d$ unbounded.

If $\mathcal{T} = \mathbb{R}^d$ we are done, otherwise, we may assume after a translation that $0 \notin \mathcal{T}$. Consider the inversion $\mathtt{i} : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \setminus \{0\}$, $x \mapsto \frac{x}{\|x\|}$, which is a Nash involution of $\mathbb{R}^d \setminus \{0\}$. Thus, at this point $\mathcal{T} \subset \mathbb{R}^d$ is a semi-algebraic set of dimension d such that $0 \in \operatorname{Cl}(\mathcal{T}_d) \setminus \mathcal{T}$. Observe that $\operatorname{Reg}(\mathcal{T}_d)$ is an open subset of $\mathbb{R}^d$ adherent to the origin.

By the Nash curve selection lemma (see [BCR, Prop.8.1.13]) there exists a Nash arc

$$\alpha := (\alpha_1, \ldots, \alpha_d) : [0, 1] \to \operatorname{Reg}(\mathcal{T}_d) \cup \{0\}$$

such that $\alpha((0, 1]) \subset \operatorname{Reg}(\mathcal{T}_d)$ and $\alpha(0) = 0$. After a linear change of coordinates we may assume that $\alpha((0, 1]) \cap \{\mathtt{x}_1 = 0\} = \varnothing$ (here we are using that $d \geq 2$). Consider now the Nash map

$$g : \mathbb{R}^d \setminus \{0\} \to \{\mathtt{x}_1 > 0\},\ (x_1, \ldots, x_d) \mapsto (\|x\|, x_2, \ldots, x_d).$$

As $g|_{\mathbb{R}^d \setminus \{\mathtt{x}_1 = 0\}}$ is a local diffeomorphism and in particular is open, $g(\operatorname{Reg}(\mathcal{T}_d))$ contains an open semi-algebraic set $U \subset \{\mathtt{x}_1 > 0\}$ adherent to the origin such that $g(\alpha((0,1])) \subset U$. After substituting α by $g\circ\alpha$ and reparameterizing we may assume $\alpha_1 = \mathtt{t}^p$, each α_i is an algebraic series in the variable $\mathtt{t}$ and the order of α_1 is smaller than or equal to the order of α_i for $i = 2,\dots,d$. The previous conditions hold because because the α_i are algebraic Puiseux series at the origin and the first component of $g\circ\alpha$ is $\sqrt{\alpha_1^2+\dots+\alpha_d^2}$. By Lemma 3.3.5 we may assume that $\alpha_i \in \mathbb{R}[\mathtt{t}]$ for $i = 2,\dots,d$. We substitute $\mathcal{T} \subset \mathbb{R}^d \setminus \{0\}$ by $g(\mathcal{T}) \subset \{\mathtt{x}_1 > 0\}$, which is a semi-algebraic subset of $\mathbb{R}^d$ of dimension d such that $0 \in \mathrm{Cl}(g(\mathcal{T})_d) \setminus g(\mathcal{T})$.

Consider the Nash diffeomorphism

$$\begin{aligned} h &: \{\mathtt{x}_1 > 0\} \to \{\mathtt{x}_1 > 0\},\\ (x_1,x_2,\dots,x_d) &\mapsto (\sqrt[p]{x_1}, x_2 - \alpha_2(\sqrt[p]{x_1}),\dots,x_d - \alpha_d(\sqrt[p]{x_1})) \end{aligned}$$

and observe that $h\circ\alpha = (t,0,\dots,0)$.

Let us fix an $\varepsilon > 0$ such that $(0,\varepsilon] \times \{(0,\dots,0)\} \subset h(U)$. Observe that $h(\mathcal{T}) \subset \{\mathtt{x}_1 > 0\}$ because $\mathcal{T} \subset \{\mathtt{x}_1 > 0\}$. Let

$$\delta : (0,\varepsilon] \to (0,+\infty),\ t \mapsto \mathrm{dist}((t,0,\dots,0), \mathbb{R}^d \setminus h(U))$$

and let $\xi : (0,\varepsilon] \to (0,+\infty)$ be a Nash function such that $|\frac{\delta}{2} - \xi| < \frac{\delta}{4}$, so $\frac{\delta}{4} < \xi < \frac{3\delta}{4}$. Write $x' := (x_2,\dots,x_d)$ and consider the open semi-algebraic set

$$\mathcal{U} := \{(x_1,x') \in (0,\varepsilon) \times \mathbb{R}^{d-1} :\ \|x'\|^2 < \xi^2(x_1)\} \subset h(\mathcal{T}).$$

Observe that ξ is a Puiseux series at the origin. The map

$$f_\ell : \{\mathtt{x}_1 > 0\} \to \{\mathtt{x}_1 > 0\},\ (x_1,x') \mapsto \Big(\frac{1}{x_1}, \frac{x'}{x_1^\ell}\Big)$$

is a Nash involution for each $\ell \geq 1$. Fix two positive numbers $N_1, N_2 > 0$ and consider the semi-algebraic set $\mathcal{F} := \{N_1 + N_2\|\mathtt{x}'\|^2 \leq \mathtt{x}_1\}$. Observe that $(y_1,y') \in f_\ell(\mathcal{F})$ if and only if $f_\ell(y_1,y') \in \mathcal{F}$, so

$$\begin{aligned} f_\ell(\mathcal{F}) &= \Big\{\|x'\|^2 \leq \frac{1}{N_2}\mathtt{x}_1^{2\ell-1}(1 - N_1\mathtt{x}_1)\Big\}\\ &\subset \{0\} \cup \Big\{(x_1,x') \in \Big(0,\frac{1}{N_1}\Big) \times \mathbb{R}^{d-1}\ :\ \|\mathtt{x}'\|^2 \leq \frac{1}{N_2}\mathtt{x}_1^{2\ell-1}\Big\}. \end{aligned}$$

If N_1, N_2, ℓ are large enough, $f_\ell(\mathcal{F}) \subset \{0\} \cup \mathcal{U}$, so $\mathcal{F} \subset (f_\ell \circ h)(\mathcal{T})$ (recall that f_ℓ is an involution). Consider the polynomial map

$$P_1 : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1,x') \mapsto (x_1 - (N_1 + N_2\|x'\|^2), x')$$

that maps $\mathcal{F}$ onto $\{\mathtt{x}_1 \geq 0\}$. Let $P_2 : \mathbb{R}^d \to \{\mathtt{x}_1 \geq 0\},\ (x_1,x') \mapsto (x_1^2, x')$. Observe that

$$\begin{aligned} \{\mathtt{x}_1 \geq 0\} = P_2(\{\mathtt{x}_1 \geq 0\}) &= (P_2 \circ P_1)(\mathcal{F})\\ &\subset (P_2 \circ P_1 \circ f_\ell \circ h)(\mathcal{T}) \subset P_2(\mathbb{R}^d) = \{\mathtt{x}_1 \geq 0\}, \end{aligned}$$

so $(P_2 \circ P_1 \circ f_\ell \circ h)(\mathcal{T}) = \{\mathtt{x}_1 \geq 0\}$. Denote $x'' := (x_3,\dots,x_d)$ and consider the polynomial map

$$P_3 : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1,x_2,x'') \mapsto (x_1^2 - x_2^2, 2x_1x_2, x'')$$

that maps $\{\mathtt{x}_1 \geq 0\}$ to $\mathbb{R}^d$ (again we have used here that $d \geq 2$). Consequently, $(P_3 \circ P_2 \circ P_1 \circ f_\ell \circ h)(\mathcal{T}) = \mathbb{R}^d$, as required. $\square$

The following example shows that Theorem 3.7.2 is no longer true if $d = 1$.

Example 3.7.3. Let $f : [0, +\infty) \to \mathbb{R}$ be a non-constant Nash map and consider its derivative $f' : [0, +\infty) \to \mathbb{R}$. Observe that $\{f' = 0\}$ is a finite set. Define $a := \max\{f' = 0\}$ and assume that f' is strictly positive on $(a, +\infty)$, so f is strictly increasing on $(a, +\infty)$. This means that $f([a, +\infty)) = [f(a), b)$ for some $b \in \mathbb{R} \cup \{+\infty\}$. As $f([0, a])$ is a connected compact set, $f([0, a]) = [c, d]$, so $f([0, +\infty)) = [c, d] \cup [f(a), b)$, which is not an open interval. Thus, $f([0, +\infty))$ is a proper subset of $\mathbb{R}$.

Putting together all these results, we obtain the following characterization for Nash images of general semi-algebraic sets.

Theorem 3.7.4. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set of dimension $d \geq 2$ connected by analytic paths. For each semi-algebraic set $\mathcal{T} \subset \mathbb{R}^m$ with $d \leq \dim(\mathcal{T})$, such that $\operatorname{Cl}(\mathcal{T}_e) \cap \mathcal{T}$ is non-compact for some $d \leq e \leq \dim(\mathcal{T})$ in case $\mathcal{S}$ is non-compact, there exists a Nash map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that $f(\mathcal{T}) = \mathcal{S}$.*

3.8 Surjective Nash maps between general semi-algebraic sets

Once established a satisfactory classification (both for the compact and non-compact case) of the possible models to represent semi-algebraic sets connected by analytic paths as Nash images, a natural question at this point is to determine until what extend we can represent *general* semi-algebraic sets as Nash images. We introduce first the analytic path-connected components of a semi-algebraic set [Fe4, §9].

3.8.1. Analytic path-connected components. A semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *connected by analytic paths* if for each $x, y \in \mathcal{S}$ there exists $\sigma : [0, 1] \to \mathcal{S}$ analytic such that $\sigma(0) = x$ and $\sigma(1) = y$. Observe that if $\mathcal{S} \subset \mathbb{R}^n$ is connected by analytic paths and $f : \mathcal{S} \to \mathbb{R}^n$ is a Nash map, then $f(\mathcal{S})$ is also connected by analytic paths. Thus, $\mathcal{T} := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0, \mathtt{z} \geq 0\}$, which is image of the polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(s, t) \mapsto (st, t, s^2)$, is connected by analytic paths, whereas its Zarisky closure $W := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0\}$ is not because there is no analytic path between a point in the stick $\{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} < 0\}$ and a point in $W \setminus \{\mathtt{x} = 0, \mathtt{y} = 0\}$. To take advantage of the full strength of our results (in particular Theorem 3.2) we introduce the analytic path-connected components of a semi-algebraic set.

Definition 3.8.1. A semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ admits a decomposition into *analytic path-connected components* if there exist semi-algebraic sets $\mathcal{S}_1, \ldots, \mathcal{S}_r \subset \mathcal{S}$ such that:

(i) Each $\mathcal{S}_i$ is connected by analytic paths.

(ii) If $\mathcal{T} \subset \mathcal{S}$ is a semi-algebraic set connected by analytic paths that contains $\mathcal{S}_i$, then $\mathcal{S}_i = \mathcal{T}$.

(iii) $\mathcal{S}_i \not\subset \bigcup_{j \neq i} S_j$.

(iv) $\mathcal{S} = \bigcup_{i=1}^r \mathcal{S}_i$.

In [Fe4, Thm.9.2] Fernando shows the following characterization for analytic path-connected components of a semi-algebraic set.

Theorem 3.8.2 ([Fe4, Thm.9.2])**.** *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set. Then $\mathcal{S}$ admits a decomposition into analytic path-connected components and this decomposition is unique. In addition, the analytic path-connected components of a semi-algebraic set are closed in $\mathcal{S}$.*

Example 3.8.3. (i) Let $\mathcal{S} := \{\mathtt{z}(\mathtt{x}^2+\mathtt{y}^2)-\mathtt{x}^3=0\} \subset \mathbb{R}^3$ be Cartan's umbrella. The analytic path-connected components of $\mathcal{S}$ are

$$\mathcal{S}_1 := (\mathcal{S} \setminus \{\mathtt{x}=0,\mathtt{y}=0\}) \cup \{0\} \ \text{ and } \ \mathcal{S}_2 := \{\mathtt{x}=0,\mathtt{y}=0\}.$$

In fact, $\mathcal{S}_1$ is image of $\mathbb{R}^2$ through the analytic map

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto (u\cos v, u\sin v, u\cos^3 v).$$

(ii) Let $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \subset \mathbb{R}^3$, where

$$\mathcal{S}_1 := [-1,1] \times \{0\} \times [-1,1],\ \mathcal{S}_2 := \{\mathtt{z}=0,\mathtt{x}\geq 1\},\ \mathcal{S}_3 := \{\mathtt{z}=0,\mathtt{x}\leq -1\}.$$

The analytic path-connected components of $\mathcal{S}$ are $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$. In contrast, $\mathcal{S}$ has two irreducible components, which are $\mathcal{S} \cap \{\mathtt{y}=0\}$ and $\mathcal{S} \cap \{\mathtt{z}=0\}$.

Recall that if $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ are irreducible components of a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$, then $\dim(\mathcal{S}_1^* \cap \mathcal{S}_2^*) < \min\{\dim(\mathcal{S}_1^*), \dim(\mathcal{S}_2^*)\}$ (see Remark 2.4.9). In the case of the analytic path-connected components of $\mathcal{S}$ this inequality is no longer true.

Example 3.8.4. Let $W := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0\} \subset \mathbb{R}^3$ be Whitney's umbrella. Its analytic path-connected components are

$$W_1 := W \cap \{\mathtt{z} \geq 0\} \ \text{ and } \ W_2 := \{\mathtt{x}=0,\mathtt{y}=0\}.$$

It holds $W_1 \cap W_2 = \{\mathtt{x}=0,\mathtt{y}=0,\mathtt{z}\geq 0\}$, which has dimension 1, so

$$\dim(W_1 \cap W_2) = 1 = \dim(W_2).$$

However, we have the following result:

Lemma 3.8.5. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a semi-algebraic set and let $\{\mathcal{S}_i\}_{i=1}^r$ be its family of analytic path-connected components. Then*

$$\dim(\mathcal{S}_i \cap \mathcal{S}_j) < \max\{\dim(\mathcal{S}_i), \dim(\mathcal{S}_j)\}$$

for $1 \leq i < j \leq r$.

Proof. If $\dim(\mathcal{S}_i) < \dim(\mathcal{S}_j)$, then

$$\dim(\mathcal{S}_i \cap \mathcal{S}_j) \leq \dim(\mathcal{S}_i) < \dim(\mathcal{S}_j) \leq \max\{\dim(\mathcal{S}_i), \dim(\mathcal{S}_j)\}.$$

Suppose next $e := \dim(\mathcal{S}_i) = \dim(\mathcal{S}_j)$ and $\dim(\mathcal{S}_i \cap \mathcal{S}_j) = e$. Let $\mathcal{S}_0 := \mathcal{S}_i \cup \mathcal{S}_j$ and observe that $\operatorname{Sing}(\mathcal{S}_0)$ has dimension $\leq e-1$. Thus, there exists $y \in (\mathcal{S}_1 \cap \mathcal{S}_2) \setminus \operatorname{Sing}(\mathcal{S}_0)$. Let $x_k \in \mathcal{S}_k$ for $k=1,2$. As each $\mathcal{S}_k$ is connected by analytic paths, by Theorem 3.4.2 each $\mathcal{S}_k$ is connected by Nash paths. Thus, there exist Nash paths $\alpha_k : [0,1] \to \mathcal{S}_k$ such that $\alpha_k(0) = x_k, \alpha_k(1) = y$. The continuous semi-algebraic path $\alpha := \alpha_1 * \alpha_2^{-1}$ connects the points x_1 and x_2 and satisfies $\eta(\alpha) \subset \{y\} \subset \operatorname{Reg}(\mathcal{S}_0)$ (see Section 3.4 for the definition of $\eta(\alpha)$). Consequently, $\mathcal{S}_0$ is well-welded, so $\mathcal{S}_0$ is connected by analytic paths by Theorem 3.4.2. As $\mathcal{S}_i, \mathcal{S}_j$ are analytic path-connected components of $\mathcal{S}$, we conclude $\mathcal{S}_i = \mathcal{S}_0 = \mathcal{S}_j$, which is a contradiction. Thus,

$$\dim(\mathcal{S}_i \cap \mathcal{S}_j) < e = \max\{\dim(\mathcal{S}_i), \dim(\mathcal{S}_j)\},$$

as required. □

3.8.2. Nash interpolation. As a consequence of Theorem 3.1 and the existence of a decomposition of semi-algebraic sets into analytic path-connected components we obtain the following interpolation result for Nash maps:

Corollary 3.8.6. *Let $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{T} \subset \mathbb{R}^m$ be semi-algebraic sets and let $\mathcal{T}^*$ be an analytic path-connected component of $\mathcal{T}$. Given $p_1, \ldots, p_k \in \mathcal{S}$ and $q_1, \ldots, q_k \in \mathcal{T}^*$ (non necessarily distinct), there exists a Nash map $F : \mathcal{S} \to \mathcal{T}$ such that $F(p_i) = q_i$ for each $i = 1, \ldots, k$.*

Proof. Let $d := \dim(\mathcal{T}^*)$. As $\mathcal{T}^*$ is connected by analytic paths, there exists by Theorem 3.1 a surjective Nash map $f : \mathbb{R}^d \to \mathcal{T}^*$. For each $i = 1, \ldots, k$ fix $s_i \in f^{-1}(q_i)$. By Lagrange's interpolation there exists a polynomial map $g : \mathbb{R}^n \to \mathbb{R}^d$ such that $g(p_i) = s_i$. Thus, the Nash map $F := f \circ g|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$ satisfies $F(p_i) = q_i$ for each $i = 1, \ldots, k$. □

3.8.3. General surjective Nash maps. In view of the previous results it is natural to wonder given arbitrary semi-algebraic sets $\mathcal{S} \subset \mathbb{R}^m$ and $\mathcal{T} \subset \mathbb{R}^n$ whether there exists a surjective Nash map $f : \mathcal{S} \to \mathcal{T}$. Recall that the image of a semi-algebraic set connected by analytic paths under a Nash map is connected by analytic paths. In addition, the image of an irreducible semi-algebraic set under a Nash map is an irreducible semi-algebraic set [FG3, §3.1].

Thus, obstructions to construct such a Nash map $f : \mathcal{S} \to \mathcal{T}$ concentrate on the configuration of the intersections of pairwise different analytic path-connected components $\{\mathcal{S}_i\}_{i=1}^r$ (resp. irreducible components $\{\mathcal{S}_j^*\}_{j=1}^\ell$) of $\mathcal{S}$ and the configuration of their images, which are semi-algebraic subsets $\mathcal{T}_i := f(\mathcal{S}_i)$ of $\mathcal{T}$ connected by analytic paths (resp. irreducible semi-algebraic subsets $\mathcal{T}_j^* := f(\mathcal{S}_j^*)$ of $\mathcal{T}$). Namely, if the intersection $\mathcal{S}_{i_1} \cap \cdots \cap \mathcal{S}_{i_k}$ (for $1 \leq i_1 < \cdots < i_k \leq r$) is non-empty, then

$$f(\mathcal{S}_{i_1} \cap \cdots \cap \mathcal{S}_{i_k}) \subset f(\mathcal{S}_{i_1}) \cap \cdots \cap f(\mathcal{S}_{i_k}) \subset \mathcal{T}_{i_1} \cap \cdots \cap \mathcal{T}_{i_k}$$

and the analytic path-connected components of $\mathcal{S}_{i_1} \cap \cdots \cap \mathcal{S}_{i_k}$ are mapped into analytic path-connected components of $\mathcal{T}_{i_1} \cap \cdots \cap \mathcal{T}_{i_k}$. Analogously, if the intersection $\mathcal{S}_{j_1}^* \cap \cdots \cap \mathcal{S}_{j_p}^*$ (for $1 \leq j_1 < \cdots < j_p \leq \ell$) is non-empty, then

$$f(\mathcal{S}_{j_1}^* \cap \cdots \cap \mathcal{S}_{j_p}^*) \subset f(\mathcal{S}_{j_1}^*) \cap \cdots \cap f(\mathcal{S}_{j_p^*}) \subset \mathcal{T}_{j_1} \cap \cdots \cap \mathcal{T}_{j_p}^*$$

and the irreducible components of $\mathcal{S}_{j_1} \cap \cdots \cap \mathcal{S}_{j_p}$ are mapped into irreducible components of $\mathcal{T}_{j_1} \cap \cdots \cap \mathcal{T}_{j_p}$.

Examples 3.8.7. (i) Let $\mathcal{S} := \{\mathtt{z} = 0\} \cup \{\mathtt{x} = 0, \mathtt{y} = 0\} \cup \{\mathtt{x} - \mathtt{z} = 0, \mathtt{y} = 0\} \subset \mathbb{R}^3$ and $\mathcal{T} := \{\mathtt{z} = 0\} \cup \{\mathtt{x} = 0, \mathtt{y} = 0\} \cup \{\mathtt{x} = 1, \mathtt{y} = 0\} \subset \mathbb{R}^3$. We claim: *There exists no surjective Nash map $f : \mathcal{S} \to \mathcal{T}$.*

The analytic path-connected components of $\mathcal{S}$ are

$$\mathcal{S}_1 := \{\mathtt{z} = 0\},\ \mathcal{S}_2 := \{\mathtt{x} = 0, \mathtt{y} = 0\} \text{ and } \mathcal{S}_3 := \{\mathtt{x} - \mathtt{z} = 0, \mathtt{y} = 0\},$$

whereas the analytic path-connected components of $\mathcal{T}$ are $\mathcal{T}_1 := \{\mathtt{z} = 0\}$, $\mathcal{T}_2 := \{\mathtt{x} = 0, \mathtt{y} = 0\}$ and $\mathcal{T}_3 := \{\mathtt{x} = 1, \mathtt{y} = 0\}$. Suppose there exists a surjective Nash map $f : \mathcal{S} \to \mathcal{T}$. Using straightforward dimensional arguments $f(\mathcal{S}_1) = \mathcal{T}_1$ and either $f(\mathcal{S}_2) = \mathcal{T}_2$ and $f(\mathcal{S}_3) = \mathcal{T}_3$ or $f(\mathcal{S}_2) = \mathcal{T}_3$ and $f(\mathcal{S}_3) = \mathcal{T}_2$. However, this is not possible because $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 = \{(0, 0, 0)\}$, whereas $\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{T}_3 = \varnothing$ and $f(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3) \subset \mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{T}_3$.

(ii) Let $\mathcal{S} := \{\mathtt{y} \geq 0\} \cup \{\mathtt{x} = 0\} \subset \mathbb{R}^2$ and $\mathcal{T} := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0\} \subset \mathbb{R}^3$, which are both irreducible. We claim: *There exists no surjective Nash map $f : \mathcal{S} \to \mathcal{T}$.*

The analytic path-connected components of $\mathcal{S}$ are $\mathcal{S}_1 := \{\mathtt{y} \geq 0\}$ and $\mathcal{S}_2 := \{\mathtt{x} = 0\}$, whereas the analytic path-connected components of $\mathcal{T}$ are

$$\mathcal{T}_1 := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0, \mathtt{z} \geq 0\} \text{ and } \mathcal{T}_2 := \{\mathtt{x} = 0, \mathtt{y} = 0\}.$$

Suppose there exists a surjective Nash map $f : \mathcal{S} \to \mathcal{T}$. Using straightforward dimensional arguments $f(\mathcal{S}_1) = \mathcal{T}_1$ and $\{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} < 0\} \subset f(\mathcal{S}_2) \subset \mathcal{T}_2$. As f is Nash, there exist a connected open semi-algebraic neighbourhood $U \subset \mathbb{R}^2$ and a Nash extension $F : U \to \mathbb{R}^3$. As U is an open connected semi-algebraic subset of $\mathbb{R}^2$, it is an irreducible semi-algebraic set of dimension 2. Thus, $F(U)$ is an irreducible semi-algebraic subset of $\mathbb{R}^3$ of dimension ≤ 2. In particular, its Zariski closure is an irreducible algebraic set of dimension ≤ 2. As $f(\mathcal{S}_1) = \mathcal{T}_1$ has dimension 2 and the Zariski closure of $\mathcal{T}_1$ is $\mathcal{T}$ (which is irreducible), we conclude that the Zariski closure of $F(U)$ is $\mathcal{T}$. As connected open semi-algebraic sets are connected by analytic paths (because they are connected Nash manifolds), we deduce that $\mathcal{T}_1 = f(\mathcal{S}_1) \subset F(U) \subset \mathcal{T}_1$, so

$$\{\mathtt{x} = 0, \mathtt{y} = 0, \mathtt{z} < 0\} \subset f(\mathcal{S}_2) \subset F(U) = \mathcal{T}_1,$$

which is a contradiction.

(iii) However, there exists a surjective Nash map $f : \mathcal{T} \to \mathcal{S}$ where $\mathcal{T} := \{\mathtt{x}^2 - \mathtt{z}\mathtt{y}^2 = 0\} \subset \mathbb{R}^3$ and $\mathcal{S} := \{\mathtt{y} \geq 0\} \cup \{\mathtt{x} = 0\} \subset \mathbb{R}^2$. It is enough to take $f(\mathtt{x}, \mathtt{y}, \mathtt{z}) = (\mathtt{y}, \mathtt{z})$.

Recall that if $\mathcal{S} \subset \mathbb{R}^m$ has dimension d, the set $\mathcal{S}^{(d)}$ of points of $\mathcal{S}$ of dimension d is a closed semi-algebraic subset of $\mathcal{S}$. In order to soften the obstructions quoted at the beginning of this section we will assume that each irreducible component $\mathcal{S}_i^*$ of $\mathcal{S}$ is mapped onto an analytic path-connected component $\mathcal{T}_i$ of $\mathcal{T}$ and that $\bigcap_{i=1}^r f(\mathcal{T}_i) \neq \varnothing$. Under this type of assumptions we propose the following characterization.

Theorem 3.8.8 (Surjective Nash maps). *Let $\mathcal{S} \subset \mathbb{R}^m$ and $\mathcal{T} \subset \mathbb{R}^n$ be semi-algebraic sets, let $\{\mathcal{S}_i^*\}_{i=1}^r$ be the irreducible components of $\mathcal{S}$ and let $\{\mathcal{T}_i\}_{i=1}^r$ be a family of (non-necessarily distinct) semi-algebraic subsets of $\mathcal{T}$ connected by analytic paths such that $\bigcap_{i=1}^r \mathcal{T}_i \neq \varnothing$. Denote $d_i := \dim(\mathcal{S}_i^*)$ and assume that the set $\mathcal{S}_i^{*,(d_i)}$ of points of $\mathcal{S}_i^*$ of dimension d_i is non-compact if $\mathcal{T}_i$ is non-compact* *for $i=1,\dots,r$. Then, there exists a Nash map $f:\mathcal{S}\to\mathcal{T}$ such that $f(\mathcal{S}_i^*)=\mathcal{T}_i$ for $i=1,\dots,r$ if and only if $e_i:=\dim(\mathcal{T}_i)\leq\dim(\mathcal{S}_i^*)=:d_i$ for $i=1,\dots,r$.*

Before proving this result we need the following preliminary one.

Lemma 3.8.9. *Let $\mathcal{S}\subset\mathbb{R}^m$ be a semi-algebraic set and let $\{\mathcal{S}_i^*\}_{i=1}^s$ be the family of the irreducible components of $\mathcal{S}$ that are non-compact. Denote $d_i:=\dim(\mathcal{S}_i^*)$ and $\mathcal{S}_i^{*,(d_i)}$ the set of points of $\mathcal{S}_i^*$ of dimension d_i, which we assume non-compact for each $i=1,\dots,s$. Let U be an open semi-algebraic subset of $\mathbb{R}^m$ that contains $\mathcal{S}$ and let $X_1,\dots,X_s$ be Nash subsets of U such that $\mathcal{S}_i^*\setminus X_i\neq\varnothing$ for each i. Up to take a smaller U if necessary, there exist a Nash manifold $M\subset\mathbb{R}^p$, a Nash diffeomorphism $\varphi:M\to U$ and a Nash function $g_i:M\to\mathbb{R}$ whose zero set contains $\varphi^{-1}(X_i)$ and the corresponding Nash map*

$$G_i:M\to\mathbb{R}^{p+1},\ x\mapsto(x\cdot g_i(x),g_i(x))$$

satisfies $0\in G_i(\varphi^{-1}(\mathcal{S}_i^*))=G_i(\varphi^{-1}(\mathcal{S}_i^{*,(d_i)}))$ and $G_i(\varphi^{-1}(\mathcal{S}_i^{*,(d_i)}))$ is pure dimensional of dimension d_i and non-compact for $i=1,\dots,s$.

Proof. We may apply the Nash diffeomorphism

$$\psi_0:\mathbb{R}^m\to\mathcal{B}_m(0,1),\ x\mapsto\frac{x}{\sqrt{1+\|x\|^2}}$$

to $\mathcal{S}$ and assume that $\mathcal{S}$ is bounded. As $\mathcal{S}_i^*\setminus X_i\neq\varnothing$, $\mathcal{S}_i^*$ is irreducible and X_i is the zero-set of a Nash function on U, we deduce by [FG3, Lem.3.6] that $\dim(\mathcal{S}_i^*\cap X_i)<\dim(\mathcal{S}_i^*)=\dim(\mathcal{S}_i^{*,(d_i)})$ for each $i=1,\dots,r$. Pick a point $q_i\in\mathcal{S}_i^{*,(d_i)}$. Let Z_i be the Zariski closure of $(\operatorname{Cl}(\mathcal{S}_i^*)\cap\operatorname{Cl}(X_i))\cup\operatorname{Cl}(\mathcal{S}_i^*\setminus\mathcal{S}_i^{*,(d_i)})\cup\{q_i\}$, which has dimension strictly smaller than $\dim(\mathcal{S}_i^*)$.

Indeed, as $\mathcal{S}_i^*$ is closed in $\mathcal{S}$ and X_i is closed in U, we deduce that

$$\operatorname{Cl}(\mathcal{S}_i^*)\cap\operatorname{Cl}(X_i)\cap\mathcal{S}=\mathcal{S}_i^*\cap X_i,$$

so $(\operatorname{Cl}(\mathcal{S}_i^*)\cap\operatorname{Cl}(X_i))\setminus\operatorname{Cl}(\mathcal{S}_i^*\cap X_i)\subset\operatorname{Cl}(\mathcal{S}_i^*)\setminus\mathcal{S}_i^*$, which has dimension strictly smaller that $\dim(\mathcal{S}_i^*)$. Thus, $\operatorname{Cl}(\mathcal{S}_i^*)\cap\operatorname{Cl}(X_i^*)$ has dimension strictly smaller than $\dim(\mathcal{S}_i^*)$. In addition, $\operatorname{Cl}(\mathcal{S}_i^*\setminus\mathcal{S}_i^{*,(d_i)})$ has dimension strictly smaller that $\dim(\mathcal{S}^*)$, because $\dim(\mathcal{S}_i^*\setminus\mathcal{S}_i^{*,(d_i)})<d_i=\dim(\mathcal{S}_i^*)$.

As $\mathcal{S}_i^{*,(d_i)}$ is bounded and non-compact and $\mathcal{S}_i^{*,(d_i)}$ is closed in $\mathcal{S}$ (because it is a closed subset of $\mathcal{S}_i^*$, which is a closed subset of $\mathcal{S}$), there exists $p_i\in\operatorname{Cl}(\mathcal{S}_i^{*,(d_i)})\setminus\mathcal{S}$ (because otherwise $\operatorname{Cl}(\mathcal{S}_i^{*,(d_i)})\subset\mathcal{S}$ and $\mathcal{S}_i^{*,(d_i)}=\operatorname{Cl}(\mathcal{S}_i^{*,(d_i)})\cap\mathcal{S}=\operatorname{Cl}(\mathcal{S}_i^{*,(d_i)})$ would be compact). As $p_i\notin\mathcal{S}$, up to replace U by $U':=U\setminus\{p_1,\dots,p_r\}$ and X_i by $U'\cap X_i$ if necessary, we may assume $p_i\notin X_i$. As $\mathcal{S}_i^{*,(d_i)}\setminus Z_i$ is dense in $\mathcal{S}_i^{*,(d_i)}$ (because $\mathcal{S}_i^{*,(d_i)}$ is pure dimensional), there exists by the Nash curve selection lemma (see [BCR, 8.1.13]) a Nash curve $\alpha_i:(-1,1)\to\mathbb{R}^m$ such that $\alpha_i((0,1))\subset\mathcal{S}_i^{*,(d_i)}\setminus Z_i$ and $\alpha_i(0)=p_i$. Let $Q_i\in\mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_n]$ be a polynomial whose zero set is Z_i.

Case 1. If $Q_i(p_i)\neq0$, we take a bounded Nash function g_i on U whose zero set is the union of X_i and the smallest Nash subset of U that contains

$\operatorname{Cl}(\mathcal{S}_i^*\setminus\mathcal{S}_i^{*,(d_i)})$. Observe that the limit $\lim_{t\to 0^+} g_i\circ\alpha_i(t)$ exists and it is non-zero, because otherwise either p_i belongs to the Zariski closure of $\operatorname{Cl}(\mathcal{S}_i^*\setminus\mathcal{S}_i^{*,(d_i)})\subset Z_i=\{Q_i=0\}$ or $p_i\in\operatorname{Cl}(\mathcal{S}_i^{*,(d_i)})\cap\operatorname{Cl}(X_i)\subset Z_i=\{Q_i=0\}$, which is a contradiction.

Consider the Nash map $G_i:U\to\mathbb{R}^{m+1},\ x\mapsto(x\cdot g_i(x),g_i(x))$, whose restriction to $U\setminus\{g_i=0\}$ is a Nash diffeomorphism between $U\setminus\{g_i=0\}$ and $G_i(U)\setminus\{0\}$, whose inverse is $H_i:G_i(U)\setminus\{0\}\to U\setminus\{g_i=0\},\ (y,t)\mapsto\frac{y}{t}$. If $G_i(\mathcal{S}_i^{*,(d_i)})$ is compact, then $\lim_{t\to 0^+}(\alpha_i(t)\cdot g_i\circ\alpha_i(t),g_i\circ\alpha_i(t))\in G_i(\mathcal{S}_i^{*,(d_i)})$. As

$$G_i|_{\mathcal{S}_i^{*,(d_i)}\setminus\{g_i=0\}}:\mathcal{S}_i^{*,(d_i)}\setminus\{g_i=0\}\to G_i(\mathcal{S}_i^{*,(d_i)})\setminus\{0\}$$

is a Nash diffeomorphism, we conclude that $\lim_{t\to 0^+}\alpha_i(t)\cdot g_i\circ\alpha_i(t)=0$ (because $p_i\notin\mathcal{S}_i^{*,(d_i)}$), which is a contradiction because $\lim_{t\to 0^+}g_i\circ\alpha_i(t)$ exists and it is non-zero. Consequently, $G_i(\mathcal{S}_i^{*,(d_i)})$ is non-compact. Again as the restriction $G_i|_{\mathcal{S}_i^{*,(d_i)}\setminus\{g_i=0\}}$ is a Nash diffeomorphism,

$$G_i(\mathcal{S}_i^{*,(d_i)})\setminus\{0\}=G_i(\mathcal{S}_i^{*,(d_i)}\setminus\{g_i=0\})$$

is pure dimensional of dimension d_i. As $q_i\in\mathcal{S}_i^{*,(d_i)}\cap\{g_i=0\}$, we conclude $0\in\operatorname{Cl}(G_i(\mathcal{S}_i^{*,(d_i)}\setminus\{g_i=0\}))$, so $G_i(\mathcal{S}_i^{*,(d_i)})$ is pure dimensional of dimension d_i. In addition,

$$0\in G(\mathcal{S}_i^{*,(d_i)})=G(\mathcal{S}_i^{*,(d_i)})\cup\{0\}=G(\mathcal{S}_i^{*,(d_i)}\cup(\mathcal{S}_i\cap\{g_i=0\}))=G(\mathcal{S}_i^*).$$

Case 2. If $Q_i(p_i)=0$, we have $Q_i\circ\alpha_i\in\mathbb{R}[[\mathtt{t}]]_{\mathrm{alg}}$ is a non-zero series. Let (Y_i,ϕ_i) be the blow-up of $\mathbb{R}^m$ at p_i. The restriction $\phi_i:Y_i\setminus\{\phi_i^{-1}(p_i)\}\to\mathbb{R}^m\setminus\{p_i\}$ is a Nash diffeomorphism and $p_i\notin\mathcal{S}\cup X_i$, so $\phi_i^{-1}(\mathcal{S})$ is Nash diffeomorphic to $\mathcal{S}$ and $\phi_i^{-1}(X_i)$ is Nash diffeomorphic to X_i. The series

$$Q_i\circ\alpha_i=(Q_i\circ\phi_i)\circ(\phi_i^{-1}\circ\alpha_i)$$

and let $(Q_i\circ\phi_i)^*$ be the strict transform of $(Q_i\circ\phi_i)$. The order of the series $(Q_i\circ\phi_i)^*\circ(\phi_i^{-1}\circ\alpha_i)$ is strictly smaller than the order of $Q_i\circ\alpha_i$, because we have eliminated from $(Q_i\circ\phi_i)$ a power of an equation of the exceptional divisor. Let $p_i':=\lim_{t\to 0^+}(\phi_i^{-1}\circ\alpha_i)(t)$. If $(Q_i\circ\phi_i)^*(p_i')\neq 0$ we have finished with this index i. Otherwise, we repeat the previous process with the point p_i'. In each step the order of the strict transform of the corresponding polynomial substituted in the corresponding curve has strictly smaller order, so in finitely many steps we achieve order 0 and the corresponding polynomial does not vanish at the limit point.

After composing all the involved blow-ups (corresponding to all the indices $i=1,\ldots,r$) and taking suitable restrictions we find a Nash manifold $M\subset\mathbb{R}^p$, a Nash diffeomorphism $\varphi:M\to U$ and Nash functions $g_i:M\to\mathbb{R}$ such that $\varphi^{-1}(X_i)\subset\{g_i=0\}$ and the corresponding Nash maps

$$G_i:M\to\mathbb{R}^{p+1},\ x\mapsto(x\cdot g_i(x),g_i(x))$$

satisfies $0\in G_i(\varphi^{-1}(\mathcal{S}_i^*))=G_i(\varphi^{-1}(\mathcal{S}_i^{*,(d_i)}))$ and $G_i(\varphi^{-1}(\mathcal{S}_i^{*,(d_i)}))$ is pure dimensional of dimension d_i and non-compact for $i=1,\ldots,s$, as required. $\square$

We are ready to prove Theorem 3.8.8.

Proof of Theorem 3.8.8. The only if condition are straightforward. The proof of the converse is conducted in several steps:

STEP 1. Suppose $\mathcal{S}_i^*$ is non-compact for $i = 1, \ldots, s$ and $\mathcal{S}_i^*$ is compact for $i = s+1, \ldots, r$. For each $i = 1, \ldots, r$ let $f_i : \mathcal{S} \to \mathbb{R}$ be a Nash function on $\mathcal{S}$ such that $\mathcal{S}_i^* = \{f_i = 0\}$ (see [FG3, Lem.2.4, Thm. 4.3]). Let U be an open semi-algebraic neighborhood of $\mathcal{S}$ in $\mathbb{R}^m$ to which all the Nash funcions f_i extend as Nash functions $F_i : U \to \mathbb{R}$. Define $X_i := \bigcup_{j\neq i}\{F_j = 0\}$ and observe that $\mathcal{S}_i^* \cap X_i = \mathcal{S}_i^* \cap \bigcup_{j\neq i} \mathcal{S}_j^*$ is a semi-algebraic subset of $\mathcal{S}_i^*$ of dimension strictly smaller than d_i. We distinguish two cases:

CASE 1. NON-COMPACT IRREDUCIBLE COMPONENTS. By Lemma 3.8.9 we may assume (up to a suitable Nash diffeomorphism) that for each $i = 1, \ldots, s$ there exist a Nash function $g_i : U \to \mathbb{R}$ whose respective zero set $\{g_i = 0\}$ contains X_i and the corresponding Nash map

$$G_i : U \to \mathbb{R}^{m+1},\ x \mapsto (x \cdot g_i(x), g_i(x))$$

satisfies $0 \in G_i(\mathcal{S}_i^*) = G_i(\mathcal{S}_i^{*,(d_i)})$ and $G_i(\mathcal{S}_i^{*,(d_i)})$ is non-compact and pure dimensional of dimension d_i for $i = 1, \ldots, s$. In addition,

$$G_i(\mathcal{S}) = G_i(\mathcal{S}_i^* \cup \bigcup_{j\neq i} \mathcal{S}_j^*) = G_i(\mathcal{S}_i^*) \cup \bigcup_{j\neq i} G_i(\mathcal{S}_j^*) = G_i(\mathcal{S}_i^*) \cup \{0\} = G_i(\mathcal{S}_i^{*,(d_i)}).$$

CASE 2. COMPACT IRREDUCIBLE COMPONENTS. For each $i = s+1, \ldots, r$ let $q_i \in \mathcal{S}_i^{*,(d_i)}$ and let h_i be a polynomial whose zero set is the union of $\{q_i\}$ and the Zariski closure Y_i of $\operatorname{Cl}(\mathcal{S}_i^* \setminus \mathcal{S}_i^{*,(d_i)})$. Define $g_i := h_i \prod_{j\neq i} F_j : U \to \mathbb{R}$ and observe that $\{g_i = 0\} = \{q_i\} \cup (Y_i \cap \mathcal{S}_i^*) \cup \bigcup_{j\neq i} \mathcal{S}_j^*$. As $\mathcal{S}_i^*$ is irreducible and g_i does not vanish identically on $\mathcal{S}_i^*$, the intersection $\{g_i = 0\} \cap \mathcal{S}_i^*$ has dimension $< d_i := \dim(\mathcal{S}_i^*)$. Consider the Nash map

$$G_i : U \to \mathbb{R}^{m+1},\ x \mapsto (x \cdot g_i(x), g_i(x)),$$

whose restriction to $U \setminus \{g_i = 0\}$ is a Nash diffeomorphism between $U \setminus \{g_i = 0\}$ and $G_i(U) \setminus \{0\}$. Observe that $G_i(\mathcal{S}_j^*) = \{0\}$ if $i \neq j$ and $\mathcal{S}_i' := G_i(\mathcal{S}) = G_i(\mathcal{S}_i^{*,(d_i)})$ *is pure dimensional of dimension* d_i.

As $G_i|_{\mathcal{S}_i^{*,(d_i)} \setminus \{g_i=0\}} : \mathcal{S}_i^{*,(d_i)} \setminus \{g_i = 0\} \to G_i(\mathcal{S}_i^{*,(d_i)}) \setminus \{0\}$ is a Nash diffeomorphism, $G_i(\mathcal{S}_i^{*,(d_i)}) \setminus \{0\} = G_i(\mathcal{S}_i^{*,(d_i)} \setminus \{g_i = 0\})$ is pure dimensional of dimension d_i. As $q_i \in \mathcal{S}_i^{*,(d_i)} \cap \{g_i = 0\}$, we conclude $0 \in \operatorname{Cl}(G_i(\mathcal{S}_i^{*,(d_i)} \setminus \{g_i = 0\}))$, so $G_i(\mathcal{S}_i^{*,(d_i)})$ is pure dimensional of dimension d_i. In addition,

$$0 \in G(\mathcal{S}_i^{*,(d_i)}) = G(\mathcal{S}_i^{*,(d_i)}) \cup \{0\} = G(\mathcal{S}_i^{*,(d_i)} \cup (\mathcal{S}_i \cap \{g_i = 0\})) = G(\mathcal{S}_i^*).$$

Moreover,

$$G_i(\mathcal{S}) = G_i(\mathcal{S}_i^* \cup \bigcup_{j\neq i} \mathcal{S}_j^*) = G_i(\mathcal{S}_i^*) \cup \bigcup_{j\neq i} G_i(\mathcal{S}_j^*) = G_i(\mathcal{S}_i^*) \cup \{0\} = G_i(\mathcal{S}_i^{*,(d_i)})$$

for $i = s+1, \ldots, r$.

STEP 2. Define $\mathcal{S}_i' := G_i(\mathcal{S})$ for $i = 1, \dots, r$ and

$$G : \mathcal{S} \to \mathbb{R}^{(m+1)r}, \ x \mapsto (G_1(x), \dots, G_r(x)).$$

Observe that

$$G(\mathcal{S}_i^*) = \{0\} \times \cdots \times \{0\} \times \overset{(i)}{\mathcal{S}_i'} \times \{0\} \times \cdots \times \{0\}$$

and $G(\mathcal{S}) = \bigcup_{i=1}^{s} G(\mathcal{S}_i^*)$. In addition, $G(\mathcal{S}_i^*) \cap G(\mathcal{S}_j^*) = \{(0, \dots, 0)\}$ if $i \neq j$.

We distinguish two cases:

CASE 1. If $\mathcal{S}_i^*$ is non-compact, $\mathcal{S}_i'$ is non-compact. By Theorem 3.7.2 there exists a Nash map $H_i : \mathbb{R}^{m+1} \to \mathbb{R}^{d_i}$ such that $H_i(\mathcal{S}_i') = \mathbb{R}^{d_i}$. We may assume in addition $H_i(0) = 0$.

CASE 2. If $\mathcal{S}_i$ is compact, also $\mathcal{S}_i'$ is compact and there exists by Theorem 3.7.1 a Nash map $H_i : \mathbb{R}^{m+1} \to \mathbb{R}^{d_i}$ such that $H_i(\mathcal{S}_i') = \overline{\mathcal{B}}_{d_i}$. Following the proof of Theorem 3.7.1 the reader can check that we may assume $H_i(0) = 0$.

STEP 3. Let $q \in \bigcap_{i=1}^{r} \mathcal{T}_i$ and assume q is the origin of $\mathbb{R}^n$. Let $F_i : \mathbb{R}^{d_i} \to \mathbb{R}^n$ be a Nash map such that $F_i(\overline{\mathcal{B}}_{d_i}) = \mathcal{T}_i$ for $i = 1, \dots, s$ and $F_i(\mathbb{R}^{d_i}) = \mathcal{T}_i$ for $i = s+1, \dots, r$. We may assume in addition $F_i(0) = 0$ for $i = 1, \dots, r$. We have

$$(F_i \circ H_i \circ G_i)(\mathcal{S}_j) = \begin{cases} F_i(\overline{\mathcal{B}}_{d_i}) = \mathcal{T}_i & \text{if } j = i, \\ F_i(\{0\}) = \{0\} & \text{if } j \neq i. \end{cases}$$

Define $E_i = \mathbb{R}^{d_i}$ if $\mathcal{S}_i'$ is non-compact $(i = 1, \dots, s)$ and $E_i = \overline{\mathcal{B}}_{d_i}$ if $\mathcal{S}_i'$ is compact $(i = s+1, \dots, r)$. Observe that

$$\begin{aligned}
((F_1 \circ H_1, \dots, F_r \circ H_r) \circ G)(\mathcal{S}_i) &= (F_1 \circ H_1 \circ G_1, \dots, F_r \circ H_r \circ G_r)(\mathcal{S}_i) \\
&= (F_1 \circ H_1)(\{0\}) \times \ldots \times (F_{i-1} \circ H_{i-1})(\{0\}) \times (F_i \circ H_i)(\mathcal{S}_i') \\
&\qquad \times (F_{i+1} \circ H_{i+1})(\{0\}) \times \ldots \times (F_r \circ H_r)(\{0\}) \\
&= F_1(\{0\}) \times \ldots \times F_{i-1}(\{0\}) \times F_i(E_i) \times F_{i+1}(\{0\}) \times \ldots \times F_r(\{0\}) \\
&= \{0\} \times \cdots \times \{0\} \times \overset{(i)}{\mathcal{T}_i} \times \{0\} \times \cdots \times \{0\}.
\end{aligned}$$

Thus, if

$$F := \sum_{i=1}^{r} (F_i \circ H_i \circ G_i) : \mathcal{S} \to \mathcal{T},$$

we have $F(\mathcal{S}_i) = \mathcal{T}_i$ for $i = 1, \dots, r$, so

$$F(\mathcal{S}) = F\left(\bigcup_{i=1}^{r} \mathcal{S}_i\right) = \bigcup_{i=1}^{r} F(\mathcal{S}_i) = \bigcup_{i=1}^{r} \mathcal{T}_i = \mathcal{T},$$

as required. □

Recall that the analytic path-connected components of $\mathcal{S}$ are irreducible semi-algebraic sets. Thus, each of them is contained in an irreducible component of $\mathcal{S}$. If $\mathcal{S}_i^*$ is the irreducible component of $\mathcal{S}$ that contains $\mathcal{S}_i$ for $i = 1, \dots, r$ it may happen that $\mathcal{S}_i^* = \mathcal{S}_j^*$ for some $i \neq j$ or $\mathcal{S}_i^* \neq \mathcal{S}_j^*$ whereas $\mathcal{S}_i \subsetneq \mathcal{S}_i^*$ and $\mathcal{S}_j \subsetneq \mathcal{S}_j^*$.

Examples 3.8.10. (i) Define $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \subset \mathbb{R}^2$ where $\mathcal{S}_1 := \{\mathtt{x} \geq 1\}$, $\mathcal{S}_2 := \{\mathtt{y} = 0\}$ and $\mathcal{S}_3 := \{\mathtt{x} \leq -1\}$. Observe that $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are the analytic path-connected components of $\mathcal{S}$, whereas $\mathcal{S}$ is irreducible. Thus, $\mathcal{S}_1^* = \mathcal{S}_2^* = \mathcal{S}_3^*$.

(ii) Define $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \subset \mathbb{R}^3$ where

$$\mathcal{S}_1 := \{\mathtt{x} = 0, \mathtt{y} \geq 0\} \quad \text{and} \quad \mathcal{S}_2 := \{\mathtt{y} \leq 0, \mathtt{z} = 0\}.$$

Observe that $\mathcal{S}_1$ and $\mathcal{S}_2$ are the analytic path-connected components of $\mathcal{S}$, whereas $\mathcal{S}_1^* = \mathcal{S}_1 \cup \{\mathtt{x} = 0, \mathtt{z} = 0\}$ and $\mathcal{S}_2^* = \mathcal{S}_2 \cup \{\mathtt{x} = 0, \mathtt{z} = 0\}$ are the irreducible components of $\mathcal{S}$.

Remarks 3.8.11. (i) Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set and let $\{\mathcal{S}_i\}_{i=1}^r$ be the analytic path-connected components of $\mathcal{S}$. Let $\mathcal{S}_i^*$ be the irreducible component of $\mathcal{S}$ that contains $\mathcal{S}_i$ for $i = 1, \ldots, r$ and assume $\mathcal{S}_i^* \neq \mathcal{S}_j^*$ for $1 \leq i < j \leq r$. Denote $d_i := \dim(\mathcal{S}_i^*)$. We claim:

(1) $\{\mathcal{S}_i^*\}_{i=1}^r$ is the collection of the irreducible components of $\mathcal{S}$.

(2) $\mathcal{S}_i^{*,(d_i)} = \mathcal{S}_i$ for $i = 1, \ldots, r$.

As $\mathcal{S} = \bigcup_{i=1}^r \mathcal{S}_i \subset \bigcup_{i=1}^r \mathcal{S}_i^* \subset \mathcal{S}$, we deduce that $\{\mathcal{S}_i^*\}_{i=1}^r$ is the collection of the irreducible components of $\mathcal{S}$, because $\mathcal{S}_i^* \neq \mathcal{S}_j^*$ if $i \neq j$. Thus, (1) holds.

Let us check that (2). To that end, we prove first: $\dim(\mathcal{S}_j \cap \mathcal{S}_i^*) < \dim(\mathcal{S}_i^*)$ if $j \neq i$.

Otherwise, there exists $\mathcal{S}_j$ with $j \neq i$ such that $\dim(\mathcal{S}_j \cap \mathcal{S}_i^*) = \dim(\mathcal{S}_i^*)$, so each Nash function that vanishes identically on $\mathcal{S}_j^*$ vanishes also identically on $\mathcal{S}_i^*$. Thus, $\mathcal{S}_i^* \subset \mathcal{S}_j^*$ and $i = j$, which is a contradiction.

Consequently,

$$\mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j \subset \mathcal{S}_i^* \setminus \bigcup_{j\neq i} \mathcal{S}_j \subset \mathcal{S} \setminus \bigcup_{j\neq i} \mathcal{S}_j = \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$$

and $\mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j = \mathcal{S}_i^* \setminus \bigcup_{j\neq i} \mathcal{S}_j$ is non-empty and has dimension d_i. As $\mathcal{S}_i$ is pure dimensional of dimension d_i and $\bigcup_{j\neq i} \mathcal{S}_i^* \cap \mathcal{S}_j$ has dimension $< d_i$, we deduce that $\mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$ is dense in $\mathcal{S}_i$. In addition, $\mathcal{S}_i \subset \mathcal{S}_i^{*,(d_i)}$ (because $\mathcal{S}_i$ is pure dimensional of dimension d_i) and $\mathcal{S}_i^{*,(d_i)} \setminus \bigcup_{j\neq i} \mathcal{S}_j$ is dense in $\mathcal{S}_i^{*,(d_i)}$. As

$$\mathcal{S}_i^{*,(d_i)} \setminus \bigcup_{j\neq i} \mathcal{S}_j \subset \mathcal{S}_i^* \setminus \bigcup_{j\neq i} \mathcal{S}_j = \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j,$$

we conclude taking closures in $\mathcal{S}$ that $\mathcal{S}_i^{*,(d_i)} = \mathcal{S}_i$ (because both $\mathcal{S}_i^{*,(d_i)}$ and $\mathcal{S}_i$ are closed in $\mathcal{S}$).

(ii) Observe that Theorems 3.7.1 and 3.7.2 are particular cases of Theorem 3.8.8 when $\mathcal{T}$ is connected by analytic paths.

As a straightforward consequence of Theorem 3.8.8 and Remark 3.8.11(i), we have the following:

Corollary 3.8.12. *Let $\mathcal{S} \subset \mathbb{R}^m$ and $\mathcal{T} \subset \mathbb{R}^n$ be semi-algebraic sets, let $\{\mathcal{S}_i\}_{i=1}^r$ be the family of analytic path-connected components of $\mathcal{S}$ and let $\mathcal{S}_i^*$ be the irreducible component of $\mathcal{S}$ that contains $\mathcal{S}_i$ for $i = 1, \ldots, r$. Assume $\mathcal{S}_i^* \neq \mathcal{S}_j^*$*

for $1 \leq i < j \leq r$. Let $\{\mathcal{T}_i\}_{i=1}^r$ be a family of (non-necessarily distinct) semi-algebraic subsets of $\mathcal{T}$ connected by analytic paths and assume $\bigcap_{i=1}^r \mathcal{T}_i \neq \varnothing$. Then, there exists a Nash map $f : \mathcal{S} \to \mathcal{T}$ such that $f(\mathcal{S}_i) = \mathcal{T}_i$ for $i = 1, \ldots, r$ if and only if $e_i := \dim(\mathcal{T}_i) \leq \dim(\mathcal{S}_i) =: d_i$ and $\mathcal{T}_i$ is compact in case $\mathcal{S}_i$ is compact for $i = 1, \ldots, r$.

3.9 Two consequences

In this section we present two remarkable consequences of Theorem 3.2. The first one about representation of pure dimensional compact irreducible arc-symmetric semi-algebraic sets as Nash images of closed balls. As a second consequence we show that a compact semi-algebraic set is the projection of a non-singular compact algebraic set with the simplest possible topology (a disjoint union of spheres).

3.9.1. Representation of arc-symmetric compact semi-algebraic sets. Arc-symmetric semi-algebraic sets were introduced by Kurdyka in [K] and subsequently studied by many authors. Recall that a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *arc-symmetric* if for each analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\gamma((-1, 0)) \subset \mathcal{S}$ it holds that $\gamma((-1, 1)) \subset \mathcal{S}$. In particular arc-symmetric semi-algebraic sets are closed subsets of $\mathbb{R}^n$. An arc-symmetric semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *irreducible* if it cannot be written as the union of two proper arc-symmetric semi-algebraic subsets [K, §2]. An arc-symmetric semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ irreducible as semi-algebraic set (in the sense of Definition 2.4.5) is not necessarily irreducible as arc-symmetric set (in the sense of [K, §2]), as shown in the following example.

Example 3.9.1. Let $X := \{\mathtt{z}(\mathtt{x}^2 + \mathtt{y}^2) - \mathtt{x}^3 = 0\} \subset \mathbb{R}^3$ be Cartan's umbrella. As X is an irreducible real analytic set, the ring $\mathcal{N}(X)$ is an integral domain. Thus, X is irreducible as semi-algebraic set. Let us show that X is reducible as arc-symmetric set. We claim: *Let $f(x, y) := \frac{x^3}{x^2+y^2}$. For each analytic arc $\gamma := (\gamma_1, \gamma_2) : (-1, 1) \to \mathbb{R}^2$ the composition $f \circ \gamma$ is analytic.* As f is regular on $\mathbb{R}^2 \setminus \{0\}$, we may assume $\gamma(0) = 0$. Denote $\operatorname{ord}_0(\gamma_i)$ the order of vanishing of γ_i at 0 for $i \in \{1, 2\}$. As $3 \operatorname{ord}_0(\gamma_1) > 2 \min\{\operatorname{ord}_0(\gamma_1), \operatorname{ord}_0(\gamma_2)\}$, the composition $f \circ \gamma$ is analytic, as required.

We can write

$$\mathtt{z}(\mathtt{x}^2 + \mathtt{y}^2) - \mathtt{x}^3 = (\mathtt{x}^2 + \mathtt{y}^2)\Big(\mathtt{z} - \frac{\mathtt{x}^3}{\mathtt{x}^2 + \mathtt{y}^2}\Big).$$

Let $X_1 := \{\mathtt{z} - f(\mathtt{x}, \mathtt{y}) = 0\}$ and let $\gamma : (-1, 1) \to \mathbb{R}^3$ be an analytic arc such that $\gamma(-1, 0) \subset X_1$. As $(\mathtt{z} - f(\mathtt{x}, \mathtt{y})) \circ \gamma$ is analytic, by the identity principle for analytic functions we deduce $\gamma(-1, 1) \subset X_1$. Thus, the semi-algebraic set X_1 is arc-symmetric. As $X = X_1 \cup X_2$, where $X_2 := \{\mathtt{x}^2 + \mathtt{y}^2 = 0\}$, we conclude that X is a reducible arc-symmetric set.

It follows from Theorem 3.2 and [K, Cor.2.8] that a pure dimensional compact irreducible arc-symmetric semi-algebraic set is a Nash image of $\overline{\mathcal{B}}_d$ where $d := \dim(\mathcal{S})$.

Corollary 3.9.2. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a pure dimensional compact irreducible arc-symmetric semi-algebraic set of dimension d. Then $\mathcal{S}$ is a Nash image of $\overline{\mathcal{B}}_d$.*

Proof. Let X be the Zariski closure of $\mathcal{S}$ and let $\pi : \widetilde{X} \to X$ be a resolution of the singularities of X (see Theorem 2.4.2). Assume $\widetilde{X} \subset \mathbb{R}^p$ and π is the restriction to $\widetilde{X}$ of a polynomial map $\Pi : \mathbb{R}^p \to \mathbb{R}^n$. By [K, Thm.2.6] applied to the irreducible arc-symmetric set $\mathcal{S}$ there exists a connected component E of $\widetilde{X}$ such that $\pi(E) = \mathrm{Cl}(\mathrm{Reg}(\mathcal{S})) = \mathcal{S}$ (recall that $\mathcal{S}$ is pure dimensional and compact). As π is proper and $\mathcal{S}$ is compact, also E is compact (because it is a closed subset of the compact set $\pi^{-1}(\mathcal{S})$). Thus, E is a connected compact Nash manifold. By Theorem 3.2 there exists a Nash map $f_0 : \mathbb{R}^d \to \mathbb{R}^p$ such that $f_0(\overline{\mathcal{B}}_d) = E$. Consequently, the Nash map $f := \pi \circ f_0 : \mathbb{R}^d \to \mathbb{R}^n$ satisfies $f(\overline{\mathcal{B}}_d) = \pi(f_0(\overline{\mathcal{B}}_d)) = \pi(E) = \mathcal{S}$, as required. $\square$

3.9.2. Elimination of inequalities. A converse problem to Tarski's theorem is to find an algebraic set in $\mathbb{R}^{n+k}$ whose projection is a given semi-algebraic subset of $\mathbb{R}^n$. This is known as the *problem of eliminating inequalities.* Motzkin proved in [Mo] that this problem always has a solution for $k = 1$. However, his solution is rather complicated and is generally a reducible algebraic set. In another direction Andradas and Gamboa proved in [AG1, AG2] that if $\mathcal{S} \subset \mathbb{R}^n$ is a closed semi-algebraic set whose Zariski closure is irreducible, then $\mathcal{S}$ is the projection of an irreducible algebraic set in some $\mathbb{R}^{n+k}$. In [P] Pecker gives some improvements on both results: for the first by finding a construction of an algebraic set in $\mathbb{R}^{n+1}$ that projects onto the given semi-algebraic subset of $\mathbb{R}^n$, far simpler than the original construction of Motzkin; for the second by proving that if $\mathcal{S}$ is a locally closed semi-algebraic subset of $\mathbb{R}^n$ with an interior point, then $\mathcal{S}$ is the projection of an irreducible algebraic subset of $\mathbb{R}^{n+1}$. In [Fe4] it is proved that each semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ is the projection of a non-singular algebraic set $X \subset \mathbb{R}^{n+k}$ whose connected components are Nash diffeomorphic to affine spaces (maybe of different dimensions). Here we improve the previous result if $\mathcal{S}$ is compact and we prove that there exists an algebraic set $X \subset \mathbb{R}^{2d+1}$, where $d := \dim(\mathcal{S})$, that is Nash diffeomorphic to a finite pairwise disjoint union of spheres (maybe of different dimensions) that project onto $\mathcal{S}$. To guarantee that $X \subset \mathbb{R}^{2d+1}$ we use implicitly in the last part of the proof the weak version of the Whitney's immersion theorem.

Corollary 3.9.3. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact semi-algebraic set of dimension d. We have:*

(i) If $\mathcal{S}$ is connected by analytic paths, it is the projection of an irreducible compact non-singular algebraic set $X \subset \mathbb{R}^{n+k}$ (for some $k \geq 0$) that has at most two connected components Nash diffeomorphic to the d-dimensional sphere $\mathbb{S}^d$. In addition,

- *(1) Each connected component of X projects onto $\mathcal{S}$.*
- *(2) There exists an automorphism of X that swaps both connected components of X.*

(ii) In general $\mathcal{S}$ is the projection of an algebraic set $X \subset \mathbb{R}^{n+k}$ (for some $k \geq 0$) of dimension d that is Nash diffeomorphic to a finite pairwise disjoint union of spheres (maybe of different dimensions).

Even for dimension 1, it is not possible to impose the connectedness of X (see Lemma 3.9.5 and Example 3.9.6). Contrast the previous result with [Fe4, Cor.1.8].

To prove Corollary 3.9.3 we recall first the following well-known separation result, that we include here for the sake of completeness.

Lemma 3.9.4 (Separation)**.** *Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$ be semi-algebraic sets such that $\mathcal{S}_1$ is compact, $\mathcal{S}_2$ is closed and $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$. Then, there exists $f \in \mathbb{R}[\mathtt{x}]$ such that $\mathcal{S}_1 \subset \{f < 0\}$ and $\mathcal{S}_2 \subset \{f > 0\}$.*

Proof. We may assume $\mathcal{S}_1 \subset \mathcal{B}_n(0, \frac{1}{2})$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $\mathcal{S}_1 \subset \{g < 0\}$ and $\mathcal{S}_2 \subset \{g > 0\}$. Let

$$\varepsilon := \operatorname{dist}(\mathcal{S}_1, \mathcal{S}_2) := \min\{\operatorname{dist}(x_1, x_2) : \ x_1 \in \mathcal{S}_1, x_2 \in \mathcal{S}_2\} > 0.$$

By Weierstrass' approximation theorem there exists a polynomial $f_0 \in \mathbb{R}[\mathtt{x}]$ such that

$$\max\{|g(x) - f_0(x)| : \ x \in \overline{\mathcal{B}}_n(0,1)\} < \frac{\varepsilon}{3}.$$

By [BCR, Prop.2.6.2] there exists a constant $c > 0$ and $m \geq 1$ such that

$$|f_0(x)| < c(1 + \|x\|^2)^m$$

on $\mathbb{R}^n$. Thus, $|f_0(x)| < 2^m c\|x\|^{2m}$ on $\mathbb{R}^n \setminus \overline{\mathcal{B}}_n(0,1)$. Denote $c' := 2^m c$ and let $k \geq m$ be such that $\frac{c'}{2^{2k}} < \frac{\varepsilon}{3}$. Define $f := f_0 + c'\|\mathtt{x}\|^{2k} \in \mathbb{R}[\mathtt{x}]$. The reader can check that $\mathcal{S}_1 \subset \{f < 0\}$ and $\mathcal{S}_2 \subset \{f > 0\}$, as required. $\square$

Proof of Corollary 3.9.3. (i) By Theorem 3.2 and Proposition 3.1.1 there exists a Nash map $f : \mathbb{R}^{d+1} \to \mathbb{R}^n$ such that $f(\mathbb{S}^d) = \mathcal{S}$. By Artin-Mazur's description of Nash maps [BCR, Thm.8.4.4] there exist $s \geq 1$ and a non-singular irreducible algebraic set $Z \subset \mathbb{R}^{d+1+n+s}$ of dimension d, a connected component $\mathcal{M}$ of Z and a Nash diffeomorphism $g : \mathbb{S}^d \to \mathcal{M}$ such that the following diagram is commutative.

$$\begin{tikzcd}
Z \arrow[rr, hook] & & \mathbb{R}^{d+1} \times \mathbb{R}^n \times \mathbb{R}^s \equiv \mathbb{R}^m \arrow[ddll, "\pi_1"'] \arrow[dd, "\pi_2"] \\
\mathcal{M} \arrow[u, hook] & & \\
\mathbb{S}^d \arrow[u, "g"', "\cong"] \arrow[rr, "f"] \arrow[dr, "f"'] & & \mathbb{R}^n \\
& \mathcal{S} \arrow[ur, hook] &
\end{tikzcd}$$

We denote the projection of $\mathbb{R}^{d+1} \times \mathbb{R}^n \times \mathbb{R}^s$ onto the first space $\mathbb{R}^{d+1}$ with π_1 and the projection of $\mathbb{R}^{d+1} \times \mathbb{R}^n \times \mathbb{R}^s$ onto the second space $\mathbb{R}^n$ with π_2. Write $m := d+1+n+s$. As $\mathcal{M}$ is compact, there exists by Lemma 3.9.4 a polynomial $f : \mathbb{R}^m \to \mathbb{R}$ such that $\mathcal{M} = Z \cap \{f > 0\}$. Observe that $\mathcal{M}$ is the projection of the algebraic set

$$Y := \{(z,t) \in Z \times \mathbb{R} : \ f(z)t^2 - 1 = 0\}$$

under $\pi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m, \ (z,t) \mapsto z$. Fix $\epsilon = \pm 1$ and let $\mathcal{M}_\epsilon := Y \cap \{\epsilon t > 0\}$. Consider the Nash diffeomorphism

$$\varphi_\epsilon : \mathcal{M} \to \mathcal{M}_\epsilon, \ x \mapsto \Big(x, \epsilon \frac{1}{\sqrt{f(x)}}\Big)$$

whose inverse map is the restriction to $\mathcal{M}_\epsilon$ of the projection π.

Observe that $\{\mathcal{M}_\epsilon\}_{\epsilon\in\{-1,1\}}$ is the collection of the connected components of Y. As $\pi(\mathcal{M}_\epsilon)=\mathcal{M}$ and using the diagram above, we deduce

$$(\pi_2\circ\pi)(\mathcal{M}_\epsilon)=\pi_2(\mathcal{M})=(f\circ\pi_1)(\mathcal{M})=f(\mathbb{S}^d)=\mathcal{S}.$$

In addition, each $\mathcal{M}_\epsilon$ is Nash diffeomorphic to $\mathbb{S}^d$ and for $\varepsilon\neq\varepsilon'$ the polynomial map

$$\phi:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^m\times\mathbb{R},\ (x,t)\mapsto(x,-t)$$

induces an involution of Y such that $\phi(\mathcal{M}_\epsilon)=\mathcal{M}_{\epsilon'}$. As Z is non-singular, also Y is non-singular. Let X be the irreducible component of Y that contains $\mathcal{M}_{+1}$.

Then $k:=d+s+2$ and the non-singular algebraic set X satisfy the requirements in the statement.

In addition, X has at most two connected components and each of them is Nash diffeomorphic to $\mathbb{S}^d$. Thus, X is Nash diffeomorphic to $\mathbb{S}^d\times\{1,s\}$, where $s=1,2$ is the number of connected components of X.

(ii) Let $\mathcal{S}_1,\dots,\mathcal{S}_r$ be the (compact) analytic path-connected components of $\mathcal{S}$, which satisfy $\mathcal{S}=\bigcup_{i=1}^r\mathcal{S}_i$. By (i) there exist $m\geq 1$ and for each $i=1,\dots,r$ a non-singular algebraic set $X_i\subset\mathbb{R}^m$ that is Nash diffeomorphic to a disjoint union of at most two spheres of $\mathbb{R}^{d+1}$ (each of them isometric to $\mathbb{S}^{d_i}$ where $d_i:=\dim(\mathcal{S}_i)\leq d=\dim(\mathcal{S})$) and satisfies $\pi(X_i)=\mathcal{S}_i$, where

$$\pi:\mathbb{R}^n\times\mathbb{R}^{m-n}\to\mathbb{R}^n,\ (x,y)\mapsto x$$

is the projection onto the first n coordinates. Consider the pairwise disjoint union $X:=\bigsqcup_{i=1}^r X_i\times\{i\}\subset\mathbb{R}^{m+1}$ and the projection

$$\pi':\mathbb{R}^n\times\mathbb{R}^{m+1-n}\times\mathbb{R}\to\mathbb{R}^n,\ (x,y,t)\mapsto x.$$

Then X is a non-singular algebraic set, which is Nash diffeomorphic to a finite pairwise disjoint union of spheres of dimension $\leq d$ and satisfies $\pi(X)=\mathcal{S}$, as required. $\square$

The following lemma together with the subsequent example shows that Corollary 3.9.3 is somehow sharp.

Lemma 3.9.5. *Let $Z\subset\mathbb{R}^m$ be a non-singular irreducible algebraic set and let $\mathcal{M}$ one of its connected components of maximal dimension d. Suppose that there exists an irreducible algebraic set $Y\subset\mathbb{R}^p$ of dimension d and a rational map $\varphi:\mathbb{R}^p\dashrightarrow\mathbb{R}^m$ such that $\varphi|_Y:Y\to\mathcal{M}$ is bijective. Then $\mathcal{M}$ is the unique connected component of Z of dimension d.*

Proof. Let $\widetilde{Y}\subset\mathbb{C}^p$ be the complexification of Y and let $\widetilde{Z}\subset\mathbb{C}$ be the complexification of Z. Observe that $\widetilde{Y},\widetilde{Z}$ are irreducible algebraic sets of (complex) dimension d. Consider the rational map $\widetilde{\varphi}:\mathbb{C}^p\dashrightarrow\mathbb{C}^m$ that extends φ. As $\varphi(Y)=\mathcal{M}\subset Z\subset\widetilde{Z}$, the Zariski closure of $\varphi(Y)$ in $\mathbb{C}^m$ is contained in $\widetilde{Z}$. As $\mathcal{M}$ has (real) dimension d and $\widetilde{Z}$ is an irreducible algebraic set of $\mathbb{C}^n$ of (complex) dimension d, we deduce that $\widetilde{Z}$ is the Zariski closure of $\varphi(Y)$. As $\widetilde{\varphi}:\mathbb{C}^p\dashrightarrow\mathbb{C}^m$ is continuous for the Zariski topology, $\widetilde{Y}$ is the Zariski closure of Y and $\widetilde{Z}$ is the

Zariski closure of $\varphi(Y) = \widetilde{\varphi}(Y)$, we conclude $\widetilde{\varphi}(\widetilde{Y}) \subset \widetilde{Z}$ and the Zariski closure of $\widetilde{\varphi}(\widetilde{Y})$ is $\widetilde{Z}$. Thus, $\widetilde{\varphi}|_{\widetilde{Y}} : \widetilde{Y} \dashrightarrow \widetilde{Z}$ is a dominant rational map. Denote $\mathcal{M}(\widetilde{Y})$ the field of meromorphic functions on $\widetilde{Y}$ and $\mathcal{M}(\widetilde{Z})$ the field of meromorphic functions on $\widetilde{Z}$. The map

$$\widetilde{\varphi}^* : \mathcal{M}(\widetilde{Z}) \to \mathcal{M}(\widetilde{Y}),\ f \mapsto f \circ \widetilde{\varphi}$$

is a homomorphism of fields of the same transcendence degree d over $\mathbb{C}$. Consequently, $\mathcal{M}(\widetilde{Y})$ is an algebraic extension of $\mathcal{M}(\widetilde{Z})$ of finite degree m. By [Ha, Prop.7.16] the number of points in a general fiber of $\widetilde{\varphi}$ is equal to m. As $\mathcal{M}$ has (real) dimension d, there exists a point $p \in \mathcal{M}$ such that the fiber $\widetilde{\varphi}^{-1}(p)$ has exactly m points. As Y is a (real) algebraic set, $\widetilde{Y} \cap \mathbb{R}^p = Y$. As φ is a real rational map and $\varphi|_Y : Y \to \mathcal{M}$ is bijective, we conclude that m is odd, because if $z \in \widetilde{Y} \setminus Y$ and $\widetilde{\varphi}(z) = p$, then $\overline{z} \in \widetilde{Y} \setminus Y$ and $\widetilde{\varphi}(\overline{z}) = p$.

Suppose Z has another connected component $\mathcal{M}'$ of dimension d. Then there exists $q \in \mathcal{M}'$ such that $\widetilde{\varphi}^{-1}(q)$ has exactly m points. As $\widetilde{Y} \cap \mathbb{R}^p = Y$ and $\widetilde{\varphi}(Y) = \mathcal{M}$, we conclude that $\widetilde{\varphi}^{-1}(q) \subset \widetilde{Y} \setminus Y$. As $q \in Z = \widetilde{Z} \cap \mathbb{R}^n$, we deduce that if $z \in \widetilde{\varphi}^{-1}(q)$, also $\overline{z} \in \widetilde{\varphi}^{-1}(q)$. Thus, $\widetilde{\varphi}^{-1}(q)$ consists of an even number of elements, which is a contradiction because m is odd. Consequently, $\mathcal{M}$ is the unique connected component of Z of dimension d, as required. $\square$

Example 3.9.6. Let $X := \{\mathtt{y}^2 = -(\mathtt{x}-1)(\mathtt{x}-2)(\mathtt{x}+1)\} \subset \mathbb{R}^2$ which is an irreducible non-singular cubic with two connected components of dimension 1, one is bounded (that we denote C_1) and the other one is unbounded (that we denote C_2). Consider the polynomial $\mathtt{x}$, which satisfies $X \cap \{\mathtt{x} > 0\} = C_1$ and $X \cap \{\mathtt{x} < 0\} = C_2$. Let $Y := \{(x,y,z) \in X \times \mathbb{R} :\ xz^2 - 1 = 0\} \subset \mathbb{R}^3$, which has exactly two connected componentes $\mathcal{M}_1 := Y \cap \{\mathtt{z} > 0\}$ and $\mathcal{M}_2 := Y \cap \{\mathtt{z} < 0\}$ and both have dimension 1.

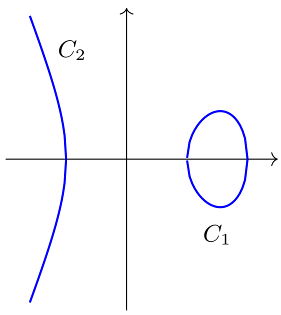

Figure 3.10: The cubic curve $\mathtt{y}^2 = -(\mathtt{x}-1)(\mathtt{x}-2)(\mathtt{x}+1)$.

Suppose there exists an algebraic set $Z \subset \mathbb{R}^p$ of dimension 1 and a polynomial map $\varphi : \mathbb{R}^p \to \mathbb{R}^3$ such that $\varphi|_Z : Z \to \mathcal{M}_1$ is bijective. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$, which satisfies $\pi|_{\mathcal{M}_1} : \mathcal{M}_1 \to C_1$ is bijective. Thus, the composition $\pi \circ f : \mathbb{R}^p \to \mathbb{R}^2$ is a polynomial map that satisfies $\pi|_Z : Z \to C_1$ is bijective, but this contradicts Lemma 3.9.5. Consequently, there does not exist the couple (φ, Z).

The previous example suggests that in Corollary 3.9.3(i) two connected components Nash diffeomorphic to $\mathbb{S}^d$ are needed in many cases.

Chapter 4

Resolution of semi-algebraic sets connected by analytic paths

Once achieved a complete characterization of Nash images of closed balls in Theorem 3.1, a natural question at this point is to determine until what extend we can represent semi-algebraic sets connected by analytic paths using polynomial maps. Polynomial images of models connected by polynomial paths (e.g. Euclidean spaces, closed balls etc.) are connected by polynomial paths. In general, semi-algebraic sets do not contain rational paths. By [C, V] a generic complex hypersurface Z of $\mathbb{CP}^m$ of degree $d \geq 2m-2$ for $m \geq 4$ and of degree $d \geq 2m-1$ for $m = 2, 3$ does not contain rational curves. If $\mathcal{S}$ is a semi-algebraic set whose Zariski closure in $\mathbb{RP}^m$ is a generic hypersurface of high enough degree, then its Zariski closure Z in $\mathbb{CP}^m$ does not contains rational curves, so $\mathcal{S}$ cannot contain rational paths. This means in particular that general semi-algebraic sets do not contain polynomial paths.

If $\mathcal{S} \subset \mathbb{R}^m$ is a closed semi-algebraic set connected by analytic paths, we show that $\mathcal{S}$ is the image under a proper polynomial map of a Nash manifold with corners of the same dimension. In fact, there exists an algebraic set of smaller dimension such that the restriction of the polynomial map to the Nash manifold with corners minus this algebraic set is a Nash diffeomorphism onto its image.

Theorem 4.1. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a d-dimensional closed semi-algebraic set connected by analytic paths. Then there exist:*

(i) *A d-dimensional non-singular irreducible algebraic set $X \subset \mathbb{R}^n$ and a normal-crossings divisor $Y \subset X$.*

(ii) *A connected Nash manifold with corners $\mathcal{Q} \subset X$ (which is a closed subset of X) whose boundary $\partial\mathcal{Q}$ has Y as its Zariski closure.*

(iii) *A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that the restriction $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{S}$ is proper and $f(\mathcal{Q}) = \mathcal{S}$.*

(iv) *A closed semi-algebraic set* $\mathcal{R} \subset \mathcal{S}$ *of dimension strictly smaller than* d *such that* $\mathcal{S} \setminus \mathcal{R}$ *and* $\mathcal{Q} \setminus f^{-1}(\mathcal{R})$ *are Nash manifolds and the polynomial map* $f|_{\mathcal{Q} \setminus f^{-1}(\mathcal{R})} : \mathcal{Q} \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ *is a Nash diffeomorphism.*

If $\mathcal{S} \subset \mathbb{R}^m$ is a general semi-algebraic set connected by analytic paths, one can wonder if it is possible to provide a similar result that also works for $\mathcal{S}$. As the chosen Nash manifold with corners $\mathcal{Q}$ is closed in its Zariski closure and the chosen polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ restricts to a proper map $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^m$, its image $\mathcal{S}$ is a closed subset of $\mathbb{R}^m$. Thus, if $\mathcal{S}$ is not closed in $\mathbb{R}^m$, we should change the type of domain and/or the type of map. The second approach considering general Nash maps non-necessarily proper has been developed in [Fe4, Proof of Thm.1.4, §8.C.12] and it is shown that if the involved Nash map is not necessarily proper, then there exists a Nash manifold H with smooth boundary and a surjective Nash map $f : H \to \mathcal{S}$. If one wants to keep the properness condition, it is not possible to keep as domains Nash manifolds $\mathcal{Q}$ with corners because they are locally compact and images of locally compact subset of $\mathbb{R}^n$ under proper maps are locally compact subsets of $\mathbb{R}^m$. Thus, we have to change the type of involved domains and we will consider semi-algebraic sets $\mathcal{T} \subset \mathbb{R}^n$ whose closure is a Nash manifold with corners $\mathcal{Q} \subset \mathbb{R}^n$ and $\mathcal{Q} \setminus \mathcal{T}$ is a union of some of the strata of the a suitable stratification of $\partial\mathcal{Q}$. A Nash quasi-manifold with corners is a Nash manifold with corners with some faces erased (the precise definition is included in Section 4.3).

Theorem 4.2. *Let* $\mathcal{S} \subset \mathbb{R}^m$ *be a* d*-dimensional semi-algebraic set connected by analytic paths. Then there exist:*

(i) *A* d*-dimensional connected compact non-singular algebraic set* $M \subset \mathbb{R}^n$ *and a normal-crossings divisor* $Y \subset M$*.*

(ii) *A connected Nash quasi-manifold with corners* $\mathcal{S}^\bullet \subset M$ *that is a checkerboard set and whose closure in* M *is a compact connected Nash manifold with corners* $\mathcal{Q}^\bullet \subset M$ *whose boundary* $\partial\mathcal{Q}^\bullet$ *has* Y *as its Zarsiki closure.*

(iii) *A Nash map* $f : \mathbb{R}^n \to \mathbb{R}^m$ *such that the restriction* $f|_{\mathcal{S}^\bullet} : \mathcal{S}^\bullet \to \mathcal{S}$ *is proper and* $f(\mathcal{S}^\bullet) = \mathcal{S}$*.*

(iv) *A closed semi-algebraic set* $\mathcal{R} \subset \mathcal{S}$ *of dimension strictly smaller than* d *such that* $\mathcal{S} \setminus \mathcal{R}$ *and* $\mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R})$ *are Nash manifolds and the Nash map* $f|_{\mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R})} : \mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ *is a Nash diffeomorphism.*

4.1 Drilling blow-up

In [Fe4] Fernando introduced the concept of *drilling blow-up* of a Nash manifold M with center a closed Nash submanifold N. We refer the reader to [S, Hi2] for the oriented blow-up of a real analytic space with center a closed subspace, which is the counterpart of the construction of Fernando in the real analytic setting. In [HPV, §5] appears a presentation of the oriented blow-up in the analytic case closer to the drilling blow-up described by Fernando. The authors consider there the case of the oriented blow-up of a real analytic manifold M with center a closed real analytic submanifold N whose vanishing ideal inside M is finitely generated (this happens for instance if N is compact). In [Fe3, §3] it is presented a similar construction in the semi-algebraic setting, which is used to 'appropriately embed' semi-algebraic sets in Euclidean space. In this section we will describe the construction made by Fernando [Fe4, §5] of the drilling blow-up with the main properties. We add some results that we need in the following sections.

4.1.1. Local structure of the drilling blow-up. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. As we are interested in the local structure, assume that there exists a Nash diffeomorphism $u := (u_1, \dots, u_d) : M \to \mathbb{R}^d$ such that

$$N = \{u_{e+1} = 0, \dots, u_d = 0\}.$$

Denote $\psi := u^{-1} : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to M$. Let $\zeta_{e+1}, \dots, \zeta_d : \mathbb{R}^d \to \mathbb{R}^k$ be Nash maps such that the vectors $\zeta_{e+1}(y,0), \dots, \zeta_d(y,0)$ are linearly independent for each $y \in \mathbb{R}^e$. Write $z \in \mathbb{R}^{d-e}$ as $z := (z_{e+1}, \dots, z_d)$. Consider the Nash maps

$$\begin{aligned}
&\varphi : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to \mathbb{R}^k,\ (y,z) \mapsto \zeta_{e+1}(y,z)z_{e+1} + \ldots + \zeta_d(y,z)z_d,\\
&\phi : \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1} \to \mathbb{R}^k,\ (y,\rho,w) \mapsto \zeta_{e+1}(y,\rho w)w_{e+1} + \cdots + \zeta_d(y,\rho w)w_d
\end{aligned}$$

and assume that $\varphi(y,z) = 0$ if and only if $z = 0$. Consider the projections

$$\begin{aligned}
&\theta_1 : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to \mathbb{R}^e,\ (y,z) \mapsto y,\\
&\theta_2 : \mathbb{R}^d \equiv \mathbb{R}^e \times \mathbb{R}^{d-e} \to \mathbb{R}^{d-e},\ (y,z) \mapsto z.
\end{aligned}$$

Consider the (well-defined) Nash map:

$$\Phi : \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1} \to M \times \mathbb{S}^{k-1},\ (y,\rho,w) \mapsto \Big(\psi(y,\rho w), \frac{\phi(y,\rho,w)}{\|\phi(y,\rho,w)\|}\Big).$$

Fact 4.1.1. *Fix $\epsilon = \pm$ and denote*

$$I_\epsilon := \begin{cases} [0,+\infty) & \text{if } \epsilon = +,\\ (-\infty, 0] & \text{if } \epsilon = -. \end{cases}$$

The closure $\widetilde{M}_\epsilon$ in $M \times \mathbb{S}^{k-1}$ of the set

$$\Gamma_\epsilon := \Big\{\Big(\psi(y,z), \epsilon\frac{\varphi(y,z)}{\|\varphi(y,z)\|}\Big) \in M \times \mathbb{S}^{k-1} :\ z \neq 0\Big\}$$

is a Nash manifold with boundary such that:

(i) $\widetilde{M}_\epsilon \subset \operatorname{im}(\Phi)$.

(ii) *The restriction of Φ to $\mathbb{R}^e \times I_\epsilon \times \mathbb{S}^{d-e-1}$ induces a Nash diffeomorphism between $\mathbb{R}^e \times I_\epsilon \times \mathbb{S}^{d-e-1}$ and $\widetilde{M}_\epsilon$. Consequently,*

$$\partial\widetilde{M}_\epsilon = \Phi(\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})$$

and $\Gamma_\epsilon = \operatorname{Int}(\widetilde{M}_\epsilon) = \Phi(\mathbb{R}^e \times (I_\epsilon \setminus \{0\}) \times \mathbb{S}^{d-e-1})$.

Fact 4.1.2. *Denote $\mathcal{R} := \partial\widetilde{M}_+ = \partial\widetilde{M}_-$ and $\widehat{M} := \widetilde{M}_+ \cup \widetilde{M}_- = \Gamma_+ \sqcup \mathcal{R} \sqcup \Gamma_-$. Then Φ induces a Nash diffeomorphism between $\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$ and $\widehat{M}$, which is the Nash closure of $\widetilde{M}_+$ and $\widetilde{M}_-$ in $M \times \mathbb{S}^{k-1}$. In addition, the Nash map $\sigma : M \times \mathbb{S}^{k-1} \to M \times \mathbb{S}^{k-1}, (a, b) \to (a, -b)$ induces a Nash involution on $\widehat{M}$ without fixed points such that $\sigma(\widetilde{M}_+) = \widetilde{M}_-$ and $\Phi(y, -\rho, -w) = (\sigma \circ \Phi)(y, \rho, w)$ for each $(y, \rho, w) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$.*

Fact 4.1.3. *Consider the projection $\pi : M \times \mathbb{S}^{k-1} \to M$ onto the first factor and denote $\pi_\epsilon := \pi|_{\widetilde{M}_\epsilon}$. Then*

(i) *π_ϵ is proper, $\pi_\epsilon(\widetilde{M}_\epsilon) = M$ and $\mathcal{R} = \pi_\epsilon^{-1}(N)$.*

(ii) *The restriction $\pi_\epsilon|_{\Gamma_\epsilon} : \Gamma_\epsilon \to M \setminus N$ is a Nash diffeomorphism.*

(iii) *For each $q \in N$ it holds $\pi_\epsilon^{-1}(q) = \{q\} \times \mathbb{S}_q^{d-e-1}$ where $\mathbb{S}_q^{d-e-1}$ is the sphere of dimension $d-e-1$ obtained when intersecting the sphere $\mathbb{S}^{k-1}$ with the linear subspace L_q generated by $(\zeta_{e+1} \circ u)(q), \ldots, (\zeta_d \circ u)(q)$.*

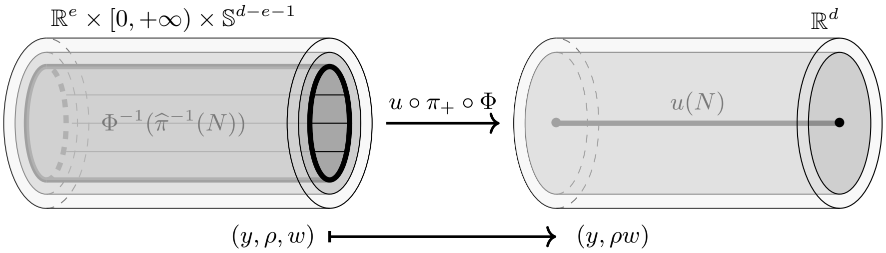

Figure 4.1: Local structure of the drilling blow-up $\widetilde{M}_+$ of M of center N (figure borrowed from [Fe4, Fig.3]).

Denote $\widehat{\pi} := \pi|_{\widehat{M}}$ and consider the commutative diagram.

$$
\begin{tikzcd}
\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1} \arrow[r, "\Phi", "\cong"'] \arrow[d, "u\circ\widehat{\pi}\circ\Phi"'] & \widehat{M} \arrow[d, "\widehat{\pi}"] \\
\mathbb{R}^d & M \arrow[l, "u"', "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(y,\rho,w) \arrow[r, mapsto] \arrow[d, mapsto] & \Phi(y,\rho,w) \arrow[d, mapsto] \\
(y,\rho w) & \psi(y,\rho w) \arrow[l, mapsto]
\end{tikzcd}
\tag{4.1.1}
$$

As a consequence, we have: *The Nash maps π_ϵ and $\widehat{\pi}$ have local representations*

$$(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_e, x_{e+1}, x_{e+1}x_{e+2}, \ldots, x_{e+1}x_d)$$

in an open neighbourhood of each point $p \in \mathcal{R}$. In addition, $d\pi_p(T_p\widehat{M}) \not\subset T_{\pi(p)}N$.

Proof ([Fe4, 5.A.5]). After a change of coordinates in $\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$, we may assume that $p \in \mathcal{R}$ is the image of the point $(0, 0, (1, 0, \ldots, 0))$. Consider the local parametrization around $(0, 0, (1, 0, \ldots, 0))$ of the set $\mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1}$ given by

$$\eta : \mathbb{R}^e \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}^e \times \mathbb{R} \times \mathbb{S}^{d-e-1},$$
$$(y, \rho, v := (v_{e+2}, \ldots, v_d)) \mapsto (y, \rho, (\sqrt{1 - \|v\|^2}, v))$$

where $\mathcal{B}$ is the open ball of center the origin and radius 1 in $\mathbb{R}^{d-e-1}$. It holds

$$u \circ \widehat{\pi} \circ \Phi \circ \eta : \mathbb{R}^e \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}^d,\ (y, \rho, v) \mapsto (y, \rho\sqrt{1 - \|v\|^2}, \rho v).$$

Consider the Nash diffeomorphism

$$f : \mathbb{R}^e \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}^d,\ (y, \rho, v) \to \Big(y, \rho\sqrt{1 - \|v\|^2}, \frac{v}{\sqrt{1 - \|v\|^2}}\Big),$$

whose inverse is

$$f^{-1} : \mathbb{R}^d \to \mathbb{R}^e \times \mathbb{R} \times \mathcal{B},\ (y, \rho', v') \mapsto \Big(y, \rho'\sqrt{1 + \|v'\|^2}, \frac{v'}{\sqrt{1 + \|v'\|^2}}\Big).$$

The Nash map

$$\pi' := u \circ \widehat{\pi} \circ \Phi \circ \eta \circ f^{-1} : \mathbb{R}^d \to \mathbb{R}^d,\ (y, \rho', v') \mapsto (y, \rho', \rho' v').$$

represents $\widehat{\pi}$ locally around p and the restriction

$$\pi'_\epsilon := \pi'|_{\{\epsilon\rho' \geq 0\}} : \{\epsilon\rho' \geq 0\} \to \mathbb{R}^d,\ (y, \rho', v') \mapsto (y, \rho', \rho' v').$$

represents π_ϵ locally around p.

To prove that $d\pi_p(T_p\widehat{M}) \not\subset T_{\pi(p)}N$, it is enough to show $d\pi'_0(\mathbb{R}^d) \not\subset u(N)$. It holds, $d\pi'_0(\mathtt{e}_{e+1}) = \mathtt{e}_{e+1} \notin u(N)$, as required. □

Remarks 4.1.4. Denote $g := u \circ \widehat{\pi} \circ \Phi$ and

$$g_+ := g|_{\mathbb{R}^e \times [0,+\infty) \times \mathbb{S}^{d-e-1}} = u \circ \pi_+ \circ \Phi|_{\mathbb{R}^e \times [0,+\infty) \times \mathbb{S}^{d-e-1}}.$$

Consider the Nash normal-crossings divisor $Z := \{\mathtt{y}_{e+1} \cdots \mathtt{y}_d = 0\} \subset \mathbb{R}^d$. Consider coordinates $(w_{e+1}, \ldots, w_d)$ in $\mathbb{R}^{d-e}$ and the sphere

$$\mathbb{S}^{d-e-1} = \{\mathtt{w}_{e+1}^2 + \ldots + \mathtt{w}_d^2 = 1\}.$$

(i) Write $Z_k := \{\mathtt{y}_k = 0\}$ for $k = e+1, \ldots, d$ and observe that

$$\begin{aligned} g^{-1}(Z_k) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1}) \cup (\mathbb{R}^e \times \mathbb{R} \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_k = 0\})), \\ g_+^{-1}(Z_k) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1}) \cup (\mathbb{R}^e \times [0,+\infty) \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_k = 0\})) \end{aligned}$$

for $k = e+1, \ldots, d$. Thus,

$$\begin{aligned} g^{-1}(Z) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1}) \cup \bigcup_{k=e+1}^{d} (\mathbb{R}^e \times \mathbb{R} \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_k = 0\})), \\ g_+^{-1}(Z) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1}) \cup \bigcup_{k=e+1}^{d} (\mathbb{R}^e \times [0,+\infty) \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_k = 0\})) \end{aligned}$$

are Nash normal-crossings divisors.

(ii) Let $\epsilon := (\epsilon_{e+1}, \ldots, \epsilon_d)$ where $\epsilon_k = \pm 1$ and denote

$$\mathcal{Q}_\epsilon := \{\epsilon_{e+1}\mathtt{y}_{e+1} \geq 0, \ldots, \epsilon_d \mathtt{y}_d \geq 0\}.$$

Write $-\epsilon := (-\epsilon_{e+1}, \dots, -\epsilon_d)$. We have:

$$\mathrm{Cl}(g_+^{-1}(\mathcal{Q}_\epsilon \setminus Z)) = \mathbb{R}^e \times \{\rho \geq 0\} \times (\mathbb{S}^{d-e-1} \cap \{\epsilon_{e+1}\mathtt{w}_{e+1} \geq 0, \dots, \epsilon_d \mathtt{w}_d \geq 0\}).$$

Consequently,

$$\mathrm{Cl}(g_+^{-1}(\mathcal{Q}_\epsilon \setminus Z)) \cap \mathrm{Cl}(g_+^{-1}(\mathcal{Q}_{-\epsilon} \setminus Z)) = \varnothing.$$

More generally, if

$$\epsilon := (\epsilon_{e+1}, \dots, \epsilon_m, \epsilon_{m+1}, \dots, \epsilon_d),\ \epsilon' := (\epsilon_{e+1}, \dots, \epsilon_m, -\epsilon_{m+1}, \dots, -\epsilon_d)$$

where $e < m < d$, then

$$\mathcal{Q}_\epsilon \cap \mathcal{Q}_{\epsilon'} = \{\epsilon_{e+1}\mathtt{y}_{e+1} \geq 0, \dots,\ \epsilon_m \mathtt{y}_m \geq 0,\ \mathtt{y}_{m+1} = 0, \dots, \mathtt{y}_d = 0\},$$

which has dimension $e + (d-e) - (d-m) = m$. In addition,

$$\begin{aligned}&\mathrm{Cl}(g_+^{-1}(\mathcal{Q}_\epsilon \setminus Z)) \cap \mathrm{Cl}(g_+^{-1}(\mathcal{Q}_{\epsilon'} \setminus Z))\\ &= \mathbb{R}^e \times \{\rho \geq 0\} \times (\mathbb{S}^{d-e-1} \cap \{\epsilon_{e+1}\mathtt{w}_{e+1} \geq 0, \dots, \epsilon_m \mathtt{w}_m \geq 0, \mathtt{w}_{m+1} = 0, \dots, \mathtt{w}_d = 0\}),\end{aligned}$$

which has dimension $e + 1 + (d - e - 1 - (d - m)) = m$.

(iii) Let Y_1, Y_2 be intersections of dimension $e+1$ of irreducible components of Z that contain N. We may assume $Y_1 = \{\mathtt{y}_{e+1} = 0, \dots, \mathtt{y}_{d-1} = 0\}$ and $Y_2 = \{\mathtt{y}_{e+1} = 0, \dots, \mathtt{y}_{d-2} = 0, \mathtt{y}_d = 0\}$, so $Y_1 \cap Y_2 = \{\mathtt{y}_{e+1} = 0, \dots, \mathtt{y}_d = 0\} = N$. Thus,

$$\begin{aligned}
g^{-1}(Y_1) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})\\
&\quad \cup (\mathbb{R}^e \times \mathbb{R} \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_{e+1} = 0, \dots, \mathtt{w}_{d-1} = 0\})),\\
g_+^{-1}(Y_1) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})\\
&\quad \cup (\mathbb{R}^e \times [0, +\infty) \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_{e+1} = 0, \dots, \mathtt{w}_{d-1} = 0\})),\\
g^{-1}(Y_2) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})\\
&\quad \cup (\mathbb{R}^e \times \mathbb{R} \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_{e+1} = 0, \dots, \mathtt{w}_{d-2} = 0, \mathtt{w}_d = 0\})),\\
g_+^{-1}(Y_2) &= (\mathbb{R}^e \times \{0\} \times \mathbb{S}^{d-e-1})\\
&\quad \cup (\mathbb{R}^e \times [0, +\infty) \times (\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_{e+1} = 0, \dots, \mathtt{w}_{d-2} = 0, \mathtt{w}_d = 0\})).
\end{aligned}$$

As the intersection

$$\mathbb{S}^{d-e-1} \cap \{\mathtt{w}_{e+1} = 0, \dots, \mathtt{w}_{d-1} = 0\} \cap \{\mathtt{w}_{e+1} = 0, \dots, \mathtt{w}_{d-2} = 0, \mathtt{w}_d = 0\}$$

is empty, we conclude that the intersection

$$g^{-1}(Y_1) \cap \mathrm{Cl}(g^{-1}(Y_1 \setminus N)) \cap g^{-1}(Y_2) \cap \mathrm{Cl}(g^{-1}(Y_2 \setminus N))$$

of the strict transforms of Y_1, Y_2 under g is also empty. Analogously, the intersection

$$g_+^{-1}(Y_1) \cap \mathrm{Cl}(g_+^{-1}(Y_1 \setminus N)) \cap g_+^{-1}(Y_2) \cap \mathrm{Cl}(g_+^{-1}(Y_2 \setminus N))$$

of the strict transforms of Y_1, Y_2 under g_+ is empty. □

We analyze next all we know about the local structure of drilling blow-up when N has dimension $d-1$.

Fact 4.1.5. Assume N has dimension $e = d-1$. The Nash diffeomorphism $u := (u_1, \dots, u_d) : M \to \mathbb{R}^d$ satisfies $N = \{u_d = 0\}$. Recall that

$$\psi := u^{-1} : \mathbb{R}^d \equiv \mathbb{R}^{d-1} \times \mathbb{R} \to M$$

and $\zeta_d : \mathbb{R}^d \to \mathbb{R}^k$ is a Nash map that does not vanish and define the Nash map

$$\Phi : \mathbb{R}^{d-1} \times \mathbb{R} \times \{\pm 1\} \to M \times \mathbb{S}^{k-1},\ (y, \rho, \pm 1) \mapsto \Big(\psi(y, \pm\rho), \pm \frac{\zeta_d(y, \pm\rho)}{\|\zeta_d(y, \pm\rho)\|}\Big).$$

Fix $\epsilon = \pm$ and denote

$$I_\epsilon := \begin{cases} [0, +\infty) & \text{if } \epsilon = +, \\ (-\infty, 0] & \text{if } \epsilon = -. \end{cases}$$

The closure $\widetilde{M}_\epsilon$ in $M \times \mathbb{S}^{k-1}$ of the set

$$\Gamma_\epsilon := \Big\{ \Big(\psi(y, z), \epsilon \frac{z}{|z|} \frac{\zeta_d(y, z)}{\|\zeta_d(y, z)\|}\Big) \in M \times \mathbb{S}^{k-1} : z \neq 0 \Big\}$$

is a Nash manifold with boundary such that:

(i) $\widetilde{M}_\epsilon \subset \operatorname{im}(\Phi)$.

(ii) The restriction of Φ to $\mathbb{R}^{d-1} \times I_\epsilon \times \{\pm 1\}$ induces a Nash diffeomorphism between $\mathbb{R}^{d-1} \times I_\epsilon \times \{\pm 1\}$ and $\widetilde{M}_\epsilon$. Consequently,

$$\partial \widetilde{M}_\epsilon = \Phi(\mathbb{R}^{d-1} \times \{0\} \times \{\pm 1\})$$

and $\Gamma_\epsilon = \operatorname{Int}(\widetilde{M}_\epsilon) = \Phi(\mathbb{R}^{d-1} \times (I_\epsilon \setminus \{0\}) \times \{\pm 1\})$.

Denote $\mathcal{R} := \partial \widetilde{M}_+ = \partial \widetilde{M}_-$ and $\widehat{M} := \widetilde{M}_+ \cup \widetilde{M}_- = \Gamma_+ \sqcup \mathcal{R} \sqcup \Gamma_-$. Then Φ induces a Nash diffeomorphism between $\mathbb{R}^{d-1} \times \mathbb{R} \times \{\pm 1\}$ and $\widehat{M}$, which is the Nash closure of $\widetilde{M}_+$ and $\widetilde{M}_-$ in $M \times \mathbb{S}^{k-1}$. In addition, the Nash map $\sigma : M \times \mathbb{S}^{k-1} \to M \times \mathbb{S}^{k-1}, (a, b) \to (a, -b)$ induces a Nash involution on $\widehat{M}$ without fixed points such that $\sigma(\widetilde{M}_+) = \widetilde{M}_-$ and $\Phi(y, -\rho, \pm 1) = (\sigma \circ \Phi)(y, \rho, \mp 1)$ for each $(y, \rho, \pm 1) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \{\pm 1\}$.

Consider the projection $\pi : M \times \mathbb{S}^{k-1} \to M$ onto the first factor and denote $\pi_\epsilon := \pi|_{\widetilde{M}_\epsilon}$. Then

(i) π_ϵ is proper, $\pi_\epsilon(\widetilde{M}_\epsilon) = M$ and $\mathcal{R} = \pi_\epsilon^{-1}(N)$.

(ii) The restriction $\pi_\epsilon|_{\Gamma_\epsilon} : \Gamma_\epsilon \to M \setminus N$ is a Nash diffeomorphism.

(iii) For each $q \in N$ it holds $\pi_\epsilon^{-1}(q) = \{q\} \times \{\pm \frac{\{(\zeta_d \circ u)(q)\}}{\|\{(\zeta_d \circ u)(q)\}\|}\}$, that is, each point $q \in N$ has exactly two preimages under π_ϵ.

Denote $\widehat{\pi} := \pi|_{\widehat{M}}$ and consider the commutative diagram.

$$\begin{tikzcd} \mathbb{R}^{d-1} \times \mathbb{R} \times \{\pm 1\} \arrow[r, "\Phi", "\cong"'] \arrow[d, "u \circ \widehat{\pi} \circ \Phi"'] & \widehat{M} \arrow[d, "\widehat{\pi}"] \\ \mathbb{R}^d & M \arrow[l, "u"', "\cong"] \end{tikzcd} \qquad \begin{tikzcd} (y, \rho, \pm 1) \arrow[r, mapsto] \arrow[d, mapsto] & \Phi(y, \rho, \pm 1) \arrow[d, mapsto] \\ (y, \pm\rho) & \psi(y, \pm\rho) \arrow[l, mapsto] \end{tikzcd} \tag{4.1.2}$$

As a consequence, we have: *The Nash maps π_ϵ and $\widehat{\pi}$ have local representations*

$$(x_1,\ldots,x_d)\mapsto(x_1,\ldots,x_d)$$

in an open neighbourhood of each point $p\in\mathcal{R}$. In addition, $d\pi_p(T_p\widehat{M})\not\subset T_{\pi(p)}N$.

4.1.2. Global definition. Let $M\subset\mathbb{R}^m$ be a d-dimensional Nash manifold and $N\subset M$ a closed e-dimensional Nash submanifold. Let $f_1,\ldots,f_k\in\mathcal{N}(M)$ be a finite system of generators of the ideal $\mathcal{I}(N)$ of Nash functions on M vanishing identically on N. Consider the Nash map

$$F:M\setminus N\to\mathbb{S}^{k-1},x\mapsto\frac{(f_1(x),\ldots,f_k(x))}{\|(f_1(x),\ldots,f_k(x))\|}.$$

We have:

Fact 4.1.6. *Fix $\epsilon=\pm$. The closure $\widetilde{M}_\epsilon$ in $M\times\mathbb{S}^{k-1}$ of the graph*

$$\Gamma_\epsilon:=\{(x,\epsilon F(x))\in M\times\mathbb{S}^{k-1}:\ x\in M\setminus N\}$$

is a Nash manifold with boundary. Denote $\mathcal{R}:=\partial\widetilde{M}_+=\partial\widetilde{M}_-$ and $\widehat{M}:=\widetilde{M}_+\cup\widetilde{M}_-=\Gamma_+\sqcup\mathcal{R}\sqcup\Gamma_-$, which is the Nash closure of $\widetilde{M}_+$ and $\widetilde{M}_-$ in $M\times\mathbb{S}^{k-1}$ if M is connected. In addition, $\widehat{M}$ is a Nash manifold and the Nash map

$$\sigma:M\times\mathbb{S}^{k-1}\to M\times\mathbb{S}^{k-1},\ (a,b)\to(a,-b)$$

induces a Nash involution on $\widehat{M}$ without fixed points that maps $\widetilde{M}_+$ onto $\widetilde{M}_-$.

Fact 4.1.7. *Consider the projection $\pi:M\times\mathbb{S}^{k-1}\to M$ onto the first factor. Denote $\pi_\epsilon:=\pi|_{\widetilde{M}_\epsilon}$ and $\widehat{\pi}:=\pi|_{\widehat{M}}$. We have:*

- *(i) π_ϵ is proper, $\pi_\epsilon(\widetilde{M}_\epsilon)=M$ and $\mathcal{R}=\pi_\epsilon^{-1}(N)$.*
- *(ii) The restriction $\pi_\epsilon|_{\Gamma_\epsilon}:\Gamma_\epsilon\to M\setminus N$ is a Nash diffeomorphism.*
- *(iii) Consider the Nash map $f:=(f_1,\ldots,f_k):M\to\mathbb{R}^k$ (whose coordinates generate $\mathcal{I}(N)$). Fix $q\in N$ and let E_q be any complementary linear subspace of T_qN in T_qM. Then $\pi_\epsilon^{-1}(q)=\{q\}\times\mathbb{S}_q^{d-e-1}$, where $\mathbb{S}_q^{d-e-1}$ denotes the sphere of dimension $d-e-1$ obtained when intersecting $\mathbb{S}^{k-1}$ with the $(d-e)$-dimensional linear subspace $d_qf(E_q)$. In case $e=d-1$, each $q\in N$ has exactly two preimages under π_ϵ.*
- *(iv) The Nash maps π_ϵ and $\widehat{\pi}$ have local representations of the type*

$$(x_1,\ldots,x_d)\mapsto(x_1,\ldots,x_e,x_{e+1},x_{e+1}x_{e+2},\ldots,x_{e+1}x_d)$$

in an open neighbourhood of each point $p\in\mathcal{R}$. In case $e=d-1$ the previous local representations are $(x_1,\ldots,x_d)\mapsto(x_1,\ldots,x_d)$. In addition, $d\pi_p(T_p\widehat{M})\not\subset T_{\pi(p)}N$.

Fact 4.1.8. *Up to Nash diffeomorphisms compatible with the respective projections, the pairs $(\widetilde{M}_\epsilon,\pi_\epsilon)$ and $(\widehat{M},\widehat{\pi})$ do not depend on the generators $f_1,\ldots,f_k$ of $\mathcal{I}(N)$. Moreover, such Nash diffeomorphisms are unique.*

Definition 4.1.9. The pair $(\widetilde{M}_+, \pi_+)$ is the *drilling blow-up* of the Nash manifold M with center the closed Nash submanifold $N \subset M$ and $(\widehat{M}, \widehat{\pi})$ is the *twisted Nash double* of $(\widetilde{M}_+, \pi_+)$.

Remark 4.1.10. Let $N \subset M' \subset M$ be Nash manifolds such that N, M' are closed in M. Let $(\widetilde{M}_+, \pi_+)$ is the drilling blow-up of M with center N and let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of $(\widetilde{M}_+, \pi_+)$. Denote $M'^* := \mathrm{Cl}(\pi_+^{-1}(M' \setminus N))$ and $M'^\bullet := \mathrm{Cl}(\widehat{\pi}(M' \setminus N))$. Then $(M'^*, \pi_+|_{M'^*})$ is the drilling blow-up of M' with center N and $(M'^\bullet, \widehat{\pi}|_{M'^\bullet})$ is the twisted Nash double of $(M'^*, \pi_+|_{M'^*})$.

In fact, a finite system of generators $f_1, \dots, f_k \in \mathcal{N}(M')$ of the ideal $\mathcal{I}(N)$ can be obtained considering a finite system of generators $g_1, \dots, g_k \in \mathcal{N}(M)$ of the ideal $\mathcal{I}(N)$ by defining $f_j := g_j|_{M'}$ for $j = 1, \dots, k$.

4.1.3. Alternative description of the drilling blow-up. Fernando extended the construction in diagram (4.1.1) to an open semi-algebraic neighbourhood of the center N of the drilling blow-up of M (see [Fe4, 5.C]). This construction gives a global picture of the drilling blow-up $(\widetilde{M}_+, \pi_+)$ and justifies the first part of the name (see also Figure 4.2).

Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. Let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{M}_+, \pi_+)$ of M with center N.

Lemma 4.1.11. *Let $U_1 \subset M$ and $U_2 \subset \widehat{M}$ be respective open semi-algebraic neighbourhoods of N and $\mathcal{R} := \widehat{\pi}^{-1}(N)$. Then there exist Nash tubular neighbourhoods (V_1, θ_1) of N in U_1 and (V_2, θ_2) of $\mathcal{R}$ in U_2 such that $\widehat{\pi}(V_2) = V_1$ and V_2 is the Nash double of a collar of $\mathcal{R}$ in $\widetilde{M}_+$.*

Theorem 4.1.12 (Alternative description of the drilling blow-up)**.** *Let $M \subset \mathbb{R}^m$ be a Nash manifold, let $N \subset M$ be a closed Nash submanifold and let U be an open semi-algebraic neighbourhood of N in M. Then there exist a Nash tubular neighbourhood (V, θ) of N in U such that $M \setminus V$ is a Nash manifold with boundary ∂V and a Nash diffeomorphism $g : M \setminus V \to \widetilde{M}_+$.*

4.1.4. Relationship between drilling blow-up and classical blow-up. Let $M \subset \mathbb{R}^m$ be a Nash manifold of dimension d and let $N \subset M$ be a closed Nash submanifold of dimension e. Let $f_1, \dots, f_k \in \mathcal{N}(M)$ be a system of generators of the ideal $\mathcal{I}(N)$. Define

$$\Gamma' := \{(x, (f_1(x) : \dots : f_k(x))) \in M \times \mathbb{RP}^{k-1} : \ x \in M \setminus N\}.$$

The closure $B(M, N)$ of Γ' in $M \times \mathbb{RP}^{k-1}$ together with the restriction π' to $B(M, N)$ of the projection $M \times \mathbb{RP}^{k-1} \to M$ is the classical *blow-up* of M with center N.

Corollary 4.1.13. *Let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{M}_+, \pi_+)$. Let $\sigma : \widehat{M} \to \widehat{M},\ (a, b) \mapsto (a, -b)$ be the involution of $\widehat{M}$ without fixed points. Consider the Nash map*

$$\Theta : M \times \mathbb{S}^{k-1} \to M \times \mathbb{RP}^{k-1}, (p, q) \to (p, [q])$$

and its restriction $\theta : \widehat{M} \to B(M, N)$. We have

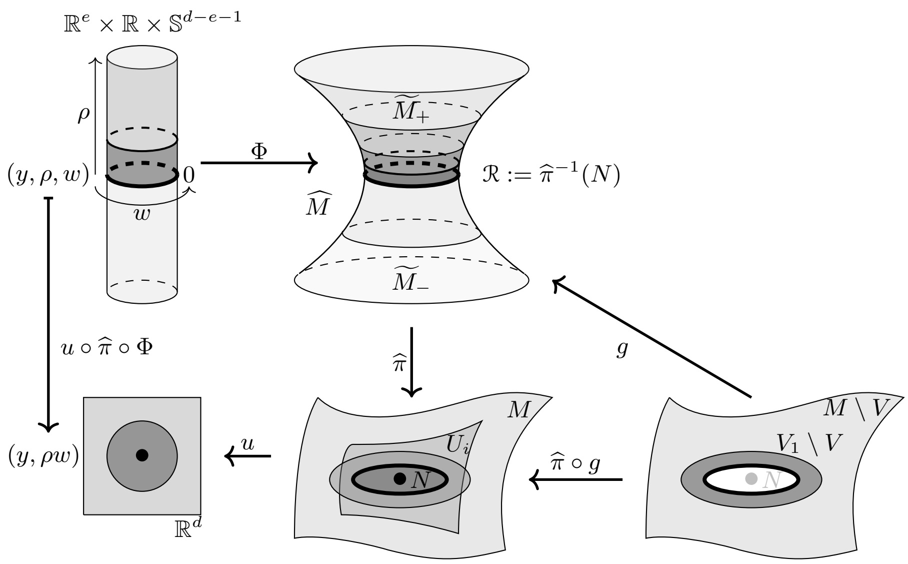

Figure 4.2: Full picture of the drilling blow-up $\widetilde{M}_+$ of M with center N (figure borrowed from [Fe4, Fig.4]).

(i) $\theta(\widehat{M}) = B(M,N)$, $\theta \circ \sigma = \theta$, $\pi' \circ \theta = \widehat{\pi}$ *and* $\theta^{-1}(a,[b]) = \{(a,b),(a,-b)\}$ *for each* $(a,[b]) \in B(M,N)$.

(ii) θ *is an unramified two to one Nash covering of* $B(M,N)$.

4.1.5. Algebraic description of drilling blow-up. Let us analyse an enough general situation for which we can guarantee that the drilling blow-up is a constructible set and its twisted (Nash) double an algebraic set. Let $X \subset \mathbb{R}^n$ be a non-singular d-dimensional algebraic set and let $Y \subset X$ be a non-singular e-dimensional algebraic subset. Let $f_1,\dots,f_r \in \mathbb{R}[\mathtt{x}]$ be a system of generators of the ideal $\mathcal{I}(Y)$ of polynomials vanishing identically on Y. Then $\widetilde{X}_+$ is the (topological) closure of

$$\Gamma_+ = \Big\{(x,u) \in (X \setminus Y) \times \mathbb{S}^{r-1} : \ \mathrm{rk}\begin{pmatrix} u_1 & \cdots & u_r \\ f_1(x) & \cdots & f_r(x)\end{pmatrix} = 1, \quad u_1 f_1(x) + \cdots + u_r f_r(x) > 0\Big\}$$

in $X \times \mathbb{S}^{r-1}$. In addition, $\widehat{X}$ is the (topological) closure of

$$\Gamma = \Big\{(x,u) \in (X \setminus Y) \times \mathbb{S}^{r-1} : \ \mathrm{rk}\begin{pmatrix} u_1 & \cdots & u_r \\ f_1(x) & \cdots & f_r(x)\end{pmatrix} = 1\Big\}$$

or equivalently the union of the irreducible components of the algebraic set

$$\Big\{(x,u) \in X \times \mathbb{S}^{r-1} : \ \mathrm{rk}\begin{pmatrix} u_1 & \cdots & u_r \\ f_1(x) & \cdots & f_r(x)\end{pmatrix} = 1\Big\}$$

different from $Y \times \mathbb{S}^{r-1}$ and it holds $\widetilde{X}_+ = \widehat{X} \cap \{u_1 f_1 + \cdots + u_r f_r \geq 0\}$. Thus, the Zariski closure of $\widetilde{X}_+$ is a union of irreducible components of $\widehat{X}$.

4.2 Resolution of closed checkerboard sets

If $\mathcal{S} \subset \mathbb{R}^m$ is a semi-algebraic set, we denote $\partial\mathcal{S} := \mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})$. Recall (see Section 3.5) that a pure dimensional semi-algebraic set $\mathcal{T} \subset \mathbb{R}^n$ is called a *checkerboard set* if it satisfies the following properties:

- $\overline{\mathcal{T}}^{\mathrm{zar}}$ is a non-singular algebraic set.
- $\overline{\partial\mathcal{T}}^{\mathrm{zar}}$ is a normal-crossings divisor of $\overline{\mathcal{T}}^{\mathrm{zar}}$.
- $\mathrm{Reg}(\mathcal{T})$ is connected.

In [Fe4] Fernando proved Theorem 3.5.2. Even if it is not explicitly quoted in the statement of [Fe4, Thm.8.4] he actually proved more. So, looking at his proof, we can reformulate the statement of Theorem 3.5.2 in the following way (recall that by Theorem 3.4.2 a semi-algebraic set $\mathcal{S}$ is well-welded if and only if it is connected by analytic paths):

Theorem 4.2.1. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set connected by analytic paths of dimension $d \geq 2$ and denote $X := \overline{\mathcal{S}}^{\mathrm{zar}}$. Then there exists a checkerboard set $\mathcal{T} \subset \mathbb{R}^n$ of dimension d and a proper polynomial map $f : Y := \overline{\mathcal{T}}^{\mathrm{zar}} \to X$ such that $f(\mathcal{T}) = \mathcal{S}$ and the restriction $f|_{\mathcal{T}} : \mathcal{T} \to \mathcal{S}$ is also proper. Moreover, there exists a semi-algebraic set $\mathcal{R} \subset \mathcal{S}$ of dimension strictly smaller than d such that $f^{-1}(\mathcal{R}) \subset \overline{\partial\mathcal{T}}^{\mathrm{zar}}$, $\mathcal{S} \setminus \mathcal{R}$ and $\mathcal{T} \setminus f^{-1}(\mathcal{R})$ are Nash manifolds and $f|_{\mathcal{T}\setminus f^{-1}(\mathcal{R})} : \mathcal{T} \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism.*

4.2.1. Closed checkerboard sets. Thus, in order to prove Theorem 4.1, we are reduced to the case when $\mathcal{T} \subset \mathbb{R}^n$ is a closed checkerboard set. We need some preliminary notations and results on checkerboard sets.

Given a non-singular algebraic set $X \subset \mathbb{R}^n$ of dimension d and a normal-crossings divisor $Z \subset X$, we denote, for $1 \leq \ell \leq d$,

$$\mathrm{Sing}_0(Z) := Z,$$
$$\mathrm{Sing}_\ell(Z) := \mathrm{Sing}(\mathrm{Sing}_{\ell-1}(Z)).$$

Observe that if $\mathrm{Sing}_\ell(Z) \neq \varnothing$, then $\dim(\mathrm{Sing}_\ell(Z)) = d - \ell - 1$. In addition, if $\mathrm{Sing}_\ell(Z) = \varnothing$, then $\mathrm{Sing}_k(Z) = \varnothing$ for $k \geq \ell$. In particular, $\mathrm{Sing}_d(Z) = \varnothing$.

Let $\mathcal{T} \subset \mathbb{R}^n$ be a closed checkerboard set and denote $X := \overline{\mathcal{T}}^{\mathrm{zar}}$. For each point $x \in \mathcal{T}$ there exists a coordinate system $(\mathtt{u}_1, \ldots, \mathtt{u}_d)$ of the Nash manifold X at x and an integer $0 \leq r_x \leq d$ such that either $\overline{\partial\mathcal{T}}^{\mathrm{an}}_x = \{\mathtt{u}_1 \cdots \mathtt{u}_{r_x} = 0\}_x$ if $r_x \geq 1$ or $x \in \mathrm{Reg}(\mathcal{T}) = \mathcal{T} \setminus \partial\mathcal{T}$ if $r_x = 0$. We denote with $e_x := e_x(\mathcal{T}) \leq r_x$ the number of indices $1 \leq i \leq r_x$ such that the germ $\mathcal{T}_x \setminus \{\mathtt{u}_i = 0\}_x$ is disconnected. If $r_x \leq 1$, then $e_x = 0$.

Lemma 4.2.2. *$e_x = 0$ if and only if $\mathcal{T}_x$ is the germ at x of a Nash manifold with corners.*

Proof. The if implication is clear because after changing the sign of some of the variables if necessary, we may assume either $x \in \mathrm{Reg}(\mathcal{T})$ or

$$\mathcal{T}_x = \{\mathtt{u}_1 \geq 0, \ldots, \mathtt{u}_{r_x} \geq 0\}_x$$

for some $1 \le r_x \le d$, so $e_x = 0$.

Conversely, if $e_x = 0$, then after changing the sign of some of the variables if necessary, we may assume either $x \in \operatorname{Reg}(\mathcal{T})$ or $\mathcal{T}_x \subset \{\mathtt{u}_1 \ge 0, \dots, \mathtt{u}_{r_x} \ge 0\}_x$ for some $1 \le r_x \le d$. As

$$\mathcal{T}_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}} = \operatorname{Reg}(\mathcal{T})_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}}$$

is an open and closed germ, $\mathcal{T}_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}}$ is a union of connected components of $\{\mathtt{u}_1 \cdots \mathtt{u}_{r_x} \neq 0\}_x$ contained in $\{\mathtt{u}_1 > 0, \dots, \mathtt{u}_{r_x} > 0\}_x$, so

$$\mathcal{T}_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}} = \{\mathtt{u}_1 > 0, \dots, \mathtt{u}_{r_x} > 0\}_x.$$

As $\mathcal{T}_x$ is closed and pure dimensional and $\dim(\overline{\partial \mathcal{T}_x}^{\mathrm{an}}) < \dim(\mathcal{T}_x)$, we conclude $\mathcal{T}_x = \{\mathtt{u}_1 \ge 0, \dots, \mathtt{u}_{r_x} \ge 0\}_x$ is the germ at x of a Nash manifold with corners, as required. □

It follows from the previous statement: *A closed checkerboard set is a Nash manifold with corners if and only if $e_x(\mathcal{T}) = 0$ for each $x \in \mathcal{T}$.*

Lemma 4.2.3. *Let $\mathcal{T} \subset \mathbb{R}^n$ be a closed checkerboard set. Then $e_x \neq 1$, for each $x \in \partial \mathcal{T}$.*

Proof. Let $X := \overline{\mathcal{T}}^{\mathrm{zar}}$. For each $x \in \partial \mathcal{T}$ there exists an open semi-algebraic set $U \subset X$ equipped with a Nash diffeomorphism $u := (u_1, \dots, u_d) : U \to \mathbb{R}^d$ and an integer $1 \le r_x \le d$ such that $u(x) = 0$ and $\overline{\partial \mathcal{T}_x}^{\mathrm{an}} = \{\mathtt{u}_1 \cdots \mathtt{u}_{r_x} = 0\}_x$.

Suppose that $e_x \ge 1$ for some $x \in \partial \mathcal{T}$. As $e_x \neq 0$, then $r_x \ge 2$, because otherwise $\mathcal{T}_x$ is the germ of a Nash manifold with boundary and $e_x = 0$. Up to rename the variables if necessary, we may assume $\mathcal{T}_x \setminus \{\mathtt{u}_1 = 0\}_x$ is disconnected. Suppose that for each $2 \le i \le r_x$ the germ $\mathcal{T}_x \setminus \{\mathtt{u}_i = 0\}_x$ is connected. After changing the signs of some of the variables if necessary, we may assume

$$\mathcal{T}_x \subset \{\mathtt{u}_2 \ge 0, \dots, \mathtt{u}_{r_x} \ge 0\}_x.$$

Proceeding as in the proof of Lemma 4.2.2, as $\mathcal{T}_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}} = \operatorname{Reg}(\mathcal{T})_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}}$ is an open and closed germ, $\mathcal{T}_x \setminus \overline{\partial \mathcal{T}_x}^{\mathrm{an}}$ is a union of connected components of $\{\mathtt{u}_1 \cdots \mathtt{u}_{r_x} \neq 0\}_x$ contained in $\{\mathtt{u}_2 > 0, \dots, \mathtt{u}_{r_x} > 0\}_x$. As $\mathcal{T}_x$ is closed and pure dimensional and $\dim(\overline{\partial \mathcal{T}_x}^{\mathrm{an}}) < \dim(\mathcal{T}_x)$, we have only two possibilities:

- $\mathcal{T}_x = \{\mathtt{u}_1 \ge 0, \dots, \mathtt{u}_{r_x} \ge 0\}_x$ (up to changing the sign of the germ $\mathtt{u}_1$ if necessary),
- $\mathcal{T}_x = \{\mathtt{u}_2 \ge 0, \dots, \mathtt{u}_{r_x} \ge 0\}_x$.

In the first case $e_x = 0$, which contradicts the fact that $\mathcal{T}_x \setminus \{\mathtt{u}_1 = 0\}_x$ is disconnected, whereas in the second case $\{\mathtt{u}_1 = 0\}_x \not\subset \overline{\partial \mathcal{T}}^{\mathrm{an}}$, which contradicts the fact that $\overline{\partial \mathcal{T}}^{\mathrm{an}} = \{\mathtt{u}_1 \cdots \mathtt{u}_{r_x} = 0\}_x$. Thus, there exists $2 \le i \le r_x$ such that $\mathcal{T}_x \setminus \{\mathtt{u}_i = 0\}_x$ is disconnected, so $e_x \ge 2$ as required. □

We show next that the function $e(\mathcal{T})$ is a semi-algebraic function.

Lemma 4.2.4 (Semi-algebricity of $e(\mathcal{T})$)**.** *Let $\mathcal{T} \subset \mathbb{R}^n$ be a d-dimensional closed checkerboard set and let $0 \leq e \leq d$. The set*

$$\mathcal{T}_e := \{x \in \mathcal{T} \, : \, e_x = e\}$$

is a semi-algebraic set and $\mathcal{T}_0$ is an open subset of $\mathcal{T}$. In addition, if Z is the Zariski closure of $\partial\mathcal{T}$ and C is a connected component of $\operatorname{Sing}_\ell(Z) \setminus \operatorname{Sing}_{\ell+1}(Z)$ for some $0 \leq \ell \leq d-1$, then $e(\mathcal{T})$ is constant on C.

Proof. The boundary $\partial\mathcal{T}$ is a closed semi-algebraic subset of the Nash manifold $X := \overline{\mathcal{T}}^{\text{zar}}$. For each $x \in \partial\mathcal{T}$ there exists a coordinate system $(\mathtt{u}_1, \ldots, \mathtt{u}_d)$ of X at x and an integer $1 \leq r_x \leq d$ such that $\overline{\partial\mathcal{T}}^{\text{an}}_x = \{\mathtt{u}_1 \cdots \mathtt{u}_{r_x} = 0\}_x$. By [FGR, Prop.4.4, Prop.4.6] there exist finitely many open semi-algebraic sets $\{U_i\}_{i=1}^s$ equipped with Nash diffeomorphism $u_i := (u_{i1}, \ldots, u_{id}) : U_i \to \mathbb{R}^d$ and integers $r_i \geq 1$ such that $\overline{\partial\mathcal{T}}^{\text{an}}_x = \{\mathtt{u}_{i1} \cdots \mathtt{u}_{ir_i} = 0\}_x$ for all $x \in \mathcal{T} \cap U_i$.

Fix $i \in \{1, \ldots, s\}$ and $J \subset \{1, \ldots, r_i\}$. Reordering the variables if necessary, we may assume $J = \{1, \ldots, m\}$ for some $1 \leq m \leq r_i$. Let V be a connected component of $U_i \setminus \{\mathtt{u}_{m+1} \cdots \mathtt{u}_{r_i} = 0\}$. After changing the signs of some of the variables if necessary, we may assume $V := \{\mathtt{u}_{i,m+1} > 0, \ldots, \mathtt{u}_{i,r_i} > 0\}$. Consider the semi-algebraic set $\mathcal{T}' := \mathcal{T} \cap U_i \cap V$ and the projection

$$\pi_i : \mathbb{R}^d \equiv \mathbb{R}^m \times \mathbb{R}^{d-m} \to \mathbb{R}^m$$

onto the first m coordinates. We take coordinates $(x_1, \ldots, x_m)$ on $\mathbb{R}^m$ and $(x_{m+1}, \ldots, x_d)$ on $\mathbb{R}^{d-m}$. Denote

$$\Lambda_i := \{\mathtt{x}_{i,m+1} > 0, \ldots, \mathtt{x}_{i,r_i} > 0\} \subset \mathbb{R}^{d-m}.$$

As $u_i(\mathcal{T} \cap U_i)$ is the union of connected components of $\mathbb{R}^d \setminus \{\mathtt{x}_1 \ldots \mathtt{x}_{r_i} = 0\}$, there exist $\varepsilon_{i,1}, \ldots, \varepsilon_{i,k} \in \{-1, 1\}^{r_i}$ such that

$$u_i(\mathcal{T} \cap U_i) = \bigcup_{p=1}^{k} \{\varepsilon_{1p_1}\mathtt{x}_1 \geq 0, \ldots, \varepsilon_{ip_{r_i}}\mathtt{x}_{r_i} \geq 0\}$$

where $\varepsilon_{ip} := (\varepsilon_{ip_1}, \ldots, \varepsilon_{ip_{r_i}})$. Consequently,

$$\begin{aligned} u_i(\mathcal{T}') &= u_i(\mathcal{T} \cap U_i \cap V) \\ &= \bigcup_{p=1}^{k} \{\varepsilon_{1p_1}\mathtt{x}_1 \geq 0, \ldots, \varepsilon_{ip_{r_i}}\mathtt{x}_{r_i} \geq 0, \mathtt{x}_{m+1} \geq 0, \ldots, \mathtt{x}_{r_i} \geq 0\} \end{aligned}$$

Observe that

$$\begin{aligned} &\{\varepsilon_{1p_1}\mathtt{x}_1 \geq 0, \ldots, \varepsilon_{ip_{r_i}}\mathtt{x}_{r_i} \geq 0, \mathtt{x}_{m+1} \geq 0, \ldots, \mathtt{x}_{r_i} \geq 0\} \\ &\qquad = \begin{cases} \{\varepsilon_{1p_1}\mathtt{x}_1 \geq 0, \ldots, \varepsilon_{ip_m}\mathtt{x}_m \geq 0, \mathtt{x}_{m+1} \geq 0, \ldots, \mathtt{x}_{r_i} \geq 0\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \varepsilon_{ip_{m+1}} = \ldots = \varepsilon_{ip_{r_i}} = 1, \\ \varnothing, \ \text{otherwhise.} \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} u_i(\mathcal{T}') &= \bigcup_{\substack{p \in \{1, \ldots, k\} \\ (\varepsilon_{ip_{m+1}}, \ldots, \varepsilon_{ip_{r_i}}) = (1, \ldots, 1)}} \{\varepsilon_{1p_1}\mathtt{x}_1 \geq 0, \ldots, \varepsilon_{ip_m}\mathtt{x}_m \geq 0, \mathtt{x}_{m+1} \geq 0, \ldots, \mathtt{x}_{r_i} \geq 0\} \\ &= \pi_i(u_i(\mathcal{T}')) \times \Lambda_i. \end{aligned}$$

If $x \in \{\mathtt{u}_{i1} = 0, \dots, \mathtt{u}_{im} = 0\} \cap V \cap \mathcal{T}$, then $e_x(\mathcal{T}) = e_0(\pi_i(u_i(\mathcal{T} \cap U_i \cap V)))$ and in particular $e_x(\mathcal{T})$ is constant on $\{\mathtt{u}_{i1} = 0, \dots, \mathtt{u}_{im} = 0, \mathtt{u}_{i,m+1} > 0, \dots, \mathtt{u}_{ir_i} > 0\}$. As each $x \in \mathcal{T} \cap U_i \cap \{\mathtt{u}_{i1} \dots \mathtt{u}_{ir_i} = 0\}$ belongs to a set of the type

$$\{\mathtt{u}_{ij} = 0, j \in J\} \cup \{\varepsilon_{ij}\mathtt{u}_{ij} > 0, j \notin J\}$$

where $J = \{1, \dots, r_i\}$ and $\varepsilon_{ij} \in \{-1, 1\}$, the function $e(\mathcal{T})$ provides a semi-algebraic partition of $\partial\mathcal{T} \cap U_i$ for each $i = 1, \dots, s$. In particular, each set $\mathcal{T}_e$ is semi-algebraic. As the condition 'to be a Nash manifold with corners' is a local open condition, we deduce $\mathcal{T}_0$ is an open semi-algebraic subset of $\mathcal{T}$.

We have proved above that if $Z_i' := \{\mathtt{u}_{i1} \cdots \mathtt{u}_{ir_i} = 0\}$ and C_i' is a connected component of $\operatorname{Sing}_\ell(Z_i') \setminus \operatorname{Sing}_{\ell+1}(Z_i')$ for some $0 \leq \ell \leq d-1$, then $e(\mathcal{T})$ is constant on C'. That means that if C is a connected component of $\operatorname{Sing}_\ell(Z) \setminus \operatorname{Sing}_{\ell+1}(Z)$, there exists a finite semi-algebraic open covering W_C of C such that $e(\mathcal{T})$ is constant on each semi-algebraic open subset of the covering. If $x, y \in C$, there exists $W_1, \dots, W_g \in W_C$ such that $x \in W_1$, $y \in W_g$ and $W_j \cap W_{j+1} \neq \varnothing$ for $j = 1, \dots, g-1$. As $e(\mathcal{T})|_{W_j}$ is constant we deduce recursively that $e_x(\mathcal{T}) = e_y(\mathcal{T})$, as required. □

We show next that the function $e(\mathcal{T})$ is upper semi-continuous.

Lemma 4.2.5 (Upper semi-continuity of $e(\mathcal{T})$)**.** *Let $\mathcal{T} \subset \mathbb{R}^d$ be a d-dimensional closed checkerboard set and let $x \in \partial\mathcal{T}$. Then $e_x \geq e_y$ for each $y \in \mathcal{T}$ close enough to x.*

Proof. Let k be the maximum of the values $e \geq 0$ such that $x \in \operatorname{Cl}(\mathcal{T}_e)$. It is enough to check that $e_x \geq k$. Consider the Nash manifold $X := \overline{\mathcal{T}}^{\text{zar}}$ and let $U \subset X$ be an open semi-algebraic neighbourhood of x equipped with a Nash diffeomorphism $u := (u_1, \dots, u_d) : U \to \mathbb{R}^d$ such that $u(x) = 0$ and $\overline{\partial\mathcal{T}}_x^{\text{an}} = \{\mathtt{u}_1 \cdots \mathtt{u}_r = 0\}_x$. If $e_x < k$, we may assume $\mathcal{T}_x \subset \{\mathtt{u}_k \geq 0, \dots, \mathtt{u}_d \geq 0\}_x$. Shrinking U if necessary, we have

$$\mathcal{T} \cap U \subset \{\mathtt{u}_k \geq 0, \dots, \mathtt{u}_d \geq 0\}.$$

Thus, $e_y < k$ for each $y \in U$, which is a contradiction because $x \in \operatorname{Cl}(\mathcal{T}_k)$. □

Remark 4.2.6. As $e(\mathcal{T})$ is upper semi-continuous, the set $\bigcup_{k \geq e} \mathcal{T}_k$ is a closed subset of $\partial\mathcal{T}$ for each $1 \leq e \leq d$.

4.2.2. Closed checkerboard sets and drilling blow-up. We want to study now how the quantity $e_x(\mathcal{T})$ changes after performing a drilling blow-up. We show the following:

Lemma 4.2.7. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a d-dimensional closed checkerboard set. Denote $X := \overline{\mathcal{S}}^{\text{zar}}$ and $Z := \overline{\partial\mathcal{S}}^{\text{zar}}$. Let $Z_1, \dots, Z_r$ be the irreducible components of Z and Y an irreducible component of $Z_1 \cap \cdots \cap Z_\ell$ for some $2 \leq \ell \leq r$. Let $(\widehat{X}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{X}, \pi_+)$ of X with center Y. Let $\mathcal{T} := \operatorname{Cl}(\pi_+^{-1}(\mathcal{S} \setminus Y)) = \pi_+^{-1}(\mathcal{S}) \cap \operatorname{Cl}(\pi_+^{-1}(\mathcal{S} \setminus Y))$ be the strict transform of $\mathcal{S}$. Then $\mathcal{T}$ is d-dimensional closed checkerboard set, $e_y(\mathcal{T}) \leq e_{\pi_+(y)}(\mathcal{S})$ for each $y \in \partial\mathcal{T}$ and $e_y(\mathcal{T}) < e_{\pi_+(y)}(\mathcal{S})$ for each $y \in \partial\mathcal{T} \cap \pi_+^{-1}(Y)$ such that $\mathcal{S}_{\pi_+(y)} \setminus Z_{i,\pi_+(y)}$ is not connected for $i = 1, \dots, \ell$.*

Proof. As $\ell \geq 2$, we have $\dim(Y) \leq d-2$, so $\operatorname{Reg}(\mathcal{S}) \setminus Y$ is connected because $\operatorname{Reg}(\mathcal{S})$ is a connected Nash manifold of dimension d. As $\mathcal{S}$ and $\mathcal{T}$ are both pure dimensional, we have

$$\pi_+^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus Y) \subset \operatorname{Reg}(\mathcal{T}) \subset \operatorname{Cl}(\pi_+^{-1}(\mathcal{S} \setminus Y)) = \operatorname{Cl}(\pi_+^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus Y)).$$

Thus, $\operatorname{Reg}(\mathcal{T})$ is connected and $\mathcal{T}$ is a checkerboard set, because: it is closed, $\overline{\mathcal{T}}^{\text{zar}} = \widehat{X}$ is a non-singular algebraic set and the Zariski closure of $\partial\mathcal{T}$ is a union of irreducible components of $\widehat{\pi}^{-1}(\overline{\partial\mathcal{S}}^{\text{zar}})$, which is a normal-crossings divisor by Remarks 4.1.4 and 4.1.10.

As $\pi_+|_{\widetilde{X}\setminus\pi_+^{-1}(Y)} : \widetilde{X} \setminus \pi_+^{-1}(Y) \to X \setminus Y$ is a Nash diffeomorphism, it holds $e_y(\mathcal{T}) = e_{\pi_+(y)}(\mathcal{S})$ for each $y \in \partial\mathcal{T} \setminus \pi_+^{-1}(Y)$. Let us see what happens at the points of $\partial\mathcal{T}\cap\pi_+^{-1}(Y)$. Fix a point $y \in \partial\mathcal{T}\cap\pi_+^{-1}(Y)$ and denote $x := \pi_+(y) \in Y$.

Assume that the irreducible components of Z that contain x are $Z_1, \ldots, Z_{r'}$ for some $2 \leq \ell \leq r' \leq r$. Let $U \subset X$ be an open semi-algebraic neighbourhood of x equipped with a Nash diffeomorphism $u := (u_1, \ldots, u_d) : U \to \mathbb{R}^d$ such that

$$u(Z \cap U) = \{\mathtt{u}_1 \cdots \mathtt{u}_{r'} = 0\} \text{ and } u(Y \cap U) = \{\mathtt{u}_1 = 0, \ldots, \mathtt{u}_\ell = 0\}$$

Write $e := \dim(Y) = d - \ell$ and assume that $e_x(\mathcal{S}) = k \leq r$. Reordering the variables and changing their signs if necessary, we may assume

$$\mathcal{S} \cap U \subset \{\mathtt{u}_1 \geq 0, \ldots, \mathtt{u}_m \geq 0, \mathtt{u}_{\ell+1} \geq 0, \ldots, \mathtt{u}_s \geq 0\} \tag{4.2.1}$$

for some $0 \leq m \leq \ell$ and $\ell \leq s \leq r'$ and both m and s are maximal satisfying (4.2.1). If $m = 0$, then

$$\mathcal{S} \cap U \subset \{\mathtt{u}_{\ell+1} \geq 0, \ldots, \mathtt{u}_s \geq 0\},$$

whereas if $s = \ell$, then $\mathcal{S} \cap U \subset \{\mathtt{u}_1 \geq 0, \ldots, \mathtt{u}_m \geq 0\}$. As $e_x(\mathcal{S}) = k$, we have $k = (\ell - m) + (r' - s)$. By Remarks 4.1.4 we can choose coordinates in $\widehat{X}$ such that π_+ behaves (with the already taken coordinates in X) as the Nash map

$$g_+ : [0, +\infty) \times \mathbb{S}^{d-e-1} \times \mathbb{R}^e \to \mathbb{R}^d, \ (\rho, w, z) \mapsto (\rho w, z),$$

where $w := (w_1, \ldots, w_\ell)$ and $z := (z_{\ell+1}, \ldots, z_d) \in \mathbb{R}^e = \mathbb{R}^{d-\ell}$. We have

$$g_+^{-1}(u(Z \cap U)) = \{\rho^\ell \mathtt{w}_1 \cdots \mathtt{w}_\ell \mathtt{z}_{\ell+1} \cdots \mathtt{z}_{r'} = 0, \ \mathtt{w}_1^2 + \cdots + \mathtt{w}_\ell^2 = 1\},$$
$$g_+^{-1}(u(Y \cap U)) = \{\rho^\ell \mathtt{w}_1 \cdots \mathtt{w}_\ell = 0, \ \mathtt{w}_1^2 + \cdots + \mathtt{w}_\ell^2 = 1\}$$

and

$$g_+^{-1}(u(\mathcal{S}\cap U\setminus Y)) \subset \{\rho \geq 0, \mathtt{w}_1 \geq 0, \ldots, \mathtt{w}_m \geq 0, \mathtt{z}_{\ell+1} \geq 0, \ldots, \mathtt{z}_s \geq 0, \mathtt{w}_1^2+\cdots+\mathtt{w}_\ell^2 = 1\}.$$

If $m = 0$, then $\mathcal{S} \cap U \subset \{\mathtt{u}_{\ell+1} \geq 0, \ldots, \mathtt{u}_s \geq 0\}$ and

$$g_+^{-1}(u(\mathcal{S} \cap U \setminus Y)) \subset \{\rho \geq 0, \mathtt{z}_{\ell+1} \geq 0, \ldots, \mathtt{z}_s \geq 0, \ \mathtt{w}_1^2 + \cdots + \mathtt{w}_\ell^2 = 1\}.$$

Thus, $e_y(\mathcal{T}) \leq \ell - 1 + r' - s = k - 1 < k = e_x(\mathcal{S})$ for each $y \in g^{-1}(x)$. The condition $m = 0$ means that $\mathcal{S}_x \setminus Z_{i,x}$ is not connected for $i = 1, \ldots, \ell$.

We assume in the following $m \geq 1$. Observe that $\mathcal{S}_x \setminus Z_{i,x}$ is not connected for $i = m+1, \ldots, \ell$. Let us show $e_y(\mathcal{T}) \leq e_x(\mathcal{S})$ for each $y \in g^{-1}(x)$. It may happen that for some $y \in g^{-1}(x)$ the previous inequality is strict even if $\mathcal{S}_x \setminus Z_{i,x}$ is connected for the indices $i = 1, \dots, m$. If some $\mathtt{w}_i(y) \neq 0$, this variable has no relevance in the description of $\mathcal{T}$ locally around y and $\mathtt{w}_i$ behaves as $\pm\sqrt{1 - \sum_{j \neq i} \mathtt{w}_i^2}$. Analogously, if some $\mathtt{z}_j(y) \neq 0$, this variable has no relevance in the description of $\mathcal{T}$ locally around y. As $\mathtt{w}_1^2 + \cdots + \mathtt{w}_\ell^2 = 1$, there exists an index $1 \leq i \leq \ell$ such that $\mathtt{w}_i(y) \neq 0$:

Case 1. If $1 \leq i \leq m$, we may assume $i = m$. Thus, $\mathtt{w}_m(y) > 0$ and $\mathtt{w}_m = +\sqrt{1 - \sum_{j=1}^{\ell-1} \mathtt{w}_i^2}$. Thus, we consider coordinates

$$(\rho, \mathtt{w}_1, \dots, \mathtt{w}_{m-1}, \mathtt{w}_{m+1}, \dots, \mathtt{w}_\ell, \mathtt{z}_{\ell+1}, \dots, \mathtt{z}_d)$$

and $e_y(\mathcal{T}) \leq \ell - (m - 1 + 1) + (r' - \ell) - (s - \ell) = (\ell - m) + (r' - s) = k = e_x(\mathcal{S})$.

Case 2. If $m + 1 \leq i \leq \ell$, we may assume $i = \ell$. Thus, $\mathtt{w}_\ell(y) \neq 0$ and $\mathtt{w}_\ell = \pm\sqrt{1 - \sum_{j=1}^{\ell-1} \mathtt{w}_i^2}$. Thus, we consider coordinates

$$(\rho, \mathtt{w}_1, \dots, \mathtt{w}_{\ell-1}, \mathtt{z}_{\ell+1}, \dots, \mathtt{z}_d)$$

and

$$\begin{aligned} e_y(\mathcal{T}) &\leq 1 + (\ell - 1) - (m + 1) + (r' - \ell) - (s - \ell) \\ &= (\ell - m) + (r' - s) - 1 = k - 1 < k = e_x(\mathcal{S}), \end{aligned}$$

as required. $\square$

4.2.3. Closed checkerboard sets and Nash manifolds with corners. We are ready to prove Theorem 4.1. After the previous preparation (in particular Theorem 4.2.1) we are reduced to the case when $\mathcal{S}$ is a d-dimensional closed checkerboard set.

Proof of Theorem 4.1. Let us construct the Nash manifold with corners $\mathcal{Q}$ first. Let $\mathcal{S} \subset \mathbb{R}^m$ be a d-dimensional closed checkerboard set and denote $X_0 := \overline{\mathcal{S}}^{\mathrm{zar}}$. Let $Z := \overline{\partial\mathcal{S}}^{\mathrm{zar}}$ and let $Z_1, \dots, Z_r$ be its irreducible components. Define

$$e := \max\{e_x(\mathcal{S}) : \ x \in \partial\mathcal{S}\}.$$

If $e = 0$, we conclude by Lemma 4.2.2 that $\mathcal{S}$ is already a Nash manifold with corners. Otherwise, by Lemma 4.2.3 $e \geq 2$. By Remark 4.2.6 $\mathcal{S}_e$ is a closed subset of $\partial\mathcal{S}$. By Lemma 4.2.4 $\mathcal{S}_e$ is a union of connected components of the semi-algebraic sets $\operatorname{Sing}_\ell(Z) \setminus \operatorname{Sing}_{\ell+1}(Z)$ for $1 \leq \ell \leq d-1$ (recall that $e \geq 2$).

Pick a point $x \in \mathcal{S}_e$ and assume that $Z_1, \dots, Z_e$ are the irreducible components of Z such that the germ $\mathcal{S}_x \setminus Z_{i,x}$ is not connected for $i = 1, \dots, e$. Then, there exists an open semi-algebraic neighbourhood $U \subset X$ of x equipped with a Nash diffeomorphism $u := (u_1, \dots, u_d) : U \to \mathbb{R}^d$ such that $u(x) = 0$, $Z \cap U = \{\mathtt{u}_1 \cdots \mathtt{u}_r = 0\}$, $Z_i \cap U = \{\mathtt{u}_i = 0\}$ and

$$\mathcal{S} \cap U \subset \{\mathtt{u}_{e+1} \geq 0, \dots, \mathtt{u}_r \geq 0\}.$$

Thus, $\{\mathtt{u}_1 = 0, \dots, \mathtt{u}_e = 0, \mathtt{u}_{e+1} \geq 0, \dots, \mathtt{u}_r \geq 0\} \subset \mathcal{S}_e$. By Lemma 4.2.5 the connected component $\mathcal{C}$ of $(Z_1 \cap \cdots \cap Z_e) \setminus \bigcup_{i=e+1}^r Z_i$ that contains $\{\mathtt{u}_1 =$

$0,\dots,\mathtt{u}_e = 0, \mathtt{u}_{e+1} \geq 0,\dots,\mathtt{u}_r \geq 0\}$ is contained in $\mathcal{S}_e$. Thus, the Zariski closure of $\mathcal{C}$ is the irreducible component of $Z_1 \cap \cdots \cap Z_e$ that contains x. As we can repeat the previous argument for each $x \in \mathcal{S}_e$, we conclude that the Zariski closure of $\mathcal{S}_e$ is a union of irreducible components of the algebraic set $\bigcup_{\{i_1,\dots,i_e\}\subset\{1,\dots,r\}} \bigcap_{j=1}^{e} Z_{i_j}$.

In addition, for each $x \in \mathcal{S}_e$ there exists irreducible components $Z_{i_1},\dots,Z_{i_e}$ of Z such that the germ $\mathcal{S}_x \setminus Z_{i_j,x}$ is not connected for $j = 1,\dots,e$. We proceed by double induction on e and the number m of irreducible components of $\mathcal{S}_e$.

Let W be an irreducible component of the Zariski closure of $\mathcal{S}_e$. Let $(\widehat{X}_0, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{X}_0, \pi_+)$ of X_0 with center W, which is by Section 4.1.5 an algebraic set. Let

$$\mathcal{T} := \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus W)) = \pi_+^{-1}(\mathcal{S}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus W))$$

be the strict transform of $\mathcal{S}$ (recall that $\mathcal{S}$ is closed). As $\mathcal{S}$ is pure dimensional and $\mathcal{S}_e \subset \overline{\partial\mathcal{S}}^{\mathrm{zar}}$ has dimension strictly smaller, $\mathcal{S} \setminus W$ is dense in $\mathcal{S}$ so

$$\pi_+(\mathcal{T}) = \pi_+(\mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus W))) = \mathrm{Cl}(\pi_+(\pi_+^{-1}(\mathcal{S} \setminus W))) = \mathrm{Cl}(\mathcal{S} \setminus W) = \mathcal{S},$$

because $\pi_+ : \widetilde{X}_0 \to X_0$ is proper and surjective. By Lemma 4.2.7 $\mathcal{T}$ is a checkerboard set, $e_y(\mathcal{T}) \leq e_{\pi_+(y)}(\mathcal{S})$ for each $y \in \partial\mathcal{T}$ and $e_y(\mathcal{T}) < e_{\pi_+(y)}(\mathcal{S})$ for each $y \in \partial\mathcal{T} \cap \pi_+^{-1}(W)$ such that $\mathcal{S}_{\pi_+(y)} \setminus Z_{i_j,\pi_+(y)}$ is not connected for $j = 1,\dots,e$.

If $\max\{e(\mathcal{T})_y :\ y \in \partial\mathcal{T}\} < e$, by induction hypothesis the statement holds for $\mathcal{T}$ so it also holds for $\mathcal{S}$. If $\max\{e(\mathcal{T})_y :\ y \in \partial\mathcal{T}\} = e$, the Zariski closure of $\mathcal{T}_e$ is contained in $\mathrm{Cl}(\widehat{\pi}(\overline{\mathcal{S}_e}^{\mathrm{zar}} \setminus W))$ and it has $m-1$ irreducible components. As by Lemma 4.2.2 $e = 0$ if and only if $\mathcal{T}$ is a Nash manifold with corners, our inductive argument is consistent. Thus, by induction hypothesis the statement holds for $\mathcal{T}$ so it also holds for $\mathcal{S}$.

Let $\mathcal{Q} \subset \mathbb{R}^n$ be the Nash manifold with corners obtained by our inductive process. We have constructed the manifold $\mathcal{Q}$, starting from $\mathcal{S}$, with a finite number of drilling blow-ups. Namely, we have constructed a finite number of tuples $\{\mathcal{T}_i, (\widetilde{X}_{\mathcal{T}_i}, \pi_{+,i}), (\widehat{X}_{\mathcal{T}_i}, \widehat{\pi}_i)\}_{i=0}^{s}$ where:

- $\mathcal{T}_0 := \mathcal{S}$, $\widetilde{X}_{\mathcal{T}_0} := X_0 = \overline{\mathcal{S}}^{\mathrm{zar}}$, $\pi_{+,0} := \mathrm{id}_{X_0}$, $\widehat{X}_{\mathcal{T}_0} := X_0$ and $\widehat{\pi}_0 := \mathrm{id}_{X_0}$.
- $(\widetilde{X}_{\mathcal{T}_{i+1}}, \pi_{+,i+1})$ is the drilling blow-up of the irreducible algebraic manifold $\overline{\mathcal{T}_i}^{\mathrm{zar}}$ with center a (suitable) irreducible algebraic submanifold W_i of $\overline{\mathcal{T}_i}^{\mathrm{zar}}$.
- $(\widehat{X}_{\mathcal{T}_{i+1}}, \widehat{\pi}_{i+1})$ is the twisted Nash double of $(\widetilde{X}_{\mathcal{T}_{i+1}}, \pi_{+,i+1})$.
- $\mathcal{T}_{i+1} := \mathrm{Cl}(\pi_{+,i+1}^{-1}(\mathcal{T}_i \setminus W_i))$ is the strict transform of $\mathcal{T}_i$ and
$$\widehat{\pi}_{i+1}(\mathcal{T}_{i+1}) = \pi_{+,i+1}(\mathcal{T}_{i+1}) = \mathcal{T}_i.$$
- $\mathcal{T}_s = \mathcal{Q}$.

By Section 4.1.5 $\overline{\mathcal{T}_i}^{\mathrm{zar}}$ is an irreducible component of the algebraic set $\widehat{X}_{\mathcal{T}_i}$. In particular $\overline{\mathcal{Q}}^{\mathrm{zar}} \subset \widehat{X}_{\mathcal{T}_s}$. As by Lemma 4.2.7 the Nash manifold with corners

$\mathcal{Q}$ is a checkerboard set, $X := \overline{\mathcal{Q}}^{\text{zar}}$ is a d-dimensional non-singular irreducible algebraic set and $Y := \overline{\partial\mathcal{Q}}^{\text{zar}}$ is a normal-crossings divisor of X. Thus, (i) and (ii) hold.

Consider the polynomial map $f := \widehat{\pi}_s \circ \ldots \circ \widehat{\pi}_1 : \widehat{X}_{\mathcal{T}_s} \to X_0$. By Fact 4.1.7(i), $f : \widehat{X}_{\mathcal{T}_s} \to X_0$ is composition of proper maps, so it is proper. Moreover, as $\mathcal{S}$ is closed and $\mathcal{Q}$ is obtained from $\mathcal{S}$ after a finite number of drilling blow-ups taking strict transforms in each step, $\mathcal{Q}$ is a closed subset of X. Thus, the restriction $f|_{\mathcal{Q}} : \mathcal{Q} \to X_0$ is proper. In addition, as $\widehat{\pi}_{i+1}(\mathcal{T}_{i+1}) = \mathcal{T}_i$ for $i = 0, \ldots, s-1$, we conclude that $f(\mathcal{Q}) = \mathcal{S}$.

Let us show (iv). The semi-algebraic set $\mathcal{R} := f(\partial\mathcal{Q}) \cup (\mathcal{S} \cap \overline{\partial\mathcal{S}}^{\text{zar}}) \subset \mathcal{S}$ is closed, because both $\mathcal{S}$ and $\overline{\partial\mathcal{S}}$ are closed and $f(\partial\mathcal{Q})$ is closed as f is proper and $\partial\mathcal{Q}$ is closed. As both $\partial\mathcal{Q}$ and $\mathcal{S} \cap \overline{\partial\mathcal{S}}^{\text{zar}}$ have dimension not greater than $d-1$, we have $\dim \mathcal{R} \leq d-1 < d$. The semi-algebraic sets $\mathcal{S} \setminus \mathcal{R}$ and $\mathcal{Q} \setminus f^{-1}(\mathcal{R})$ are Nash manifolds, because $\mathcal{S} \setminus \mathcal{R}$ is an open semi-algebraic subset of the Nash manifold $\operatorname{Reg}(\mathcal{S})$ and $\mathcal{Q} \setminus f^{-1}(\mathcal{R})$ is an open semi-algebraic subset of the Nash manifold $\operatorname{Int}(\mathcal{Q})$. Looking at the procedure developed to construct the Nash manifold with corners $\mathcal{Q}$ starting from the checkerboard set $\mathcal{T}_0 = \mathcal{S}$ using drilling blow-ups and taking into account the nature of the centre W_i of each step that is contained in $\overline{\partial\mathcal{T}_i}^{\text{zar}}$ and satisfies $\pi_{+,i+1}^{-1}(W_i) \cap \mathcal{T}_{i+1} \subset \partial\mathcal{T}_{i+1}$, we conclude $f|_{\mathcal{Q}\setminus f^{-1}(\mathcal{R})} : \mathcal{Q} \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism (see Facts 4.1.6 and 4.1.7(ii)), as required. $\square$

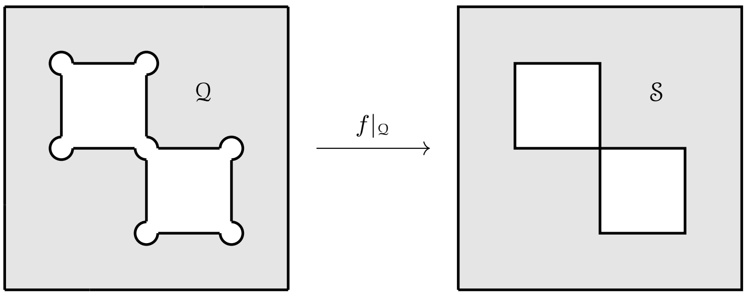

Figure 4.3: Resolution of the closed checkerboard set $\mathcal{S}$ (right) by the Nash manifold with corners $\mathcal{Q}$ (left).

Remark 4.2.8. As we have considered in each step of the inductive procedure in the proof of Theorem 4.1 the Zariski closure of $\mathcal{T}_{i,e_i}$ instead of $\mathcal{T}_{i,e_i}$, it may happen that $f(\partial\mathcal{Q}) \not\subset \mathcal{S} \cap \overline{\partial\mathcal{S}}^{\text{zar}}$ and we have to add it in $\mathcal{R}$. If we change in the statement of Theorem 4.1 polynomial maps by Nash maps, then it holds $f(\partial\mathcal{Q}) \subset \mathcal{S} \cap \overline{\partial\mathcal{S}}^{\text{zar}}$ and $\mathcal{R} = \mathcal{S} \cap \overline{\partial\mathcal{S}}^{\text{zar}}$.

4.3 Resolution of general checkerboard sets

If $\mathcal{S} \subset \mathbb{R}^m$ is a general semi-algebraic set connected by analytic paths, one can wonder if it is possible to provide a similar result to Theorem 4.1 that also works for $\mathcal{S}$. As the chosen Nash manifold with corners $\mathcal{Q}$ is closed in its Zariski closure and the chosen polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ restricts to a proper map $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^m$, its image $\mathcal{S}$ is a closed subset of $\mathbb{R}^m$. Thus, if $\mathcal{S}$ is not closed in $\mathbb{R}^m$, we should change the type of domain and/or the type of map. The second approach considering general Nash maps non-necessarily proper has been developed in [Fe4, Proof of Thm.1.4, §8.C.12] and it is shown that if the involved Nash map is not necessarily proper, then there exists a Nash manifold H with smooth boundary and a surjective Nash map $f : H \to \mathcal{S}$. If one wants to keep the properness condition, it is not possible to keep as domains Nash manifolds $\mathcal{Q}$ with corners because they are locally compact and images of locally compact subset of $\mathbb{R}^n$ under proper maps are locally compact subsets of $\mathbb{R}^m$. Thus, we have to change the type of involved domains and we will consider semi-algebraic sets $\mathcal{T} \subset \mathbb{R}^n$ whose closure is a Nash manifold with corners $\mathcal{Q} \subset \mathbb{R}^n$ and $\mathcal{Q} \setminus \mathcal{T}$ is a union of some of the strata of the a suitable stratification of $\partial\mathcal{Q}$. Let us recall the definition of (Nash) stratification of a semi-algebraic set.

Definition 4.3.1. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set. A (Nash) *stratification* of $\mathcal{S}$ is a finite partition $\{\mathcal{S}_\alpha\}_{\alpha\in A}$ of $\mathcal{S}$, where each $\mathcal{S}_\alpha$ is a connected Nash submanifold of $\mathbb{R}^m$ and the following property is satisfied: if $\mathcal{S}_\alpha \cap \mathrm{Cl}(\mathcal{S}_\beta) \neq \varnothing$ and $\alpha \neq \beta$, then $\mathcal{S}_\alpha \subset \mathrm{Cl}(\mathcal{S}_\beta)$ and $\dim(\mathcal{S}_\alpha) < \dim(\mathcal{S}_\beta)$. The $\mathcal{S}_\alpha$ are called the *strata* of the stratification and if $d := \dim(\mathcal{S}_\alpha)$, then $\mathcal{S}_\alpha$ is a *d-stratum*.

Note that the condition $\dim(\mathcal{S}_\alpha) < \dim(\mathcal{S}_\beta)$ follows from [BCR, Prop.2.8.13] because $\mathcal{S}_\alpha \subset \mathrm{Cl}(\mathcal{S}_\beta) \setminus \mathcal{S}_\beta$.

Given a d-dimensional semi-algebraic set $\mathcal{S} \subset \mathbb{R}^m$, we consider the following partition of $\mathcal{S}$. Recall that $\mathrm{Sth}(\mathcal{S})$ is the set of points $x \in \mathcal{S}$ at which the germ $\mathcal{S}_x$ is the germ of a Nash manifold (see Section 2.4.1). Define $\Gamma_1 := \mathrm{Sth}(\mathcal{S})$ and $\Gamma_k := \mathrm{Sth}(\mathcal{S} \setminus \bigcup_{j=1}^{k-1} \Gamma_j)$ for $k \geq 2$. Let $s \geq 1$ be the largest index k such that $\Gamma_k \neq \varnothing$. For each $k \geq 1$ let $\Gamma_{k\ell}$ (for $\ell = 1, \ldots, r_k$) be the connected components of Γ_k. The collection

$$\mathfrak{G}(\mathcal{S}) := \{\Gamma_{k\ell} : \ 1 \leq k \leq s, 1 \leq \ell \leq r_k\}$$

is a partition of $\mathcal{S}$. We say that a semi-algebraic set $\mathcal{T} \subset \mathcal{S}$ is *compatible with* $\mathfrak{G}(\mathcal{S})$ if it is the union of some of the $\Gamma_{k\ell}$.

Examples 4.3.2. (i) The semi-algebraic partition $\mathfrak{G}(\mathcal{S})$ of a semi-algebraic set is not in general a stratification of $\mathcal{S}$. Consider for instance the semi-algebraic set $\mathcal{S} := \{\mathtt{y}^2 - \mathtt{x}^3 = 0\} \cap (\{\mathtt{z} > 0\} \cup \{\mathtt{z} \leq 0, \mathtt{y} \geq 0\}) \subset \mathbb{R}^3$. Then

$$\Gamma_{11} := \{\mathtt{y}^2 - \mathtt{x}^3 = 0, \mathtt{y} > 0\}, \Gamma_{12} := \{\mathtt{y}^2 - \mathtt{x}^3 = 0, \mathtt{y} < 0, \mathtt{z} > 0\},$$
$$\Gamma_{21} = \{\mathtt{x} = 0, \mathtt{y} = 0\}$$

and $\mathfrak{G}(\mathcal{S}) = \{\Gamma_{11}, \Gamma_{12}, \Gamma_{21}\}$. Observe that $\Gamma_{21} \cap \mathrm{Cl}(\Gamma_{12})$, but $\Gamma_{21} \not\subset \mathrm{Cl}(\Gamma_{12})$. Thus, $\mathfrak{G}(\mathcal{S})$ is not a stratification of $\mathcal{S}$.

(ii) If $M \subset \mathbb{R}^n$ is a d-dimensional Nash manifold and $X \subset M$ is a normal-crossings divisor, then $\mathfrak{G}(X)$ is a stratification of X.

It is enough to consider local models, that is, $X := \{\mathtt{x}_1 \cdots \mathtt{x}_r = 0\} \subset \mathbb{R}^d$. In fact, we may assume $r = d$, because $X = Y \times \mathbb{R}^{d-r}$ where

$$Y := \{\mathtt{x}_1 \cdots \mathtt{x}_r = 0\} \subset \mathbb{R}^r.$$

The semi-algebraic partition $\mathfrak{G}(X)$ of X is the collection of semi-algebraic sets $\Gamma_\ell := \{\mathtt{x}_1 *_1 0, \ldots, \mathtt{x}_d *_d 0\}$ where $*_i \in \{<, =, >\}$ and one of the $*_i$ values $=$.

The closure of each Γ_ℓ is a union of finitely many Γ_k and consequently $\mathfrak{G}(X)$ is a stratification of X.

In this case $\operatorname{Sth}(X) = \operatorname{Reg}(X)$ and $\operatorname{Sth}(\operatorname{Sing}_\ell(X)) = \operatorname{Reg}(\operatorname{Sing}_\ell(X))$ for each $\ell \geq 1$.

(iii) Let $M \subset \mathbb{R}^n$ be a Nash manifold such that $\operatorname{Reg}(M) = M$, $X \subset M$ is a normal-crossings divisor and $\mathcal{S}$ the closure of a union of connected components of $M \setminus X$. Then $\mathfrak{G}(\mathcal{S})$ is a stratification of $\mathcal{S}$ and $\mathfrak{G}(X)$ is compatible with $\partial\mathcal{S} = \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})$ and $\mathcal{S} \cap X$.

Also in this case it is enough to consider local models and we may assume $X := \{\mathtt{x}_1 \cdots \mathtt{x}_r = 0\} \subset \mathbb{R}^d$. Again we suppose $r = d$, because $X = Y \times \mathbb{R}^{d-r}$ where $Y := \{\mathtt{x}_1 \cdots \mathtt{x}_r = 0\} \subset \mathbb{R}^r$. Thus, $\mathcal{S} := \bigcup_{\varepsilon \in \mathfrak{F}} \mathcal{Q}_\varepsilon$, where $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_d)$,

$$\mathcal{Q}_\varepsilon := \{\varepsilon_1 \mathtt{x}_1 \geq 0, \ldots, \varepsilon_d \mathtt{x}_d \geq 0\}$$

and $\mathfrak{F} \subset \{-1, 1\}^d$. The semi-algebraic partition $\mathfrak{G}(\mathcal{S})$ of $\mathcal{S}$ is a collection of the type $\{\mathtt{x}_{i_1} *_{i_1} 0, \ldots, \mathtt{x}_{i_\ell} *_{i_\ell} 0\}$ where $0 \leq \ell \leq d$, $1 \leq i_1 < \cdots < i_\ell \leq d$ and $*_{i_k} \in \{<, =, >\}$ for $k = 1, \ldots, \ell$. The closure of each Γ_ℓ is a union of finitely many Γ_k, so $\mathfrak{G}(\mathcal{S})$ is a stratification of $\mathcal{S}$. Observe that $\partial\mathcal{S} = \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})$ and $\mathcal{S} \cap X$ are unions of finitely many of the sets $\{\mathtt{x}_{i_1} *_{i_1} 0, \ldots, \mathtt{x}_{i_\ell} *_{i_\ell} 0\}$ with the condition that some of the $*_{i_k}$ is equal to $=$, that is, all of them belong to $\mathfrak{G}(X)$ and $\mathfrak{G}(X)$ is compatible with $\partial\mathcal{S} = \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})$.

In this case $\operatorname{Sth}(\mathcal{S}) = \operatorname{Reg}(\mathcal{S})$ and $\operatorname{Sth}(\partial^\ell\mathcal{S}) = \operatorname{Reg}(\partial^\ell\mathcal{S})$, where $\partial^\ell\mathcal{S} := \partial(\partial^{\ell-1}\mathcal{S})$ for each $\ell \geq 2$. This is so because $\partial\mathcal{S} \subset \operatorname{Sing}(X)$ and $\partial^\ell\mathcal{S} \subset \operatorname{Sing}_\ell(X)$ for each $\ell \geq 2$.

(iv) If $\mathcal{Q} \subset \mathbb{R}^n$ is a d-dimensional Nash manifold with corners, $\mathfrak{G}(\mathcal{Q})$ is a stratification of $\mathcal{Q}$.

It is enough to apply Theorem 2.5.12 and (iii).

A subset $\mathcal{T} \subset \mathbb{R}^n$ is a *Nash quasi-manifold with corners* if $\mathcal{Q} := \operatorname{Cl}(\mathcal{T})$ is a Nash manifold with corners and $\mathcal{Q} \setminus \mathcal{T}$ is a union of elements of the stratification $\mathfrak{G}(\partial\mathcal{Q})$.

We are ready to prove Theorem 4.2. That is, we show that if $\mathcal{S} \subset \mathbb{R}^m$ is a d-dimensional semi-algebraic set connected by analytic paths, then there exist:

(i) A d-dimensional connected compact non-singular algebraic set $M \subset \mathbb{R}^n$ and a normal-crossings divisor $Y \subset M$.

(ii) A connected Nash quasi-manifold with corners $\mathcal{S}^\bullet \subset M$ that is a checkerboard set and whose closure in M is a compact connected Nash manifold with corners $\mathcal{Q}^\bullet \subset M$ whose boundary $\partial\mathcal{Q}^\bullet$ has Y as its Zarsiki closure.

(iii) A Nash map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that the restriction $f|_{\mathcal{S}^\bullet} : \mathcal{S}^\bullet \to \mathcal{S}$ is proper and $f(\mathcal{S}^\bullet) = \mathcal{S}$.

(iv) A closed semi-algebraic set $\mathcal{R} \subset \mathcal{S}$ of dimension strictly smaller than d such that $\mathcal{S} \setminus \mathcal{R}$ and $\mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R})$ are Nash manifolds and the Nash map $f|_{\mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R})} : \mathcal{S}^\bullet \setminus f^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism.

Proof of Theorem 4.2. The proof is conducted in several steps and subsequent reductions:

STEP 1. INITIAL PREPARATION. We embed $\mathbb{R}^m$ in $\mathbb{RP}^m$ and the latter in $\mathbb{R}^N$ for N large enough, so we can suppose $\mathcal{S}$ is in addition bounded, so $\operatorname{Cl}(\mathcal{S})$ is compact, and the Zariski closure of $\mathcal{S}$ is compact. By Theorem 4.2.1 we may assume that $\mathcal{S} \subset \mathbb{R}^m$ is a checkerboard set, whose Zariski closure X is a d-dimensional non-singular compact algebraic subset of $\mathbb{R}^m$. In particular, $\operatorname{Reg}(\mathcal{S})$ is connected.

Let Z_1 be the Zariski closure of $\operatorname{Cl}(\mathcal{S}) \setminus \operatorname{Reg}(\mathcal{S})$, which is a normal crossing divisor of the Zariski closure X of $\mathcal{S}$ (and has dimension $d-1$). We construct next a semi-algebraic partition of $\operatorname{Cl}(\mathcal{S})$ as a finite union of Nash manifolds of different dimensions compatible with $\mathcal{T}_1 := \operatorname{Cl}(\mathcal{S}) \setminus \mathcal{S}$ and $\mathcal{T}_2 := \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})$. Observe that $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint semi-algebraic sets, they have dimensions $\leq d-1$ and $\partial\mathcal{S} := \operatorname{Cl}(\mathcal{S}) \setminus \operatorname{Reg}(\mathcal{S}) = \mathcal{T}_1 \sqcup \mathcal{T}_2$.

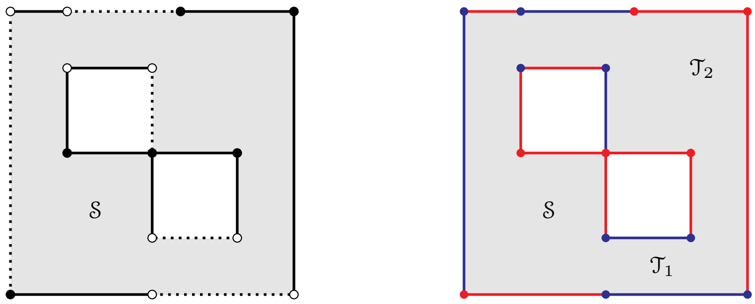

Figure 4.4: A bounded checkerboard set $\mathcal{S}$ (left) and $\mathcal{S}$ with the sets $\mathcal{T}_1$ (blue) and $\mathcal{T}_2$ (red) coloured (right).

Let N_1 be the union of the connected components of $\operatorname{Reg}(\mathcal{T}_1) \sqcup \operatorname{Reg}(\mathcal{T}_2)$ of dimension $d-1$. Note that the connected components of dimension $d-1$ of $\operatorname{Reg}(\mathcal{T}_1) \sqcup \operatorname{Reg}(\mathcal{T}_2)$ are in general different from the connected components of dimension $d-1$ of $\operatorname{Reg}(\partial\mathcal{S})$. As $\dim(\mathcal{T}_i \setminus \operatorname{Reg}(\mathcal{T}_i)) \leq d-2$, the semi-algebraic set $\partial\mathcal{S} \setminus N_1$ has dimension $\leq d-2$. Let Z_2 be the Zariski closure of $\partial\mathcal{S} \setminus N_1$. Each connected component of the Nash manifold $N_1' := \partial\mathcal{S} \setminus Z_2 = N_1 \setminus Z_2$ has dimension $d-1$ and it is contained in either $\mathcal{T}_1$ or $\mathcal{T}_2$. In addition, $\partial\mathcal{S} \setminus N_1' \subset Z_2$ has dimension $\leq d-2$ and $\operatorname{Cl}(N_1') \subset \partial\mathcal{S}$.

Let us construct recursively pairwise disjoint semi-algebraic sets N_k' that are either Nash manifolds of dimension $d-k$, whose connected components are contained in either $\mathcal{T}_1$ or $\mathcal{T}_2$, or the empty set and algebraic sets Z_k of dimension $\leq d-k$ such that $N_k' \subset Z_k$, $N_k' \cap Z_{k+1} = \varnothing$, $Z_{k+1} \subset Z_k$, $\partial\mathcal{S} \setminus (N_1' \cup \cdots \cup N_k') \subset Z_{k+1}$ and $\operatorname{Cl}(N_k') \subset \partial\mathcal{S} \setminus (N_1' \cup \cdots \cup N_{k-1}')$.

Suppose we have constructed the Nash manifolds $N_1', \ldots, N_{k-1}'$ and the algebraic sets $Z_1, \ldots, Z_k$ satisfying the required conditions and let us construct N_k' and Z_{k+1}. Let N_k be the union of the connected components of dimension $d-k$ of $\operatorname{Reg}(\mathcal{T}_1 \cap Z_k) \sqcup \operatorname{Reg}(\mathcal{T}_2 \cap Z_k)$ or the empty set if $\dim((Z_k \cap \operatorname{Cl}(\mathcal{S})) \setminus \mathcal{S}) < d-k$. As $\dim((\mathcal{T}_i \cap Z_k) \setminus \operatorname{Reg}(\mathcal{T}_i \cap Z_k)) \leq d-(k+1)$, the semi-algebraic set $\partial\mathcal{S} \setminus (N_1' \cup \cdots \cup N_{k-1}' \cup N_k) \subset \partial\mathcal{S} \cap Z_k \setminus N_k$ has dimension $\leq d-(k+1)$. Let Z_{k+1} be the Zariski closure of $\partial\mathcal{S} \setminus (N_1' \cup \cdots \cup N_{k-1}' \cup N_k)$, which has dimension $\leq d-(k+1)$. In case $N_k = \varnothing$, then $\dim(Z_k) < d-k$ and $Z_{k+1} = Z_k$. Each connected component of the Nash manifold $N_k' := \partial\mathcal{S} \setminus (N_1' \cup \cdots \cup N_{k-1}' \cup Z_{k+1}) = N_k \setminus Z_{k+1}$ has dimension $d-k$ and it is contained in either $\mathcal{T}_1$ or $\mathcal{T}_2$. In addition,

$\partial\mathcal{S} \setminus (N'_1 \cup \cdots \cup N'_k) \subset Z_{k+1}$, so it has dimension $\leq d - (k+1)$.

As $Z_\ell \subset Z_k$ if $\ell < k$ and $N'_\ell \cap Z_{\ell+1} = \varnothing$ for $\ell < k$, we deduce $N'_\ell \cap Z_k = \varnothing$ if $\ell < k$, so $\mathrm{Cl}(N'_k) \subset Z_k$ does not meet $N'_1 \cup \cdots \cup N'_{k-1}$, so

$$\mathrm{Cl}(N'_k) \subset \partial\mathcal{S} \setminus (N'_1 \cup \cdots \cup N'_{k-1}).$$

Thus, we have constructed the Nash manifolds N'_k of dimension $d-k$ (or the emptyset) and the algebraic sets Z_k of dimension $\leq d-k$ satisfying the required conditions for $k = 1, \ldots, d-1$. In particular,

$$\mathrm{Cl}(N'_k) \setminus N'_k \subset \partial\mathcal{S} \setminus (N'_1 \cup \cdots \cup N'_k) \subset Z_{k+1}$$

for $k = 1, \ldots, d$ (and $Z_{d+1} := \varnothing$).

Define the algebraic set $T_k := Z_{d-k}$, which has dimension $\leq k$, and the Nash manifold $M_k := N'_{d-k}$, which is either empty or has dimension k, for $k = 0, \ldots, d-1$. The algebraic set T_k is the Zariski closure of the Nash manifold M_k if $M_k \neq \varnothing$. If $M_0 \neq \varnothing$, then M_0 is a finite set and $M_0 = T_0$. Otherwise, if m is the least k such that $M_k \neq \varnothing$, then M_m is a (finite) union of connected components of T_m of dimension m and $M_k = \varnothing$ for $0 \leq k < m$.

Observe that $\dim(T_k) \leq d-k$, $T_k \subset T_{k+1}$, $M_k \subset T_k$, $M_k \cap T_{k-1} = \varnothing$, $\mathrm{Cl}(M_k) \setminus M_k \subset M_0 \sqcup \cdots \sqcup M_{k-1} \subset T_{k-1}$. In addition,

$$\begin{aligned}
\mathrm{Cl}(\mathcal{S}) \cap T_k = \mathcal{S} \cap Z_{d-k} &= (\mathrm{Reg}(\mathcal{S}) \cap Z_{d-k}) \cup (\partial\mathcal{S} \cap Z_{d-k}) \\
&= N'_{d-k} \sqcup \cdots \sqcup N'_d = M_0 \sqcup \cdots \sqcup M_k, \\
(\mathrm{Cl}(\mathcal{S}) \setminus \mathcal{S}) \cap T_k &= (M_0 \cap \mathcal{T}_1) \sqcup \cdots \sqcup (M_k \cap \mathcal{T}_1), \\
\mathcal{S} \cap T_k &= (M_0 \cap \mathcal{T}_2) \sqcup \cdots \sqcup (M_k \cap \mathcal{T}_2),
\end{aligned}$$

where $M_\ell \cap \mathcal{T}_i$ is the union of the connected components of the Nash manifold M_ℓ contained in $\mathcal{T}_i$ for $i = 1, 2$. Observe that $M_\ell = (M_\ell \cap \mathcal{T}_1) \sqcup (M_\ell \cap \mathcal{T}_2)$.

Step 2. Initial algebraic resolution procedure. Let $g_0 : X_0 \to X$ be the blowing-up of X of center T_0 and let $E_0 := g_0^{-1}(T_0)$ be the exceptional divisor of g_0. Recall that $g_0|_{X_0 \setminus E_0} : X_0 \setminus E_0 \to X \setminus T_0$ is a Nash diffeomorphism whose inverse map is a regular map and X_0 is a non-singular (compact) algebraic set. Denote $T_{0i} := g_0^{-1}(T_i) \cap \mathrm{Cl}(g_0^{-1}(T_i \setminus T_0))$ the strict transform of T_i under g_0, which is an algebraic set of the same dimension as T_i and denote $Y_1 := T_{01}$. Observe that no T_{0i} is contained in the algebraic set E_0 for $i \geq 1$. In particular, $\dim(T_{0i} \cap E_0) < \dim(T_{0i})$ for each $i \geq 1$ and no irreducible component of Y_1 is contained in E_0. We desingularize next $E_0 \cup Y_1$.

By Theorem 2.4.4 there exists a non-singular (compact) algebraic set X_1 and a proper surjective polynomial map $g_1 : X_1 \to X_0$ such that $E_1 := g_1^{-1}(E_0 \cup Y_1)$ is a normal-crossings divisor of X_1 and the restriction

$$g_1|_{X_1 \setminus E_1} : X_1 \setminus E_1 \to X_0 \setminus (E_0 \cup Y_1)$$

is a Nash diffeomorphism whose inverse map is a regular map. In fact, g_1 is a composition of finitely many blowing-ups whose non-singular centers have dimension $\leq \min\{\dim(Y_1), d-2\}$. Denote

$$T_{1i} := g_1^{-1}(T_{0i}) \cap \mathrm{Cl}(g_1^{-1}(T_{0i} \setminus (E_0 \cup Y_1)))$$

the strict transform of T_{0i}, which is an algebraic set of the same dimension as T_i and denote $Y_2 := T_{12}$. Observe that no T_{1i} is contained in the algebraic set E_1 for $i \geq 2$. In particular, $\dim(T_{1i} \cap E_1) < \dim(T_{1i})$ for each $i \geq 2$ and no irreducible component of Y_2 is contained in E_1. We desingularize next $E_1 \cup Y_2$.

We proceed recursively and in the step $k \leq d-1$ we find by Theorem 2.4.4 a non-singular (compact) algebraic set X_k and a proper surjective polynomial map $g_k : X_k \to X_{k-1}$ such that $E_k := g_k^{-1}(E_{k-1} \cup Y_k)$ is a normal-crossings divisor of X_k and the restriction $g_k|_{X_k \setminus E_k} : X_k \setminus E_k \to X_{k-1} \setminus (E_{k-1} \cup Y_k)$ is a Nash diffeomorphism whose inverse map is a regular map. In fact, g_k is a composition of finitely many blowing-ups whose non-singular centers have dimension $\leq \min\{\dim(Y_k), d-2\}$. Denote

$$T_{ki} := g_k^{-1}(T_{k-1,i}) \cap \operatorname{Cl}(g_k^{-1}(T_{k-1,i} \setminus (E_{k-1} \cup Y_k)))$$

the strict transform of $T_{k-1,i}$, which is an algebraic set of the same dimension as T_i and let $Y_{k+1} := T_{k,k+1}$. Observe that no T_{ki} is contained in the algebraic set E_k for $i \geq k+1$. In particular, $\dim(T_{ki} \cap E_k) < \dim(T_{ki})$ for each $i \geq k+1$ and no irreducible component of Y_{k+1} is contained in E_k. Observe that $Y_d = \varnothing$ (because $T_d = \varnothing$, so $T_{d-1,d} = \varnothing$) and $E_{d-1} = (g_0 \circ \cdots \circ g_{d-1})^{-1}(T_{d-1})$ is a normal-crossing divisor.

Step 3. Properties of the strict transform. For each $k = 0, \ldots, d-1$ the polynomial map $g_k : X_k \to X_{k-1}$ (where $X_{-1} := X$) is the composition of finitely many blow-ups whose centers have dimensions $\leq d-2$. Recall that the blow-up $b : \widehat{V} \to V$ of a d-dimensional non-singular algebraic set V with center a non-singular algebraic subset W of dimension $\leq d-2$ provides a Nash diffeomorphism $b|_{\widehat{V} \setminus b^{-1}(W)} : \widehat{V} \setminus b^{-1}(W) \to V \setminus W$. As the image of a semi-algebraic set of dimension $\leq d-2$ is a semi-algebraic set of dimension $\leq d-2$, we conclude that there exists a semi-algebraic set $R_{k-1} \subset X_{k-1}$ of dimension $\leq d-2$ such that $g_k|_{X_k \setminus g_k^{-1}(R_{k-1})} : X_k \setminus g_k^{-1}(R_{k-1}) \to X_{k-1} \setminus R_{k-1}$ is a Nash diffeomorphism. Substituting R_k by its Zariski closure we may assume R_k is an algebraic set.

Consider the composition $g := g_0 \circ \cdots \circ g_{d-1} : X_{d-1} \to X$. Let R be the Zariski closure of $R_{-1} \cup \bigcup_{k=0}^{d-2} (g_0 \circ \cdots \circ g_k)(R_k)$, which is an algebraic set of dimension $\leq d-2$. The restriction $g|_{X_{d-1} \setminus g^{-1}(R)} : X_{d-1} \setminus g^{-1}(R) \to X \setminus R$ is a Nash diffeomorphism.

Define $\mathcal{S}^* := g^{-1}(\mathcal{S}) \cap \operatorname{Cl}(g^{-1}(\mathcal{S}) \setminus E_{d-1})$ the strict transform of $\mathcal{S}$ under g and $\mathcal{S}_k := g_k^{-1}(\mathcal{S}_{k-1}) \cap \operatorname{Cl}(g_k^{-1}(\mathcal{S}_{k-1}) \setminus E_k)$ the strict transform of $\mathcal{S}_{k-1}$ under g_k for $k = 0, \ldots, d-1$, where $\mathcal{S}_{-1} := \mathcal{S}$. We claim: $\mathcal{S}^* = \mathcal{S}_d$.

Let us prove by induction on ℓ that

$$\mathcal{S}_\ell^* := (g_0 \circ \cdots \circ g_\ell)^{-1}(\mathcal{S}) \cap \operatorname{Cl}((g_0 \circ \cdots \circ g_\ell)^{-1}(\mathcal{S}) \setminus E_\ell)$$

equals $\mathcal{S}_\ell$ for each $\ell = 0, \ldots, d-1$. If $\ell = 0$, then $\mathcal{S}_0^* = \mathcal{S}_0$. Suppose the result true for $\ell - 1$, that is, $\mathcal{S}_{\ell-1}^* = \mathcal{S}_{\ell-1}$ and let us check $\mathcal{S}_\ell^* = \mathcal{S}_\ell$. Denote $h_{\ell-1} := g_0 \circ \cdots \circ g_{\ell-1}$. We have

$$\begin{aligned}\mathcal{S}_\ell^* = (g_0 \circ \cdots \circ g_\ell)^{-1}(\mathcal{S}) \cap \operatorname{Cl}((g_0 \circ \cdots \circ g_\ell)^{-1}(\mathcal{S}) \setminus E_\ell)&\\ = g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \cap \operatorname{Cl}(g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell)&\end{aligned}$$

is the strict transform of $h_{\ell-1}^{-1}(\mathcal{S})$ under g_ℓ. It holds

$$\mathcal{S}_{\ell-1}^* = h_{\ell-1}^{-1}(\mathcal{S}) \cap \operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})$$

and recall that $E_\ell = g_\ell^{-1}(E_{\ell-1} \cup Y_\ell)$.

The strict transform of $\mathcal{S}_{\ell-1}^*$ under g_ℓ is

$$\begin{aligned} g_\ell^{-1}(\mathcal{S}_{\ell-1}^*) \cap \operatorname{Cl}(g_\ell^{-1}(\mathcal{S}_{\ell-1}^*) \setminus E_\ell) &= g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \cap g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})) \\ &\cap \operatorname{Cl}((g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell) \cap (g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1}) \setminus E_\ell)) \end{aligned}$$

As $h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1} \subset \operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})$ and $g_\ell^{-1}(E_{\ell-1}) \subset E_\ell$, we deduce

$$g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell \subset g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})).$$

Consequently,

$$\begin{aligned} \operatorname{Cl}(g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell) &\subset g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})), \\ g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell &\subset g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})) \setminus E_\ell. \end{aligned}$$

Thus,

$$\begin{aligned} \operatorname{Cl}((g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell) \cap (g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1}) \setminus E_\ell)) &= \operatorname{Cl}(g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell), \\ \operatorname{Cl}(g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell) \cap g_\ell^{-1}(\operatorname{Cl}(h_{\ell-1}^{-1}(\mathcal{S}) \setminus E_{\ell-1})) &= \operatorname{Cl}(g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell). \end{aligned}$$

We conclude

$$g_\ell^{-1}(\mathcal{S}_{\ell-1}^*) \cap \operatorname{Cl}(g_\ell^{-1}(\mathcal{S}_{\ell-1}^*) \setminus E_\ell) = g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \cap \operatorname{Cl}(g_\ell^{-1}(h_{\ell-1}^{-1}(\mathcal{S})) \setminus E_\ell) = \mathcal{S}_\ell^*.$$

As by induction hypothesis $\mathcal{S}_{\ell-1}^* = \mathcal{S}_{\ell-1}$, we have

$$\mathcal{S}_\ell^* = g_\ell^{-1}(\mathcal{S}_{\ell-1}) \cap \operatorname{Cl}(g_\ell^{-1}(\mathcal{S}_{\ell-1}) \setminus E_\ell) = \mathcal{S}_\ell,$$

as claimed.

We prove next: $\operatorname{Reg}(\mathcal{S}^*)$ *is a connected d-dimensional Nash manifold and contains* $g^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus R)$ *as connected dense open semi-algebraic subset.*

As $\operatorname{Reg}(\mathcal{S})$ is a connected Nash manifold and $\dim(R) \le d-2$, also $\operatorname{Reg}(\mathcal{S}) \setminus R$ is a connected Nash manifold. As the restriction

$$g|_{X_{d-1} \setminus g^{-1}(R)} : X_{d-1} \setminus g^{-1}(R) \to X \setminus R$$

is a Nash diffeomorphism, $g^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus R)$ is a connected Nash manifold. As $\dim(R) \le d-2$, $\mathcal{S}$ is pure dimensional of dimension d and $\operatorname{Reg}(\mathcal{S})$ is dense in $\mathcal{S}$, we deduce $\operatorname{Reg}(\mathcal{S}) \setminus R$ is a dense open semi-algebraic subset of $\mathcal{S}$, so $g^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus R)$ is a dense open semi-algebraic subset of $g^{-1}(\mathcal{S} \setminus R)$. As $\dim(E_{d-1}) = d-1$, $g^{-1}(R) \subset E_{d-1}$ and $g^{-1}(\mathcal{S} \setminus R)$ is pure dimensional of dimension d, we deduce that $g^{-1}(\operatorname{Reg}(\mathcal{S})) \setminus E_{d-1} = g^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus R) \setminus E_{d-1}$ is a dense open semi-algebraic subset of $g^{-1}(\mathcal{S}) \setminus E_{d-1}$, so

$$\begin{aligned} \operatorname{Cl}(g^{-1}(\operatorname{Reg}(\mathcal{S}) \setminus R) \setminus E_{d-1}) &= \operatorname{Cl}(g^{-1}(\operatorname{Reg}(\mathcal{S})) \setminus E_{d-1}) \\ &= \operatorname{Cl}(g^{-1}(\mathcal{S}) \setminus E_{d-1}). \end{aligned} \tag{4.3.1}$$

As $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R)$ is a d-dimensional Nash manifold and $\dim(E_{d-1}) = d-1$, we deduce that $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \setminus E_{d-1}$ is dense in $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R)$, so

$$\mathrm{Cl}(g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R)) = \mathrm{Cl}(g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \setminus E_{d-1}).$$

Consequently,

$$g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) = g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \cap \mathrm{Cl}(g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \setminus E_{d-1}) \subset \mathcal{S}^*.$$

As $\mathcal{S}$ is connected by analytic paths, we deduce by [Fe4, Lem.7.16] that also $\mathcal{S}^*$ is connected by analytic paths, so in particular $\mathcal{S}^*$ is pure dimensional (of dimension d). Thus,

$$\mathcal{S}^* \setminus E_{d-1} = g^{-1}(\mathcal{S}) \setminus E_{d-1} \tag{4.3.2}$$

is a dense open semi-algebraic subset of $\mathcal{S}^*$. As $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R)$ is a dense open semi-algebraic subset of $g^{-1}(\mathcal{S} \setminus R)$ and $g^{-1}(R) \subset E_{d-1}$, we deduce that $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R)$ is a dense open semi-algebraic subset of $\mathcal{S}^*$. As $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R)$ is a d-dimensional connected Nash manifold, $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \subset \mathrm{Reg}(\mathcal{S}^*)$, so $\mathrm{Reg}(\mathcal{S}^*)$ is connected because it contains a dense connected subset.

Let us prove: $\partial\mathcal{S}^* \subset E_{d-1}$.

It holds $\partial\mathcal{S} = \mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S}) \subset \bigcup_{k=0}^{d-1} T_k$ and

$$g^{-1}(\partial\mathcal{S}) = g^{-1}(\mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})) \subset \bigcup_{k=0}^{d-1} g^{-1}(T_k).$$

Recall that T_{ki} is the strict transform of $T_{k-1,i}$ under g_k for $i \geq k$, Y_k is the Zariski closure of $T_{k-1,k}$, $E_k = g_k^{-1}(E_{k-1} \cup Y_k)$ for $k \geq 1$ and $E_0 = g_0^{-1}(T_0)$. Thus,

$$(g_0 \circ \cdots \circ g_{k-1})^{-1}(T_k) \subset T_{k-1,k} \cup E_{k-1} \subset Y_k \cup E_{k-1},$$

so $(g_0 \circ \cdots \circ g_k)^{-1}(T_k) \subset g_k^{-1}(Y_k \cup E_{k-1}) = E_k$ and

$$g^{-1}(T_k) = (g_0 \circ \cdots \circ g_{d-1})^{-1}(T_k) \subset E_{d-1}$$

for each $k = 0, \ldots, d-1$. Thus,

$$g^{-1}(\partial\mathcal{S}) = g^{-1}(\mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})) \subset \bigcup_{k=0}^{d-1} g^{-1}(T_k) \subset E_{d-1}. \tag{4.3.3}$$

We deduce using that $g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \subset \mathrm{Reg}(\mathcal{S}^*)$

$$\begin{aligned}
\mathrm{Cl}(\mathcal{S}^*) \setminus E_{d-1} &\subset \mathrm{Cl}(g^{-1}(\mathcal{S})) \cap \mathrm{Cl}(g^{-1}(\mathcal{S}) \setminus E_{d-1}) \setminus E_{d-1} \\
&= \mathrm{Cl}(g^{-1}(\mathcal{S})) \setminus E_{d-1} \subset g^{-1}(\mathrm{Cl}(\mathcal{S})) \setminus E_{d-1} \\
&= (g^{-1}(\mathrm{Reg}(\mathcal{S})) \setminus E_{d-1}) \cup (g^{-1}(\partial\mathcal{S}) \setminus E_{d-1})) \\
&= g^{-1}(\mathrm{Reg}(\mathcal{S})) \setminus E_{d-1} \subset g^{-1}(\mathrm{Reg}(\mathcal{S}) \setminus R) \subset \mathrm{Reg}(\mathcal{S}^*),
\end{aligned}$$

so $\partial\mathcal{S}^* = \mathrm{Cl}(\mathcal{S}^*) \setminus \mathrm{Reg}(\mathcal{S}^*) \subset E_{d-1}$.

Define $T_{-1} := \varnothing$. Let E_{d-1}^k be the Zariski closure of $g^{-1}(T_k \setminus T_{k-1})$ for $k = 0, \ldots, d-1$, which is the union of the irreducible components of $g^{-1}(T_k)$ that are non contained in $g^{-1}(T_{k-1})$. We have $E_{d-1} = \bigcup_{k=0}^{d-1} E_{d-1}^k$ and each irreducible component of E_{d-1} is an irreducible component of E^k_{d-1} for exactly one $k = 0,\dots,d-1$. Conversely, each irreducible component of E^k_{d-1} is an irreducible component of E_{d-1}. If H is an irreducible component of E^k_{d-1}, then $g(H) \subset T_k$ and $g(H) \not\subset T_\ell$ if $\ell < k$. As

$$\mathcal{S}^* \setminus E_{d-1} = g^{-1}(\mathcal{S}) \setminus E_{d-1} = g^{-1}(\operatorname{Reg}(\mathcal{S})) \setminus E_{d-1} = g^{-1}(\operatorname{Cl}(\mathcal{S})) \setminus E_{d-1}$$

is an open and closed subset of $X_{d-1} \setminus E_{d-1}$, it is a union of connected components of $X_{d-1} \setminus E_{d-1}$.

Let $\mathcal{T}^*_1 := \operatorname{Cl}(\mathcal{S}^*) \setminus \mathcal{S}^*$ and $\mathcal{T}^*_2 := \mathcal{S}^* \setminus \operatorname{Reg}(\mathcal{S}^*)$. Let us check: $g(\mathcal{T}^*_i) \subset \mathcal{T}_i$ *for* $i = 1, 2$.

Recall that by (4.3.1) and (4.3.3) we have

$$\operatorname{Cl}(g^{-1}(\mathcal{S}) \setminus E_{d-1}) = \operatorname{Cl}(g^{-1}(\operatorname{Cl}(\mathcal{S})) \setminus E_{d-1}) = \operatorname{Cl}(g^{-1}(\operatorname{Reg}(\mathcal{S})) \setminus E_{d-1}).$$

As $\mathcal{S}$ is connected by analytic paths, $\operatorname{Cl}(\mathcal{S})$ is also connected by analytic paths [Fe4, Lem.7.4]. Thus, the strict transform $\operatorname{Cl}(\mathcal{S})^*$ of $\operatorname{Cl}(\mathcal{S})$ under g is connected by analytic paths [Fe4, Lem.7.16], so it is pure dimensional of dimension d. Thus,

$$\operatorname{Cl}(\mathcal{S})^* \setminus E_{d-1} = g^{-1}(\operatorname{Cl}(\mathcal{S})) \setminus E_{d-1} = g^{-1}(\mathcal{S}) \setminus E_{d-1} \subset \mathcal{S}^* \subset \operatorname{Cl}(\mathcal{S}^*)$$

is a dense subset of $\operatorname{Cl}(\mathcal{S})^*$. As $\operatorname{Cl}(\mathcal{S})^*$ is a closed set that contains $\mathcal{S}^*$, we conclude $\operatorname{Cl}(\mathcal{S}^*) = \operatorname{Cl}(\mathcal{S})^*$ is the strict transform under g of $\operatorname{Cl}(\mathcal{S})$. Consequently, $\mathcal{T}^*_1 = \operatorname{Cl}(\mathcal{S}^*) \setminus \mathcal{S}^* = g^{-1}(\operatorname{Cl}(\mathcal{S}) \setminus \mathcal{S}) \cap \operatorname{Cl}(g^{-1}(\mathcal{S}) \setminus E_{d-1}) \subset g^{-1}(\mathcal{T}_1)$, so $g(\mathcal{T}^*_1) \subset \mathcal{T}_1$.

In addition, the strict transform $\operatorname{Reg}(\mathcal{S})^*$ of $\operatorname{Reg}(\mathcal{S})$ under g is

$$g^{-1}(\operatorname{Reg}(\mathcal{S})) \cap \operatorname{Cl}(g^{-1}(\operatorname{Reg}(\mathcal{S})) \setminus E_{d-1}),$$

As $g^{-1}(\operatorname{Reg}(\mathcal{S}))$ is pure dimensional of dimension d, because it is an open semi-algebraic subset of X_{d-1}, and E_{d-1} has dimension $d-1$, we deduce that $g^{-1}(\operatorname{Reg}(\mathcal{S})) \setminus E_{d-1}$ is dense in $g^{-1}(\operatorname{Reg}(\mathcal{S}))$, so $\operatorname{Reg}(\mathcal{S})^* = g^{-1}(\operatorname{Reg}(\mathcal{S}))$ is an open semi-algebraic subset of X_{d-1}. Thus, $\operatorname{Reg}(\mathcal{S})^* \subset \operatorname{Reg}(\mathcal{S}^*)$ and

$$\begin{aligned}\mathcal{T}^*_2 = \mathcal{S}^* \setminus \operatorname{Reg}(\mathcal{S}^*) \subset \mathcal{S}^* \setminus \operatorname{Reg}(S)^* \\ = g^{-1}(\mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})) \cap \operatorname{Cl}(g^{-1}(\mathcal{S}) \setminus E_{d-1}) \subset g^{-1}(\mathcal{T}_2),\end{aligned}$$

so $g(\mathcal{T}^*_2) \subset \mathcal{T}_2$.

Next, let $W \in \mathfrak{G}(\partial\mathcal{S}^*)$ and let L be the Zariski closure of W. As W is a connected Nash manifold, L is an irreducible algebraic set. Observe that L is an irreducible component of $\operatorname{Sing}_j(E_{d-1})$ for some $j \geq 1$ and W is a connected component of $\operatorname{Reg}(\operatorname{Sing}_j(E_{d-1}))$. Let $k \geq 0$ be such that $g(L) \subset T_k$, but $g(L) \not\subset T_{k-1}$. Then $L \subset E^\ell_{d-1}$ for $\ell \geq k$, but $L \not\subset \bigcup_{\ell=0}^{k-1} E^\ell_{d-1} = g^{-1}(T_{k-1})$.

Observe that W is a connected component of $L \setminus g^{-1}(T_{k-1})$, because

$$L \cap \operatorname{Sing}(\operatorname{Sing}_j(E_{d-1})) = L \cap g^{-1}(T_{k-1})$$

(recall that $L \subset E^\ell_{d-1}$ for $\ell \geq k$). Then $g(W) \subset \partial\mathcal{S} \cap T_k \setminus T_{k-1} = M_k$ is connected. As each connected component of M_k is contained in either $\mathcal{T}_1$ or $\mathcal{T}_2$, we deduce either $g(W) \subset \mathcal{T}_1$ or $g(W) \subset \mathcal{T}_2$. If $W \cap \mathcal{T}^*_i \neq \varnothing$, then $W \subset \mathcal{T}^*_i$, because otherwise also $W \cap \mathcal{T}_j^* \neq \varnothing$ for $j \in \{1,2\} \setminus \{i\}$ and $g(W)$ meets $\mathcal{T}_1$ and $\mathcal{T}_2$, which is a contradiction because $g(W)$ is contained in either $\mathcal{T}_1$ or $\mathcal{T}_2$. Thus, W is contained in either $\mathcal{T}_1^*$ or $\mathcal{T}_2^*$.

Consequently, $\mathfrak{G}(E_{d-1})$ is compatible with $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$, that is, these sets are unions of elements of $\mathfrak{G}(E_{d-1})$. As $\mathrm{Cl}(\mathcal{S}^*)$ is the closure of a union of connected components of $X_{d-1} \setminus E_{d-1}$, then $\mathcal{T}_3^* := \mathrm{Cl}(\mathcal{S}^*) \setminus \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*))$ is a $(d-1)$-dimensional semi-algebraic subset contained in E_{d-1}. In fact, using local coordinates one realizes that both $\mathcal{T}_3^*$ and $\mathrm{Cl}(\mathcal{S}^*) \cap E_{d-1}$ are unions of elements of $\mathfrak{G}(E_{d-1})$ (see Examples 4.3.2(iii)). Thus,

$$\begin{aligned}\mathcal{T}_1^* \cap \mathcal{T}_3^* &= (\mathrm{Cl}(\mathcal{S}^*) \setminus \mathcal{S}^*) \cap (\mathrm{Cl}(\mathcal{S}^*) \setminus \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*)))\\ &= \mathrm{Cl}(\mathcal{S}^*) \setminus (\mathcal{S}^* \cup \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*)))\\ &= (\mathrm{Cl}(\mathcal{S}^*) \cap E_{d-1}) \setminus ((\mathcal{S}^* \cup \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*))) \cap E_{d-1})\end{aligned}$$

is a union of elements of $\mathfrak{G}(E_{d-1})$, so $(\mathcal{S}^* \cup \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*))) \cap E_{d-1}$ is also a union of elements of $\mathfrak{G}(E_{d-1})$. Thus,

$$\mathcal{T}_4^* := \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*)) \setminus \mathcal{S}^* = ((\mathcal{S}^* \cup \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*))) \cap E_{d-1}) \setminus \mathcal{T}_2$$

is a union of elements of $\mathfrak{G}(E_{d-1})$.

Let Z be the Zariski closure of $\partial\mathcal{S}^* := \mathrm{Cl}(\mathcal{S}^*) \setminus \mathrm{Reg}(\mathcal{S}^*)$ in the non-singular (compact) algebraic set $\overline{\mathcal{S}^*}^{\mathrm{zar}}$. We have proved the following: *Z is a normal-crossings divisor of $X_{d-1} = \overline{\mathcal{S}^*}^{\mathrm{zar}}$, $\mathcal{S}^*$ is a checkerboard set and the semi-algebraic sets*

$$\begin{aligned}&\mathcal{T}_1^* := \mathrm{Cl}(\mathcal{S}^*) \setminus \mathcal{S}^*,\ \mathcal{T}_2^* := \mathcal{S}^* \setminus \mathrm{Reg}(\mathcal{S}^*),\ \mathcal{T}_3^* := \mathrm{Cl}(\mathcal{S}^*) \setminus \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*)),\\ &\mathcal{T}_1^* \cap \mathcal{T}_3^* = \mathrm{Cl}(\mathcal{S}^*) \setminus (\mathcal{S}^* \cup \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*)))\ \textit{and}\ \mathcal{T}_4^* := \mathrm{Reg}(\mathrm{Cl}(\mathcal{S}^*)) \setminus \mathcal{S}^*\end{aligned}$$

are unions of elements of the stratification $\mathfrak{G}(Z)$.

The Zariski closure of $\mathcal{S}^*$ is X_{d-1}, which is a non-singular (compact) algebraic set, the Zariski closure of $\partial\mathcal{S}^*$ is a union of connected components of E_{d-1}, which is a normal-crossings divisor of X_{d-1}. As $\mathcal{S}^*$ is the strict transform of $\mathcal{S}$ under g and $\mathcal{S}$ is pure dimensional of dimension d, the restriction $g|_{\mathcal{S}^*} : \mathcal{S}^* \to \mathcal{S}$ is a proper surjective map. Take $\mathcal{R} := g(E_{d-1}) \cap \mathcal{S}$, which has dimension $\leq d-1$ and observe that $g|_{\mathcal{S}^* \setminus g^{-1}(\mathcal{R})} : \mathcal{S}^* \setminus g^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism, because $g|_{X_{d-1} \setminus E_{d-1}} : X_{d-1} \setminus E_{d-1} \to X \setminus T_{d-1}$. As $\mathcal{S}^* \setminus g^{-1}(\mathcal{R}) = \mathcal{S}^* \setminus E_{d-1} \subset \mathcal{S}^* \setminus \partial\mathcal{S}^*$ is a Nash manifold, so its image $\mathcal{S} \setminus \mathcal{R}$ under $g|_{X_{d-1} \setminus E_{d-1}}$ is also a Nash manifold. Thus, it only remains to modify the construction to achieve that $\mathrm{Cl}(\mathcal{S}^*)$ is a Nash manifold with corners.

We assume that the initial situation is the one quoted above concerning $\mathcal{S}^*$. For the sake of simplicity we reset all the previous notations above to continue the proof.

Step 4. First drilling resolution procedure. Thus, we may assume in the following: *$\mathcal{S}$ is a checkerboard set (and in particular $\mathrm{Reg}(\mathcal{S})$ is a connected Nash manifold), $\mathrm{Cl}(\mathcal{S})$ is compact, the Zarsiki closure X of $\mathcal{S}$ is a non-singular (compact) algebraic set, the Zarsiki closure Z of $\mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathcal{S})$ is a normal crossing divisor of X, the semi-algebraic sets*

$$\begin{aligned}&\mathcal{T}_1 := \mathrm{Cl}(\mathcal{S}) \setminus \mathcal{S},\ \mathcal{T}_2 := \mathcal{S} \setminus \mathrm{Reg}(\mathcal{S}),\ \mathcal{T}_3 := \mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Reg}(\mathrm{Cl}(\mathcal{S})),\\ &\mathcal{T}_1 \cap \mathcal{T}_3 = \mathrm{Cl}(\mathcal{S}) \setminus (\mathcal{S} \cup \mathrm{Reg}(\mathrm{Cl}(\mathcal{S})))\ \textit{and}\ \mathcal{T}_4 := \mathrm{Reg}(\mathrm{Cl}(\mathcal{S})) \setminus \mathcal{S}\end{aligned}$$

are unions of elements of the stratification $\mathfrak{G}(Z)$.

By Theorem 4.1 applied to $\operatorname{Cl}(\mathcal{S})$ there exist:

(i) A d-dimensional non-singular irreducible algebraic set X' and a normal-crossings divisor $Z' \subset X'$.

(ii) A connected Nash manifold with corners $\mathcal{Q} \subset X'$ (which is a closed subset of X') whose boundary $\partial\mathcal{Q}$ has Z' as its Zariski closure.

(iii) A polynomial map $g : \mathbb{R}^n \to \mathbb{R}^m$ such that the restriction $g|_{\mathcal{Q}} : \mathcal{Q} \to \operatorname{Cl}(\mathcal{S})$ is proper and $g(\mathcal{Q}) = \operatorname{Cl}(\mathcal{S})$.

(iv) A closed semi-algebraic set $\mathcal{R} \subset \operatorname{Cl}(\mathcal{S})$ of dimension strictly smaller than d such that $\operatorname{Cl}(\mathcal{S}) \setminus \mathcal{R}$ and $\mathcal{Q} \setminus g^{-1}(\mathcal{R})$ are Nash manifolds and the polynomial map $g|_{\mathcal{Q}\setminus g^{-1}(\mathcal{R})} : \mathcal{Q} \setminus g^{-1}(\mathcal{R}) \to \operatorname{Cl}(\mathcal{S}) \setminus \mathcal{R}$ is a Nash diffeomorphism.

Let $\mathcal{S}^* := g^{-1}(\mathcal{S}) \cap \operatorname{Cl}(g^{-1}(\mathcal{S}) \setminus \mathcal{R})$ be the strict transform of $\mathcal{S}$ under g. By the properties of the drilling blow-up and specially Remarks 4.1.4 (one can almost reproduce the procedure already developed in Step 3 taking into the proof of Theorem 4.1 applied to $\operatorname{Cl}(\mathcal{S})$) in addition the semi-algebraic sets

$$\mathcal{T}_1^* := \operatorname{Cl}(\mathcal{S}^*) \setminus \mathcal{S}^*,\ \mathcal{T}_2^* := \mathcal{S}^* \setminus \operatorname{Reg}(\mathcal{S}^*),\ \mathcal{T}_3^* := \operatorname{Cl}(\mathcal{S}^*) \setminus \operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*)),$$
$$\mathcal{T}_1^* \cap \mathcal{T}_3^* = \operatorname{Cl}(\mathcal{S}^*) \setminus (\mathcal{S}^* \cup \operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*))) \text{ and } \mathcal{T}_4^* := \operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*)) \setminus \mathcal{S}^*$$

are unions of elements of the stratification $\mathfrak{G}(Z')$. If $\operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*)) = \operatorname{Reg}(\mathcal{S}^*)$, then $\mathcal{T}_3^* = \operatorname{Cl}(\mathcal{S}^*) \setminus \operatorname{Reg}(\mathcal{S}^*) = \partial\mathcal{S}^*$ and $\mathcal{T}_4 = \varnothing$. Thus, $\mathcal{S}^*$ is a quasi Nash manifolds with corners. Consequently, to continue we suppose $\operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*)) \neq \operatorname{Reg}(\mathcal{S}^*)$. This means that $\mathcal{T}_4^* \neq \varnothing$, because otherwise $\operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*)) \subset \mathcal{S}^*$ and consequently $\operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*)) = \operatorname{Reg}(\mathcal{S}^*)$ (because $\operatorname{Reg}(\operatorname{Cl}(\mathcal{S}^*))$ is an open semi-algebraic subset of X' contained in $\mathcal{S}^*$ that contains $\operatorname{Reg}(\mathcal{S}^*)$).

Step 5. Second drilling resolution procedure. We assume in the following that: *$\mathcal{S}$ is a checkerboard set (and in particular* $\operatorname{Reg}(\mathcal{S})$ *is a connected Nash manifold),* $\mathcal{Q} := \operatorname{Cl}(\mathcal{S})$ *is a compact Nash manifold with corners, the Zarsiki closure X of $\mathcal{S}$ is a non-singular (compact) algebraic set, the Zarsiki closure Z of* $\mathcal{Q} \setminus \operatorname{Reg}(\mathcal{S})$ *is a normal crossing divisor of X, the semi-algebraic sets*

$$\mathcal{T}_1 := \mathcal{Q} \setminus \mathcal{S},\ \mathcal{T}_2 := \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S}),\ \mathcal{T}_3 := \mathcal{Q} \setminus \operatorname{Reg}(\mathcal{Q}),$$
$$\mathcal{T}_1 \cap \mathcal{T}_3 = \mathcal{Q} \setminus (\mathcal{S} \cup \operatorname{Reg}(\mathcal{Q})) \text{ and } \mathcal{T}_4 := \operatorname{Reg}(\mathcal{Q}) \setminus \mathcal{S} \neq \varnothing$$

are unions of elements of the stratification $\mathfrak{G}(Z)$. Let us prove: *We may assume in addition* $\operatorname{Reg}(\mathcal{T}_4)$ *is a pure dimensional semi-algebraic set of dimension $d-1$ and* $\partial\mathcal{T}_4 = \operatorname{Cl}(\mathcal{T}_4) \setminus \operatorname{Reg}(\mathcal{T}_4) \subset \partial\mathcal{Q}$.

As $\mathcal{T}_4$ is a union of elements of the stratification $\mathfrak{G}(Z)$, then $\operatorname{Cl}(\mathcal{T}_4)$ is also a union of elements of the stratification $\mathfrak{G}(Z)$. If $\mathcal{T}_4$ has dimension $\leq d-2$, then it is contained in $\operatorname{Sing}(Z)$ and $\operatorname{Cl}(\mathcal{T}_4) \cap \operatorname{Reg}(\mathcal{Q}) \neq \varnothing$. Otherwise, $\mathcal{T}_4$ has dimension $d-1$ and let $\mathcal{M}_4$ be the union of the connected components of $\operatorname{Reg}(\mathcal{T}_4)$ of dimension $d-1$. Observe that both $\mathcal{M}_4$ and $\operatorname{Cl}(\mathcal{T}_4) \setminus \mathcal{M}_4$ are unions of elements of the stratification $\mathfrak{G}(Z)$. Define

$$\mathcal{A}(\mathcal{S}) := \begin{cases} \operatorname{Cl}(\mathcal{T}_4) \cap \operatorname{Reg}(\mathcal{Q}) & \text{if } \dim(\mathcal{T}_4) \leq d-2, \\ (\operatorname{Cl}(\mathcal{T}_4) \setminus \mathcal{M}_4) \cap \operatorname{Reg}(\mathcal{Q}) & \text{if } \dim(\mathcal{T}_4) = d-1, \end{cases}$$

which is empty if and only if $\dim(\mathcal{T}_4) = d-1$ and $\mathrm{Cl}(\mathcal{T}_4) \setminus \mathcal{M}_4 \subset \partial\mathcal{Q}$. As $\mathcal{T}_4 = \mathrm{Reg}(\mathcal{Q}) \setminus \mathcal{S}$, this means that $\mathrm{Reg}(\mathcal{T}_4) = \mathcal{M}_4$ is a pure dimensional semi-algebraic set of dimension $d-1$. Our purpose it to develop a procedure to reduce to this case.

Let Y be the Zariski closure of $\mathcal{A}(\mathcal{S})$, which is a union of irreducible components of $\mathrm{Sing}_\ell(Z)$ for $\ell = 1,\dots,d-1$, maybe of different dimensions. Let e be the dimension of Y and let Y_{e-k} be the union of $\mathrm{Sing}_k(Y)$ and the irreducible components of Y of dimension $e-k$ for $k = 0,\dots,e$. Let $\ell := \ell(\mathcal{S}) \leq e$ be the minimum value k such that $Y_k = \varnothing$ and $m := m(\mathcal{S})$ the number of irreducible components of Y_ℓ. Observe that Y_ℓ is a pure dimensional non-singular (compact) algebraic set. We proceed by double induction on ℓ and m.

Let W be an irreducible component of Y_ℓ. Let $(\widehat{X}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{X}, \pi_+)$ of X with center W, which is by Section 4.1.5 an algebraic set. Let

$$\mathcal{Q}^* := \pi_+^{-1}(\mathcal{Q}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{Q} \setminus W))$$

be the strict transform of $\mathcal{Q}$. As $\mathcal{Q}$ is pure dimensional and $Y_\ell \subset Z$ has dimension strictly smaller, $\mathcal{Q} \setminus W$ is dense in $\mathcal{Q}$, so $\pi_+(\mathcal{Q}^*) = \mathcal{Q}$, because $\pi_+ : \widetilde{X} \to X$ is proper and surjective. By Lemma 4.2.7 $\mathcal{Q}^*$ is a checkerboard set and a Nash manifold with corners such that $\pi_+^{-1}(W) \cap \mathcal{Q}^* \subset \partial\mathcal{Q}^*$. Let $\mathcal{S}^* := \pi_+^{-1}(\mathcal{S}) \cap \mathrm{Cl}(\pi_+^{-1}(\mathcal{S} \setminus W))$ be the strict transform of $\mathcal{S}^*$, which keep the same properties required to $\mathcal{S}$ (to check this fact one proceeds similarly as we have done in Steps 3 and 4). Observe that $\mathcal{A}(\mathcal{S}^*) = \pi_+^{-1}(\mathcal{A}(\mathcal{S}) \setminus W)$, so $m(\mathcal{S}^*) = m(\mathcal{S}) - 1$ and

$$\ell(\mathcal{S}^*) \begin{cases} > \ell(\mathcal{S}) & \text{if } m(\mathcal{S}) = 1, \\ = \ell(\mathcal{S}) & \text{if } m(\mathcal{S}) > 1. \end{cases}$$

Observe that $\pi_+|_{\mathcal{S}^*} : \mathcal{S}^* \to \mathcal{S}$ is a surjective proper polynomial map and if $\mathcal{R} := \partial\mathcal{S} \cup W$, the restriction $\pi_+|_{\mathcal{S}^* \setminus \pi_+^{-1}(\mathcal{R})} : \mathcal{S}^* \setminus \pi_+^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism. We proceed inductively and after finitely many steps we may assume $\mathcal{A}(\mathcal{S}) = \varnothing$.

Step 6. Final drilling resolution procedure. We assume in the following that: $\mathcal{S}$ *is a checkerboard set (and in particular* $\mathrm{Reg}(\mathcal{S})$ *is a connected Nash manifold),* $\mathcal{Q} := \mathrm{Cl}(\mathcal{S})$ *is a compact Nash manifold with corners, the Zarsiki closure* X *of* $\mathcal{S}$ *is a non-singular (compact) algebraic set, the Zarsiki closure* Z *of* $\mathcal{Q} \setminus \mathrm{Reg}(\mathcal{S})$ *is a normal crossing divisor of* X*, the semi-algebraic sets*

$$\mathcal{T}_1 := \mathcal{Q} \setminus \mathcal{S},\ \mathcal{T}_2 := \mathcal{S} \setminus \mathrm{Reg}(\mathcal{S}),\ \mathcal{T}_3 := \mathcal{Q} \setminus \mathrm{Reg}(\mathcal{Q}),$$
$$\mathcal{T}_1 \cap \mathcal{T}_3 = \mathcal{Q} \setminus (\mathcal{S} \cup \mathrm{Reg}(\mathcal{Q}))\ \textit{and}\ \mathcal{T}_4 := \mathrm{Reg}(\mathcal{Q}) \setminus \mathcal{S} \neq \varnothing$$

are unions of elements of the stratification $\mathfrak{G}(Z)$*. In addition,* $\mathrm{Reg}(\mathcal{T}_4)$ *is a pure dimensional semi-algebraic set of dimension* $d-1$ *and*

$$\partial\mathcal{T}_4 = \mathrm{Cl}(\mathcal{T}_4) \setminus \mathrm{Reg}(\mathcal{T}_4) \subset \partial\mathcal{Q}.$$

In order to finish we will take advantage of Fact 4.1.5. Until this step all the involved maps are polynomials, however in this step as we will perform the drilling blow-up of a Nash submanifold of dimension $d-1$, we have to proceed with care in order to not disconnect the regular locus of $\mathcal{S}$. We prove first that $\operatorname{Reg}(\mathcal{T}_4)=\mathcal{T}_4$. As $\mathcal{T}_4=\operatorname{Reg}(\mathcal{Q})\setminus\mathcal{S}\subset\operatorname{Reg}(\mathcal{Q})$ and $\partial\mathcal{T}_4=\operatorname{Cl}(\mathcal{T}_4)\setminus\operatorname{Reg}(\mathcal{T}_4)\subset\partial\mathcal{Q}$, we deduce $\mathcal{T}_4\cap\partial\mathcal{T}_4\subset\operatorname{Reg}(\mathcal{Q})\cap\partial\mathcal{Q}=\varnothing$, so $\mathcal{T}_4=\operatorname{Reg}(\mathcal{T}_4)$. Thus, $\operatorname{Reg}(\mathcal{T}_4)$ is a union of elements of the stratification $\mathfrak{G}(Z)$. Consequently, each connected component C of $\operatorname{Reg}(\mathcal{T}_4)$ is a union of elements of the stratification $\mathfrak{G}(Z)$. In particular, if C_1 and C_2 are connected components of $\operatorname{Reg}(\mathcal{T}_4)$, then $\operatorname{Cl}(C_1)\cap C_2=\varnothing$, because if $x\in\operatorname{Cl}(C_1)\cap C_2$, then x has no neighbourhood in $\mathcal{T}_4$ Nash diffeomorphic to a Nash manifold. Let $C_1,\dots,C_m$ be the connected components of $\operatorname{Reg}(\mathcal{T}_4)$. As $\operatorname{Reg}(C_i)=C_i$, we have

$$\partial\mathcal{T}_4=\operatorname{Cl}(\mathcal{T}_4)\setminus\operatorname{Reg}(\mathcal{T}_4)=\operatorname{Cl}\Big(\bigcup_{i=1}^m C_i\Big)\setminus\bigcup_{i=1}^m C_i=\bigcup_{i=1}^m\operatorname{Cl}(C_i)\setminus C_i=\bigcup_{i=1}^m\partial C_i.$$

We claim: $\operatorname{Cl}(C_i)\cap\operatorname{Cl}(C_j)=\varnothing$ if $i\neq j$.

Assume $\operatorname{Cl}(C_1)\cap\operatorname{Cl}(C_2)\neq\varnothing$. As $\operatorname{Cl}(C_i)\cap C_j=\varnothing$ if $i\neq j$, we deduce

$$\operatorname{Cl}(C_1)\cap\operatorname{Cl}(C_2)=(\operatorname{Cl}(C_1)\setminus C_1)\cap(\operatorname{Cl}(C_2)\setminus C_2)=\partial C_1\cap\partial C_2\subset\partial\mathcal{Q}.$$

Pick $x\in\operatorname{Cl}(C_1)\cap\operatorname{Cl}(C_2)$ and let $U\subset X$ be an open semi-algebraic neighbourhood of x such that $Z\cap U$ has coordinates $\{\mathtt{x}_1\cdots\mathtt{x}_r=0\}$. We may assume $\mathcal{Q}\cap U=\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0\}$ for some $1\leq s\leq r-2$, $C_1\cap U\subset\{\mathtt{x}_{r-1}=0\}$ and $C_2\cap U\subset\{\mathtt{x}_r=0\}$ (recall that $C_1,C_2\subset\operatorname{Reg}(\mathcal{Q})$ and $\mathfrak{G}(Z)$ is compatible with C_1,C_2). As $\mathfrak{G}(Z)$ is compatible with C_1 and C_2, we may assume

$$\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0,\mathtt{x}_{s+1}\geq 0,\dots,\mathtt{x}_{r-2}\geq 0,\mathtt{x}_{r-1}=0,\mathtt{x}_r\geq 0\}\subset\operatorname{Cl}(C_1)\cap U,$$
$$\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0,\mathtt{x}_{s+1}*_{s+1}0,\mathtt{x}_{r-2}*_{r-2}0,\mathtt{x}_{r-1}*_{r-1}0,\mathtt{x}_r=0\}\subset\operatorname{Cl}(C_1)\cap U,$$

where $*_j\in\{\geq,\leq\}$ for $j=s+1,\dots,r-1$. Thus,

$$\begin{aligned}\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0,\mathtt{x}_{s+1}=0,\dots,\mathtt{x}_r=0\}&\subset\operatorname{Cl}(C_1)\cap\operatorname{Cl}(C_2)\cap U\\ &\subset\partial\mathcal{Q}\cap U=\bigcup_{i=1}^s\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0,\mathtt{x}_i=0\},\end{aligned}$$

which is a contradiction. Consequently, $\operatorname{Cl}(C_1)\cap\operatorname{Cl}(C_2)=\varnothing$, as claimed.

Let Γ be a stratum of $\mathfrak{G}(Z)$ contained in $\mathcal{Q}$ such that Γ is not contained in $\operatorname{Cl}(C_i)$. We prove next: *If the Zariski closure of* Γ *is contained in the Zariski closure of* C_i*, then* $\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)=\varnothing$. As Γ is not contained in $\operatorname{Cl}(C_i)$ and the stratification $\mathfrak{G}(Z)$ is compatible with C_i, we have $\operatorname{Cl}(\Gamma)\cap C_i=\varnothing$, so

$$\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)=\operatorname{Cl}(\Gamma)\cap(\operatorname{Cl}(C_i)\setminus C_i)=\operatorname{Cl}(\Gamma)\cap\partial C_i.$$

Suppose $\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)\neq\varnothing$, pick $x\in\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)$ and let $U\subset X$ be an open semi-algebraic neighbourhood of x such that $Z\cap U$ has coordinates $\{\mathtt{x}_1\cdots\mathtt{x}_r=0\}$. We may assume $\mathcal{Q}\cap U=\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0\}$ for some $1\leq s\leq r-1$ and $C_i\cap U\subset\{\mathtt{x}_r=0\}$. As the Zariski closures of Γ is contained in the Zariski closure of C_i and $x\in\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)\cap U$, we deduce $\operatorname{Cl}(\Gamma)\cap U\subset\{\mathtt{x}_r=0\}$. In addition, $\partial C_i\cap U\subset\partial\mathcal{Q}\cap U=\bigcup_{i=1}^s\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0,\mathtt{x}_i=0\}$. As $\mathfrak{G}(Z)$ is compatible with C_i and $\partial C_i\subset\partial\mathcal{Q}$, we deduce

$$C_i\cap U=\{\mathtt{x}_1>0,\dots,\mathtt{x}_s>0,\mathtt{x}_r=0\}=\operatorname{Reg}(\mathcal{Q})\cap U\cap\{\mathtt{x}_r=0\},$$
$$\operatorname{Cl}(C_i)\cap U=\{\mathtt{x}_1\geq 0,\dots,\mathtt{x}_s\geq 0,\mathtt{x}_r=0\}=\mathcal{Q}\cap U\cap\{\mathtt{x}_r=0\},$$

so $\operatorname{Cl}(\Gamma)\cap U\subset \mathcal{Q}\cap U\cap\{\mathtt{x}_r=0\}=\operatorname{Cl}(C_i)\cap U$, which is a contradiction. Thus, $\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)=\varnothing$.

For each connected component C_i of $\operatorname{Reg}(\mathcal{T}_4)$ the Zariski closure of C_i is the irreducible component Z_i of Z that contains C_i. The semi-algebraic set $Z_i\cap\mathcal{Q}\setminus\operatorname{Cl}(C_i)$ is a union of elements of $\mathfrak{G}(Z)$ and it is closed, because otherwise there exists a stratum Γ of $\mathfrak{G}(Z)$ contained in $\mathcal{Q}$ such that $\Gamma\not\subset\operatorname{Cl}(C_i)$ but $\operatorname{Cl}(\Gamma)\cap\operatorname{Cl}(C_i)\neq\varnothing$, which is a contradiction. Consider the closed semi-algebraic set

$$K_i:=(Z_i\cap\mathcal{Q}\setminus\operatorname{Cl}(C_i))\cup\bigcup_{j\neq i}\operatorname{Cl}(C_j)$$

and observe that $K_i\cap\operatorname{Cl}(C_i)=\varnothing$. As both semi-algebraic sets are compact and disjoint,

$$\varepsilon:=\frac{1}{2}\min\{\operatorname{dist}(K_i,\operatorname{Cl}(C_i)):\ i=1,\dots,m\}>0.$$

Define $U_i:=\{x\in Z_i:\ \operatorname{dist}(x,\operatorname{Cl}(C_i))<\varepsilon\}$. We claim: $N:=\bigcup_{i=1}^m U_i$ *is a closed Nash submanifold of the Nash manifold* $M:=X\setminus\bigcup_{i=1}(\operatorname{Cl}(U_i)\setminus U_i)$, $\mathcal{Q}\subset M$ *and* $\mathcal{Q}\cap N=\bigcup_{i=1}^m\operatorname{Cl}(C_i)=\operatorname{Cl}(\mathcal{T}_4)$. It is clear that N is a closed subset of M. As each U_i is an open semi-algebraic subset of the Nash manifold Z_i, to prove that $N\subset M$ is a Nash manifold, it is enough to show that $\operatorname{Cl}(U_i)\cap\operatorname{Cl}(U_j)=\varnothing$ if $i\neq j$. If there exists $x\in\operatorname{Cl}(U_i)\cap\operatorname{Cl}(U_j)$, then

$$\begin{aligned}\operatorname{dist}(\operatorname{Cl}(C_i),\operatorname{Cl}(C_j))&\le\operatorname{dist}(x,\operatorname{Cl}(C_i))+\operatorname{dist}(x,\operatorname{Cl}(C_j))\\&<2\varepsilon\le\operatorname{dist}(K_i,\operatorname{Cl}(C_i))\le\operatorname{dist}(\operatorname{Cl}(C_i),\operatorname{Cl}(C_j)),\end{aligned}$$

which is a contradiction. Consequently, the semi-algebraic sets $\operatorname{Cl}(U_i)$ for $i=1,\dots,m$ are pairwise disjoint. We check next: $\mathcal{Q}\subset M$. Suppose there exists $x\in\mathcal{Q}\cap(\operatorname{Cl}(U_i)\setminus U_i)$, so $x\in Z_i\cap\mathcal{Q}\cap(\operatorname{Cl}(U_i)\setminus U_i)\subset(Z_i\cap\mathcal{Q}\setminus\operatorname{Cl}(C_i))\cap\operatorname{Cl}(U_i)$, because $\operatorname{Cl}(C_i)\subset U_i$. As $x\in\operatorname{Cl}(U_i)$ and $x\in Z_i\cap\mathcal{Q}\setminus\operatorname{Cl}(C_i)$, we have

$$\begin{aligned}\operatorname{dist}(x,\operatorname{Cl}(C_i))&\le\varepsilon\le\frac{1}{2}\operatorname{dist}(K_i,\operatorname{Cl}(C_i))\\&\le\frac{1}{2}\operatorname{dist}(Z_i\cap\mathcal{Q}\setminus\operatorname{Cl}(C_i),\operatorname{Cl}(C_i))\le\frac{1}{2}\operatorname{dist}(x,\operatorname{Cl}(C_i)),\end{aligned}$$

which is a contradiction. Consequently, $\mathcal{Q}\cap(\operatorname{Cl}(U_i)\setminus U_i)=\varnothing$ for each $i=1,\dots,m$. Thus, $\mathcal{Q}\subset X\setminus\bigcup_{i=1}(\operatorname{Cl}(U_i)\setminus U_i)=M$. To prove that $\mathcal{Q}\cap N=\operatorname{Cl}(\mathcal{T}_4)$, it is enough to check: $\mathcal{Q}\cap U_i=\operatorname{Cl}(C_i)$ or, equivalently, $\mathcal{Q}\cap(U_i\setminus\operatorname{Cl}(C_i))=\varnothing$. If $x\in\mathcal{Q}\cap(U_i\setminus\operatorname{Cl}(C_i))\subset Z_i$, then $x\in U_i$ and $x\in Z_i\cap\mathcal{Q}\setminus\operatorname{Cl}(C_i)$, which is a contradiction as we have seen in the previous paragraph. Thus, $\mathcal{Q}\cap N=\operatorname{Cl}(\mathcal{T}_4)$, as claimed.

As $\partial\mathcal{T}_4\subset\partial\mathcal{Q}$, we have $\operatorname{Cl}(\mathcal{T}_4)\cap\operatorname{Reg}(\mathcal{Q})=\operatorname{Reg}(\mathcal{T}_4)\cap\operatorname{Reg}(\mathcal{Q})$, so $\operatorname{Cl}(\mathcal{T}_4)\cap\operatorname{Reg}(\mathcal{Q})=\mathcal{T}_4\cap\operatorname{Reg}(\mathcal{Q})$. As $\mathcal{T}_4=\operatorname{Reg}(\mathcal{Q})\setminus\mathcal{S}$,

$$\operatorname{Reg}(\mathcal{Q})\setminus\operatorname{Cl}(\mathcal{T}_4)=\operatorname{Reg}(\mathcal{Q})\setminus\mathcal{T}_4=\mathcal{S}\cap\operatorname{Reg}(\mathcal{Q})\subset\mathcal{S}$$

is an open semi-algebraic subset of X contained in $\mathcal{S}$, so $\mathcal{S}\cap\operatorname{Reg}(\mathcal{Q})\subset\operatorname{Reg}(\mathcal{S})$. As $\operatorname{Reg}(\mathcal{S})\subset\mathcal{S}\cap\operatorname{Reg}(\mathcal{Q})$, we conclude

$$\operatorname{Reg}(\mathcal{S})=\operatorname{Reg}(\mathcal{Q})\setminus\operatorname{Cl}(\mathcal{T}_4)=\operatorname{Reg}(\mathcal{Q})\setminus N.$$

In general, N is not an algebraic set and its Zariski closure Y is not an option because $\operatorname{Reg}(\mathcal{Q}) \setminus Y$ might be disconnected (see Example 4.3.3). Thus, in the following the involved drilling blow-up needs to be a Nash map. Let $(\widehat{M}, \widehat{\pi})$ be the twisted Nash double of the drilling blow-up $(\widetilde{M}, \pi_+)$ of $M := X$ with center N, which is by Section 4.1.2 a Nash manifold. In addition, as $\widehat{\pi} : \widehat{M} \to M$ is proper and surjective and M is compact, also $\widehat{M}$ is compact. We have denoted X by M in order to stress that $\widehat{M}$ is a compact Nash manifold, which is not in general a non-singular algebraic set (but only one of its compact connected components). Let

$$\mathcal{Q}^\bullet := \pi_+^{-1}(\mathcal{Q}) \cap \operatorname{Cl}(\pi_+^{-1}(\mathcal{Q} \setminus N))$$

be the strict transform of $\mathcal{Q}$. As $\mathcal{Q}$ is pure dimensional and $N \subset Z$ has dimension strictly smaller, $\mathcal{Q} \setminus N$ is dense in $\mathcal{Q}$, so $\pi_+(\mathcal{Q}^\bullet) = \mathcal{Q}$, because $\pi_+ : \widetilde{M} \to M$ is proper and surjective. By Remarks 4.1.4 and Fact 4.1.5 $\mathcal{Q}^\bullet$ is a Nash manifold with corners such that $\pi_+^{-1}(N) \cap \mathcal{Q}^\bullet \subset \partial\mathcal{Q}^\bullet$. Observe that $\mathcal{Q}^\bullet \setminus \pi_+^{-1}(N) = \pi_+^{-1}(\mathcal{Q} \setminus N)$ is Nash diffeomorphic to $\mathcal{Q} \setminus N$. Thus,

$$\begin{aligned}\operatorname{Sth}(\mathcal{Q}^\bullet) = \operatorname{Sth}(\pi_+^{-1}(\mathcal{Q} \setminus N)) &= \pi_+^{-1}(\operatorname{Reg}(\mathcal{Q} \setminus N)) \\ &= \pi_+^{-1}(\operatorname{Reg}(\mathcal{Q}) \setminus N) = \pi_+^{-1}(\operatorname{Reg}(\mathcal{S})),\end{aligned}$$

so $\operatorname{Sth}(\mathcal{Q}^\bullet)$ is connected, because $\pi_+|_{\widetilde{M}\setminus\pi_+^{-1}(N)} : \widetilde{M} \setminus \pi_+^{-1}(N) \to M \setminus N$ is a Nash diffeomorphism. Let $\mathcal{S}^\bullet := \pi_+^{-1}(\mathcal{S}) \cap \operatorname{Cl}(\pi_+^{-1}(\mathcal{S} \setminus N))$ be the strict transform of $\mathcal{S}$, which keeps the same properties required to $\mathcal{S}$ (to check this fact one proceeds similarly as we have done in Steps 3, 4 and 5) if one changes the operator $\operatorname{Reg}(\cdot)$ by the operator $\operatorname{Sth}(\cdot)$ in each case. In addition,

$$\operatorname{Sth}(\mathcal{S}^\bullet) \subset \operatorname{Sth}(\mathcal{Q}^\bullet) = \pi_+^{-1}(\operatorname{Reg}(\mathcal{S})) \subset \operatorname{Sth}(\mathcal{S}^\bullet)$$

(because $\operatorname{Reg}(\mathcal{S}) \subset M \setminus N$), so $\operatorname{Sth}(\mathcal{S}^\bullet) = \operatorname{Sth}(\mathcal{Q}^\bullet)$. Observe that $\pi_+|_{\mathcal{S}^\bullet} : \mathcal{S}^\bullet \to \mathcal{S}$ is a surjective proper Nash map and if $\mathcal{R} := \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S})$, the restriction $\pi_+|_{\mathcal{S}^\bullet\setminus\pi_+^{-1}(\mathcal{R})} : \mathcal{S}^\bullet \setminus \pi_+^{-1}(\mathcal{R}) \to \mathcal{S} \setminus \mathcal{R}$ is a Nash diffeomorphism, as required.

Recall that by [AK, Thm.1.1] the pair constituted by a compact Nash manifold and a Nash normal-crossing divisor is diffeomorphic to a pair constituted by a non-singular compact algebraic set and a normal-crossing divisor and the previous diffeomorphism preserves Nash irreducible components of the corresponding Nash normal-crossing divisors. By the proof of the approximation results [BFR, Thm.1.7 & Prop.8.2] modified to fit our situation (we have to substitute Efroymson's approximation result [Sh, Thm.II.4.1] for differentiable semialgebraic functions on a Nash manifold by Nash functions by Stone-Weierstrass aproximation for differentiable functions on differentiable manifolds by polynomial functions) we may assume that the previous diffeomorphism is in addition a Nash diffeomorphism.

Using the previous fact and [Fe4, Lem.8.3 & Lem.C.1] we may assume in addition (using a suitable Nash embedding of $\widetilde{M}$ in some affine space) that the quasi Nash manifold with corners $\mathcal{S}^\bullet$ is a checkerboard set, the Nash manifold with corners $\mathcal{Q}^\bullet = \operatorname{Cl}(\mathcal{S}^\bullet)$ is a checkerboard set, the Zariski closure $X^\bullet$ of $\mathcal{S}^\bullet$ is a connected compact non-singular irreducible algebraic set, the Zariski closure $Z^\bullet$ of

$$\partial\mathcal{S}^\bullet = \mathcal{Q}^\bullet \setminus \operatorname{Reg}(\mathcal{S}^\bullet) = \mathcal{Q}^\bullet \setminus \operatorname{Reg}(\mathcal{Q}^\bullet) = \partial\mathcal{Q}^\bullet$$

is a normal-crossings divisor of $X^\bullet$ and the stratification $\mathfrak{G}(Z^\bullet)$ is compatible with $\mathcal{S}^\bullet \setminus \operatorname{Reg}(\mathcal{S}^\bullet)$. $\square$

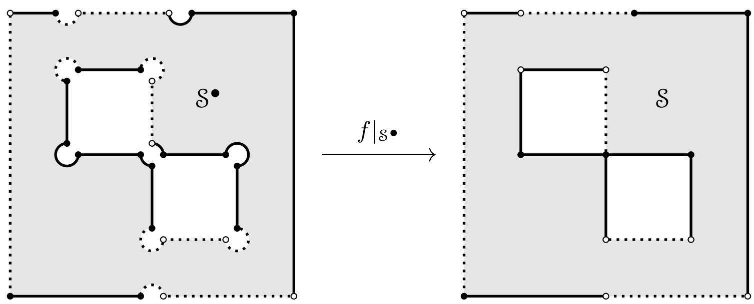

Figure 4.5: Resolution of the checkerboard set $\mathcal{S}$ (right) by the quasi Nash manifold with corners $\mathcal{S}^\bullet$ (left).

Example 4.3.3. Let $X := \{\mathtt{x}_1^2 + \cdots + \mathtt{x}_n^2 = 1\} \subset \mathbb{R}^n$ and let

$$\mathcal{S} := X \cap \{\mathtt{x}_n^2 \leq \tfrac{1}{4}\} \setminus \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\},$$

which is a checkerboard set whose Zariski closure is X. The algebraic set X is the $(n-1)$-dimensional unit sphere, so it is compact and non-singular. The closure $\operatorname{Cl}(\mathcal{S}) = X \cap \{\mathtt{x}_n^2 \leq \frac{1}{4}\}$ is a compact Nash manifold with corners. Observe that $\operatorname{Reg}(\mathcal{S}) = \mathcal{S} \cap \{\mathtt{x}_n^2 < \frac{1}{4}\}$, so

$$\operatorname{Cl}(\mathcal{S}) \setminus \operatorname{Reg}(\mathcal{S}) = (X \cap \{\mathtt{x}_n^2 = \tfrac{1}{4}\}) \cup (X \cap \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\} \cap \{\mathtt{x}_n^2 \leq \tfrac{1}{4}\}).$$

The Zariski closure of $\operatorname{Cl}(\mathcal{S}) \setminus \operatorname{Reg}(\mathcal{S})$ is

$$Z := (X \cap \{\mathtt{x}_n = \tfrac{1}{2}\}) \cup (X \cap \{\mathtt{x}_n = -\tfrac{1}{2}\}) \cup (X \cap \{\mathtt{x}_{n-1} = 0\}),$$

which is a normal-crossings divisor of X. Denote $\mathcal{Q} := \operatorname{Cl}(\mathcal{S})$. The semi-algebraic sets

$$\begin{aligned}
\mathcal{T}_1 :&= \mathcal{Q} \setminus \mathcal{S} = X \cap \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\} \cap \{\mathtt{x}_n^2 \leq \tfrac{1}{4}\},\\
\mathcal{T}_2 :&= \mathcal{S} \setminus \operatorname{Reg}(\mathcal{S}) = X \cap \{\mathtt{x}_n^2 = \tfrac{1}{4}\} \setminus \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\},\\
\mathcal{T}_3 :&= \mathcal{Q} \setminus \operatorname{Reg}(\mathcal{Q}) = X \cap \{\mathtt{x}_n^2 = \tfrac{1}{4}\},\\
\mathcal{T}_1 \cap \mathcal{T}_3 &= \mathcal{Q} \setminus (\mathcal{S} \cup \operatorname{Reg}(\mathcal{Q})) = X \cap \{\mathtt{x}_n^2 = \tfrac{1}{4}\} \cap \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\},\\
\mathcal{T}_4 :&= \operatorname{Reg}(\mathcal{Q}) \setminus \mathcal{S} = X \cap \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\} \setminus \{\mathtt{x}_n^2 = \tfrac{1}{4}\} \neq \varnothing
\end{aligned}$$

are unions of elements of the stratification $\mathfrak{G}(Z)$. In addition, $\operatorname{Reg}(\mathcal{T}_4)$ is a pure dimensional semi-algebraic set of dimension $d-1$ and

$$\partial\mathcal{T}_4 = \operatorname{Cl}(\mathcal{T}_4) \setminus \operatorname{Reg}(\mathcal{T}_4) = X \cap \{\mathtt{x}_n^2 = \tfrac{1}{4}\} \cap \{\mathtt{x}_{n-2} \leq 0, \mathtt{x}_{n-1} = 0\} \subset \partial\mathcal{Q}.$$

Thus, we are under the hypothesis of Step 6 of the Proof of Theorem 4.2. We can take as $N := X \cap \{\mathtt{x}_{n-2} < 0, \mathtt{x}_{n-1} = 0\}$, but we can not take its Zariski closure $X \cap \{\mathtt{x}_{n-1} = 0\}$, because it disconnects $\operatorname{Reg}(\mathcal{S})$.

Chapter 5

Folding Nash manifolds

In the article [FGR] Fernando, Gamboa and Ruiz proved that given a Nash manifold $\mathcal{Q} \subset \mathbb{R}^n$ with corners it is contained as a closed subset in a Nash manifold $M \subset \mathbb{R}^n$ of the same dimension and the behaviour of the Nash closure of its boundary is the suitable one. The main purpose of this chapter is to show that the Nash manifold M can be 'folded' to reconstruct the manifold with corners $\mathcal{Q}$. That is, there exists a surjective Nash map $M \to \mathcal{Q}$ such that the restriction to $\mathcal{Q}$ is close to the identity and preserves the stratification of the boundary $\partial\mathcal{Q}$. The construction we present, even if it requires some technicalities, is geometrical and neat. This construction, that has interest by its own, has remarkable consequences. A first consequence is that this construction provides an approximation result for (proper) continuous semi-algebraic maps by Nash maps, when the target space is a Nash manifold with corners.

A second consequence of our construction is a variant of Theorem 4.1. A similar result changing $\mathcal{Q}$ by a Nash manifold with boundary seems difficult to be achieved if we want to keep that the map f is polynomial, so we will show that a closed semi-algebraic set $\mathcal{S}$ connected by analytic paths can be 'resolved' by a Nash manifold with boundary, up to consider Nash maps instead of polynomial ones. Moreover, we will provide an alternative characterization of the Nash images of the closed ball, taking advantage of this new technique of 'resolution' of semi-algebraic sets by Nash manifolds with boundary.

5.1 Folding boundaries to construct Nash manifolds with corners.

In this section we deal with the main construction of this chapter. We show how to 'fold' the Nash manifold M to reconstruct the manifold with corners $\mathcal{Q} \subset M$. We will use some of the standard tools for manifold with boundary (boundary equations, doubling, collars etc.). These standard constructions have been done, for the Nash category, by Shiota [Sh, Ch.VI] in the compact case and by Fernando [Fe4, §4] in the general case. We adapt their proofs (especially those of Fernando) in order to obtain constructions compatible with an assigned Nash normal-crossings divisor. Eventually, we show that our construction is canonical and does not depend on the order of the 'foldings along the facets of $\mathcal{Q}$'.

5.1.1. Compatible Nash retractions. Let $M \subset \mathbb{R}^n$ be a connected Nash manifold of dimension d and let $Y \subset M$ be a Nash normal-crossings divisor. For each $\ell \geq 2$ define inductively

$$\begin{aligned}\operatorname{Sing}_1(Y) &:= \operatorname{Sing} Y,\\ \operatorname{Sing}_\ell(Y) &:= \operatorname{Sing}_{\ell-1}(\operatorname{Sing}(Y)).\end{aligned}$$

In order to lighten the exposition, we write $\operatorname{Sing}_0(Y) := Y$ and $\operatorname{Sing}_{-1}(Y) := M$. The irreducible components of $\operatorname{Sing}_\ell(Y)$ are Nash manifolds for each $\ell \geq 1$ such that $\operatorname{Sing}_\ell(Y) \neq \varnothing$. In fact, if $Y_{\ell,1}, \ldots, Y_{\ell,s_\ell}$ are the irreducible components of $\operatorname{Sing}_\ell(Y)$, then

$$\operatorname{Sing}_{\ell+1}(Y) = \bigcup_{i \neq j} (Y_{\ell,i} \cap Y_{\ell,j}).$$

If $\operatorname{Sing}_\ell(Y) \neq \varnothing$, then $\dim(\operatorname{Sing}_\ell(Y)) = d - \ell - 1$.

For each $\ell \geq 1$ we have $\operatorname{Sing}_\ell(Y) \subset \operatorname{Sing}_{\ell-1}(Y)$, so there exists an $r \geq 0$ such that $\operatorname{Sing}_r(Y) \neq \varnothing$, but $\operatorname{Sing}_{r+1}(Y) = \varnothing$, whereas $\operatorname{Sing}_t(Y) \neq \varnothing$ for each $0 \leq t \leq r$. Let Z be an irreducible component of $\operatorname{Sing}_t(Y)$.

Definition 5.1.1. A Nash retraction $\rho : W \to Z$, where $W \subset M$ is an open semi-algebraic neighbourhood of Z, is *compatible with* Y if

$$\rho(Y_i \cap W) = Y_i \cap Z,$$

for each irreducible component Y_i of Y such that $Y_i \cap Z \neq \varnothing$.

In [FGh, Prop. 4.1] Fernando and Ghiloni proved the following result, which is a powerful tool to make constructions compatible with an assigned Nash normal-crossings divisor. We will take advantage of this result in the rest of the chapter.

Proposition 5.1.2 (Compatible Nash retractions, [FGh, Prop. 4.1])**.** *There exist an open semi-algebraic neighbourhood $W \subset M$ of Z and a Nash retraction $\rho : W \to Z$ that is compatible with Y. In addition*

$$\rho(X \cap W) = X \cap Z,$$

for each irreducible component X of $\operatorname{Sing}_\ell(Y)$ such that $X \cap Z \neq \varnothing$ and $\ell \geq 1$.

5.1.2. Compatible Nash collars. Nash collars for Nash manifolds with non-empty boundary have been constructed by Shiota [Sh, VI.1.6] in the compact case and by Fernando [Fe4, Lem.4.2] in the general case. For our purposes we need to adapt these constructions in order to make the collars compatible with a Nash normal-crossings divisor Y that contains ∂H.

Let $M \subset \mathbb{R}^n$ be a Nash manifold and let $Y \subset M$ be a Nash normal-crossings divisor.

Proposition 5.1.3 (Compatible Nash collars)**.** *Let Y_1 be an irreducible component of Y and let $\rho : W \to Y_1$ be a Nash retraction compatible with Y_1 such that $\rho(X \cap W) = X \cap Z$ for each irreducible component X of $\operatorname{Sing}_\ell(Y)$ such that $X \cap Z \neq \varnothing$ and $\ell \geq 1$. Let h be a Nash function on W such that $\{h = 0\} = Y_1$ and $d_x h : T_x M \to \mathbb{R}$ is surjective for all $x \in Y_1$. Then there exist an open* *semi-algebraic neighbourhood* $V \subset W$ *of* Y_1 *and a strictly positive Nash function* $\varepsilon : Y_1 \to \mathbb{R}$ *such that the Nash map* $\varphi := (\rho, \frac{h}{\varepsilon \circ \rho}) : V \to Y_1 \times (-1,1)$ *is a Nash diffeomorphism and*

$$\varphi(Z \cap V) = (Z \cap Y_1) \times (-1,1)$$

for each irreducible component Z *of* $\operatorname{Sing}_\ell(Y)$ *such that* $Z \not\subset Y_1$ *and* $Z \cap Y_1 \neq \varnothing$ *and* $\ell \geq 0$.

Proof. We show first: *Define* $\phi := (\rho, h) : W \to Y_1 \times \mathbb{R}$. *Then the derivative* $d_x\phi = (d_x\rho, d_xh) : T_xM \to T_xY_1 \times \mathbb{R}$ *is an isomorphism for all* $x \in Y_1$. As $\dim(T_xM) = \dim(T_xY_1 \times \mathbb{R})$, it is enough to show: $d_x\phi$ *is surjective.*

As $\phi|_{Y_1} = (\mathrm{id}_{Y_1}, 0)$, we have $d_x\phi|_{T_xY_1} = (\mathrm{id}_{T_xY_1}, 0)$, so $T_xY_1 \times \{0\} \subset \operatorname{im}(d_x\phi)$. In addition $d_xh : T_xM \to \mathbb{R}$ is surjective, so there exists $v \in T_xM$ such that $d_xh(v) = 1$. Thus, $d_x\phi(v) = (d_x\rho(v), 1)$ and $d_x\phi$ is surjective.

Let $W' := \{x \in W : \ d_x\phi \text{ is an isomorphism}\}$, which is an open semi-algebraic neighbourhood of Y_1. Thus, $\phi|_{W'} : W' \to Y_1 \times \mathbb{R}$ is an open map and $\phi(W')$ is an open semi-algebraic neighbourhood of $Y_1 \times \{0\}$ in $Y_1 \times \mathbb{R}$. As $\phi|_{W'} : W' \to \phi(W')$ is a local homeomorphism and $\phi|_{Y_1} = (\mathrm{id}_{Y_1}, 0)$ is a homeomorphism (onto its image), there exist by [BFR, Lem.9.2] open semi-algebraic neighbourhoods $W'' \subset W'$ of Y_1 and $U \subset Y_1 \times \mathbb{R}$ of $Y_1 \times \{0\}$ such that $\phi|_{W''} : W'' \to U$ is a semi-algebraic homeomorphism.

Consider the strictly positive, continuous semi-algebraic map

$$\delta : Y_1 \to (0, +\infty), \ x \mapsto \operatorname{dist}((x,0), (Y_1 \times \mathbb{R}) \setminus U).$$

By [Sh, II.4.1] there exists a strictly positive Nash function ε on Y_1 such that $\frac{1}{2}\delta < \varepsilon < \delta$. Consider the open semi-algebraic neighbourhood

$$U' := \{(x,t) \in Y_1 \times \mathbb{R} : \ |t| < \varepsilon(x)\} \subset U$$

of $Y_1 \times \{0\}$ and define $V := (\phi|_{W''})^{-1}(U')$. The restriction $\phi|_V : V \to U'$ is a Nash diffeomorphism. Consequently,

$$\varphi : V \to Y_1 \times (-1,1), \ x \mapsto \Big(\rho(x), \frac{h(x)}{\varepsilon(\rho(x))}\Big)$$

is a Nash diffeomorphism (as it is the composition of $\phi|_V$ with a Nash diffeomorphism).

Let $\ell \geq 0$ and let Z be an irreducible component of $\operatorname{Sing}_\ell(Y)$ such that $Z \not\subset Y_1$ and $Z \cap Y_1 \neq \varnothing$. By Proposition 5.1.2 $\rho(Z \cap V) = Z \cap Y_1$, so

$$\varphi(Z \cap V) \subset (Z \cap Y_1) \times (-1,1).$$

As Y is a Nash normal-crossings divisor, $Z \cap Y_1$ is a Nash manifold, so its connected components C_i are Nash manifolds (all of the same dimension $e-1$ where $e = \dim(Z)$, because $Z \not\subset Y_1$). Thus, $C_i \times (-1,1)$ are closed connected Nash submanifolds of $Y_1 \times (-1,1)$ of dimension e.

Let Z_i be an irreducible component of $Z \cap V$ such that $C_i \subset Z_i \cap Y_1$. As Z_i is connected, also $\rho(Z_i) \subset Z \cap Y_1$ is connected, so it is contained in one of the connected components of $Z \cap Y_1$. As $C_i \subset Z_i \cap Y_1 \subset \rho(Z_i)$ is a connected component of $Z \cap Y_1$, we deduce $\rho(Z_i) = Z_i \cap Y_1 = C_i$. Then $\varphi(Z_i) \subset C_i \times (-1,1)$ is a Nash subset of $Y_1 \times (-1,1)$ of dimension e. By the identity principle, we conclude $\varphi(Z_i) = C_i \times (-1,1)$ because $C_i \times (-1,1)$ is an irreducible Nash subset of $Y_1 \times (-1,1)$ of dimension e. Consequently, $\varphi(Z \cap V) = (Z \cap Y_1) \times (-1,1)$, as required. □

5.1.3. Nash equations for boundary components. Recall that (see Section 2.5.3) a semi-algebraic set $\mathcal{Q} \subset \mathbb{R}^n$ is a *Nash manifold with corners* of dimension d if for each point $y \in \mathcal{Q}$ there exist an integer $0 \leq k \leq d$ and an open semi-algebraic neighbourhood $U \subset \mathcal{Q}$ of y equipped with a Nash diffeomorphism

$$\phi : U \to \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_k \geq 0\} \subset \mathbb{R}^d.$$

Recall that we consider only Nash manifolds with *divisorial corners* (see Section 2.5.3). Let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners of dimension $d \geq 2$. By [FGR, Thm.1.11, 1.12] there exists a d-dimensional Nash manifold $M \subset \mathbb{R}^n$, called a *Nash envelope* of $\mathcal{Q}$, that contains $\mathcal{Q}$ as a closed subset and satisfies:

(i) The Nash closure Y of $\partial\mathcal{Q}$ in M is a Nash normal-crossings divisor of M and $\mathcal{Q} \cap Y = \partial\mathcal{Q}$.

(ii) For every $x \in \partial\mathcal{Q}$ the analytic closure of the germ $\partial\mathcal{Q}_x$ is Y_x.

(iii) For each irreducible component Y_1 of Y the intersection $\mathcal{Q} \cap Y_1$ is a facet of $\mathcal{Q}$.

(iv) M can be covered by finitely many open semi-algebraic subsets U_i, for $i = 1, \dots, r$, equipped with Nash diffeomorphisms

$$u_i := (u_{i1}, \dots, u_{id}) : U_i \to \mathbb{R}^d$$

such that:

$$\begin{cases} U_i \subset \operatorname{Int}(\mathcal{Q}) \text{ or } U_i \cap \mathcal{Q} = \varnothing & \text{if } U_i \text{ does not meet } \partial\mathcal{Q}, \\ U_i \cap \mathcal{Q} = \{u_{i1} \geq 0, \dots, u_{ik_i} \geq 0\} & \text{if } U_i \text{ meets } \partial\mathcal{Q} \text{ (for a suitable } k_i \geq 1). \end{cases}$$

The following result allows us to build suitable Nash equations for the facets of $\mathcal{Q}$. The proof is strongly inspired on the proof of [Fe4, Lem.4.3].

Lemma 5.1.4 (Nash equations for the faces)**.** *Let Y_1 an irreducible component of the Nash normal-crossings divisor Y (which is the Nash closure of $\partial\mathcal{Q}$ in the Nash envelope M). Then, after shrinking M is necessary, there exists a Nash function $h_1 : M \to \mathbb{R}$ such that $Y_1 = \{h_1 = 0\}$ and $d_x h_1 : T_x M \to \mathbb{R}$ is surjective for each $x \in Y_1$. In addition, $H_1 := h_1^{-1}([0, +\infty))$ is a Nash manifold with boundary Y_1 that contains $\mathcal{Q}$ as a closed subset.*

Proof. The proof is conducted in several steps:

Step 1. We construct first an $\mathcal{S}^2$ semi-algebraic function h_1^* on M such that $Y_1 \subset \{h_1^* = 0\}$ and $d_x h_1^*(v) > 0$ for each $x \in Y_1 \cap \mathcal{Q}$ and each non-zero vector $v \in T_x M$ pointing 'inside $\mathcal{Q}$'.

Assume that Y_1 meets U_i exactly for $i = 1, \dots, s$ for some $s \leq r$. We reorder the variables u_{ij} in order to guarantee that $Y_1 \cap U_i = \{u_{i1} = 0\}$ and $\mathcal{Q} \cap U_i \subset \{u_{i1} \geq 0\}$ for $i = 1, \dots, s$. Let $\{\theta_i\}_{i=1}^{s+1} : M \to [0,1]$ be an $\mathcal{S}^2$ partition of unity subordinated to the finite covering $\{U_i\}_{i=1}^{s} \cup U_{s+1} := \{M \setminus Y_1\}$ of M and consider the $\mathcal{S}^2$ function $h_1^* := \sum_{i=1}^{s} \theta_i u_{i1}$. It holds $Y_1 \subset \{h_1^* = 0\}$.

Fix $x \in Y_1 \cap \mathcal{Q}$ and let $v \in T_x M$ be a non-zero vector pointing 'inside $\mathcal{Q}$', that is, $d_x u_{i1}(v) > 0$ if $x \in U_i$. We have $u_{i1}(x) = 0$ for $i = 1, \dots, s$ and $x \notin U_{s+1}$, so

$$\begin{aligned} d_x h_1^* &= \sum_{x \in U_i} u_{i1}(x) d_x \theta_i + \sum_{x \in U_i} \theta_i(x) d_x u_{i1} = \sum_{x \in U_i} \theta_i(x) d_x u_{i1} \\ &\rightsquigarrow\ d_x h_1^*(v) = \sum_{x \in U_i} \theta_i(x) d_x u_{i1}(v) > 0 \end{aligned}$$

because $\sum_{x \in U_i} \theta_i(x) = 1$, $\theta_i(x) \geq 0$ and $d_x u_{i1}(v) > 0$ if $x \in U_i$.

Step 2. By [BFR, Prop.8.2] there exists a Nash function h_1' on M close to h_1^* in the $\mathcal{S}^2$ topology such that $Y_1 \subset \{h_1' = 0\}$ and $d_x h_1'(v) > 0$ for each $x \in Y_1 \cap \mathcal{Q}$ and each non-zero vector $v \in T_x M$ pointing 'inside $\mathcal{Q}$'. We claim: *there exists an open semi-algebraic neighbourhood $W \subset M$ of $Y_1 \cap \mathcal{Q}$ such that*

$$\operatorname{Int}(\mathcal{Q}) \cap W \subset \{h_1' > 0\} \cap W$$

and $\{h_1' = 0\} \cap W = Y_1$.

Pick a point $x \in Y_1$ and assume $x \in U_1$. As h_1' vanishes identically at Y_1, we may write $h_1'|_{U_1} = u_{11} a_1$ where a_1 is a Nash function on U_1. Pick $y \in Y_1 \cap U_1 \cap \mathcal{Q}$ and observe that $d_y h_1' = a_1(y) d_y u_{11}$. Let $v \in T_y M$ be a non-zero vector pointing 'inside $\mathcal{Q}$'. As $d_y u_{11}(v) > 0$ and $d_y h_1'(v) > 0$, we deduce $a_1(y) > 0$. Define $W_1 := \{a_1 > 0\} \subset U_1$ and notice that $Y_1 \cap U_1 \cap \mathcal{Q} \subset W_1$, $\operatorname{Int}(\mathcal{Q}) \cap W_1 \subset \{h_1' > 0\} \cap W_1$ and $\{h_1' = 0\} \cap W_1 = Y_1 \cap W_1$. Construct analogously $W_2, \dots, W_s$ and observe that $W := \bigcup_{i=1}^{s} W_i$ satisfies the required properties.

Substitute M by $M \setminus (Y_1 \setminus W)$, which is an open semi-algebraic subset of M that contains $\mathcal{Q}$ as a closed subset. Substitute Y by the Nash closure of $\partial\mathcal{Q}$ in the new M and Y_1 by the irreducible component of the new Y that contains the facet $Y_1 \cap \mathcal{Q}$.

Step 3. Next, we construct h_1. If $W = M$, it is enough to set $h_1 := h_1'$. Suppose $W \neq M$. Let ε_0 be a (continuous) semi-algebraic function whose value is 1 on Y_1 and -1 on $M \setminus W$. Let ε be a Nash approximation of ε_0 such that $|\varepsilon - \varepsilon_0| < \frac{1}{2}$. Then

$$\varepsilon(x) \begin{cases} > \frac{1}{2} & \text{if } x \in Y_1, \\ < -\frac{1}{2} & \text{if } x \in M \setminus W. \end{cases}$$

Thus, $\{\varepsilon > 0\} \subset W$ is an open semi-algebraic neighbourhood of Y_1 in M. By [Sh, II.5.3] Y_1 is a Nash subset of M. Let f be a Nash equation of Y_1 in M. Substituting f by $\frac{f^2}{\varepsilon^2 + f^2}$ we may assume that f is non-negative and $f(x) = 1$ if $\varepsilon(x) = 0$. Consider the (continuous) semi-algebraic function on M given by

$$\delta(x) := \begin{cases} 1 & \text{if } \varepsilon(x) > 0, \\ \frac{1}{f(x)} & \text{if } \varepsilon(x) \leq 0. \end{cases}$$

Let g be a Nash function on M such that $\delta < g$ (see [BCR, Prop.2.6.2], after embedding M in $\mathbb{R}^{n+1}$ as a closed subset). Consider the Nash function

$$h_1 := h_1' + f^2 g^2 (h_1'^2 + 1)$$

and let us prove that it satisfies the required conditions.

STEP 4. We claim: h_1 *is positive on* $\operatorname{Int}(\mathcal{Q})$.

Let $x \in \operatorname{Int}(\mathcal{Q})$. If $h_1'(x) > 0$, then $h_1(x) > 0$. If $h_1'(x) \leq 0$, then $x \notin W$ (because $\operatorname{Int}(\mathcal{Q}) \cap W \subset \{h_1' > 0\} \cap W$). Thus, $\varepsilon(x) \leq 0$ and

$$\begin{aligned} h_1(x) &= h_1'(x) + g^2(x) f^2(x)(h_1'^2(x) + 1) \\ &> h_1'(x) + \frac{1}{f^2(x)} f^2(x)(h_1'^2(x) + 1) = h_1'^2(x) + h_1'(x) + 1 > 0. \end{aligned}$$

STEP 5. It holds: *There exists an open semi-algebraic neighbourhood* $W' \subset W$ *of* Y_1 *such that* $\{h_1 = 0\} \cap W' = Y_1$*,* $\operatorname{Int}(\mathcal{Q}) \cap W' \subset \{h_1 > 0\} \cap W'$ *and the differential* $d_x h_1 : T_x M \to \mathbb{R}$ *is surjective for all* $x \in Y_1$.

Recall that $W = \bigcup_{i=1}^s W_i$. We have seen in Step 2 that there exists a Nash function a_1 on U_1 such that $h_1'|_{U_1} = u_{11} a_1$ and $Y_1 \cap U_1 \subset W_1 := \{a_1 > 0\}$. As f vanishes identically at Y_1, we deduce $f|_{U_1} = u_{11} b_1$ where b_1 is a Nash function on U_1. Consequently,

$$h_1|_{U_1} = u_{11} a_1 + g^2|_{U_1} u_{11}^2 b_1^2 (u_{11}^2 a_1^2 + 1) = u_{11}(a_1 + g^2|_{U_1} u_{11} b_1^2 (u_{11}^2 a_1^2 + 1))$$

and $d_x h_1 = a_1(x) d_x u_{11} = d_x h_1'$ for $x \in Y_1 \cap U_1$. Define

$$W_1' := \{a_1 + g^2|_{U_1} u_{11} b_1^2 (u_{11}^2 a_1^2 + 1) > 0\} \cap W_1,$$

which is an open semi-algebraic subset of M. We have $Y_1 \cap U_1 \subset W_1'$ and

$$\operatorname{Int}(\mathcal{Q}) \cap W_1' \subset \{h_1 > 0\} \cap W_1'.$$

Construct analogously $W_2', \ldots, W_s'$ and observe that the open semi-algebraic subset $W' := \bigcup_{i=1}^s W_i' \subset M$ is an open neighbourhood of Y_1 that satisfies $\{h_1 = 0\} \cap W' = Y_1$, $\operatorname{Int}(\mathcal{Q}) \cap W' \subset \{h_1 > 0\} \cap W'$ and $d_x h : T_x M \to \mathbb{R}$ is surjective for all $x \in Y_1$.

Consequently, $M' := \{h_1 > 0\} \cup W'$ and $h_1|_{M'}$ satisfy all the required conditions. $\square$

Remark 5.1.5. Let $H \subset \mathbb{R}^n$ be a d-dimensional Nash manifold with non-empty boundary ∂H and $M \subset \mathbb{R}^n$ a Nash manifold that contains H as a closed subset. In this case, the previous lemma provides a *Nash equation* for the boundary ∂H. That is, up to shrink M if necessary, there exists a Nash function $h : M \to \mathbb{R}$ such that

(i) $\partial H = \{h = 0\}$,

(ii) $\operatorname{Int}(H) = \{h > 0\}$,

(iii) $d_x h : T_x H \to \mathbb{R}$ is surjective for all $x \in \partial H$.

Let $Y_1, \dots, Y_\ell$ the irreducible components of Y. As an immediate consequence of the previous lemma, we have the following:

Corollary 5.1.6. *After shrinking the manifold M if necessary, there exist Nash functions $h_i : M \to \mathbb{R}$ such that $Y_i = \{h_i = 0\}$, $d_x h_i : T_x M \to \mathbb{R}$ is surjective for each $x \in Y_i$ and*

$$\mathcal{Q} = \{h_1 \geq 0, \dots, h_\ell \geq 0\}.$$

5.1.4. Nash doubles. Doubling a smooth manifold with boundary is a very standard tool in differential topology. The Nash construction has been treated by Shiota [Sh, VI.2.1] in the compact case and by Fernando [Fe4, 4.B.1] in the general case.

Let $H \subset \mathbb{R}^n$ be a d-dimensional Nash manifold with non-empty boundary ∂H and let $h : M \to \mathbb{R}$ be a Nash equation for ∂H (as in Remark 5.1.5).

Proposition 5.1.7 (Nash double)**.** *The semi-algebraic set*

$$D(H) := \{(x,t) \in H \times \mathbb{R} : \; t^2 - h(x) = 0\} \subset \mathbb{R}^{n+1}$$

is a Nash manifold of dimension d that contains $\partial H \times \{0\}$ as the Nash subset $\{t = 0\}$.

Proof. As $H = \{x \in M \; : \; h(x) \geq 0\}$, we can describe $D(H)$ as the Nash subset

$$D(H) = \{(x,t) \in M \times \mathbb{R} : \; t^2 - h(x) = 0\} \subset M \times \mathbb{R},$$

so it is enough to check that $D(H)$ is smooth. We consider the Nash function $f : M \times \mathbb{R} \to \mathbb{R}$, defined as $f(x,t) = t^2 - h(x)$. It holds $D(H) = f^{-1}(0)$. The differential of f at $(x,t) \in M \times \mathbb{R}$ is $d_{(x,t)} f = 2t - d_x h$, that is surjective for each (x,t) (we are using Lemma 5.1.4 when $t = 0$). Thus $0 \in \mathbb{R}$ is a regular value for the Nash function f, so $D(H)$ is a smooth Nash subset of dimension d of $M \times \mathbb{R}$. The last part of the statement is clear. $\square$

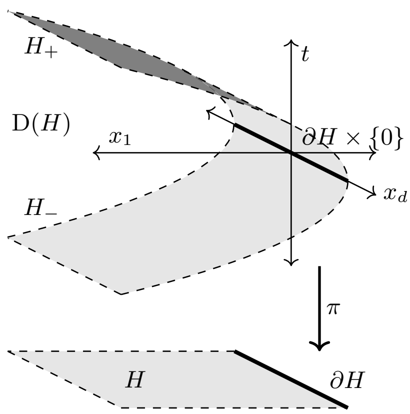

Figure 5.1: Nash double of H (figure borrowed from [Fe4, Fig.3]).

Consider now the projection $\pi : D(H) \to H$, $(x,t) \mapsto x$, that is a surjective Nash map. Fix $\epsilon = \pm$ and denote $H_\epsilon := D(H) \cap \{\epsilon t \geq 0\}$. We have the following:

Proposition 5.1.8. *The map $\pi : D(H) \to H$ verifies the following properties.*

(i) *The restriction $\pi_\epsilon := \pi|_{H_\epsilon} : H_\epsilon \to H$ is a semi-algebraic homeomorphism and the restriction $\pi|_{\mathrm{D}(H)\cap\{\epsilon t>0\}} : \mathrm{D}(H) \cap \{\epsilon t > 0\} \to \mathrm{Int}(H)$ is a Nash diffeomorphism.*

(ii) *$\pi(x, 0) = x$ for all $(x, 0) \in \partial H \times \{0\} = \mathrm{D}(H) \cap \{t = 0\}$.*

(iii) *π has local representations $(y_1, \dots, y_d) \mapsto (y_1^2, y_2, \dots, y_d)$ at each point of $\mathrm{D}(H) \cap \{t = 0\}$.*

(iv) *π is open and proper.*

Proof. (i) Observe that H_ϵ is the graph of the continuous semi-algebraic map $\epsilon\sqrt{h}$ on H, so $\pi_\epsilon : H_\epsilon \to H$ is a semi-algebraic homeomorphism.

The intersection $\mathrm{D}(H) \cap \{\epsilon t > 0\}$ is the graph of the strictly positive Nash function $\epsilon\sqrt{h}$ on $\mathrm{Int}(H)$. Consequently

$$\pi|_{\mathrm{D}(H)\cap\{\epsilon t>0\}} : \mathrm{D}(H) \cap \{\epsilon t > 0\} \to \mathrm{Int}(H)$$

is Nash diffeomorphism.

Statement (ii) is clear.

(iii) Let us first construct local coordinates at the points of $D(H) \cap \{t = 0\}$. Pick a point $x_0 \in \partial H$ and let $U \subset M$ be an open semi-algebraic neighbourhood of x_0 equipped with a Nash diffeomorphism

$$u := (u_1, \dots, u_d) : U \to (-1, 1) \times \mathbb{R}^{d-1},$$

such that $u(x_0) = 0$ and $U \cap H = \{u_1 \geq 0\}$. We may assume, after shrinking U if necessary and modifying suitably u and h, that $u_1 = h|_U$. This is so because $\{u_1 \geq 0\} = \{h \geq 0\}$, $\{u_1 = 0\} = \{h = 0\} \cap U$ and $d_x u, d_x h : T_x M \to \mathbb{R}$ are both surjective for each $x \in \{h = 0\} \cap U$, so u_1/h is a strictly positive Nash function on a neighbourhood of $\{h = 0\} \cap U$. Observe that $V := (U \cap H) \times \mathbb{R}$ is an open semi-algebraic subset of $H \times \mathbb{R}$ and

$$W := \mathrm{D}(H) \cap V = \{(x, \pm\sqrt{h(x)}) : \ x \in U \cap H\}$$

is an open semi-algebraic neighbourhood of $(x_0, 0)$ in $\mathrm{D}(H)$. Consider the Nash map

$$u' := (u'_1, \dots, u'_d) : W \to (-1, 1) \times \mathbb{R}^{d-1}, \ (x, t) \mapsto (t, u_2(x), \dots, u_d(x))$$

and let us check that it is a Nash diffeomorphism. As

$$(u'_2, \dots, u'_d)(W) = (u_2, \dots, u_d)(U \cap H) = \mathbb{R}^{d-1},$$

we have $u'(W) = (-1, 1) \times \mathbb{R}^{d-1}$, so u' is surjective.

Pick $(x_1, t_1), (x_2, t_2) \in W$ such that $u'(x_1, t_1) = u'(x_2, t_2)$. Then $t_1 = t_2$, so

$$u_1(x_1) = h(x_1) = t_1^2 = t_2^2 = h(x_2) = u_1(x_2)$$

and $u(x_1) = u(x_2)$. As u is injective, we have $x_1 = x_2$, so $(x_1, t_1) = (x_2, t_2)$. Thus, u' is injective.

Denote $u^{-1} := \phi := (\phi_1, \dots, \phi_m)$. The inverse of u' is the Nash map

$$\zeta : (-1,1) \times \mathbb{R}^{d-1} \to W,\ (t, y') := (t, y_2, \dots, y_d) \mapsto (\phi(t^2, y'), t).$$

The differential of ζ at a point $(t, y') \in (-1,1) \times \mathbb{R}^{d-1}$ is

$$\begin{pmatrix} 2t\frac{\partial \phi_1}{\partial y_1}(t^2, y') & \frac{\partial \phi_1}{\partial y_2}(t^2, y') & \cdots & \frac{\partial \phi_1}{\partial y_d}(t^2, y') \\ \vdots & \vdots & \ddots & \vdots \\ 2t\frac{\partial \phi_m}{\partial y_1}(t^2, y') & \frac{\partial \phi_m}{\partial y_2}(t^2, y') & \cdots & \frac{\partial \phi_m}{\partial y_d}(t^2, y') \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and it has rank d. Consequently, u' is a Nash diffeomorphism.

Let us see how is the local representation of π at $(t, y') \in D(H) \cap \{t = 0\}$ if we use suitable coordinates. We have

$$\begin{aligned} (-1,1) \times \mathbb{R}^{d-1} \xrightarrow{\zeta} W \xrightarrow{\pi} H \cap U \xrightarrow{u} (-1,1) \times \mathbb{R}^{d-1}, \\ (t, y') \mapsto (\phi(t^2, y'), t) \mapsto \phi(t^2, y') \mapsto (t^2, y'), \end{aligned}$$

as required.

(iv) We prove first that π is open. Let $A \subset D(H)$ be an open set and let $A_\epsilon := A \cap H_\epsilon$, which is an open subset of H_ϵ. As $\pi_\epsilon : H_\epsilon \to H$ is a semi-algebraic homeomorphism, $\pi(A_\epsilon)$ is an open subset of H. Thus, $\pi(A) = \pi(A_+) \cup \pi(A_-)$ is an open subset of H and π is open.

To show that π is proper, pick $K \subset H$ compact and observe that

$$\pi^{-1}(K) = (\pi_+)^{-1}(K) \cup (\pi_-)^{-1}(K).$$

As each $\pi_\epsilon : H_\epsilon \to H$ is a semi-algebraic homeomorphism, $(\pi_\epsilon)^{-1}(K)$ is compact for $\epsilon = \pm$, so $\pi^{-1}(K)$ is a compact subset of $D(H)$ and π is proper as required. □

5.1.5. Folding one boundary component. Let $M \subset \mathbb{R}^n$ be a d-dimensional Nash manifold that contains H as a closed subset and assume that ∂H is a Nash subset of M. Let Y be a Nash normal-crossings divisor of M, such that ∂H is a union of irreducible components of Y. Let $Y_1, \dots, Y_r$ be the irreducible components of Y that meet ∂H but are not contained in ∂H. Let $h : M \to \mathbb{R}$ be a Nash equation of ∂H. Observe that $h_i := h|_{Y_i}$ is a Nash equation of $Y_i \cap \partial H$ such that $\operatorname{Int}(H) \cap Y_i = \{h_i > 0\}$, and $d_x h_i : T_x Y_i \to \mathbb{R}$ is surjective for all $x \in Y_i \cap \partial H$. Thus, $Y_i \cap H$ is a Nash manifold with boundary $Y_i \cap \partial H$ that is contained in Y_i as a closed subset. In addition

$$D(Y_i \cap H) = \{(x, t) \in (Y_i \cap H) \times \mathbb{R} : t^2 - h_i(x) = 0\} = D(H) \cap (Y_i \times \mathbb{R}),$$

is the Nash double of $Y_i \cap H$. Define $H_\epsilon := D(H) \cap \{\epsilon t \geq 0\}$, for $\epsilon = \pm$.

We want to show that there exists a Nash embedding of M into $D(H)$, that maps the Nash normal-crossings divisor Y into $D(Y)$ component-wise. We need the following technical lemma.

Lemma 5.1.9. *Let $k \geq 1$ and $0 < a \leq 1$. Consider the $\mathcal{C}^{2k}$ semi-algebraic function $f_a := (1 - (\mathtt{t}/a)^{2k})^{2k}\mathtt{t} + (1 - (1 - (\mathtt{t}/a)^{2k})^{2k})\sqrt{\mathtt{t}}$. Then f_a is positive semidefinite $[0, a]$, it is strictly increasing on $[0, a]$, the Taylor polynomial of f_a at $t = 0$ of degree $2k$ is $\mathtt{t}$ whereas the Taylor polynomial of f_a at $t = a$ of degree $2k - 1$ is that of $\sqrt{\mathtt{t}}$ at $t = a$. In addition, $f_a(t) \leq \sqrt{t}$ on $[0, 1]$.*

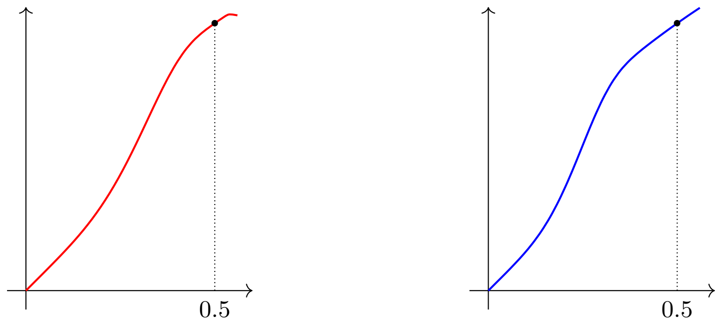

Figure 5.2: Graphs of f_a for $a = \frac{1}{2}$ and $k = 2$ (left) and $k = 6$ (right).

Proof. Write $f := f_a$ to light notations. Using Newton's binomial we have

$$f = \sum_{\ell=0}^{2k} \binom{2k}{\ell} (-1)^\ell \Big(\frac{\mathtt{t}}{a}\Big)^{2k\ell} \mathtt{t} - \sum_{\ell=1}^{2k} \binom{2k}{\ell} (-1)^\ell \Big(\frac{\mathtt{t}}{a}\Big)^{2k\ell} \sqrt{\mathtt{t}},$$

so the Taylor polynomial of f at $t = 0$ of degree $2k$ is $\mathtt{t}$. In particular, f is a $\mathcal{C}^{2k}$ semi-algebraic function. In addition,

$$f = (1 - (1 + (\mathtt{t}/a) - 1)^{2k})^{2k}\mathtt{t} + (1 - (1 - (1 + (\mathtt{t}/a) - 1)^{2k})^{2k})\sqrt{\mathtt{t}}$$

and using Newton's binomial we have

$$f = \sqrt{\mathtt{t}} - \Big(\frac{\mathtt{t}}{a} - 1\Big)^{2k} \left(\sum_{\ell=1}^{2k} \binom{2k}{\ell} \Big(\frac{\mathtt{t}}{a} - 1\Big)^{\ell-1} \right)^{2k} (\mathtt{t} - \sqrt{\mathtt{t}}),$$

so the Taylor polynomial of f at $t = a$ of degree $2k - 1$ is that of $\sqrt{\mathtt{t}}$ at $t = a$.

Define $\sigma := (1 - (\mathtt{t}/a)^{2k})^{2k}$ and observe that $\sigma(0) = 1$, $\sigma(a) = 0$ and both $\sigma, 1 - \sigma$ are positive semidefinite on $[0, a]$. Thus, $f = \sigma\mathtt{t} + (1 - \sigma)\sqrt{\mathtt{t}}$ is positive semidefinite on $[0, a]$. As $0 < a \leq 1$, it holds $\sqrt{t} - t > 0$ on $(0, a)$, so the derivative

$$\begin{aligned} f' &:= \Big(\mathtt{t} - \sqrt{\mathtt{t}}\Big)\sigma' + (1 - \sigma)\frac{1}{2\sqrt{\mathtt{t}}} + \sigma \\ &= (\sqrt{\mathtt{t}} - \mathtt{t})4k^2(1 - (\mathtt{t}/a)^{2k})^{2k-1}(\mathtt{t}/a)^{2k-1}(1/a) + (1 - \sigma)\frac{1}{2\sqrt{\mathtt{t}}} + \sigma \end{aligned}$$

is strictly positive on $(0, a)$ and f is strictly increasing on $[0, a]$. Finally, as $t \leq \sqrt{t}$ on $[0, 1]$ we deduce that $f = \sigma t + (1 - \sigma)\sqrt{t} \leq \sqrt{t}$ on $[0, 1]$, as required. $\square$

We are ready to prove the embedding theorem. Note that, as a straightforward consequence of this result, we obtain: *The Nash manifold with boundary H is Nash diffeomorphic to H_ϵ, for $\epsilon = \pm$.*

Theorem 5.1.10 (Embedding)**.** *After shrinking M if necessary, there exists a Nash embedding $\phi : M \to D(H)$ that maps H onto H_+ and Y_i into $D(Y_i \cap H)$. In addition, $\phi|_{\partial H} = \mathrm{id}_{\partial H}$, $\phi|_H$ is close to $(\pi|_{H_+})^{-1}$, $\phi(x)$ is close to $\rho(x)$ for* *each $x \in M \setminus H$, where ρ is a Nash retraction compatible with Y and there exists an open semi-algebraic neighbourhood $W \subset M$ of ∂H such that*

$$\phi|_{\operatorname{Cl}(W)\cup H} : \operatorname{Cl}(W) \cup H \to D(H)$$

is proper.

Proof. The proof is constructed in several steps.

STEP 1. We construct first suitable semi-algebraic neighbourhoods $U \subset M$ of ∂H and $V \subset D(H)$ of $\partial H \times \{0\}$.

Let $U \subset M$ be an open semi-algebraic neighbourhood of ∂H equipped with a Nash retraction $\rho : U \to \partial H$ compatible with Y (see Proposition 5.1.2). We may assume that U does not meet the irreducible components of Y that do not meet ∂H. Define the Nash map $\varphi := (\rho, h) : U \to \partial H \times \mathbb{R}$. By Proposition 5.1.3 we may assume, after shrinking U if necessary, that there exists a strictly positive Nash function $\varepsilon : \partial H \to (0, 1)$ such that

$$\begin{aligned}
&\varphi(U) = \{(y, s) \in \partial H \times \mathbb{R} : |s| < \varepsilon(y)\},\\
&\varphi(U \cap H) = \{(y, s) \in \partial H \times \mathbb{R} : 0 \leq s < \varepsilon(y)\},
\end{aligned}$$

and $\varphi : U \to \varphi(U)$ is a Nash diffeomorphism such that

$$\varphi(Z \cap U) = \{(y, s) \in (Z \cap \partial H) \times \mathbb{R} : |s| < \varepsilon(y)\}$$

for each irreducible component Z of $\operatorname{Sing}_\ell(Y)$ such that $Z \cap \partial H \neq \varnothing$, $Z \not\subset \partial H$ and $\ell \geq 0$. Define $V := \pi^{-1}(U \cap H)$.

STEP 2. Let $V' := \{(y, t) \in \partial H \times \mathbb{R} : |t| < \sqrt{\varepsilon(y)}\}$. We want to prove: *The Nash map*

$$\psi : V \to \partial H \times \mathbb{R},\ (x, t) \mapsto (\rho(x), t)$$

is a Nash diffeomorphism onto its image V', such that

$$\psi(Z \cap U \cap H) = \{(y, t) \in (Z \cap \partial H) \times \mathbb{R} : 0 \leq t < \sqrt{\varepsilon(y)}\}$$

for each irreducible component Z of $\operatorname{Sing}_\ell(Y)$ *such that $Z \cap \partial H \neq \varnothing$, $Z \not\subset \partial H$ and $\ell \geq 0$.*

(1) *ψ is injective.* If $(x_1, t_1), (x_2, t_2) \in V$ satisfy $\psi(x_1, t_1) = \psi(x_2, t_2)$, then $\rho(x_1) = \rho(x_2)$ and

$$h(x_1) = t_1^2 = t_2^2 = h(x_2).$$

Then we have $\varphi(x_1) = \varphi(x_2)$, so $x_1 = x_2$. Thus $(x_1, t_1) = (x_2, t_2)$.

(2) $\psi(V) = V'$. Fix a point $(x, t) \in V$. Then, by definition, $x \in U$ and $\varphi(x) = (\rho(x), h(x)) \in \varphi(U)$, so $t^2 = h(x) < \varepsilon(\rho(x))$ and $\psi(x, t) \in V'$. Conversely, fix a point $(y, t) \in V'$. As $(y, t^2) \in \varphi(U)$, there exists a point $x \in U$ such that $\varphi(x) = (\rho(x), h(x)) = (y, t^2)$. As $x \in U$ and $h(x) = t^2 \geq 0$, we have $x \in U \cap H$, so $(x, t) \in V$ and $(y, t) = (\rho(x), t) = \psi(x, t) \in \psi(V)$.

(3) *The differential $d_z\psi : T_zD(H) \to T_{\rho(z)}H \times \mathbb{R}$ is an isomorphism for each $z \in V$.* Write $z := (x, t)$ and notice that

$$T_zD(H) = \{(v, r) \in T_xH \times \mathbb{R} : d_xh(v) - 2tr = 0\}$$

and $d_x\psi(v,r)=(d_x\rho(v),r)$. If $t\neq 0$,

$$d_z\psi(v,r)=\left(d_x\rho(v),\frac{1}{2t}d_xh(v)\right).$$

As $d_x\varphi=(d_x\rho,d_xh)$ is an isomorphism, also $d_z\psi$ is an isomorphism. If $t=0$, that is $z=(x,0)\in\partial H\times\mathbb{R}$, then

$$T_zD(H)=\{(v,r)\in T_xH\times\mathbb{R}\,:\,d_xh(v)=0\}=T_x\partial H\times\mathbb{R}$$

and $d_z\psi(v,r)=(v,r)$ because $\rho|_{\partial H}=\mathrm{id}_{\partial H}$. So $d_z\psi$ is an isomorphism also in this case.

(4) *Let Z be an irreducible component of $\mathrm{Sing}_\ell(Y)$ such that $Z\cap\partial H\neq\varnothing$, $Z\not\subset\partial H$ and $\ell\geq 0$. Then*

$$\psi(Z\cap U\cap H)=\{(y,t)\in(Z\cap\partial H)\times\mathbb{R}\,:\,0\leq t<\sqrt{\varepsilon(y)}\}.$$

By Proposition 5.1.2 we have $\rho(Z\cap U)=Z\cap\partial H$. Thus,

$$\psi(Z\cap U\cap H)\subset\{(y,t)\in(Z\cap\partial H)\times\mathbb{R}\,:\,0\leq t<\sqrt{\varepsilon(y)}\}.$$

To prove that the previous inclusion is in fact an equality it is enough to proceed similarly to the end of the proof of Proposition 5.1.3.

Step 3. Let $a:\partial H\to\mathbb{R}$ be a strictly positive Nash function such that $a<\varepsilon$. For $\epsilon=\pm$, define

$$H^\bullet:=H\setminus\varphi^{-1}\left(\left\{(y,s)\in\partial H\times\mathbb{R}\,:\,|s|<\frac{a(y)}{4}\right\}\right)\subset H\setminus U,$$
$$H^\bullet_\epsilon:=H_\epsilon\setminus\psi^{-1}\left(\left\{(y,s)\in\partial H\times\mathbb{R}\,:\,|s|<\frac{\sqrt{a(y)}}{2}\right\}\right)\subset H_\epsilon\setminus V.$$

The restriction $\omega_{a\epsilon}:=\pi|_{H^\bullet_\epsilon}:H^\bullet_\epsilon\to H^\bullet$ is a Nash diffeomorphism for $\epsilon=\pm$.

Indeed $\omega_{a\epsilon}$ is clearly injective. Let $x\in H^\bullet$. As $x\in\mathrm{Int}(H)$, we have $h(x)>0$ and write $t:=\epsilon\sqrt{h(x)}$. It holds that $(x,t)\in D(H)$ and $\pi(x,t)=x$. We want to check that $(x,t)\in H^\bullet_\epsilon$. If $x\notin U$, then $(x,t)\in H_\epsilon\setminus V\subset H^\bullet_\epsilon$. If $x\in U$, then $\psi(x,t)=(\rho(x),t)$. As $x\in H^\bullet$, it holds

$$\frac{a(\rho(y))}{4}\leq h(x),\ \text{ so }\ \frac{\sqrt{a(\rho(y))}}{2}\leq\sqrt{h(x)}=\epsilon t.$$

Consequently, $(x,t)\in H^\bullet_\epsilon$ and $\omega_{a\epsilon}$ is surjective. In addition, by Proposition 5.1.8(i) $d_z\omega_{a\epsilon}=d_z\pi$ is an isomorphism for each $z\in H^\bullet_\epsilon\subset D(H)\cap\{\epsilon t>0\}$. Consequently $\omega_{a\epsilon}$ is a Nash diffeomorphism.

Step 4. We want to construct now a semi-algebraic embedding $\phi_a:M\to D(H)$ of class $\mathcal{C}^{2k-1}$, for $k\geq 1$ arbitrarily large. Substitute M by $H\cup U$ and define

$$F_a:\partial H\times\mathbb{R}\to\partial H\times\mathbb{R},\ (y,s)\mapsto\begin{cases}(y,s) & \text{if } s<0,\\ (y,f_{a(y)}(s)) & \text{if } 0\leq s\leq a(y),\\ (y,\sqrt{s}) & \text{if } a(y)<s,\end{cases}$$

where $f_{a(y)}$ is the $\mathcal{C}^{2k}$ semi-algebraic function introduced in Lemma 5.1.9 and we choose k large enough. Denote

$$U' := \varphi(U) = \{(y,s) \in \partial H \times \mathbb{R} : \ |s| < \varepsilon(y)\},$$
$$V' = \psi(V) = \{(y,t) \in \partial H \times \mathbb{R} \,:\, |t| < \sqrt{\varepsilon(y)}\} \text{ (already introduced in Step 2).}$$

The open semi-algebraic set $F_a(U')$ is contained in V', because if $-\varepsilon(y) < -s < 0$, then $-\sqrt{\varepsilon(y)} < -\sqrt{s} < -s < 0$ (recall that $0 < \varepsilon(y) < 1$) and $0 \le f_{a(y)}(s) \le \sqrt{s}$ if $0 \le s \le a(y) \le 1$.

The map F_a is a $\mathcal{S}^{2k-1}$ diffeomorphism because $f_{a(y)} : [0, a(y)] \to [0, \sqrt{a(y)}]$ is by Lemma 5.1.9 a $\mathcal{S}^{2k}$ diffeomorphism such that the Taylor polynomial of degree $2k$ at $t = 0$ is $\mathtt{t}$ and the Taylor polynomial of f_a of degree $2k-1$ at $t = a(y)$ is that of $\sqrt{\mathtt{t}}$.

Define

$$\phi_a : M \to D(H), \ x \mapsto \begin{cases} \omega_{a+}^{-1}(x) & \text{if } x \in M \setminus U = H \setminus U \subset H^\bullet, \\ (\psi^{-1} \circ F_a \circ \varphi)(x) & \text{if } x \in U. \end{cases}$$

It holds that ϕ_a is a $\mathcal{S}^{2k-1}$ diffeomorphism onto its image $\phi_a(M)$, whose $\mathcal{S}^{2k-1}$ inverse is

$$\phi_a^{-1} : \phi_a(M) \to M, \ x \mapsto \begin{cases} \omega_{a+}(y) & \text{if } y \in \phi_a(M) \setminus V \subset H_+^\bullet, \\ (\varphi^{-1} \circ F_a^{-1} \circ \psi)(y) & \text{if } y \in V. \end{cases}$$

In addition, it satisfies:

- $\phi_a(H) = H_+$ and $\phi_a|_{\partial H} = \mathrm{id}_{\partial H}$.
- $\phi_a(M)$ is an open semi-algebraic subset of $D(H)$, because $D(H) \setminus H^\bullet$ is an open semi-algebraic subset of $\phi_a(H) = H^+$, $\phi|_U : U \to U'$ and $\psi_V : V \to V'$ are Nash diffeomorphisms and $F_a(U') \subset V'$ is an open semi-algebraic set.
- $\phi_a(Y_i) \subset \mathrm{D}(Y_i \cap H) = D(H) \cap (Y_i \times \mathbb{R})$.

It only remains to show that if a is small enough, then $\phi_a|_H$ is close to $(\pi|_{H_+})^{-1}$.

Observe that $\pi' := \psi \circ \pi|_{H_+} \circ \varphi^{-1}(y,s) = (y, \sqrt{s})$. Let $\eta : M \to \mathbb{R}$ be a strictly positive continuous semi-algebraic function and let us choose a to guarantee that $\|\phi_a|_H - (\pi|_{H_+})^{-1}\| < \eta|_H$. In fact, as ψ, φ are Nash diffeomorphisms, it is enough to check: *If $b : \varphi(U) \to \mathbb{R}$ is a strictly positive semi-algebraic function, there exists a strictly positive semi-algebraic function $a : \partial H \to \mathbb{R}$ such that $a < \varepsilon$ and $\|F_a|_{\varphi(U)} - \pi'\| < b$.*

As F_a coincides with π' outside $\{(y,s) \in \partial H \times \mathbb{R} : \ s \in [0, \varepsilon(y))\}$, we will find $a < \frac{3}{4}\varepsilon < 1$. Let us work on $K := \{(y,s) \in \partial H \times \mathbb{R} : \ s \in [0, \frac{3}{4}\varepsilon(y)]\}$. The projection $\pi_1 : K \to \partial H$ is open, closed and surjective. The semi-algebraic map π_1 is surjective because $\partial H \times \{0\} \subset K$. Let us show that it is open.

Let $A \subset K$ be an open set and let $y_0 \in \pi_1(A)$. Let $s_0 \in [0, \frac{3}{4}\varepsilon(y_0)]$ be such that $(y_0, s_0) \in A$. As A is open, there exists $\xi > 0$ and $B \subset \partial H$ open such that $y_0 \in B$ and $(B \times [s_0 - \xi, s_0 + \xi]) \cap K \subset A$. In particular, $(y_0, s) \in A$

and $\pi_1(y_0, s) = y_0$ for each $s \in [s_0 - \xi, \frac{3}{4}\varepsilon(y_0)]$. Changing s_0 by $s_0 - \xi$ we may assume $s_0 \in [0, \frac{3}{4}\varepsilon(y_0)]$. Let ξ' be such that $s_0 + \xi' < \frac{3}{4}\varepsilon(y_0)$. As ε is continuous and $s_0 + \xi' < \frac{3}{4}\varepsilon(y_0)$, we may assume that $\frac{3}{4}\varepsilon(B) \subset [s_0 + \xi', +\infty)$. Thus, $B \times (s_0 - \xi', s_0 + \xi') \subset K$ and $A \cap (B \times (s_0 - \xi', s_0 + \xi')) \subset K$ is an open subset of $\partial H \times (s_0 - \xi', s_0 + \xi')$.

Let us show that π_1 is in fact proper. Let $C \subset \partial H$ be a compact set and let $\lambda := \max \frac{3}{4}\varepsilon|_C$. Then $\pi_1^{-1}(C) \subset C \times [0, \lambda]$, which is a compact set, so $\pi^{-1}(C)$ is also compact and π_1 is in particular closed.

By [FG4, Const.3.1] the function

$$a : \partial H \to \mathbb{R},\ y \mapsto \frac{1}{2} \min \Big\{ b(y, s)^2 :\ s \in \Big[0, \frac{3}{4}\varepsilon(y)\Big] \Big\}$$

is strictly positive, continuous and semi-algebraic. As the last component of F_a is strictly increasing on $[0, a(y)]$ and $f_a(a(y)) = \sqrt{a(y)}$, we have

$$\begin{aligned} &\|F_a(y, s) - (y, \sqrt{s})\| \\ &\quad = \begin{cases} 0 < b(y, s) & \text{if } a(y) \le s \le \frac{3}{4}\varepsilon(y), \\ |f_a(s) - \sqrt{s}| = \sqrt{s} - f_a(s) < \sqrt{s} < \sqrt{a(y)} \le b(y, s) & \text{if } 0 \le s < a(y). \end{cases} \end{aligned}$$

In addition, if $-a(y) < s < 0$,

$$\|F_a(y, s) - (\psi \circ \rho \circ \varphi^{-1})(y, s)\| = \|(y, s) - (y, 0)\| = -s < a(y) < \sqrt{a(y)} \le b(y, s).$$

We have used that $a < \frac{3}{4}\varepsilon(y) < 1$.

Step 5. Let us see now how to obtain the desired Nash embedding. Recall that $Y_1, \ldots, Y_r$ are the irreducible components of Y that meet ∂H but are not contained in ∂H. Consider the Nash normal-crossings divisor

$$\overline{Y} := \bigcup_{i=1}^{r} Y_i \subset M.$$

Let $Y_{r+1}, \ldots, Y_s$ be the irreducible components of Y that do not meet ∂H and recall that $U \cap \bigcup_{j=r+1}^{s} Y_j = \varnothing$, so

$$\phi_a|_{\bigcup_{j=r+1}^{s} Y_j} = (\pi|_{H_+})^{-1}|_{\bigcup_{j=r+1}^{s} Y_j}.$$

By [BFR, Thm.1.6, Thm 1.7], up to take k big enough, we can approximate the restriction $\phi_a|_Y : Y \to D(H)$ by a Nash map $\widehat{\phi} : Y \to D(H)$ such that

$$\widehat{\phi}|_{\partial H \cup \bigcup_{j=r+1}^{s} Y_j} = (\pi|_{H_+})^{-1}|_{\partial H \cup \bigcup_{j=r+1}^{s} Y_j}$$

and $\widehat{\phi}|_{\overline{Y}} : \overline{Y} \to D(H)$ satisfies $\widehat{\phi}(Y_i) \subset D(Y_i \cap H)$. By [BFR, Prop.8.2] we can extend $\widehat{\phi}$ to a global Nash map $\phi : M \to D(H)$, that is, up to take k big enough, close to ϕ_a in the $\mathcal{C}^1$ semi-algebraic topology. Thus by [Sh, II.1.7] the map ϕ is a Nash embedding.

Step 6. We show: *There exists an open semi-algebraic neighbourhood $W \subset M$ of ∂H such that $\phi|_{\mathrm{Cl}(W) \cup H} : \mathrm{Cl}(W) \cup H \to D(H)$ is proper.* Once this is done we conclude as it is straightforward to see that, up to take a smaller semi-algebraic function $a > 0$ if necessary, the Nash embedding $\phi : M \to D(H)$ defined in Step 5 satisfies all the other required properties.

As $\phi(M)$ is open in $D(H)$, the set $C := D(H) \setminus \phi(M)$ is a closed subset of $D(H)$ that does not intersect ∂H. Let V be an open semi-algebraic neighbourhood of ∂H in $\phi(M)$ such that $\mathrm{Cl}(V) \cap C = \varnothing$. The set $W := \phi^{-1}(V)$ is an open semi-algebraic neighbourhood of ∂H in M such that $\pi(\mathrm{Cl}(W)) = \mathrm{Cl}(V)$ because $\phi : M \to \phi(M)$ is a semi-algebraic homeomorphism. The restriction $\phi|_{\mathrm{Cl}(W)\cup H} : \mathrm{Cl}(W) \cup H \to D(H)$ satisfies $\phi(\mathrm{Cl}(W) \cup H) = \mathrm{Cl}(V) \cup H_+$ because $\phi(\mathrm{Cl}(W)) = \mathrm{Cl}(V)$ and $\phi(H) = H_+$. As $\mathrm{Cl}(V) \cup H_+$ is closed in $D(H)$, if $K \subset D(H)$ is compact then the set $K \cap (\mathrm{Cl}(V) \cup H_+)$ is compact. The map $\phi|_{\mathrm{Cl}(W)\cup H} : \mathrm{Cl}(W) \cup H \to D(H)$ is a semi-algebraic homeomorphism, so $\phi^{-1}(K \cap (\mathrm{Cl}(V) \cup H_+))$ is a compact set. We conclude that the restriction $\phi|_{\mathrm{Cl}(W)\cup H} : \mathrm{Cl}(W) \cup H \to D(H)$ is proper, as required. $\square$

As an immediate consequence of this embedding theorem, we have the following:

Corollary 5.1.11. *The composition $f := \pi \circ \phi : M \to H$ is a Nash map with the following properties:*

(i) *$f(Y_i) \subset Y_i$ for $i = 1, \dots, r$.*

(ii) *$f|_H$ is a semi-algebraic homeomorphism close to id_H. Moreover $f|_{\mathrm{Int}(H)}$ is a Nash diffeomorphism and $f|_{\partial H} = \mathrm{id}_{\partial H}$.*

(iii) *$f|_{M\setminus H} : M \setminus H \to H$ is a Nash embedding close to $\rho|_{M\setminus H}$.*

(iv) *At each point $x \in \partial H$ the map f has a local presentation of the type*

$$(y_1, \dots, y_d) \mapsto (y_1^2, y_2, \dots, y_d).$$

(v) *There exists an open semi-algebraic neighbourhood $V \subset M$ of ∂H, such that $f|_{\mathrm{Cl}(V)\cup H}$ is a proper Nash map.*

Proof. Statements (i), (ii) and (iii) follow from Theorem 5.1.10, whereas statement (iv) is a consequence of Proposition 5.1.8(iii). To prove (v) we use that π is proper (see Proposition 5.1.8(v)) and that ϕ is proper on $\mathrm{Cl}(V) \cup H$ where $V \subset M$ is a small semi-algebraic neighbourhood of ∂H in M. $\square$

5.1.6. Folding all the boundary components. Now we have all the ingredients to prove the main result of this chapter. We want to 'fold' a (small enough) Nash envelope M of $\mathcal{Q}$ to construct $\mathcal{Q}$ from M. That is, we want to construct a surjective Nash map $f : M \to \mathcal{Q}$ close to the identity when restricted to $\mathcal{Q}$ and with 'nice' properties. We start with a technical lemma to construct local models:

Lemma 5.1.12. *Consider the Nash map*

$$\varphi : \mathbb{R}^d :\to \mathbb{R}^d,\ (x_1, \dots, x_d) \mapsto (x_1^2, x_2, \dots, x_d)$$

and let $\psi_1, \psi_2 : \mathbb{R}^d \to \mathbb{R}^d$ be Nash diffeomorphisms such that

$$\psi_i(\{\mathtt{x}_j = 0\}) = \{\mathtt{x}_j = 0\},$$

for $j = 1, \dots, s$ and $i = 1, 2$. Then there exist Nash functions $f_j : \mathbb{R}^d \to \mathbb{R}$ each one strictly positive around $\{\mathtt{x}_j = 0\}$ and a Nash map $g : \mathbb{R}^d \to \mathbb{R}^{d-s}$ such that $g|_{\{\mathtt{x}_1=0,\dots,\mathtt{x}_s=0\}} : \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\} \to \mathbb{R}^{d-s}$ is a Nash diffeomorphism and $\psi_2 \circ \varphi \circ \psi_1 = ((\mathtt{x}_1 f_1)^2, \mathtt{x}_2 f_2, \dots, \mathtt{x}_s f_s, g)$.

Proof. As $\psi_i(\{\mathtt{x}_1 = 0\}) = \{\mathtt{x}_1 = 0\}$ and ψ_i is a Nash diffeomorphism, then $\psi_i|_{\{\mathtt{x}_1=0,\dots,\mathtt{x}_s=0\}} : \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\} \to \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\}$ is a Nash diffeomorphism. Thus

$$\psi_2 \circ \varphi \circ \psi_1|_{\{\mathtt{x}_1=0,\dots,\mathtt{x}_s=0\}} : \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\} \to \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\}$$

is a Nash diffeomorphism. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-s},\ (x_1, \dots, x_n) \to (x_{s+1}, \dots, x_n)$ and let $g := \pi \circ \psi_2 \circ \varphi \circ \psi_1$. We have

$$g|_{\{\mathtt{x}_1=0,\dots,\mathtt{x}_s=0\}} : \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\} \to \mathbb{R}^{d-s}$$

is a Nash diffeomorphism and $\psi_2 \circ \varphi \circ \psi_1 = (g_1, \dots, g_s, g)$ for some Nash functions $g_i : \mathbb{R}^d \to \mathbb{R}$. As $\psi_i(\{\mathtt{x}_j = 0\}) = \{\mathtt{x}_j = 0\}$, its jth component ψ_{ij} is divisible by $\mathtt{x}_j$ thus, there exists a Nash function $h_{ij} : \mathbb{R}^d \to \mathbb{R}$ that is strictly positive around $\{\mathtt{x}_j = 0\}$ such that $\psi_{ij} = \mathtt{x}_j h_{ij}$ for $i = 1, 2$ and $j = 1, \dots, s$. Thus,

$$\begin{aligned}
&\psi_2 \circ \varphi \circ \psi_1 = \\
&(\psi_{11}^2 h_{21}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d}), \psi_{12} h_{22}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d}), \cdots, \psi_{1s} h_{2s}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d}), g) \\
&\quad = ((\mathtt{x}_1 h_{11})^2 h_{21}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d}), \mathtt{x}_2 h_{12} h_{22}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d}), \dots \\
&\qquad\qquad \dots, \mathtt{x}_s h_{1s} h_{2s}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d}), g).
\end{aligned}$$

It is enough to take $f_1 := h_{11}\sqrt{h_{21}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d})}$ and

$$f_j := h_{1j} h_{2j}(\psi_{11}^2, \psi_{12}, \dots, \psi_{1d})$$

for $j = 2, \dots, s$. □

Theorem 5.1.13 (Folding Nash manifolds)**.** *Let $\mathcal{Q} \subset \mathbb{R}^n$ be a d-dimensional Nash manifold with corners. Then, there exist*

(i) A d-dimensional Nash manifold $M \subset \mathbb{R}^n$ that contains $\mathcal{Q}$ as a closed subset.

(ii) A Nash normal-crossings divisor $Y \subset M$ that is the smallest Nash subset of M that contains $\partial\mathcal{Q}$ and satisfies $\mathcal{Q} \cap Y = \partial\mathcal{Q}$.

(iii) A Nash map $f : M \to \mathcal{Q}$ such that $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is a semi-algebraic homeomorphism close to the identity map and $f|_{\mathrm{Int}(\mathcal{Q})} : \mathrm{Int}(\mathcal{Q}) \to \mathrm{Int}(\mathcal{Q})$ is a Nash diffeomorphism.

In addition, for each $x \in \partial\mathcal{Q}$ there exist open semi-algebraic neighbourhoods $U, V \subset M$ of x equipped with Nash diffeomorphisms $\varphi : U \to \mathbb{R}^d$ and $\psi : V \to \mathbb{R}^d$ and $1 \le s \le d$ such that

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1, \dots, x_d) \mapsto (x_1^2, \dots, x_s^2, x_{s+1}, \dots, x_d).$$

Proof. Let M be a Nash envelope of $\mathcal{Q}$ and let $Y_1,\dots,Y_\ell$ be the irreducible components of the Nash closure Y of $\partial\mathcal{Q}$ in M (see Section 5.1.3). By Lemma 5.1.4 there exist, after shrinking M if necessary, Nash equations $h_i : M \to \mathbb{R}$ of Y_i such that $d_x h_i : T_x M \to \mathbb{R}$ is surjective for each $x \in Y_i$ and $H_i := h_i^{-1}([0,+\infty))$ is a Nash manifold with boundary whose boundary is Y_i for each $i = 1,\dots,\ell$. By Corollary 5.1.11 we have for each index i a proper Nash map $f_i : \mathrm{Cl}(M) \to H_i$ such that:

(i) $f_i(Y_j) \subset Y_j$ for $j = 1,\dots,\ell$ and $f_i|_{Y_i} = \mathrm{id}_{Y_i}$,

(ii) $f_i|_{H_i} : H_i \to H_i$ is a semi-algebraic homeomorphism close to the identity map, whose restriction $f_i|_{\mathrm{Int}(H_i)} : \mathrm{Int}(H_i) \to \mathrm{Int}(H_i)$ is a Nash diffeomorphism and $f_i|_{\partial H_i} = \mathrm{id}_{\partial H_i}$,

(iii) $f_i|_{M\setminus H_i} : M \setminus H_i \to \mathrm{Int}(H_i)$ is a semi-algebraic embedding close to $\rho_i|_{M\setminus H_i} : M \setminus H_i \to \partial H_i$,

(iv) f_i has local representations $(y_1,\dots,y_d) \mapsto (y_1^2, y_2,\dots,y_d)$ at each point $x \in Y_i$.

Let $\{C_k\}_k$ be the connected components of $M\setminus Y$ such that $\mathrm{Cl}(C_k)\cap H_i = \varnothing$ for some $i = 1,\dots,\ell$. As $\mathcal{Q} = H_1 \cap \ldots \cap H_\ell$, we deduce that $M \setminus \bigcup_{k=1} \mathrm{Cl}(C_k)$ is an open semi-algebraic neighbourhood of $\mathcal{Q}$ in M. Let us substitute M by $M' := M \setminus \bigcup_k \mathrm{Cl}(C_k)$ and Y by $Y \cap M'$. Observe that now the connected components of $M \setminus Y$ satisfy $\mathrm{Cl}(C) \cap H_i \neq \varnothing$ for each $i = 1,\dots,\ell$.

Consider the proper Nash map $f := f_\ell \circ \cdots \circ f_1 : \mathrm{Cl}(M) \to \mathrm{Cl}(M)$. Let us check that f satisfies the following properties:

(1) $f(Y_i) \subset Y_i$ for each $i = 1,\dots,\ell$.

(2) $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is a semi-algebraic homeomorphism close to the identity map, whose restriction $f|_{\mathrm{Int}(\mathcal{Q})} : \mathrm{Int}(\mathcal{Q}) \to \mathrm{Int}(\mathcal{Q})$ is a Nash diffeomorphism,

(3) $f(\mathrm{Cl}(M)) = \mathcal{Q}$,

(4) f has local representations $(y_1,\dots,y_d) \mapsto (y_1^2,\dots,y_s^2, y_{s+1},\dots,y_d)$ at each point $x \in \partial\mathcal{Q}$. The integer $s \geq 1$ depends on the point x and corresponds to the number of irreducible components of Y that passes through x.

Property (1) follows straightforwardly from (i). To prove (2) we show first: $f_i(\mathcal{Q}) = \mathcal{Q}$ *and* $f_i(\mathrm{Int}(\mathcal{Q})) = \mathrm{Int}(\mathcal{Q})$ *for* $i = 1,\dots,\ell$. Once this is done, as $\mathcal{Q} = H_1 \cap \cdots \cap H_\ell$, $f_i|_{H_i} : H_i \to H_i$ is a semi-algebraic homomorphism close to the identity map for each i and $f_i|_{\mathrm{Int}(H_i)} : \mathrm{Int}(H_i) \to \mathrm{Int}(H_i)$ is a Nash diffeomorphism, we deduce that $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is a semi-algebraic homeomorphism close to the identity map and its restriction $f|_{\mathrm{Int}(\mathcal{Q})} : \mathrm{Int}(\mathcal{Q}) \to \mathrm{Int}(\mathcal{Q})$ is a Nash diffeomorphism.

As f_i is proper and $\mathcal{Q} = \mathrm{Cl}(\mathrm{Int}(\mathcal{Q}))$, we have

$$f_i(\mathcal{Q}) = f_i(\mathrm{Cl}(\mathrm{Int}(\mathcal{Q}))) = \mathrm{Cl}(f_i(\mathrm{Int}(\mathcal{Q})))$$

so it is enough to prove $f_i(\mathrm{Int}(\mathcal{Q})) = \mathrm{Int}(\mathcal{Q})$.

As the map $f_i|_{\operatorname{Int}(H_i)} : \operatorname{Int}(H_i) \to \operatorname{Int}(H_i)$ is a Nash diffeomorphism and $\operatorname{Int}(\mathcal{Q}) \subset \operatorname{Int}(H_i)$, we have $f_i(\operatorname{Int}(\mathcal{Q})) \subset \operatorname{Int}(H_i)$. As $f_i(Y_j) \subset Y_j$, we deduce $f_i(Y_j \cap \operatorname{Int}(H_i))$ is a closed Nash submanifold of the Nash manifold $Y_j \cap \operatorname{Int}(H_i)$ and both have the same dimension. Consequently, the image of each connected component of $f_i(Y_j \cap \operatorname{Int}(H_i))$ is a connected component of $Y_j \cap \operatorname{Int}(H_i)$. As f_i is close to the identity map $f_i(D) = D$ for each connected component of $Y_j \cap \operatorname{Int}(H_i)$. Thus, $f_i(Y_j \cap \operatorname{Int}(H_i)) = Y_j \cap \operatorname{Int}(H_i)$ and $f_i(Y \cap \operatorname{Int}(H_i)) = Y \cap \operatorname{Int}(H_i)$. Observe that $\operatorname{Int}(\mathcal{Q})$ is a union of connected components of $\operatorname{Int}(H_i) \setminus Y$ because $\mathcal{Q} \cap Y = \partial\mathcal{Q}$. Again as $f_i|_{\operatorname{Int}(H_i)} : \operatorname{Int}(H_i) \to \operatorname{Int}(H_i)$ is a Nash diffeomorphism, $f_i(\operatorname{Int}(\mathcal{Q}))$ is also a union of connected components of $\operatorname{Int}(H_i) \setminus Y$. As $f_i|_{\operatorname{Int}(\mathcal{Q})}$ is close to the identity map, $f_i(C) = C$ for each connected component of $\operatorname{Int}(\mathcal{Q})$. In particular, $f_i(\operatorname{Int}(\mathcal{Q})) = \operatorname{Int}(\mathcal{Q})$, as claimed.

Next, we prove (3). Let us show: *$f_i(H_j) \subset H_j$ for each i, j.* Once this is proved, we have $f_i(H_j) \subset H_i \cap H_j$ for each i, j (because $f_i(\operatorname{Cl}(M)) \subset H_i$) and:

$$\begin{aligned} f(\operatorname{Cl}(M)) = (f_\ell \circ \cdots \circ f_1)(\operatorname{Cl}(M)) &= (f_\ell \circ \cdots \circ f_2)(H_1) \\ &\subset (f_\ell \circ \cdots \circ f_3)(H_1 \cap H_2) \subset \cdots \subset H_1 \cap \cdots \cap H_\ell = \mathcal{Q}, \end{aligned}$$

as $f(\mathcal{Q}) = \mathcal{Q}$, we conclude $f(\operatorname{Cl}(M)) = \mathcal{Q}$.

For simplicity we prove $f_2(H_1) \subset H_1$. As f_2 is continuous, it is enough to prove: $f_2(\operatorname{Int}(H_1 \cap H_2)) \subset \operatorname{Int}(H_1 \cap H_2)$ and $f_2(\operatorname{Int}(H_1) \setminus H_2) \subset \operatorname{Int}(H_1)$. For the first part, it is enough to consider the Nash manifold with corners $H_1 \cap H_2$ and to proceed analogously as above with f_i and $\mathcal{Q}$.

As $f_2|_{M \setminus H_2} : M \setminus H_2 \to \operatorname{Int}(H_2)$ is a Nash embedding and $f_2(Y_1) \subset Y_1$, we deduce that each connected component C of $\operatorname{Int}(H_1) \setminus H_2$ is mapped under f_2 into a connected component of $\operatorname{Int}(H_2) \setminus Y_1$. Let C be a connected component of $\operatorname{Int}(H_1) \setminus H_2 \subset M \setminus (Y_1 \cup Y_2)$. As $M \setminus Y_2 = (M \setminus H_2) \sqcup \operatorname{Int}(H_2)$ and $\operatorname{Int}(H_1) = H_1 \setminus Y_1$, we deduce that $\operatorname{Int}(H_1) \setminus H_2$ is open and closed in $M \setminus (Y_1 \cup Y_2)$, so C is a connected component of $M \setminus (Y_1 \cup Y_2)$.

If $\operatorname{Cl}(C) \cap Y_2 \neq \varnothing$, we pick a point $x \in \operatorname{Cl}(C) \cap (Y_2 \setminus Y_1)$. This is possible because C is a connected component of $M \setminus (Y_1 \cup Y_2)$ and $Y_1 \cup Y_2$ is a Nash normal-crossings divisor. Let V be a semi-algebraic neighbourhood of x in M with compact closure K that does not meet Y_1. As $x \in \operatorname{Cl}(C)$ and $C \subset \operatorname{Int}(H_1)$, we deduce $V \subset \operatorname{Int}(H_1)$. As $\rho_2(x) = x \in \operatorname{Int}(H_1)$, and $f_2|_{M \setminus H_2} : M \setminus H_2 \to \operatorname{Int}(H_2)$ is close to $\rho_2|_{M \setminus H_2} : M \setminus H_2 \to \partial H_2$, we may assume $f_2(K) \subset \operatorname{Int}(H_1)$. Thus, $f_2(C)$ meets $\operatorname{Int}(H_1)$. As $f_2(C)$ is a connected component of $\operatorname{Int}(H_2) \setminus H_1$ and $\operatorname{Int}(H_1) \cap \operatorname{Int}(H_2)$ is open and closed in $\operatorname{Int}(H_2) \setminus Y_1$, we deduce that

$$f_2(C) \subset \operatorname{Int}(H_1) \cap \operatorname{Int}(H_2) \subset \operatorname{Int}(H_1).$$

Suppose now $\operatorname{Cl}(C) \cap Y_2 = \varnothing$. As $C \cap H_2 = \varnothing$, we deduce that

$$\begin{aligned} \operatorname{Cl}(C) \cap H_2 = \operatorname{Cl}(C) \cap (Y_2 \cup \operatorname{Int}(H_2)) &= \operatorname{Cl}(C) \cap \operatorname{Int}(H_2) \\ &= \operatorname{Cl}(C \cap \operatorname{Int}(H_2)) \cap \operatorname{Int}(H_2) \subset \operatorname{Cl}(C \cap H_2) = \varnothing. \end{aligned}$$

Observe that C is the closure in $M \setminus (Y_1 \cup Y_2)$ of some connected components of $M \setminus Y$, which is a contradiction because by construction all the connected components of $M \setminus Y$ meet H_2. Thus $\operatorname{Cl}(C) \cap Y_2 \neq \varnothing$.

So, it holds $f_2(C) \subset \operatorname{Int}(H_1)$ for each connected component C of $\operatorname{Int}(H_1)\setminus H_2$. Consequently, $f_2(\operatorname{Int}(H_1) \setminus H_2) \subset \operatorname{Int}(H_1)$.

Finally, we prove (4). Pick a point $x \in \partial\mathcal{Q}$ and assume that x belongs exactly to $Y_1, \dots, Y_s$. Recall that the analytic closure of $\partial\mathcal{Q}_x$ is $Y_{1,x} \cup \dots \cup Y_{s,x}$ and there exists an open semi-algebraic neighbourhood $U \subset M$ of x equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^d$ such that $u(x) = 0$, $u(Y_j \cap U) = \{\mathtt{x}_j = 0\}$ and $u(\mathcal{Q} \cap U) = \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_s \geq 0\}$.

Observe that $\eta := u \circ f \circ u^{-1} = \eta_\ell \circ \dots \circ \eta_1$ where $\eta_k := u \circ f_k \circ u^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ is either a Nash diffeomorphism or a Nash map that has local representations $(y_1, \dots, y_d) \mapsto (y_1^2, y_2, \dots, y_d)$ at each point $x \in Y_k$. We may assume that this local representations preserve the local representations of $Y_1, \dots, Y_s$. Note that $Y_1, \dots, Y_s$ correspond to coordinates hyperplanes in these coordinates. By Lemma 5.1.12 we may assume that $\eta = \sigma_s \circ \dots \circ \sigma_1$ is a composition of Nash maps of the type $\sigma_m : \mathbb{R}^d \to \mathbb{R}^d$,

$$(x_1, \dots, x_d) \mapsto (x_1 f_{1m}, \dots, x_{m-1} f_{m-1,m}, x_m^2 f_{mm}^2, x_{m+1} f_{m+1,m}, \dots, x_s f_{sm}, g_m)$$

where $f_{jm} : \mathbb{R}^d \to \mathbb{R}$ is a Nash function that does not vanish around $\{\mathtt{x}_j = 0\}$ and $g_m : \mathbb{R}^d \to \mathbb{R}^{d-s}$ satisfies $g_m|_{\{\mathtt{x}_1=0,\dots,\mathtt{x}_s=0\}} : \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\} \to \mathbb{R}^{d-s}$ is a Nash diffeomorphism. Thus, there exist Nash functions $f_j : \mathbb{R}^d \to \mathbb{R}$ that does not vanish around $\{\mathtt{x}_j = 0\}$ and a Nash map $g : \mathbb{R}^d \to \mathbb{R}^{d-s}$ such that $g|_{\{\mathtt{x}_1=0,\dots,\mathtt{x}_s=0\}} : \{\mathtt{x}_1 = 0, \dots, \mathtt{x}_s = 0\} \to \mathbb{R}^{d-s}$ is a Nash diffeomorphism satisfying

$$\eta : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1, \dots, x_d) \mapsto (x_1^2 f_1^2, \dots, x_s^2 f_s^2, g).$$

Observe that the map

$$\psi : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1, \dots, x_d) \mapsto (x_1 f_1, \dots, x_s f_s, g)$$

is a Nash diffeomorphism around the origin, because the determinant of its Jacobian matrix

$$J_\psi(x) = \left[\begin{array}{ccc|c} f_1(x) & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & f_s(x) & 0 \\ \hline 0 & \dots & 0 & J_g(x) \end{array}\right]$$

does not vanish at the origin. Thus, after shrinking the open semi-algebraic neighbourhood $U \subset M$ of x we may assume that there exists a Nash diffeomorphism $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\psi \circ \phi = \mathrm{id}_{\mathbb{R}^d}$. As $\eta = \theta \circ \psi$, where

$$\theta : \mathbb{R}^d \to \mathbb{R}^d,\ (x_1, \dots, x_d) \mapsto (x_1^2, \cdots, x_s^2, x_{s+1}, \dots, x_d),$$

we deduce that $\eta \circ \phi = \theta$, so f has local representation

$$(y_1, \dots, y_d) \mapsto (y_1^2, \dots, y_s^2, y_{s+1}, \dots, y_d)$$

at $x \in \partial\mathcal{Q}$. Recall that $s \geq 1$ corresponds to the number of irreducible components of Y that passes through x, as required. $\square$

5.1.7. A 'canonical' folding. Let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners and let M be a Nash envelope of $\mathcal{Q}$ such that the Nash closure Y of $\partial\mathcal{Q}$ is a Nash normal-crossings divisor of M and $\mathcal{Q} \cap Y = \partial\mathcal{Q}$. Let $Y_1, \ldots, Y_\ell$ be the irreducible components of Y and $h_i : M \to \mathbb{R}$ a Nash equations of Y_i for $i = 1, \ldots, \ell$ as in Lemma 5.1.4. We can consider the following two constructions.

Doubling after doubling. By Lemma 5.1.4 the sets $H_i := h_i^{-1}([0, +\infty))$ are Nash manifolds with boundary Y_i, that contain $\mathcal{Q}$ as a closed subset. Write (see Corollary 5.1.6):

$$\mathcal{Q} = H_1 \cap \ldots \cap H_\ell = \{x \in M : h_1(x) \geq 0, \ldots, h_\ell(x) \geq 0\} \subset M \subset \mathbb{R}^n,$$

and let $(D(H_1), \pi_1)$ be the Nash double of H_1. Let

$$\begin{aligned}\mathcal{Q}_1 :&= \pi_1^{-1}(H_2) \cap \ldots \cap \pi_1^{-1}(H_\ell)\\ &= \{(x, t_1) \in M \times \mathbb{R} : t_1^2 - h_1(x) = 0, h_2(x) \geq 0, \ldots, h_\ell(x) \geq 0\} \subset D(H_1) \subset \mathbb{R}^{n+1},\end{aligned}$$

which is a Nash manifolds with corners. It is described as intersection of $\ell - 1$ Nash manifolds with boundary, which are $\pi_1^{-1}(H_2), \ldots, \pi_1^{-1}(H_\ell)$. By Theorem 5.1.10 there exists a Nash embedding $\phi_1 : M \to D(H_1)$ such that $\phi_1(H_1) = H_{1+}$ and $M_1 := \phi_1(M)$ is an open semi-algebraic subset of $D(H_1)$ that contains $\mathcal{Q}_1 \cap M_1$ as a closed subset. In addition $\phi_1|_{Y_1} = \mathrm{id}_{Y_1}$ if we identify Y_1 with $Y_1 \times \{0\}$, $\phi_1(Y_i) \subset D(Y_i \cap H_1)$ for $i = 2, \ldots, \ell$ and $\phi_1|_{H_1} : H_1 \to H_{1+}$ is close to $(\pi_1|_{H_{1+}})^{-1}$. We claim: $\phi_1(\mathcal{Q}) = \mathcal{Q}_1 \cap H_{1+}$.

To that end we prove that $\phi_1(H_1 \cap H_i) = H_{1+} \cap \pi_1^{-1}(H_i)$ for $i = 2, \ldots, \ell$. As $\phi_1 : M \to M_1$ is a Nash diffeomorphism it is enough to check

$$\phi_1(\mathrm{Int}(H_1) \cap \mathrm{Int}(H_i)) = \mathrm{Int}(H_{1+}) \cap \pi_1^{-1}(\mathrm{Int}(H_i)).$$

Observe that $\mathrm{Int}(H_1) \cap \mathrm{Int}(H_i)$ is a union of connected component of $M \setminus (Y_1 \cup Y_2)$. In addition $\phi_1(Y_1) = Y_1 \times \{0\}$ and $\phi_1(Y_i) \subset D(Y_i \cap H_1)$, so

$$\phi_1(Y_i \cap \mathrm{Int}(H_1)) \subset D(Y_i \cap H_i) \cap \mathrm{Int}(H_{i+}).$$

As $\phi_1|_{\mathrm{Int}(H_1)} : \mathrm{Int}(H_1) \to \mathrm{Int}(H_{1+})$ is a Nash diffeomorphism we deduce that $\phi_1(Y_i \cap \mathrm{Int}(H_1))$ is a closed Nash submanifold of the Nash manifold $D(Y_i \cap H_1) \cap \mathrm{Int}(H_{1+})$ of the same dimension. As ϕ_1 is close to $(\pi_1|_{H_1,+})^{-1}$, both Nash manifolds have the same number of connected components and

$$\phi_1(Y_1 \cap \mathrm{Int}(H_1)) = D(Y_i \cap H_1) \cap \mathrm{Int}(H_{1+}).$$

Note that $\mathrm{Int}(H_{1+}) \cap \pi_1^{-1}(H_i)$ is a union of connected components of $\mathrm{Int}(H_{1+}) \setminus D(Y_i \cap H_1)$. As π_1 is close to $(\pi_1|_{H_{1+}})^{-1}$ we conclude that ϕ_1 establish a bijection between the connected components of $\mathrm{Int}(H_1) \cap \mathrm{Int}(H_i)$ and the connected components of $\mathrm{Int}(H_{1+}) \cap \pi_1^{-1}(H_i)$, so that

$$\phi_1(\mathrm{Int}(H_1) \cap \mathrm{Int}(H_i)) = \mathrm{Int}(H_{1+}) \cap \pi_1^{-1}(H_i).$$

At this point we have the Nash manifold $N^1 := D(H_1)$, the Nash manifolds with boundary H_{1+} and $H_i^1 := \pi_1^{-1}(H_i)$ for $i = 2, \ldots, \ell$ whose boundaries are respectively $Y_1 = \partial H_1, D(Y_2 \cap H_1), \ldots, D(Y_\ell \cap H_1)$. Recall that

$$Y^1 := Y_1 \cup D(Y_2 \cap H_1) \cup \ldots \cup D(Y_\ell \cap H_1)$$

is a Nash normal-crossings divisor of $D(H_1)$ and that

$$\begin{aligned}\mathcal{Q}^1 :&= \mathcal{Q}_1 \cap H_{1+} = H_{1+} \cap H_2^1 \cap \ldots \cap H_\ell^1 \\ &= \{(x,t_1) \in M \times \mathbb{R} : t_1^2 - h_1(x) = 0, t_1 \geq 0, h_2(x) \geq 0, \ldots, h_\ell(x) \geq 0\} \subset D(H_1),\end{aligned}$$

is a Nash manifold with corners. The Nash closure of $\partial\mathcal{Q}^1$ is Y^1 and $M_1 = \phi_1(M)$ is an open semi-algebraic neighbourhood of $\mathcal{Q}^1$ in $D(H_1)$. In addition $\phi_1(H_i \cap H_1) = H_i^1 \cap H_{1+}$. Let us check: *The Nash map $h_2^1 : D(H_1) \to \mathbb{R}$ defined as $h_2^1(x,t_1) := h_2(x)$ is a Nash equation of $D(Y_2 \cap H_1)$ in $D(H_1)$ such that*

$$d_{(x,t_1)}h_2^1 : T_{(x,t_1)}D(H_1) \to \mathbb{R},$$

is surjective.

It is clear that

$$\begin{aligned}D(Y_2 \cap H_1) &= \{(x,t_1) \in M \times \mathbb{R} : t_1^2 - h_1(x) = 0, h_2(x) = 0\} \\ &= \{(x,t_1) \in Y_2 \times \mathbb{R} : t_1^2 - h_1(x) = 0\},\end{aligned}$$

so $h_2^1 = 0$ is a Nash equation of $D(Y_2 \cap H_1)$ in $D(H_1)$. The tangent space of $D(H_1)$ at a point (x,t_1) is

$$T_{(x,t_1)}D(H_1) = \{(v,s) \in T_xM \times \mathbb{R} : d_xh_1(v) - 2t_1s = 0\}$$

and $d_{(x,t_1)}h_2^1(v,s) = d_xh_2(v)$. Let us check that it is surjective.

CASE 1: $t_1 \neq 0$. As $T_{(x,t_1)}D(H_1) = \{(v,s) \in T_xM \times \mathbb{R} : d_xh_1(v) - 2t_1s = 0\}$ we pick a vector $v \in T_xM$ such that $d_xh_2(v) \neq 0$ and $\left(v, \frac{d_xh_1(v)}{2t_1}\right) \in T_{(x,t_1)}D(H_1)$. We have,

$$d_{(x,t_1)}h_2^1\left(v, \frac{d_xh_1(v)}{2t_1}\right) = d_xh_2(v) \neq 0.$$

CASE 2: $t_1 = 0$. We have $T_{(x,0)}D(H_1) = \{(v,s) \in T_xM \times \mathbb{R} : d_xh_1(v) = 0\}$. As $Y_1 \cup Y_2$ is a normal-crossings divisor, we have that d_xh_1 and d_xh_2 are linearly independent, so there exists $v \in T_xM$ such that $d_xh_2(v) \neq 0$ and $d_xh_1(v) = 0$. Pick $(v,0) \in T_{(x,0)}D(H_1)$ such that

$$d_{(x,t_1)}h_2^1(v,0) = d_xh_2^1(v) \neq 0.$$

Thus, in both cases $d_{(x,t_1)}h_2^1 : T_{(x,t_1)}D(H_1) \to \mathbb{R}$ is surjective and h_2^1 satisfies the required properties.

Observe that the same happens with $h_3^1, \ldots, h_\ell^1$. In addition,

$$h_1^1 : D(H_1) \to \mathbb{R},\ (x,t) \mapsto t$$

is a Nash equation of $Y_1 \times \{0\}$ in $D(H_1)$. Observe that

$$T_{(x,0)}D(H_1) = \{(v,s) \in T_xM \times \mathbb{R} : d_xh_1(v) = 0\} = T_xY_1$$

and $d_{(x,t)}h_1^1 : T_xY_1 \times \mathbb{R} \to \mathbb{R}$, $(v,s) \mapsto s$ is surjective.

We construct next the double of H_2^1 (with respect to h_2^1) which is a Nash manifold with boundary $Y_2^1 = D(Y_2 \cap H_1)$ contained in $D(H_1)$ as a closed subset. Thus, at this point we have the Nash manifold

$$\begin{aligned} N^2 := D(H_2^1) &= \{(x,t_1,t_2) \in (M \times \mathbb{R}) \times \mathbb{R} : t_1^2 - h_1(x) = 0, t_2^2 - h_2^1(x,t_1) = 0\} \\ &= \{(x,t_1,t_2) \in M \times \mathbb{R}^2 : t_1^2 - h_1(x) = 0, t_2^2 - h_2(x) = 0\} \end{aligned}$$

and the projection $\pi_2 : D(H_2^1) \to H_2^1$. We have also the Nash manifolds with boundary

$$H_1^2 := \pi_2^{-1}(H_{1+}), H_3^2 := \pi_2^{-1}(H_3^1), \ldots, H_\ell^2 := \pi_2^{-1}(H_\ell^1)$$

and $H_2^2 := H_{2+}^1$. Consider also the Nash manifold with corners

$$\mathcal{Q}^2 := H_1^2 \cap \ldots \cap H_\ell^2.$$

The boundary of H_2^1 is $Y_2^1 \times \{0\}$ and the boundary of H_i^2 for $i = 1, 3, \ldots, \ell$ is $Y_i^2 := D(Y_i^1 \cap H_2^1)$ where $Y_1^1 := Y_1 \times \{0\}$. By theorem 5.1.10 there exists a Nash embedding $\phi_2 : M_1 \to D(H_2^1)$ that maps the Nash manifold with corners $\mathcal{Q}^1 = H_1^1 \cap \ldots \cap H_\ell^1$ onto the Nash manifold with corners $\mathcal{Q}^2 = H_1^2 \cap \ldots \cap H_\ell^2$ (where $H_2^2 = H_{2+}$), see the claim above concerning $\phi_1 : M \to M_1$, that maps $\mathcal{Q}$ onto $\mathcal{Q}^1$. Observe also that $Y^2 := Y_1^2 \cup \ldots \cup Y_\ell^2$ is a Nash normal-crossings divisor of $D(H_2^1)$ and it is the Nash closure of $\partial\mathcal{Q}^2$ in $N^2 := D(H_2^1)$. Note also that

$$h_i^2 : N^2 \to \mathbb{R},\ (x,t_1,t_2) \mapsto h_i^1(x,t_1)$$

is a Nash equation of Y_i^2 fot $i = 1, 3, \ldots, \ell$ such that

$$d_{(x,t_1,t_2)} h_i^2 : T_{(x,t_1,t_2)} N^2 \to \mathbb{R}$$

is surjective for each $(x,t_1,t_2) \in Y_i^2$. In addition, $h_2^2 : N^2 \to \mathbb{R}$, $(x,t_1,t_2) \mapsto t_2$ is a Nash equation of $Y_2^2 := Y_2^1 \times \{0\}$ such that

$$d_{(x,t_1,0)} h_2^2 : T_{(x,t_1,0)} N^2 = T_{(x,t_1)} Y_2^1 \times \mathbb{R} \to \mathbb{R},\ (v,s_1,s_2) \mapsto s_2$$

is surjective. The proofs of all these facts are similar to the ones included in the first step and the concrete details are left to the reader.

We proceed recursively with $H_3^3, \ldots, H_\ell^\ell$. In the last step, we have the Nash manifold

$$N^\ell := D(H_\ell^{\ell-1}) = \{(x,t) \in M \times \mathbb{R}^\ell : t_1^2 - h_1(x) = 0, \ldots, t_\ell^2 - h_\ell(x) = 0\}$$

the projection $\pi_\ell : D(H_\ell^{\ell-1}) \to H_\ell^{\ell-1}$ and the Nash manifolds with boundary $H_i^\ell := \pi_\ell^{-1}(H_i^{\ell-1})$ for $i = 1, \ldots, \ell-1$ and $H_\ell^\ell = H_{\ell+}^{\ell-1}$. The boundary of H_i^ℓ is $Y_i^\ell := D(Y_i^{\ell-1} \cap H_\ell^{\ell-1})$ and $\mathcal{Q}^\ell := H_1^\ell \cap \ldots \cap H_\ell^\ell$ is a Nash manifold with corners.

By Theorem 5.1.10 and proceeding similarly to the first step, there exists a Nash embedding $\phi_\ell : M_{\ell-1} \to D(H_\ell^{\ell-1})$ such that $\phi_\ell(\mathcal{Q}^{\ell-1}) = \mathcal{Q}^\ell$ and $M^\ell = \phi_\ell(M_{\ell-1})$ is an open semi-algebraic neighbourhood of $\mathcal{Q}^\ell$ in N^ℓ.

Let $Y_i^\ell := D(Y_i^{\ell-1} \cap H_\ell^{\ell-1})$ for $i = 1, \ldots, \ell-1$ and $Y_\ell^\ell := Y_\ell^{\ell-1} \times \{0\}$. It holds that

$$h_i^\ell : N^\ell \to \mathbb{R},\ (x,t) \mapsto h_i^{\ell-1}(x,t_1,\ldots,t_{\ell-1})$$

is a Nash equation of Y_i^ℓ in N^ℓ such that $d_{(x,t)}h_i^\ell : T_{(x,t)}N^\ell \to \mathbb{R}$ is surjective for each $(x,t) \in Y_i^\ell$ and $i = 1,\dots,\ell-1$. In addition, $h_\ell^\ell : N^\ell \to \mathbb{R}$, $(x,t) \mapsto t_\ell$ is a Nash equation of Y_ℓ^ℓ such that

$$d_{(x,t_1,\dots,t_{\ell-1},0)}h_\ell^\ell : T_{(x,t_1,\dots,t_{\ell-1},0)}N^\ell \to \mathbb{R}$$

is surjective. We have $Y^\ell := \bigcup_{i=1}^\ell Y_i^\ell$ is the Nash closure of $\partial\mathcal{Q}^\ell$ in N^ℓ.

Define

$$D(\mathcal{Q}) := N^\ell = \{(x,t) \in \mathcal{Q} \times \mathbb{R}^\ell \,:\, t_1^2 - h_1(x) = 0,\dots,t_\ell^2 - h_\ell(x) = 0\}$$

and $\pi := \pi_\ell \circ \dots \circ \pi_1 : D(\mathcal{Q}) \to \mathcal{Q}$. Define also $g_i : D(\mathcal{Q}) \to \mathbb{R}$, $(x,t) \mapsto t_i$, which satisfies $\{g_i = 0\} = Y_i^\ell$ and $d_{(x,t)}g_i : T_{(x,t)}D(\mathcal{Q}) \to \mathbb{R}$ is surjective for each $(x,t) \in Y_i$ and $\phi := \phi_\ell \circ \dots \circ \phi_1 : M \to M^\ell$, which is a Nash diffeomorphism that maps $\mathcal{Q} := \{h_1 \geq 0,\dots,h_\ell \geq 0\}$ onto

$$\begin{aligned}\mathcal{Q}^\ell &= \{(x,t) \in M \times \mathbb{R}^\ell : t_1^2 - h_1(x) = 0,\dots,t_\ell^2 - h_\ell(x) = 0, t_1 \geq 0,\dots,t_\ell \geq 0\}\\ &= \{(x,t) \in D(\mathcal{Q}) \times \mathbb{R}^\ell : t_1 \geq 0,\dots,t_\ell \geq 0\}.\end{aligned}$$

In addition,

- $D(\mathcal{Q}) = \{(x,t) \in M \times \mathbb{R}^\ell \,:\, t_1^2 - h_1(x) = 0,\dots,t_\ell^2 - h_\ell(x) = 0\}$, which only depends on $\mathcal{Q}$ and $h_1,\dots,h_\ell$, which are unique up to multiplication by strictly positive Nash function on $\mathcal{Q}$.
- $\pi : D(\mathcal{Q}) \to \mathcal{Q}$, $(x,t) \mapsto x$.
- $\pi|_{\mathcal{Q}_\ell} : \mathcal{Q}^\ell \to \mathcal{Q}$ is a semi-algebraic homeomorphism.
- $\pi|_{\mathrm{Int}(\mathcal{Q}^\ell)} : \mathrm{Int}(\mathcal{Q}^\ell) \to \mathcal{Q}$ is a Nash diffeomorphism.

Doubling everything together. Thus, we can 'double' all the irreducible components of $\partial\mathcal{Q}$ at the same time. Let us check alternatively that $D(\mathcal{Q})$ is a Nash manifold. As

$$D(\mathcal{Q}) = \{(x,t) \in M \times \mathbb{R}^\ell \,:\, t_1^2 - h_1(x) = 0,\dots,t_\ell^2 - h_\ell(x) = 0\} \subset M \times \mathbb{R}^\ell,$$

it is enough to check that $D(\mathcal{Q})$ is smooth. Consider the Nash map

$$f : M \times \mathbb{R}^\ell \to \mathbb{R}^\ell,\ (x,t) \mapsto (t_1^2 - h_1(x),\dots,t_\ell^2 - h_\ell(x)),$$

where $t := (t_1,\dots,t_\ell)$ and note that $D(\mathcal{Q}) = f^{-1}(0)$. Fixed a point $(x,t) \in D(\mathcal{Q})$ the differential of f at (x,t) is:

$$\begin{aligned}&d_{(x,t)}f : T_{(x,t)}(M \times \mathbb{R}^\ell) \equiv T_xM \times \mathbb{R}^\ell \to \mathbb{R}^\ell,\\ &\qquad\qquad (v,s) \mapsto (2t_1s_1 - d_xh_1(v),\dots,2t_\ell s_\ell - d_xh_\ell(v)),\end{aligned}$$

where $s := (s_1,\dots,s_\ell)$. If $t_i \neq 0$ for each index i the differential is surjective because

$$d_{(x,t)}f\left(0,\frac{s_1}{2t_1},\dots,\frac{s_\ell}{2t_\ell}\right) = (s_1,\dots,s_\ell)$$

for each $s = (s_1,\dots,s_\ell) \in \mathbb{R}^\ell$.

Assume now that there exist $r \geq 1$ indices such that $t_i = 0$. Up to reorder the variables, we suppose that $t_1 = \ldots = t_r = 0$ whereas $t_{r+1} \neq 0, \ldots, t_\ell \neq 0$. As Y is a Nash normal-crossings divisor, the tangent spaces

$$T_x Y_1 = \ker(d_x h_1), \ldots, T_x Y_r = \ker(d_x h_r) \subset T_x M$$

are in general position. By Lemma 5.1.4 the differentials

$$d_x h_1, \ldots, d_x h_r : T_x M \to \mathbb{R},$$

are all surjective for each $x \in Y_1 \cap \ldots \cap Y_r$. Note that $h_1(x) = \ldots = h_r(x) = 0$ because $(x,t) \in D(\mathcal{Q})$, so $x \in Y_1 \cap \ldots \cap Y_r$. As $\{d_x h_1, \ldots, d_x h_r\}$ are linearly independent for each index $i = 1, \ldots, r$, there exists a vector $v_i \in T_x M$ such that $d_x h_i(v_i) = 1$, whereas $d_x h_j(v_i) = 0$ for $j \neq i$. Thus, also in this case the differential $d_{(x,t)} f$ is surjective because given the vectors

$$s := \left(0, \ldots, 0, \frac{s_{r+1} - \sum_{i=1}^r s_i d_x h_{r+1}(v_i)}{2t_{r+1}}, \ldots, \frac{s_\ell - \sum_{i=1}^r s_i d_x h_\ell(v_i)}{2t_\ell}\right) \in \mathbb{R}^\ell$$
$$v := -s_1 v_1 - \ldots - s_r v_r \in T_x M,$$

it holds $d_{(x,t)} f(v, s) = (s_1, \ldots, s_\ell)$ for each $s = (s_1, \ldots, s_\ell) \in \mathbb{R}^\ell$.

We deduce that $0 \in \mathbb{R}^\ell$ is a regular value for the Nash map f, so $D(\mathcal{Q})$ is a smooth Nash submanifold of $M \times \mathbb{R}^\ell$ of dimension d.

Observe that $D(\mathcal{Q})$ depends only on $\mathcal{Q}$ and on the Nash functions $h_1, \ldots, h_\ell$ that are unique up to multiplication by strictly positive Nash functions on M. Observe that $D(\mathcal{Q})$ is a Nash envelope of $\mathcal{Q}^\ell = D(\mathcal{Q}) \cap \{t_1 \geq 0, \ldots, t_\ell \geq 0\}$. In fact, if $\phi : \mathcal{Q} \to \mathcal{Q}^\ell$ is a Nash diffeomorphism there exist a Nash envelope $M' \subset M$ of $\mathcal{Q}$, a Nash envelope $N \subset D(\mathcal{Q})$ of $\mathcal{Q}^\ell$ and a unique Nash diffeomorphism $\Phi : M' \to N$ that extends ϕ to M' (see [FGR, Thm.1.3]). The uniqueness of Φ follows from the identity principle.

Remark 5.1.14. The Nash manifold $D(\mathcal{Q})$ does not depend on the order we do the doublings with respect to the boundaries $Y_1, \ldots, Y_r$, because at the end the result is $(D(\mathcal{Q}), \pi)$ and

- $D(\mathcal{Q}) = \{(x,t) \in \mathcal{Q} \times \mathbb{R}^\ell : t_1^2 - h_1(x) = 0, \ldots, t_\ell^2 - h_\ell(x) = 0\}$,
- $\pi : D(\mathcal{Q}) \to \mathcal{Q}$, $(x,t) \mapsto x$,
- $\mathcal{Q}^\ell = D(\mathcal{Q}) \cap \{t_1 \geq 0, \ldots, t_\ell \geq 0\}$ is Nash diffeomorphic to $\mathcal{Q}$.

We will see in the following sections some remarkable consequences of Theorem 5.1.13 and the construction made in this section. Here we present a simple example. It is well-known that the compact orientable surface of genus g admits a Nash model. We show in the following example how to deduce straightforwardly by our construction this fact.

Example 5.1.15. Let $\mathcal{P}_n \subset \mathbb{R}^2$ be a convex polygon with n edges. As $\mathcal{P}_n$ is convex, there exist polynomials $h_1, \ldots, h_n \in \mathbb{R}[\mathtt{x}, \mathtt{y}]$ of degree 1 such that

$$\mathcal{P}_n = \{h_1 \geq 0, \ldots, h_n \geq 0\}.$$

Fix $2 + \frac{1}{2}(1 - (-1)^n) \le s \le n$ and let $\mathcal{J} := \{J_k\}_{k=1}^s$ be a partition of the set $\{1, \dots, n\}$ such that

$$\mathcal{P}_n \cap \{h_i = 0\} \cap \{h_j = 0\} = \varnothing \tag{5.1.1}$$

for each $i, j \in J_k$ with $i \neq j$. For each $k = 1, \dots, s$ define the polynomial $h_{J_k} := \prod_{i \in J_k} h_i$. As the partition $\mathcal{J}$ satisfies (5.1.1), it holds that $\mathcal{P}_n$ is a connected component of the semialgebraic set $\{x \in \mathbb{R}^2 : \ h_{J_1}(x) \ge 0, \dots, h_{J_s}(x) \ge 0\}$. Thus,

$$D_s(\mathcal{P}_n) := \{(x,t) \in \mathcal{P}_n \times \mathbb{R}^s : \ t_1^2 - h_{J_1}(x) = 0, \dots, t_s^2 - h_{J_s}(x) = 0\} \subset \mathbb{R}^{s+2}$$

is a connected compact Nash surface, which is in addition a connected component of the (maybe singular) algebraic set

$$X_{s,n} = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R}^s : \ t_1^2 - h_{J_1}(x) = 0, \dots, t_s^2 - h_{J_s}(x) = 0\} \subset \mathbb{R}^{s+2}.$$

We claim: $D_s(\mathcal{P}_n) \subset \operatorname{Reg}(X_{s,n})$.

Pick $(x,t) \in D_s(\mathcal{P}_n)$ and consider the Jacobian matrix

$$J_{s,n}(x,t) := \begin{pmatrix} 2t_1 & 0 & \cdots & 0 & \frac{\partial h_{J_1}}{\partial \mathtt{x}_1}(x) & \frac{\partial h_{J_1}}{\partial \mathtt{x}_2}(x) \\ 0 & 2t_2 & \cdots & 0 & \frac{\partial h_{J_2}}{\partial \mathtt{x}_1}(x) & \frac{\partial h_{J_2}}{\partial \mathtt{x}_2}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2t_s & \frac{\partial h_{J_s}}{\partial \mathtt{x}_1}(x) & \frac{\partial h_{J_s}}{\partial \mathtt{x}_2}(x) \end{pmatrix}$$

has rank $\le s$. As $(x,t) \in D_s(\mathcal{P}_n)$, then $x \in \mathcal{P}_n$ and there exists at most two indices k, ℓ such that $x \in \{h_{J_k} = 0, h_{J_\ell} = 0\}$. If such is the case, the vectors $(\frac{\partial h_{J_k}}{\partial \mathtt{x}_1}(x), \frac{\partial h_{J_k}}{\partial \mathtt{x}_2}(x))$ and $(\frac{\partial h_{J_\ell}}{\partial \mathtt{x}_1}(x), \frac{\partial h_{J_\ell}}{\partial \mathtt{x}_2}(x))$ are linearly independent. If there exists only one index k such that $x \in \{h_{J_k} = 0\}$, the vector $(\frac{\partial h_{J_k}}{\partial \mathtt{x}_1}(x), \frac{\partial h_{J_k}}{\partial \mathtt{x}_2}(x))$ is non-zero because the partition $\mathcal{J}$ satisfies (5.1.1). As $t_k^2 - h_{J_k}(x) = 0$ for $k = 1, \dots, s$, the matrix $J_{s,n}(x,t)$ has rank s, so $(x,t) \in \operatorname{Reg}(X_{s,n})$, as claimed.

As $D_s(\mathcal{P}_n)$ is obtained recursively by doubling orientable Nash manifolds with boundary, $D_s(\mathcal{P}_n)$ is an orientable Nash surface. By the (smooth) classification of surfaces (see for instance [H, Ch.9]) $D_s(\mathcal{P}_n)$ is diffeomorphic to a connected sum of g tori and the genus g completely characterize its diffeomorphism class.

For each $\varepsilon := (\varepsilon_1, \dots, \varepsilon_s) \in \{-1, 1\}^s$ the set

$$D_s(\mathcal{P}_n) \cap \{\varepsilon_1 t_1 \ge 0, \dots, \varepsilon_s t_s \ge 0\}$$

is Nash diffeomorphic to $\mathcal{P}_n$. Thus, $D_s(\mathcal{P}_n)$ is obtained (topologically) by glueing 2^s copies of $\mathcal{P}_n$, one for each choice of $\varepsilon \in \{-1, 1\}^s$. The polygon $\mathcal{P}_n$ has a (natural) structure of CW complex with n vertices, n edges and 1 face. This CW complex structure induces a CW complex structure on $D_s(\mathcal{P}_n)$ with $2^{s-2}n$ vertices (because each vertex belongs exactly to 4 polygons of the CW complex), $2^{s-1}n$ edges (because each edge belongs exactly to 2 polygons of the CW complex) and 2^s faces. We deduce that $D_s(\mathcal{P}_n)$ is diffeomorphic to the connected sum of $2^{s-3}(n-4)+1$ tori, whenever the number $2^{s-3}(n-4)+1$ is a non-negative integer. In particular, the compact orientable surface of genus $g \ge 1$ is diffeomorphic to the Nash surfaces $D_2(\mathcal{P}_{2g+2}) \subset \mathbb{R}^4$ and $D_3(\mathcal{P}_{g+3}) \subset \mathbb{R}^5$. Note that the genus g of the surface $D_s(\mathcal{P}_n)$ depends both on n and s (see Table 5.1).

	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
$n=3$	–	0	–	–	–	–
$n=4$	1	1	1	–	–	–
$n=5$	–	2	3	5	–	–
$n=6$	2	3	5	9	17	–
$n=7$	–	4	7	13	25	49

Table 5.1: The genus of the Nash surface $D_s(\mathcal{P}_n)$ for $n \leq 7$.

5.2 Approximation for manifolds with corners

Approximation of classes of functions by sub-classes of nicer functions is an important tool in many areas of mathematics. In particular, in geometry the possibility of approximating a certain class of functions by a dense (with respect to suitable topologies) sub-class with a better behaviour, allows (often) a deeper and better understanding of many situations.

A celebrated example, with uncountable applications, is the Whitney's approximation theorem [W] for continuous maps whose target space is a $\mathcal{C}^r$ submanifold M of $\mathbb{R}^n$, for $r \in \mathbb{N} \cup \{\infty\}$. Whitney approximation theorem has been extended in many directions, like the case of manifolds with boundary (using partitions of unit and collars). Recently Fernando and Ghiloni [FGh2] proved new results of approximation in Whitney's style when the target space has singularities, under the hypothesis that it admits some 'nice' triangulations. For example, as an application of their (much more general) results, it is possible to prove that every continuous map between a locally compact set $X \subset \mathbb{R}^m$ and a smooth manifold with corners (not necessarily divisorial corners) $\mathcal{Q} \subset \mathbb{R}^n$, can be approximated by a smooth map.

There are several and relevant results on approximation also in the semi-algebraic setting. Efoymson [Ef] showed that every continuous semi-algebraic function can be approximated by a Nash function on Nash manifolds. Shiota improved this result (see [Sh]) in many directions. He proved relative versions and results with (a strong) control on the derivatives of the approximation. Recently approximation techniques have been developed in the case where the target space has singularities. In [BFR] Baro, Fernando and Ruiz obtained results when the target space is a Nash set with monomial singularities (under some regularity assumptions on the involved maps). In another direction, Fernando and Ghiloni [FGh] proved results on differentiable approximation of continuous semi-algebraic maps, when the target space admits 'nice' triangulations. The techniques developed by Fernando and Ghiloni, make an essential use of partitions of unity, thus their results do not extend to Nash approximation.

One of the main tools in approximation is the existence of (suitable) tubular neighbourhood (together with the corresponding retractions). When the target space has singularities, we cannot take advantage of tubular neighbourhoods and retractions, as it is shown in the following example.

Example 5.2.1 ([FGh, 1.10]). There exists no $\mathcal{C}^1$ retractions from a neighbourhood U of $X := \{\mathtt{xy} = 0\} \subset \mathbb{R}^2$ onto X. Suppose that $\rho : U \to X$ is a $\mathcal{C}^1$ retraction. As ρ is the identity on X, we have that $d_0\rho = \mathrm{id}_{\mathbb{R}^2}$. Thus, by the implicit function theorem, ρ is a local $\mathcal{C}^1$ diffeomorphism at the origin, which is a contradiction.

5.2.1. Nash approximation. When one wants to approximate continuous semi-algebraic functions by Nash functions often technical difficulties arise. The rigidity of the Nash class and their algebraic nature prevents us from using the standard tools in approximation theory, such as partitions of unity, integration of vector fields etc. When the target space has singularities, the lack of the existence of retractions complicates even more the situation. Consequently, when we deal with Nash functions we have (often) to develop *ad hoc* techniques and constructions.

Using Theorem 5.1.13 we are able to prove that Nash approximation is possible for proper semi-algebraic continuous maps when the target space is a Nash manifold with corners.

Theorem 5.2.2 (Nash approximation)**.** *Let $\mathcal{S} \subset \mathbb{R}^m$ be a locally compact semi-algebraic set and let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners. Let $h : \mathcal{S} \to \mathcal{Q}$ be a proper continuous semi-algebraic map. Then there exist Nash maps $g : \mathcal{S} \to \mathcal{Q}$ arbitrarily close to h with respect to the $\mathcal{C}^0$ semi-algebraic topology.*

Proof. By [DK] there exist an open semi-algebraic neighbourhood $U \subset \mathbb{R}^m$ of $\mathcal{S}$ (in which $\mathcal{S}$ is closed) and a semi-algebraic retraction $\nu : U \to \mathcal{S}$. Consider the continuous semi-algebraic function $H := h \circ \nu : U \to \mathcal{Q}$. Let $M \subset \mathbb{R}^n$ be a d-dimensional Nash manifold that contains $\mathcal{Q}$ as a closed subset. By Theorem 5.1.13 there exists (after shrinking M if necessary) a Nash map $f : M \to \mathcal{Q}$ such that $f|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is a semi-algebraic homeomorphism close to the identity map and $f|_{\operatorname{Int}(\mathcal{Q})} : \operatorname{Int}(\mathcal{Q}) \to \operatorname{Int}(\mathcal{Q})$ is a Nash diffeomorphism. Let $\varepsilon_0 : \mathcal{S} \to \mathbb{R}$ be a strictly positive continuous semi-algebraic function and let $\varepsilon : U \to \mathbb{R}$ be a strictly positive continuous semi-algebraic extension of ε_0 to U. As the map $f_* : \mathcal{S}^0(U, M) \to \mathcal{S}^0(U, \mathcal{Q}),\ H \mapsto f \circ H$ is continuous [Sh, II.1.5], there exists $\delta : U \to \mathbb{R}$ such that if $G \in \mathcal{S}(U, M)$ and $\|G - H\| < \delta$, then $\|f \circ G - f \circ H\| < \frac{\varepsilon}{2}$. As the map $h^* : \mathcal{S}^0(\mathcal{Q}, \mathcal{Q}) \to \mathcal{S}^0(\mathcal{S}, \mathcal{Q}),\ \zeta \mapsto \zeta \circ h$ is continuous by [Sh, II.1.5], because h is proper and f is close to $\mathrm{id}_{\mathcal{Q}}$, we may assume that

$$\|h - f \circ h\| = \|\operatorname{id}_{\mathcal{Q}} \circ h - f \circ h\| < \frac{\varepsilon|_{\mathcal{S}}}{2}.$$

Let $G : U \to M$ be a Nash map such that $\|G - H\| < \delta$, so $\|f \circ G - f \circ H\| < \frac{\varepsilon}{2}$. As $H|_{\mathcal{S}} = h$, we deduce

$$\|f \circ G|_{\mathcal{S}} - h\| \leq \|f \circ G|_{\mathcal{S}} - f \circ H|_{\mathcal{S}}\| + \|f \circ h - h\| < \frac{\varepsilon|_{\mathcal{S}}}{2} + \frac{\varepsilon|_{\mathcal{S}}}{2} = \varepsilon|_{\mathcal{S}} = \varepsilon_0,$$

as required. □

When approximation results are achieved a natural question is whether it is possible to approximate also homotopies.

Question 5.2.2. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semi-algebraic set and let $\mathcal{Q} \subset \mathbb{R}^n$ be a Nash manifold with corners. Given two Nash maps $f, g : \mathcal{S} \to \mathcal{Q}$ that are homotopic through a continuous semi-algebraic homotopy $F : \mathcal{S} \times [0, 1] \to \mathcal{Q}$, are they Nash homotopic?

The previous question is open. The difficulty lies in the fact that, in order to approximate the continuous semi-algebraic homotopy $F : \mathcal{S} \times [0, 1] \to \mathcal{Q}$ by a Nash one, we need have to approximate relatively to the semi-algebraic set $\mathcal{S} \times \{0,1\}$. So far, the techniques we developed seem not to be extendible to relative approximation.

5.2.3. Smooth approximation. In [FGh2, Cor.1.10] Fernando and Ghiloni showed that, if the target space is a smooth manifold with corners (non necessarily divisorial corners), continuous maps can be approximated by smooth maps. The construction we made in this chapter and in particular Theorem 5.1.13, holds also in the smooth case if $\mathcal{Q}$ is a smooth manifold with divisorial corners. So, at the cost of (much) more restrictive hypothesis, we obtain an alternative and more direct proof of this result.

Unlike the Nash case, when working with the smooth category, we can take advantage of many and useful tools proper of this category. That allows us to go further in the smooth case and prove an approximation result for homotopies, that is, we answer positively to Question 5.2.2 in the smooth setting.

Let $\mathcal{Q} \subset \mathbb{R}^n$ be a smooth d-dimensional manifold with corners (not necessarily divisorial corners). For each $x \in \partial\mathcal{Q}$ consider a smooth chart

$$\phi_x : U_x \to \mathbb{R}^{n,k} := \{\mathtt{x}_1 \geq 0, \ldots, \mathtt{x}_k \geq 0\} \subset \mathbb{R}^n,$$

with $\phi_x(x) = 0$. On $\mathbb{R}^{n,k}$ we can consider the pointing 'inside' vector field

$$V_k := \frac{\partial}{\partial \mathtt{x}_1} + \ldots + \frac{\partial}{\partial \mathtt{x}_k},$$

and its pullback $V_x := \phi_x^*(V_k)$ on U_x, that is a vector field on U_x pointing 'inside U_x'. Let $\{U_i\}_{i\in I}$ be a locally finite refinement in $\mathcal{Q}$ of the open covering $\{U_x\}_{x\in\partial\mathcal{Q}} \cup \{\mathcal{Q} \setminus \partial\mathcal{Q}\}$ (recall that $\mathcal{Q}$ is paracompact). For each $i \in I$ satisfying $U_i \not\subset \mathcal{Q} \setminus \partial\mathcal{Q}$ let $x_i \in \partial\mathcal{Q}$ be such that $U_i \subset U_{x_i}$ and define

$$V_i := \begin{cases} 0, & \text{if } U_i \subset \mathcal{Q} \setminus \partial\mathcal{Q}, \\ V_{x_i}|_{U_i}, & \text{otherwise.} \end{cases}$$

Let $\{\rho_i\}_{i\in I}$ be a smooth partition of unity subordinated to the open covering $\{U_i\}_{i\in I}$. We can glue together the local vector fields V_i obtaining a global vector field

$$V := \sum_{i\in I} \rho_i V_i,$$

pointing 'inside $\mathcal{Q}$'.

We want to integrate pointing 'inside' vector fields on manifolds with corners. We start with compact sets: *Given any compact set $K \subset \mathcal{Q}$ and a pointing 'inside' vector field V, there exists an $\varepsilon > 0$ and a (well-known) smooth map*

$$\exp : K \times [0,\varepsilon) \to \mathcal{Q},\ (x,t) \mapsto \exp(tV)(x),$$

called exponential map, such that $\exp(0)(x) = x$ for each $x \in K$ and the curve $t \mapsto \exp(tV)(x)$ is the unique integral curve of V through x for $t = 0$.

The uniqueness of integral curves follows from the uniqueness theorem for ordinary differential equations (see [Le, Thm.17.9]) using standard arguments (see for instance the proof of [Le, Thm.17.8]). Fix a point $y \in K$. Let U be an open neighbourhood of y in K and $\phi : U \to \mathbb{R}^{n,k}$ a local chart such that $\phi(y) = 0$. In order to prove the local existence of the exponential map for the vector field V in $\mathcal{Q}$, by the uniqueness of integral curves it is sufficient to show the local existence of the exponential map for the vector field $(\phi^{-1})^*(V)$ in $\mathbb{R}^{n,k}$ (because the uniqueness of integral curves implies the invariance by changing of coordinates). As $\mathbb{R}^{n,k}$ is closed in $\mathbb{R}^n$, using smooth partition of unity we can extend the vector field $(\phi^{-1})^*(V)$ to a vector field defined on an open neighbourhood $W \subset \mathbb{R}^n$ of $\mathbb{R}^{n,k}$. By the existence and smoothness theorem for ordinary differential equations (see [Le, Thm.17.9]) there exists $\epsilon > 0$ such that the map

$$\exp' : W \times (-\epsilon, \epsilon) \to \mathbb{R}^n,\ (x,t) \mapsto \exp(t(\phi^{-1})^*V)(x),$$

is smooth and well-defined. As V is a vector field pointing 'inside $\mathcal{Q}$', the vector field $(\phi^{-1})^*V$ points 'inside $\mathbb{R}^{n,k}$', so the restriction $\exp'|_{\mathbb{R}^{n,k}\times[0,\epsilon)}$ takes values in $\mathbb{R}^{n,k}$. In particular,

$$\exp|_{U\times[0,\epsilon)} : U \times [0,\epsilon) \to \mathcal{Q},$$

because $\exp(tV)(x) = \phi^{-1}(\exp(t(\phi^{-1})^*(V))(\phi(x))$ for each $(x,t) \in U \times [0,\epsilon)$. As K is compact, there exists $\varepsilon > 0$ that verifies the required conditions (see also [Me, Cor.1.13.1]).

Let now $\{K_j\}_{j\in J}$ be a locally finite covering of $\mathcal{Q}$ made of compact sets, such that $\{\operatorname{Int} K_j\}_{j\in J}$ is still a locally finite covering of $\mathcal{Q}$. Let $\varepsilon_j > 0$ be a positive number such that the exponential map is defined on $K_j \times [0,\varepsilon_j)$ and $\exp(K_j \times (0,\varepsilon_j)) \subset \operatorname{Int}(\mathcal{Q})$. Define the function

$$\varepsilon : \mathcal{Q} \to \mathbb{R},\ x \mapsto \inf\{\varepsilon_i\ :\ x \in K_i\}.$$

As the covering $\{K_j\}_{j\in J}$ is locally finite, this function is strictly positive and takes only finitely many values on a (small) neighbourhood of each point. Thus, using a smooth partition of unity, there exists a smooth function $\epsilon : \mathcal{Q} \to \mathbb{R}$ such that $0 < \epsilon < \varepsilon$. Consider the smooth map

$$H : \mathcal{Q} \times [0,1] \to \mathcal{Q},\ (x,t) \mapsto H(x,t) := \exp(\epsilon(x)tV)(x). \tag{5.2.1}$$

For each $t \in [0,1]$ the map $H(-,t)$ is a diffeomorphism onto its image (see [Le, Lem.17.2, Thm.17.8]) and $H(\mathcal{Q} \times (0,1]) \subset \operatorname{Int} \mathcal{Q}$.

Let $X \subset \mathbb{R}^m$ be a locally compact set. We want to show: *Every continuous map $f \in \mathcal{C}^0(X,\mathcal{Q})$ is homotopic to a smooth map $g \in \mathcal{C}^\infty(X,\mathcal{Q})$.* We need the following well-known result whose proof follows straightforwardly from [H, Ex.10, pp.64-65] using standard arguments.

Lemma 5.2.3. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be locally compact sets and let $f : Y \to Y$ be a continuous map. Then the map*

$$f_* : \mathcal{C}^0(X,Y) \to \mathcal{C}^0(X,Y),\ g \mapsto f \circ g,$$

is continuous with respect to the compact-open topology.

Proposition 5.2.4. *Let $X \subset \mathbb{R}^m$ be a locally compact set, $\mathcal{Q} \subset \mathbb{R}^n$ be a d-dimensional smooth manifold with corners (not necessarily divisorial corners) and $f : X \to \mathcal{Q}$ a continuous map. Then f is homotopic to a smooth map $g : X \to \mathcal{Q}$.*

Proof. By [FGh2, Cor.1.10] the space $\mathcal{C}^0(X,\mathcal{Q})$ is dense in $\mathcal{C}^\infty(X,\mathcal{Q})$ with respect to the compact-open topology. So in order to conclude it is sufficient to prove that: *If $f, g \in \mathcal{C}^0(X,\mathcal{Q})$ are close enough, with respect to the compact-open topology, then they are homotopic.*

It is well known (see for instance [H, Thm.5.1]), that there exists a strictly positive continuous function $\varepsilon : \operatorname{Int}\mathcal{Q} \to \mathbb{R}_{>0}$, such that, if we consider the open neighbourhood in $\mathbb{R}^n$ of $\operatorname{Int}\mathcal{Q}$, defined as

$$(\operatorname{Int}\mathcal{Q})_\varepsilon := \{y \in \mathbb{R}^n \,:\, \|x-y\| < \varepsilon(x) \text{ for some } x \in \operatorname{Int}\mathcal{Q}\},$$

then

(i) each $y \in (\operatorname{Int}\mathcal{Q})_\varepsilon$ admits a unique closest point $\pi(y) \in \operatorname{Int}\mathcal{Q}$, namely, $d(y, \operatorname{Int}\mathcal{Q}) = \|y - \pi(y)\|$;

(ii) the map $\pi : (\operatorname{Int}\mathcal{Q})_\varepsilon \to \operatorname{Int}\mathcal{Q}$ is smooth.

Consider now the map H defined in (5.2.1) and $H_1 : \mathcal{Q} \to \operatorname{Int}(\mathcal{Q}),\ x \mapsto H(x,1)$. By Lemma 5.2.3 the map $(H_1)_* : \mathcal{C}^0(X,\mathcal{Q}) \to \mathcal{C}^0(X,\mathcal{Q}),\ f \mapsto H_1 \circ f$ is continuous, so the maps $H_1 \circ f, H_1 \circ g : X \to \operatorname{Int}\mathcal{Q}$ can be taken as close as needed, because we can take f close enough to g. Thus, we may assume

$$\|H_1(f(x)) - H_1(g(x))\| < \varepsilon(H_1(g(x))),$$

for all $x \in \operatorname{Int}\mathcal{Q}$. Then for each $t \in [0,1]$ we have

$$(1-t)H_1(f(x)) + tH_1(g(x)) \subset (\operatorname{Int}\mathcal{Q})_\varepsilon.$$

Thus the map, $F : X \times [0,1] \to \operatorname{Int}\mathcal{Q}$ given by

$$F(x,t) := \pi((1-t)H_1(f(x)) + tH_1(g(x))),$$

is a well defined homotopy between $H_1 \circ f$ and $H_1 \circ g$. The map $G : X \times [0,1] \to \mathcal{Q}$ defined as

$$G(x,t) := \begin{cases} H(f(x), 3t), & \text{if } 0 \le t \le \frac{1}{3}, \\ F(x, 3t-1), & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ H(g(x), 3-3t), & \text{if } \frac{2}{3} \le t \le 1, \end{cases}$$

is the desired homotopy between f and g. □

We show next that, in the smooth setting homotopies can be approximated by smooth homotopies if the target space is a smooth manifold with corners.

Theorem 5.2.5 (Differential homotopy)**.** *Let $X \subset \mathbb{R}^m$ be a locally compact set and let $\mathcal{Q} \subset \mathbb{R}^n$ be a d-dimensional smooth manifold with corners (not necessarily divisorial corners). Let $f_1, f_2 : X \to \mathcal{Q}$ be two homotopic smooth maps. Then f_1, f_2 are homotopic trough a smooth homotopy.*

Proof. Let $H_1 := H(-,1)$, where $H : \mathcal{Q} \times [0,1] \to \mathcal{Q}$ is the map defined in (5.2.1) and let $F : X \times [0,1] \to \mathcal{Q}$ be a homotopy between f_1 and f_2. Then the map $H_1 \circ F : X \times [0,1] \to \operatorname{Int}\mathcal{Q}$ is a homotopy between $H_1 \circ f_1$ and $H_1 \circ f_2$. As $\operatorname{Int}\mathcal{Q}$ is a smooth manifold, there exists a smooth homotopy $G : X \times [0,1] \to \operatorname{Int}\mathcal{Q}$

between $H_1 \circ f_1$ and $H_1 \circ f_2$. Let $\mu : [0,1] \to [0,1]$ be a smooth function that is identically 0 in a neighbourhood of 0 and identically 1 on a neighbourhood of 1. The smooth map $\widehat{F} : X \times [0,1] \to \mathcal{Q}$ defined as

$$\widehat{F}(x,t) := \begin{cases} H(f_1(x), \mu(3t)), & \text{if } 0 \le t \le \frac{1}{3}, \\ G(x, \mu(3t-1)), & \text{if } \frac{1}{3} < t \le \frac{2}{3}, \\ H(f_2(x), 1-\mu(3t-2)), & \text{if } \frac{2}{3} < t \le 1, \end{cases}$$

is a smooth homotopy between f_1 and f_2, as required. □

5.3 An alternative construction of Nash images of closed balls

The purpose of this section is to prove an alternative version to Theorem 4.1. We will show how to 'resolve' a semi-algebraic set connected by analytic paths by a Nash manifold with boundary. At the end of the section we will provide an alternative construction of Nash images of the closed ball, as a consequence of this resolution of semi-algebraic sets.

5.3.1. Resolution by Nash manifolds with boundary. A similar result to Theorem 4.1 changing the Nash manifold with corners $\mathcal{Q}$ by Nash manifolds with boundary seems difficult to be achieved if we want to keep the map f is polynomial and that the semi-algebraic set $\mathcal{R}$ has dimension strictly smaller than the dimension of $\mathcal{S}$. We propose the following statement:

Theorem 5.3.1. *Let $\mathcal{S} \subset \mathbb{R}^n$ be a d-dimensional closed semi-algebraic set connected by analytic paths and let $\varepsilon > 0$. Then there exist:*

(i) *A d-dimensional non-singular algebraic set $X \subset \mathbb{R}^m$.*

(ii) *A Nash manifold with boundary $\mathcal{H}_\varepsilon \subset \mathbb{R}^m$ such that the Zariski closure Z_ε of $\partial\mathcal{H}_\varepsilon$ is a non-singular algebraic set $Z_\varepsilon \subset X$ of dimension $d-1$ and* $\operatorname{Int}(\mathcal{H}_\varepsilon)$ *is a connected component of $X \setminus Z_\varepsilon$.*

(iii) *A proper Nash map $f : \mathcal{H}_\varepsilon \to \mathcal{S}$ such that $f(\mathcal{H}_\varepsilon) = \mathcal{S}$.*

(iv) *The restriction $f|_{\mathcal{H}_\varepsilon \setminus f^{-1}(\mathcal{T}_\varepsilon)} : \mathcal{H}_\varepsilon \setminus f^{-1}(\mathcal{T}_\varepsilon) \to \mathcal{S} \setminus \mathcal{T}_\varepsilon$ is a Nash diffeomorphism, where $\mathcal{T}_\varepsilon := \{x \in \mathcal{S} : \operatorname{dist}(x, \mathcal{R}) \le \varepsilon\}$ for a certain closed semi-algebraic set $\mathcal{R} \subset \mathcal{S}$ of dimension strictly smaller than d.*

Proof. By Theorem 4.1 we may assume that $\mathcal{S} = \mathcal{Q} \subset \mathbb{R}^n$ is a Nash manifold with corners. Recall that $X := \overline{\mathcal{Q}}^{\text{zar}} \subset \mathbb{R}^n$ is a non-singular algebraic set. Using the stereographic projection and Hironaka's desingularization (Theorem 2.4.2), we may assume in addition that X is compact (see also [Sh, I.5.11]). Fix $\varepsilon > 0$ and let $M \subset X$ be the set of points $x \in X$ such that $\operatorname{dist}(x, \mathcal{Q}) < \varepsilon$, which is a Nash manifold. By [FGR, Thm.1.11, 1.12] we may assume, up to eventually take a smaller ε, in addition:

- The Nash closure Y of $\partial\mathcal{Q}$ in M is a Nash normal-crossings divisor of M and $\mathcal{Q} \cap Y = \partial\mathcal{Q}$.

- For every $x \in \partial\mathcal{Q}$ the analytic closure of the germ $\partial\mathcal{Q}_x$ is Y_x.

By [Sh, VI.2.1] and its proof there exists a $\mathcal{C}^1$ function $g : M \to \mathbb{R}$ such that M is diffeomorphic to $g^{-1}((0,+\infty))$ and $H := g^{-1}([0,+\infty))$ is a compact manifold with boundary that contains $\mathcal{Q}$ in its interior. We may assume in addition that the differential $d_xg : T_xM \to \mathbb{R}$ is surjective for each $x \in g^{-1}(0)$ (see Lemma 5.1.4). Let $G : X \to \mathbb{R}$ be a $\mathcal{C}^1$ extension of g to X such that $G^{-1}(0) = g^{-1}(0)$. Let $F : X \to \mathbb{R}$ be a polynomial approximation of G in the $\mathcal{C}^1$ compact-open topology. As G is strictly positive on the compact set $\mathcal{Q}$, we may assume F is strictly positive on $\mathcal{Q}$, so $F^{-1}(0) \subset M \setminus \mathcal{Q}$. In addition, as we consider a $\mathcal{C}^1$ approximation, we may assume $d_xF : T_xX \to \mathbb{R}$ is surjective for each $x \in F^{-1}(0)$. Thus, $\mathcal{H}_\varepsilon := F^{-1}([0,+\infty)) \subset M$ is a compact Nash manifold with (non-singular) boundary $Z_\varepsilon := F^{-1}(0)$, which is a non-singular algebraic subset of X of dimension $d-1$.

By Theorem 5.1.13 there exists a Nash map $h : M \to \mathcal{Q}$ such that $h(M) = h(\mathcal{Q}) = \mathcal{Q}$ and $h|_{\mathcal{H}_\varepsilon \setminus h^{-1}(\mathcal{T}_\varepsilon)} : \mathcal{H}_\varepsilon \setminus h^{-1}(\mathcal{T}_\varepsilon) \to \mathcal{Q} \setminus \mathcal{T}_\varepsilon$ is a Nash diffeomorphism, where $\mathcal{T}_\varepsilon := \{x \in \mathcal{Q} :\ \operatorname{dist}(x, \partial\mathcal{Q}) < \varepsilon\}$. □

5.3.2. An alternative proof of Theorem 3.2 In order to give an alternative proof of Theorem 3.2 based on Theorem 5.3.1 we need some preliminary results. We start with the following Lemma that extends [Fe4, Lem.2.8] to Nash manifolds with non-empty boundary.

Lemma 5.3.2. *Let $H_1 \subset \mathbb{R}^m$ and $H_2 \subset \mathbb{R}^n$ be Nash manifolds with (non-singular) boundary. Let $f : H_1 \to H_2$ be a semi-algebraic homeomorphism. Then every continuous semi-algebraic map $g : H_1 \to H_2$ close to f, such that $g(\partial H_1) \subset \partial H_2$, is surjective.*

Proof. As $f : H_1 \to H_2$ is a semi-algebraic homeomorphism, $f(\partial H_1) = \partial H_2$ (use invariance of domain). Consider the Nash doubles $(D(H_i), \pi_i)$ and

$$H_{i,\epsilon} = D(H_i) \cap \{\epsilon t \geq 0\},$$

where $\epsilon = \pm$. Recall that $\pi_{i\epsilon} : H_{i,\epsilon} \to H_i$ is a semi-algebraic homeomorphism. As $f(\partial H_1) = \partial H_2$, the map

$$F : D(H_1) \to D(H_2),\ (x,t) \mapsto (\pi_{2\epsilon})^{-1} \circ f \circ \pi_{1\epsilon}(x,t)$$

is well-defined and semi-algebraic for $\epsilon = \pm$. Observe that F is bijective and

$$F^{-1} : D(H_2) \to D(H_1),\ (y,s) \mapsto (\pi_{1\epsilon})^{-1} \circ f^{-1} \circ \pi_{2\epsilon}(y,s).$$

Let us check: *F is continuous.* Once this is done, the same proof shows that F^{-1} is continuous, so f is a semi-algebraic homeomorphism.

As F is continuous on both $H_{1,+}$, $H_{1,-}$ and $\pi_1|_{H_{1+} \cap H_{1-}} = \pi_1|_{\partial H_1} = \mathrm{id}_{\partial H_1}$, we conclude by the pasting lemma that F is continuous on $D(H_1)$.

Let $g : H_1 \to H_2$ be a continuous semi-algebraic map. If $g(\partial H_1) \subset \partial H_2$, the map

$$G : D(H_1) \to D(H_2),\ (x,t) \mapsto (\pi_{2\epsilon})^{-1} \circ g \circ \pi_{1\epsilon}(x,t)$$

is well-defined, continuous and semi-algebraic for $\epsilon = \pm$. The proof is analogous to the one for F.

By [Fe4, Lem.2.8] there exists a strictly positive continuous semi-algebraic function $\varepsilon : D(H_1) \to \mathbb{R}$ such that if $\|G - F\| < \varepsilon$, then G is surjective.

For $\epsilon = \pm$, consider the strictly positive continuous semi-algebraic functions

$$\varepsilon_\epsilon := \varepsilon \circ \pi_{1\epsilon}^{-1} : H_1 \to \mathbb{R}$$

and $\varepsilon^* = \min\{\varepsilon_+, \varepsilon_-\}$. Let $\delta : H_1 \to \mathbb{R}$ be a strictly positive continuous semi-algebraic function such that if $f, g : H_1 \to H_2$ are continuous semi-algebraic maps such that $\|g - f\| < \delta$, then $\|(\pi_{2\epsilon})^{-1} \circ f - (\pi_{2\epsilon})^{-1} \circ g\| < \varepsilon^*$ for $\epsilon = \pm$ (see [Sh, II.1.5]). Thus, $\|(\pi_{2\epsilon})^{-1} \circ f \circ \pi_{1\epsilon} - (\pi_{2\epsilon})^{-1} \circ g \circ \pi_{1\epsilon}\| < \varepsilon^* \circ \pi_{1\epsilon} \leq \varepsilon$ for $\epsilon = \pm$. Consequently, $\|F - G\| < \varepsilon$, so G is surjective. Following the definition of G, we conclude that g is also surjective, as required. □

Let us prove that: *If $\mathcal{S} \subset \mathbb{R}^n$ is a compact semi-algebraic set connected by analytic paths of dimension $d \geq 2$, then there exists a Nash map $f : \mathbb{R}^d \to \mathbb{R}^n$ such that $f(\overline{\mathcal{B}}_d) = \mathcal{S}$.*

By Theorem 5.3.1 we may assume that $H := \mathcal{S}$ is a connected compact Nash manifold with smooth boundary. Let $M := D(H)$ be the Nash double of H and consider the surjective Nash map $\pi : D(H) \to H$ introduced in Proposition 5.1.8. Observe that M is a connected compact Nash manifold. Thus, we may assume that $M := \mathcal{S}$ is a connected compact Nash manifold. Let $\{U_i\}_{i=1}^r$ be a finite covering of M equipped with Nash diffeomorphisms $u_i : U_i \to \mathbb{R}^d$ such that $M = \bigcup_{i=1}^m u_i^{-1}(\Delta_d)$ where $\Delta_d := \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_d \geq 0,\ \mathtt{x}_1 + \cdots + \mathtt{x}_d \leq 1\}$. Define

$$\Delta'_d := \{\mathtt{x}_1 \geq -2, \dots, \mathtt{x}_d \geq -2,\ \mathtt{x}_1 + \cdots + \mathtt{x}_d \leq d + 1\}$$

and observe that $\Delta_d \subset \overline{\mathcal{B}}_d \subset \Delta'_d$. Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be an affine isomorphism such that $\psi(\Delta_d) = \Delta'_d$.

Lemma 5.3.3. *Let $\sigma \subset \mathbb{R}^{d-1}$ be the $(d-1)$-simplex*

$$\sigma := \{\mathtt{x}_1 \geq 0, \dots, \mathtt{x}_{d-1} \geq 0,\ \mathtt{x}_1 + \cdots + \mathtt{x}_{d-1} \leq 1\}.$$

Then the surjective Nash map $f : \sigma \times [0,1] \to \Delta'_d,\ (x,t) \mapsto \psi(((1-t)x, t))$ restricts to a Nash diffeomorphism $f|_{\sigma\times[0,1)} : \sigma \times [0,1) \to \Delta'_d \setminus \{\psi((0,\dots,0,1))\}$.

Proof. As ψ is an affine isomorphism such that $\psi(\Delta_d) = \Delta'_d$, it is enough to prove that the surjective Nash map $\sigma \times [0,1] \to \Delta_d$, $(x,t) \mapsto ((1-t)x, t)$ restricts to a Nash diffeomorphism

$$\varphi : \sigma \times [0,1) \to \Delta_d \setminus \{(0,\dots,0,1)\}.$$

This is straightforward, because the Nash map

$$\phi : \Delta_d \setminus \{(0,\dots,0,1)\} \to \sigma \times [0,1),\ (x_1,\dots,x_d) \mapsto \left(\frac{x_1}{1-x_d}, \dots, \frac{x_{d-1}}{1-x_d}, x_d\right)$$

is the inverse of φ. □

We are now ready to give an alternative proof of Theorem 3.2.

Proof of Theorem 3.2. Let us construct a surjective continuous semi-algebraic function

$$h : \sigma \times [0, 2r-1] \to M = \bigcup_{i=1}^{r} u_i^{-1}(\Delta_d) = \bigcup_{i=1}^{r} u_i^{-1}(\overline{\mathcal{B}}_d) = \bigcup_{i=1}^{r} u_i^{-1}(\Delta'_d)$$

that is Nash on $\sigma \times (\bigcup_{i=1}^{r}(2(i-1), 2(i-1)+1)$ and satisfies

$$u_i^{-1}(\overline{\mathcal{B}}_d) \subset h(\Delta_d \times (2(i-1), 2(i-1)+1)),$$

for each $i = 1, \dots, r$.

Let $f : \sigma \times [0, 1] \to \Delta'_d$ be the surjective Nash map introduced in Lemma 5.3.3 and define

$$h_i : \sigma \times [2(i-1), 2(i-1)+1] \to u_i^{-1}(\Delta'_d),\ (x,t) \mapsto u_i^{-1}(f(x, t-2(i-1))),$$

which is a surjective Nash map whose restriction to $\sigma \times \{2(i-1)+1\}$ is constant and it is restriction to $\sigma \times [2(i-1), 2(i-1)+1)$ is a Nash diffeomorphism onto its image. Write $\{p_i\} := h_i(\sigma \times \{2(i-1)+1\})$. Let $b_1, \dots, b_d$ be the vertices of Δ'_d different from $\psi(0, \dots, 0, 1)$. Define $x_d := 1 - x_1 - \cdots - x_{d-1}$ and consider the continuous semi-algebraic map

$$g'_i : \sigma \times \left[\tfrac{1}{2} + 2(i-1) + 1, 2(i-1) + 2\right] \to \mathbb{R}^d,$$
$$(x,t) \mapsto u_{i+1}^{-1}\Big(\sum_{i=1}^{d} x_i b_i 2\left(t - \tfrac{1}{2} - 2(i-1) - 1\right)\Big).$$

Let $\alpha_i : [2(i-1)+1, \frac{1}{2}+2(i-1)+1] \to M$ be a continuous semi-algebraic map such that $\alpha_i(2(i-1)+1) = p_i$ and $\alpha_i(\frac{1}{2}+2(i-1)+1) = u_{i+1}^{-1}(0)$. Consider the continuous semi-algebraic map

$$g_i : \sigma \times [2(i-1)+1, \tfrac{1}{2}+2(i-1)+1] \to M,\ (x,t) \mapsto \alpha_i(t).$$

Define $h : \sigma \times [0, 2r-1] \to M$, as

$$(x,t) \mapsto \begin{cases} h_i(x,t) & \text{if } (x,t) \in \sigma \times [2(i-1), 2(i-1)+1], \\ g_i(x,t) & \text{if } (x,t) \in \sigma \times [2(i-1)+1, \frac{1}{2}+2(i-1)+1], \\ g'_i(x,t) & \text{if } (x,t) \in \sigma \times [\frac{1}{2}+2(i-1)+1, 2(i-1)+2], \\ h_r(x,t) & \text{if } (x,t) \in \sigma \times [2(r-1), 2r-1] \end{cases}$$

for $i = 1, \dots, r-1$. Observe that h is a continuous semi-algebraic map such that for each $i = 1, \dots, r$:

- the restriction $h|_{\sigma \times (2(i-1), 2(i-1)+1)}$ is a Nash diffeomorphism onto its image.
- $h(\sigma \times [2(i-1), 2(i-1)+1]) = u_i^{-1}(\Delta'_d)$.
- $u_i^{-1}(\overline{\mathcal{B}}_d) \subset h(\sigma \times (2(i-1), 2(i-1)+1))$.

We conclude that the restriction of h to $\sigma \times (\bigcup_{i=1}^{r}(2(i-1), 2(i-1)+1)$ is Nash and surjective, because $M = \bigcup_{i=1}^{m} u_i^{-1}(\overline{\mathcal{B}}_d)$. Let $H : \mathbb{R}^d \to M$ be a continuous semi-algebraic extension of h to $\mathbb{R}^d$. Define $X_1 := \bigcup_{i=1}^r (u_i \circ h_i)^{-1}(\partial\overline{\mathcal{B}}_d)$, which is a Nash subset of

$$\Omega := \operatorname{Int}(\sigma) \times \bigcup_{i=1}^{r} (2(i-1), 2(i-1)+1),$$

and observe that H is Nash on the open semi-algebraic set Ω. By [Sh, II.5.2] there exists a Nash maps $F : \mathbb{R}^d \to M$ close to H such that

$$F|_{(u_i \circ h_i)^{-1}(\partial\overline{\mathcal{B}}_d)} = H|_{(u_i \circ h_i)^{-1}(\partial\overline{\mathcal{B}}_d)}$$

for $i = 1, \ldots, r$. By Lemma 5.3.2 the restriction $F|_{(u_i \circ h_i)^{-1}(\overline{\mathcal{B}}_d)}$ is surjective for $i = 1, \ldots, r$. Thus,

$$M = \bigcup_{i=1}^{m} u_i^{-1}(\overline{\mathcal{B}}_d) \subset F(\sigma \times [0, 2r-1]) \subset M,$$

so $F(\sigma \times [0, 2r-1]) = M$, as required. □

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